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IMPROVED SELF-ADAPTIVE ITERATIVE ALGORITHMS FOR THE SPLIT EQUALITY COMMON FIXED-POINT PROBLEM OF FIRMLY QUASI-NONEXPANSIVE OPERATORS

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Abstract. In this paper, we investigate the split equality common fixed-point problem of firmly quasi-nonexpansive operators in Hilbert spaces. We introduce new iterative algorithms with a way of selecting the step-sizes such that its implementation does not need any prior information about the operator norms. The new methods are extended from the method for solving the split common fixed-point problem. The range of the new step-sizes even can be enlarged two times. Under suitable conditions, we establish a weak convergence theorem of the proposed algorithm and a strong convergence theorem of its variant by the viscosity approximation method. Numerical results are reported to show the effectiveness of the proposed algorithm.

Keywords. Split equality common fixed-point problem; Iterative algorithm; Firmly quasi-nonexpansive operators; Féjer-monotone; Strong convergence.

1. INTRODUCTION

The split equality common fixed-point problem (SECFP), which has been extensively investigated recently, is a generalization of the split feasibility problem and the split common fixed problem. It was first introduced by Moudafi [1]. Let H_1, H_2 and H_3 be real Hilbert spaces, and let $A : H_1 \to H_3$ and $B : H_2 \to H_3$ be two bounded linear operators. The SECFP can mathematically be formulated as the problem of finding

$$x \in F(U), y \in F(T) \ \text{ such that } \ Ax = By, \tag{1.1}$$

where $U : H_1 \to H_1$ and $T : H_2 \to H_2$ are two nonlinear operators. Since every nonempty closed convex subset of a Hilbert space can be regarded as a set of fixed points of a projection. If U and T are projection operators onto the nonempty closed convex subsets C and Q of real Hilbert

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space H_1 and H_2, respectively, then the SECFP (1.1) is reduced to the split equality problem (SEP) proposed by Moudafi [1]:

$$\text{finding } \ x \in C, \ y \in Q \ \text{ such that } \ Ax = By. \tag{1.2}$$

The SECFP and SEP attract much attention due to its extraordinary utility and broad applicability in many areas of applied mathematics; see, e.g., [1, 2].

To begin with, let us recall that the split feasibility problem (SFP) is to find a point

$$x \in C \ \text{ such that } \ Ax \in Q, \tag{1.3}$$

where C and Q are nonempty closed convex subset of real Hilbert spaces H_1 and H_2, respectively, and $A: \ H_1 \to H_2$ is a bounded linear operator. For the SEP (1.2), if $H_2 = H_3$ and $B = I$, then the SEP (1.2) reduces to the SFP (1.3). The SFP was originally introduced by Censor and Elfving [3] in finite-dimensional Hilbert spaces for modeling inverse problems which arise from phase retrievals and in medical image reconstruction [4]. The SFP attracts the attention of many authors due to its application in signal processing. Many algorithms have been invented to solve it (see [5, 6, 7, 8, 9, 10] and references therein).

Note that if split feasibility problem (1.3) is consistent (i.e. (1.3) has a solution), it is no hard to see that x^* solves the SFP (1.3) if and only if it solves the fixed point equation

$$P_C(I - \gamma A^*(I - P_Q)A)x^* = x^*, \tag{1.4}$$

where P_C and P_Q are the (orthogonal) projection onto C and Q, respectively, $\gamma > 0$ is any positive constant and A^* denotes the adjoint of A. This implies that we can use fixed point algorithms (see [11, 12, 13, 14, 15]) to solve the SFP.

To solve the SFP (1.3), Byrne [4] proposed the well-known CQ algorithm which generates a sequence $\{x_n\}$ by

$$x_{n+1} = P_C(I - \gamma A^*(I - P_Q)A)x_n, \ n \geq 1,$$

where $\gamma \in (0, \frac{2}{\lambda})$ with λ being the spectral radius of the operator A^*A.

For the SECFP (1.1), if $H_2 = H_3$ and $B = I$, then the SECFP (1.1) is reduced to the following split common fixed point problem (SCFP) introduced by Censor and Segal [16]:

$$\text{finding } \ x^* \in F(U) \ \text{ such that } \ Ax^* \in F(T), \tag{1.5}$$

where $A: H_1 \to H_2$ is a bounded linear operator, $U: H_1 \to H_1$ and $T: H_2 \to H_2$ are two nonexpansive operators with nonempty fixed point sets $F(U)$ and $F(T)$.

To solve the SCFP (1.5), Censor and Segal [16] proposed and proved, in finite-dimensional spaces, the convergence of the following algorithm:

$$x_{n+1} = U(x_n + \gamma A^t(T - I)Ax_n), \ \ n \geq 1, \tag{1.6}$$

where $\gamma \in (0, \frac{2}{\lambda})$ with λ being the largest eigenvalue of the matrix $A^tA(A^t$ stands for matrix transposition).

In [17], Wang proposed the following iterative algorithm for solving SCFP (1.5) of directed operators (i.e. firmly quasi-nonexpansive operators):

$$x_{n+1} = x_n - \tau_n[(x_n - Ux_n) + A^*(I - T)Ax_n], \tag{1.7}$$

where the step-size sequence τ_n is chosen as

$$\tau_n = \frac{\|x_n - Ux_n\|^2 + \|(I - T)Ax_n\|^2}{\|(x_n - Ux_n) + A^*(I - T)Ax_n\|^2}. \tag{1.8}$$

Assume that the solution set of the SEP is nonempty, by use of the optimality conditions, Moudafi [1] obtained the following fixed-point formulation: (x^*, y^*) solves the SEP if and only if

$$\begin{cases} x^* = P_C(x^* - \gamma A^*(Ax^* - By^*)), \\ y^* = P_Q(y^* + \beta B^*(Ax^* - By^*)), \end{cases}$$

where γ and $\beta > 0$. For solving the SECFP (1.1), Moudafi [1] introduced the following alternating algorithm

$$\begin{cases} x_{n+1} = U(x_n - \gamma_n A^*(Ax_n - By_n)), \\ y_{n+1} = T(y_n + \gamma_n B^*(Ax_{n+1} - By_n)) \end{cases} \tag{1.9}$$

for firmly quasi-nonexpansive operators U and T, where non-decreasing sequence $\gamma_n \in (\varepsilon, \min(\frac{1}{\lambda_A}, \frac{1}{\lambda_B}) - \varepsilon)$, λ_A and λ_B stand for the spectral radius of A^*A and B^*B, respectively.

In [18], Moudafi introduced the following simultaneous iterative method to solve the SECFP (1.1):

$$\begin{cases} x_{n+1} = U(x_n - \gamma_n A^*(Ax_n - By_n)), \\ y_{n+1} = T(y_n + \gamma_n B^*(Ax_n - By_n)) \end{cases} \tag{1.10}$$

for firmly quasi-nonexpansive operators U and T, where $\gamma_n \in (\varepsilon, \frac{2}{\lambda_A + \lambda_B} - \varepsilon)$, λ_A and λ_B stand for the spectral radius of A^*A and B^*B, respectively. This means that in order to implement the algorithms (1.9) and (1.10), one has first to compute (or, at least, estimate) the matrix norms of A and B, which is in general not an easy work in practice. A great deal of algorithms have been proposed to solve the SECFP (1.1) and the SEP (1.2) (see [19, 20, 21, 22, 23, 24, 25, 26]).

Inspired and motivated by the works mentioned above, for solving the SECFP (1.1) of firmly quasi-nonexpansive operators, we introduce the following new algorithm in which the determination of the step-size does not need any prior information about the operator norms:

$$\begin{cases} u_n = x_n - Ux_n + A^*(Ax_n - By_n), \\ x_{n+1} = x_n - \tau_n u_n, \\ v_n = y_n - Ty_n - B^*(Ax_n - By_n), \\ y_{n+1} = y_n - \tau_n v_n, \end{cases} \tag{1.11}$$

where the step-sizes τ_n are chosen in such a way that

$$\tau_n = \gamma_n \frac{\|x_n - Ux_n\|^2 + \|y_n - Ty_n\|^2 + \|Ax_n - By_n\|^2}{\|u_n\|^2 + \|v_n\|^2} \tag{1.12}$$

with $\{\gamma_n\}_{n=1}^{\infty} \subset (0, 2)$. If $\gamma_n \equiv 1$, then the step-sizes in (1.12) are reduced to the step-sizes which can be selected similarly in (1.8).

The aim of this paper is twofold. first, we propose a new iterative algorithm with larger ranges of the step-sizes which do not depend on the operator norms. Second, we modify the proposed algorithm by the viscosity approximation methods so that it has a strongly convergent iterative sequence. The organization of this paper is as follows. Some useful definitions and results are listed for the convergence analysis of the iterative algorithms in the Section 2. In Section 3, the weak convergence theorem of the proposed iterative algorithm is obtained. In Section 4, the strong convergence theorem of the proposed viscosity iterative algorithm is obtained. Finally, some numerical experiments are provided to illustrate the efficiency of the proposed iterative algorithm in Section 5.

2. PRELIMINARIES

Throughout this paper, we always assume that H is a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. Let I denote the identity operator on H. Let $T: H \to H$ be a operator. A point $x \in H$ is said to be a fixed point of T provided $Tx = x$. In this paper, we use $F(T)$ to denote the fixed point set of T. We use $\to$ and $\rightharpoonup$ to denote the strong convergence and weak convergence, respectively. We use $\omega_w(x_n) = \{x : \exists x_{n_j} \rightharpoonup x\}$ stand for the weak ω-limit set of $\{x_n\}$.

Definition 2.1. An operator $T: H \to H$ is said to be
(i) nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in H$.
(ii) quasi-nonexpansive if $F(T) \neq \emptyset$ and $\|Tx - q\| \leq \|x - q\|$ for all $x \in H$ and $q \in F(T)$.
(iii) firmly nonexpansive if $\|Tx - Ty\|^2 \leq \|x - y\|^2 - \|(x - y) - (Tx - Ty)\|^2$ for all $x, y \in H$.
(iv) firmly quasi-nonexpansive (also called directed operator) if $F(T) \neq \emptyset$ and

$$\|Tx - q\|^2 \leq \|x - q\|^2 - \|x - Tx\|^2$$

or equivalently

$$\langle x - q, x - Tx\rangle \geq \|x - Tx\|^2$$

for all $x \in H$ and $q \in F(T)$.

Definition 2.2. An operator $T: H \to H$ is called demiclosed at the origin if for any sequence $\{x_n\}$ which weakly converges to x, and the sequence $\{Tx_n\}$ strongly converges to 0, then $Tx = 0$.

Definition 2.3. An operator $h: H \to H$ is said to be a contraction with constant $\rho \in [0, 1)$ if, for any $x, y \in H$,

$$\|h(x) - h(y)\| \leq \rho\|x - y\|.$$

Definition 2.4. An operator $h: C \subset H \to H$ is said to be η-strongly monotone if there exists a positive constant η such that

$$\langle h(x) - h(y), x - y\rangle \geq \eta\|x - y\|^2, \quad \forall x, y \in C.$$

It is obvious that if h is a ρ-contraction, then $I - h$ is a $(1-\rho)$-strongly monotone operator. Recall the variational inequality problem [27] is to find a point $x^* \in C$ such that

$$\langle Fx^*, x - x^*\rangle \geq 0, \quad \forall x \in C,$$

where C is a nonempty closed convex subset of H and $F: C \to H$ is a nonlinear operator. It is well known that [28] if $F: C \to H$ is a Lipschitzian and strongly monotone operator, then the above variational inequality problem has a unique solution.

The concept of Féjer-monotonicity plays an important role in the subsequent analysis of this paper. Recall that a sequence $\{x_n\}$ in H is said to be Féjer-monotone with respect to a nonempty closed convex subset S of H if

$$\|x_{n+1} - z\| \leq \|x_n - z\|, \ \forall n \geq 0, \ \forall z \in S.$$

Lemma 2.5. [29] *Let S be a nonempty closed convex subset in H. If the sequence $\{x_n\}$ is Féjer-monotone with respect to S, then the following holds:*
(i) $x_n \rightharpoonup x^* \in S$ if and only if $\omega_w(x_n) \subseteq S$;
(ii)the sequence $\{P_S x_n\}$ converges strongly;
(iii) if $x_n \rightharpoonup x^* \in S$, then $x^* = \lim_{n\to\infty} P_S(x_n)$.

Lemma 2.6. [30] *Assume* $\{s_n\}$ *is a sequence of nonnegative real numbers such that*

$$\begin{cases} s_{n+1} \le (1-\lambda_n)s_n + \lambda_n \delta_n, & n \ge 0, \\ s_{n+1} \le s_n - \eta_n + \mu_n, & n \ge 0, \end{cases}$$

where $\{\lambda_n\}$ *is a sequence in* $(0,1)$*,* $\{\eta_n\}$ *is a sequence of nonnegative real numbers and* $\{\delta_n\}$ *and* $\{\mu_n\}$ *are two sequences in* $\mathbb{R}$ *such that*
(i) $\Sigma_{n=1}^{\infty} \lambda_n = \infty$*;*
(ii) $\lim_{n\to\infty} \mu_n = 0$*;*
(iii) $\lim_{k\to\infty} \eta_{n_k} = 0$ *implies* $\limsup_{k\to\infty} \delta_{n_k} \le 0$ *for any subsequence* $\{n_k\} \subseteq \{n\}$*.*
Then $\lim_{n\to\infty} s_n = 0$*.*

3. The weak convergence theorem

In this paper, we will solve the SECFP (1.1) under the following assumptions:
(i) the problem is consistent, i.e., its solution set, denoted by Γ, is nonempty;
(ii) both U and T are firmly quasi-nonexpansive operators, and both $I-U$ and $I-T$ are demiclosed at origin.

Now we propose the new iterative algorithm for the SECFP (1.1), in which the choice of the step-sizes does not need any prior information of the operator norms $\|A\|$ and $\|B\|$.

Algorithm 3.1. Choose a positive sequence $\{\gamma_n\}_{n=1}^{\infty} \subset (0,2)$ and select starting points $(x_0, y_0) \in H_1 \times H_2$ arbitrarily. Assume that the nth iterate (x_n, y_n) has been constructed. If

$$\begin{cases} x_n - Ux_n + A^*(Ax_n - By_n) = 0, \\ y_n - Ty_n - B^*(Ax_n - By_n) = 0, \end{cases} \tag{3.1}$$

then stop; otherwise we calculate the (n+1)th iterate (x_{n+1}, y_{n+1}) via the formula:

$$\begin{cases} u_n = x_n - Ux_n + A^*(Ax_n - By_n), \\ x_{n+1} = x_n - \tau_n u_n, \\ v_n = y_n - Ty_n - B^*(Ax_n - By_n), \\ y_{n+1} = y_n - \tau_n v_n, \end{cases} \tag{3.2}$$

where the step-sizes τ_n are chosen in such a way that

$$\tau_n = \gamma_n \frac{\|x_n - Ux_n\|^2 + \|y_n - Ty_n\|^2 + \|Ax_n - By_n\|^2}{\|u_n\|^2 + \|v_n\|^2}. \tag{3.3}$$

Remark 3.2. On comparing with the conditions on $\{\tau_n\}$ chosen by (1.8) in algorithm (1.7) of [17], our assumptions on $\{\tau_n\}$ in Algorithm 3.1 are obviously relaxed. When $\gamma_n \equiv 1$, the step-sizes in (3.3) are reduced to the step-sizes which can be selected similarly in (1.8). We have that the range of γ_n becomes larger. To be more precise, we enlarge the range of the parameter γ_n from 1 to $(0,2)$.

Lemma 3.3. *If equality (3.1) holds for some* $n \ge 0$*, then* (x_n, y_n) *is a solution of the SECFP (1.1).*

Proof. Let $u_n = x_n - Ux_n + A^*(Ax_n - By_n)$ and $v_n = y_n - Ty_n - B^*(Ax_n - By_n)$, and fix $z = (x^*, y^*) \in \Gamma$. Since U and T are firmly quasi-nonexpansive, we have

$$\begin{aligned}
0 =& \langle u_n, x_n - x^* \rangle + \langle v_n, y_n - y^* \rangle \\
=& \langle x_n - Ux_n, x_n - x^* \rangle + \langle A^*(Ax_n - By_n), x_n - x^* \rangle + \langle y_n - Ty_n, y_n - y^* \rangle \\
& - \langle B^*(Ax_n - By_n), y_n - y^* \rangle \\
=& \langle x_n - Ux_n, x_n - x^* \rangle + \langle Ax_n - By_n, Ax_n - Ax^* \rangle + \langle y_n - Ty_n, y_n - y^* \rangle \\
& - \langle Ax_n - By_n, By_n - By^* \rangle \\
\geq& \|x_n - Ux_n\|^2 + \|y_n - Ty_n\|^2 + \langle Ax_n - By_n, Ax_n - By_n - Ax^* + By^* \rangle.
\end{aligned} \tag{3.4}$$

From $z = (x^*, y^*) \in \Gamma$, we have $Ax^* = By^*$. Then

$$\|x_n - Ux_n\|^2 + \|y_n - Ty_n\|^2 + \|Ax_n - By_n\|^2 \leq 0.$$

This shows that $x_n = Ux_n$, $y_n = Ty_n$, and $Ax_n = By_n$, i.e., (x_n, y_n) is a solution of the SECFP (1.1) □

Lemma 3.4. *If the sequence $\{(x_n, y_n)\}$ satisfies*

$$\lim_{n\to\infty} \frac{(\|x_n - Ux_n\|^2 + \|y_n - Ty_n\|^2 + \|Ax_n - By_n\|^2)^2}{\|x_n - Ux_n + A^*(Ax_n - By_n)\|^2 + \|y_n - Ty_n - B^*(Ax_n - By_n)\|^2} = 0,$$

then $\lim_{n\to\infty} \|x_n - Ux_n\| = \lim_{n\to\infty} \|y_n - Ty_n\| = \lim_{n\to\infty} \|Ax_n - By_n\| = 0.$

Proof. Since

$$\begin{aligned}
& \frac{(\|x_n - Ux_n\|^2 + \|y_n - Ty_n\|^2 + \|Ax_n - By_n\|^2)^2}{\|x_n - Ux_n + A^*(Ax_n - By_n)\|^2 + \|y_n - Ty_n - B^*(Ax_n - By_n)\|^2} \\
\geq & \frac{(\|x_n - Ux_n\|^2 + \|y_n - Ty_n\|^2 + \|Ax_n - By_n\|^2)^2}{2(\|x_n - Ux_n\|^2 + \|A\|^2\|Ax_n - By_n\|^2) + 2(\|y_n - Ty_n\|^2 + \|B\|^2\|Ax_n - By_n\|^2)} \\
= & \frac{(\|x_n - Ux_n\|^2 + \|y_n - Ty_n\|^2 + \|Ax_n - By_n\|^2)^2}{2(\|x_n - Ux_n\|^2 + \|y_n - Ty_n\|^2) + 2(\|A\|^2 + \|B\|^2)\|Ax_n - By_n\|^2} \\
\geq & \frac{(\|x_n - Ux_n\|^2 + \|y_n - Ty_n\|^2 + \|Ax_n - By_n\|^2)^2}{\max\{2, 2(\|A\|^2 + \|B\|^2)\}(\|x_n - Ux_n\|^2 + \|y_n - Ty_n\|^2 + \|Ax_n - By_n\|^2)} \\
= & \frac{\|x_n - Ux_n\|^2 + \|y_n - Ty_n\|^2 + \|Ax_n - By_n\|^2}{\max\{2, 2(\|A\|^2 + \|B\|^2)\}},
\end{aligned}$$

we have the desired assertion immediately. □

Theorem 3.5. *Let $\{(x_n, y_n)\}$ be the sequence generated by Algorithm 3.1. Suppose*

$$0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < 2.$$

Then the sequence $\{(x_n, y_n)\}$ converges weakly to a solution $(\tilde{x}, \tilde{y})$ of the SECFP (1.1), where $(\tilde{x}, \tilde{y}) = \lim_{n\to\infty} P_\Gamma(x_n, y_n)$.

Proof. First, we show that the sequence $\{(x_n, y_n)\}$ is Féjer-monotone with respect to Γ. Let $z = (x^*, y^*) \in \Gamma$. It follows from (3.4) in the proof of Lemma 3.3 that

$$\langle u_n, x_n - x^* \rangle + \langle v_n, y_n - y^* \rangle \geq \|x_n - Ux_n\|^2 + \|y_n - Ty_n\|^2 + \|Ax_n - By_n\|^2. \tag{3.5}$$

Then

$$\begin{aligned}
&\|x_{n+1}-x^*\|^2+\|y_{n+1}-y^*\|^2\\
=&\|x_n-\tau_n u_n-x^*\|^2+\|y_n-\tau_n v_n-y^*\|^2\\
=&\|x_n-x^*\|^2-2\tau_n\langle u_n,x_n-x^*\rangle+\tau_n^2\|u_n\|^2+\|y_n-y^*\|^2-2\tau_n\langle v_n,y_n-y^*\rangle+\tau_n^2\|v_n\|^2\\
=&\|x_n-x^*\|^2+\|y_n-y^*\|^2-2\tau_n(\langle u_n,x_n-x^*\rangle+\langle v_n,y_n-y^*\rangle)+\tau_n^2(\|u_n\|^2+\|v_n\|^2)\\
\le&\|x_n-x^*\|^2+\|y_n-y^*\|^2-\gamma_n(2-\gamma_n)\frac{(\|x_n-Ux_n\|^2+\|y_n-Ty_n\|^2+\|Ax_n-By_n\|^2)^2}{\|u_n\|^2+\|v_n\|^2},
\end{aligned} \tag{3.6}$$

which implies that

$$\|x_{n+1}-x^*\|^2+\|y_{n+1}-y^*\|^2\le\|x_n-x^*\|^2+\|y_n-y^*\|^2. \tag{3.7}$$

Obviously, we have

$$\sqrt{\|x_{n+1}-x^*\|^2+\|y_{n+1}-y^*\|^2}\le\sqrt{\|x_n-x^*\|^2+\|y_n-y^*\|^2}.$$

Letting $X_n=(x_n,y_n)$ in the product space $H_1\times H_2$, we have

$$\|X_{n+1}-z\|\le\|X_n-z\|.$$

This implies that $\{X_n\}$, i.e., $\{(x_n,y_n)\}$ is Féjer-monotone with respect to Γ. So

$$\lim_{n\to\infty}\|X_n-Z\|=\lim_{n\to\infty}\sqrt{\|x_n-x^*\|^2+\|y_n-y^*\|^2}$$

exists, $\{(x_n,y_n)\}$ and $\{(Ax_n,By_n)\}$ are bounded. From (3.6) and the condition on the sequence $\{\gamma_n\}$, we have

$$\sum_{n=1}^{\infty}\frac{(\|x_n-Ux_n\|^2+\|y_n-Ty_n\|^2+\|Ax_n-By_n\|^2)^2}{\|x_n-Ux_n+A^*(Ax_n-By_n)\|^2+\|y_n-Ty_n-B^*(Ax_n-By_n)\|^2}<\infty, \tag{3.8}$$

which implies

$$\lim_{n\to\infty}\frac{(\|x_n-Ux_n\|^2+\|y_n-Ty_n\|^2+\|Ax_n-By_n\|^2)^2}{\|x_n-Ux_n+A^*(Ax_n-By_n)\|^2+\|y_n-Ty_n-B^*(Ax_n-By_n)\|^2}=0. \tag{3.9}$$

From Lemma 3.4, we have

$$\lim_{n\to\infty}\|x_n-Ux_n\|=\lim_{n\to\infty}\|y_n-Ty_n\|=\lim_{n\to\infty}\|Ax_n-By_n\|=0. \tag{3.10}$$

Now we show $\omega_w(x_n,y_n)\subseteq\Gamma$. We assume that $(\bar{x},\bar{y})\in\omega_w(x_n,y_n)$, i.e., there exist a subsequence $\{(x_{n_k},y_{n_k})\}$ of $\{(x_n,y_n)\}$ such that $(x_{n_k},y_{n_k})\rightharpoonup(\bar{x},\bar{y})$. Since $I-U$ and $I-T$ are demiclosed at origin, then we get from (3.10) that $\bar{x}\in F(U)$ and $\bar{y}\in F(T)$. On the other hand, $(Ax_{n_k}-By_{n_k})\rightharpoonup(A\bar{x},B\bar{y})$ and weakly lower semicontinuity of the norm imply

$$\|A\bar{x}-B\bar{y}\|\le\liminf_{n\to\infty}\|Ax_{n_k}-By_{n_k}\|=0.$$

Hence $(\bar{x},\bar{y})\in\Gamma$. From Lemma 2.5, we can get that $\{(x_n,y_n)\}$ converges weakly to a solution $(\tilde{x},\tilde{y})$ of the SECFP (1.1), where $(\tilde{x},\tilde{y})=\lim_{n\to\infty}P_\Gamma(x_n,y_n)$. □

We now turn our attention to SEP (1.2). We assume that the solution set of SEP (1.2) is nonempty and its solution set is denoted by Ω. Since the projection operators P_C and P_Q are firmly quasi-nonexpansive, and $I-P_C$ and $I-P_Q$ are demiclosed at origin, we get the following self-adaptive algorithm and result for SEP (1.2).

Algorithm 3.6. Choose a positive sequence $\{\gamma_n\}_{n=1}^{\infty} \subset (0,2)$ and select the starting points $(x_0, y_0) \in H_1 \times H_2$ arbitrarily. Assume that the nth iterate (x_n, y_n) has been constructed. If

$$\begin{cases} x_n - P_C x_n + A^*(Ax_n - By_n) = 0, \\ y_n - P_Q y_n - B^*(Ax_n - By_n) = 0, \end{cases} \tag{3.11}$$

then stop; otherwise we calculate the (n+1)th iterate (x_{n+1}, y_{n+1}) via the formula:

$$\begin{cases} u_n = x_n - P_C x_n + A^*(Ax_n - By_n), \\ x_{n+1} = x_n - \tau_n u_n, \\ v_n = y_n - P_Q y_n - B^*(Ax_n - By_n), \\ y_{n+1} = y_n - \tau_n v_n, \end{cases} \tag{3.12}$$

where the step-sizes τ_n are chosen in such a way that

$$\tau_n = \gamma_n \frac{\|x_n - P_C x_n\|^2 + \|y_n - P_Q y_n\|^2 + \|Ax_n - By_n\|^2}{\|u_n\|^2 + \|v_n\|^2}. \tag{3.13}$$

Corollary 3.7. *Let $\{(x_n, y_n)\}$ be the sequence generated by Algorithm 3.6. Suppose*

$$0 < \liminf_{n\to\infty} \gamma_n \le \limsup_{n\to\infty} \gamma_n < 2.$$

Then the sequence $\{(x_n, y_n)\}$ converges weakly to a solution $(\tilde{x}, \tilde{y})$ of the SEP (1.2), where $(\tilde{x}, \tilde{y}) = \lim_{n\to\infty} P_\Omega(x_n, y_n)$.

4. The strong convergence theorem

In this section, we aim to modify Algorithm 3.1 so that it is strongly convergent. It is known that the viscosity approximation method is often used to approximate a fixed point of a nonexpansive operator V in a Hilbert space with strong convergence and it is defined [31] by:

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)V(x_n), \qquad n \ge 1, \tag{4.1}$$

where $\{\alpha_n\} \subseteq [0,1]$ and f is a contractive operator. Now we adapt (4.1) to get the strong convergence result.

Algorithm 4.1. Let $f_1 : H_1 \to H_1$ and $f_2 : H_2 \to H_2$ be two contractions with constants ρ_1, $\rho_2 \in [0, \frac{1}{\sqrt{2}})$, and let $\{\alpha_n\} \subseteq [0,1]$ be a sequence of real numbers. Choose a positive sequence $\{\gamma_n\}_{n=1}^{\infty} \subset (0,2)$. Select the starting points $(x_0, y_0) \in H_1 \times H_2$ arbitrarily. Assume that the nth iterate (x_n, y_n) has been constructed. If

$$\begin{cases} x_n - Ux_n + A^*(Ax_n - By_n) = 0, \\ y_n - Ty_n - B^*(Ax_n - By_n) = 0, \end{cases}$$

then stop; otherwise we calculate the (n+1)th iterate (x_{n+1}, y_{n+1}) via the formula:

$$\begin{cases} u_n = x_n - Ux_n + A^*(Ax_n - By_n), \\ x_{n+1} = \alpha_n f_1(x_n) + (1-\alpha_n)(x_n - \tau_n u_n), \\ v_n = y_n - Ty_n - B^*(Ax_n - By_n), \\ y_{n+1} = \alpha_n f_2(y_n) + (1-\alpha_n)(y_n - \tau_n v_n), \end{cases} \tag{4.2}$$

where the step-sizes τ_n are chosen in such a way that

$$\tau_n = \gamma_n \frac{\|x_n - Ux_n\|^2 + \|y_n - Ty_n\|^2 + \|Ax_n - By_n\|^2}{\|u_n\|^2 + \|v_n\|^2}. \tag{4.3}$$

Theorem 4.2. *Let $\{(x_n, y_n)\}$ be the sequence generated by Algorithm 4.1. Assume that*

$$0 < \liminf_{n\to\infty} \gamma_n \le \limsup_{n\to\infty} \gamma_n < 2,$$

$\lim_{n\to\infty} \alpha_n = 0$ *and* $\Sigma_{n=0}^{\infty} \alpha_n = +\infty$. *Then the sequence $\{(x_n, y_n)\}$ converges strongly to a solution (x^*, y^*) of the SECFP (1.1) which solves the variational inequality problem:*

$$\begin{cases} \langle (I - f_1)x^*, x - x^* \rangle \ge 0, \\ \langle (I - f_2)y^*, y - y^* \rangle \ge 0, \end{cases} \quad (x, y) \in \Gamma. \tag{4.4}$$

Proof. Let $\tilde{u}_n = x_n - \tau_n u_n$ and $\tilde{v}_n = y_n - \tau_n v_n$. Let $(x^*, y^*) \in \Gamma$ be the unique solution of variational inequality problem (4.4). Then $x^* \in F(U)$, $y^* \in F(T)$ and $Ax^* = By^*$. From (3.6), we have

$$\begin{aligned} \|\tilde{u}_n - x^*\|^2 + \|\tilde{v}_n - y^*\|^2 &\le \|x_n - x^*\|^2 + \|y_n - y^*\|^2 \\ &- \gamma_n(2 - \gamma_n) \frac{(\|x_n - Ux_n\|^2 + \|y_n - Ty_n\|^2 + \|Ax_n - By_n\|^2)^2}{\|u_n\|^2 + \|v_n\|^2}. \end{aligned} \tag{4.5}$$

In particular,

$$\|\tilde{u}_n - x^*\|^2 + \|\tilde{v}_n - y^*\|^2 \le \|x_n - x^*\|^2 + \|y_n - y^*\|^2. \tag{4.6}$$

It follows that

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &\le \alpha_n \|f_1(x_n) - f_1(x^*) + f_1(x^*) - x^*\|^2 + (1 - \alpha_n)\|\tilde{u}_n - x^*\|^2 \\ &\le 2\alpha_n[\|f_1(x_n) - f_1(x^*)\|^2 + \|f_1(x^*) - x^*\|^2] + (1 - \alpha_n)\|\tilde{u}_n - x^*\|^2 \\ &\le 2\alpha_n \rho_1^2 \|x_n - x^*\|^2 + 2\alpha_n \|f_1(x^*) - x^*\|^2 + (1 - \alpha_n)\|\tilde{u}_n - x^*\|^2 \end{aligned}$$

and

$$\|y_{n+1} - y^*\|^2 \le 2\alpha_n \rho_2^2 \|y_n - y^*\|^2 + 2\alpha_n \|f_2(y^*) - y^*\|^2 + (1 - \alpha_n)\|\tilde{v}_n - y^*\|^2.$$

Setting $\rho = \max\{\rho_1, \rho_2\}$, we have $\rho \in [0, \frac{1}{\sqrt{2}})$. Adding up the last two inequalities and using (4.6), we have

$$\begin{aligned} s_{n+1} &\le (1 - \alpha_n + 2\alpha_n \rho^2) s_n + 2\alpha_n (\|f_1(x^*) - x^*\|^2 + \|f_2(y^*) - y^*\|^2) \\ &= [1 - \alpha_n(1 - 2\rho^2)] s_n + \alpha_n (1 - 2\rho^2) \frac{2}{1 - 2\rho^2} (\|f_1(x^*) - x^*\|^2 + \|f_2(y^*) - y^*\|^2). \end{aligned} \tag{4.7}$$

where $s_n = \|x_n - x^*\|^2 + \|y_n - y^*\|^2$. Since $\lim_{n\to\infty} \alpha_n = 0$, without loss of generality, we can assume that $0 \le \alpha_n(1 - 2\rho^2) \le 1$. It follows from induction that

$$s_n \le \max\left\{ s_0, \frac{2}{1 - 2\rho^2} (\|f_1(x^*) - x^*\|^2 + \|f_2(y^*) - y^*\|^2) \right\}, \ \ n \ge 0,$$

which implies that $\{x_n\}$ and $\{y_n\}$ are bounded. It follows that $\{\tilde{u}_n\}, \{\tilde{v}_n\}, \{f_1(x_n)\}$ and $\{f_2(y_n)\}$ are bounded. From (4.2) we have

$$\begin{aligned}
\|x_{n+1}-x^*\|^2 =&\alpha_n^2\|f_1(x_n)-x^*\|^2+2\alpha_n(1-\alpha_n)\langle f_1(x_n)-f_1(x^*),\tilde{u}_n-x^*\rangle \\
&+2\alpha_n(1-\alpha_n)\langle f_1(x^*)-x^*,\tilde{u}_n-x^*\rangle+(1-\alpha_n)^2\|\tilde{u}_n-x^*\|^2 \\
\leq&\alpha_n^2\|f_1(x_n)-x^*\|^2+\alpha_n(1-\alpha_n)(\|f_1(x_n)-f_1(x^*)\|^2+\|\tilde{u}_n-x^*\|^2) \\
&+2\alpha_n(1-\alpha_n)\langle f_1(x^*)-x^*,\tilde{u}_n-x^*\rangle+(1-\alpha_n)^2\|\tilde{u}_n-x^*\|^2 \\
\leq&\alpha_n^2\|f_1(x_n)-x^*\|^2+\alpha_n(1-\alpha_n)\rho_1^2\|x_n-x^*\|^2 \\
&+(1-\alpha_n)\|\tilde{u}_n-x^*\|^2+2\alpha_n(1-\alpha_n)\langle f_1(x^*)-x^*,\tilde{u}_n-x^*\rangle.
\end{aligned} \tag{4.8}$$

Similarly, we have

$$\begin{aligned}
\|y_{n+1}-y^*\|^2 \leq&\alpha_n^2\|f_2(y_n)-y^*\|^2+\alpha_n(1-\alpha_n)\rho_2^2\|y_n-y^*\|^2 \\
&+(1-\alpha_n)\|\tilde{v}_n-y^*\|^2+2\alpha_n(1-\alpha_n)\langle f_2(y^*)-y^*,\tilde{v}_n-y^*\rangle.
\end{aligned} \tag{4.9}$$

From (4.6), (4.8) and (4.9), we have

$$\begin{aligned}
s_{n+1} \leq&[1-\alpha_n(1-(1-\alpha_n)\rho^2)]s_n+\alpha_n[\alpha_n(\|f_1(x_n)-x^*\|^2+\|f_2(y_n)-y^*\|^2) \\
&+2(1-\alpha_n)(\langle f_1(x^*)-x^*,\tilde{u}_n-x^*\rangle+\langle f_2(y^*)-y^*,\tilde{v}_n-y^*\rangle)] \\
=&(1-\lambda_n)s_n+\lambda_n\delta_n,
\end{aligned} \tag{4.10}$$

where

$$\begin{aligned}
\lambda_n =&\alpha_n(1-(1-\alpha_n)\rho^2), \\
\delta_n =&\frac{2(1-\alpha_n)(\langle f_1(x^*)-x^*,\tilde{u}_n-x^*\rangle+\langle f_2(y^*)-y^*,\tilde{v}_n-y^*\rangle)}{1-(1-\alpha_n)\rho^2} \\
&+\frac{\alpha_n(\|f_1(x_n)-x^*\|^2+\|f_2(y_n)-y^*\|^2)}{1-(1-\alpha_n)\rho^2}.
\end{aligned}$$

On the other hand, from (4.5), (4.8) and (4.9), we obtain

$$\begin{aligned}
s_{n+1} \leq&[1-\alpha_n(1-(1-\alpha_n)\rho^2)]s_n+\alpha_n^2(\|f_1(x_n)-x^*\|^2+\|f_2(y_n)-y^*\|^2) \\
&+2\alpha_n(1-\alpha_n)(\langle f_1(x^*)-x^*,\tilde{u}_n-x^*\rangle+\langle f_2(y^*)-y^*,\tilde{v}_n-y^*\rangle) \\
&-(1-\alpha_n)\gamma_n(2-\gamma_n)\frac{(\|x_n-Ux_n\|^2+\|y_n-Ty_n\|^2+\|Ax_n-By_n\|^2)^2}{\|u_n\|^2+\|v_n\|^2} \\
\leq&s_n+\alpha_n^2(\|f_1(x_n)-x^*\|^2+\|f_2(y_n)-y^*\|^2) \\
&+2\alpha_n(1-\alpha_n)(\langle f_1(x^*)-x^*,\tilde{u}_n-x^*\rangle+\langle f_2(y^*)-y^*,\tilde{v}_n-y^*\rangle) \\
&-(1-\alpha_n)\gamma_n(2-\gamma_n)\frac{(\|x_n-Ux_n\|^2+\|y_n-Ty_n\|^2+\|Ax_n-By_n\|^2)^2}{\|u_n\|^2+\|v_n\|^2}.
\end{aligned} \tag{4.11}$$

Setting

$$\begin{aligned}
\mu_n =&\alpha_n^2(\|f_1(x_n)-x^*\|^2+\|f_2(y_n)-y^*\|^2) \\
&+2\alpha_n(1-\alpha_n)(\langle f_1(x^*)-x^*,\tilde{u}_n-x^*\rangle+\langle f_2(y^*)-y^*,\tilde{v}_n-y^*\rangle), \\
\eta_n =& (1-\alpha_n)\gamma_n(2-\gamma_n)\frac{(\|x_n-Ux_n\|^2+\|y_n-Ty_n\|^2+\|Ax_n-By_n\|^2)^2}{\|u_n\|^2+\|v_n\|^2},
\end{aligned}$$

we have that (4.11) can be rewritten as the following form,

$$s_{n+1} \leq s_n - \eta_n + \mu_n, \quad n \geq 0. \tag{4.12}$$

By the assumption on $\{\alpha_n\}$, we get $\sum_{n=0}^{\infty} \lambda_n = \infty$ and $\lim_{n\to\infty} \mu_n = 0$ due to the boundedness of $\{x_n\}$ and $\{y_n\}$. To use Lemma 2.6, it suffices to verify that, for all subsequence $\{n_k\} \subseteq \{n\}$, $\lim_{k\to\infty} \eta_{n_k} = 0$ implies

$$\limsup_{k\to\infty} \delta_{n_k} \leq 0. \tag{4.13}$$

It follows from $\lim_{k\to\infty} \eta_{n_k} = 0$ that

$$\lim_{k\to\infty} \frac{(\|x_{n_k} - Ux_{n_k}\|^2 + \|y_{n_k} - Ty_{n_k}\|^2 + \|Ax_{n_k} - By_{n_k}\|^2)^2}{\|x_{n_k} - Ux_{n_k} + A^*(Ax_{n_k} - By_{n_k})\|^2 + \|y_{n_k} - Ty_{n_k} - B^*(Ax_{n_k} - By_{n_k})\|^2} = 0. \tag{4.14}$$

From Lemma 3.4, we obtain

$$\lim_{k\to\infty} \|x_{n_k} - Ux_{n_k}\| = \lim_{k\to\infty} \|y_{n_k} - Ty_{n_k}\| = \lim_{k\to\infty} \|Ax_{n_k} - By_{n_k}\| = 0. \tag{4.15}$$

In a similar way as in Theorem 3.5, we can get $\omega_w(x_{n_k}, y_{n_k}) \subseteq \Gamma$. It follows from (4.14) and

$$\begin{aligned}
&\frac{(\|x_{n_k} - Ux_{n_k}\|^2 + \|y_{n_k} - Ty_{n_k}\|^2 + \|Ax_{n_k} - By_{n_k}\|^2)^2}{(\|u_{n_k}\| + \|v_{n_k}\|)^2} \\
\leq &\frac{(\|x_{n_k} - Ux_{n_k}\|^2 + \|y_{n_k} - Ty_{n_k}\|^2 + \|Ax_{n_k} - By_{n_k}\|^2)^2}{(\|u_{n_k}\|^2 + \|v_{n_k}\|^2)}
\end{aligned}$$

that

$$\lim_{k\to\infty} \frac{(\|x_{n_k} - Ux_{n_k}\|^2 + \|y_{n_k} - Ty_{n_k}\|^2 + \|Ax_{n_k} - By_{n_k}\|^2)^2}{(\|u_{n_k}\| + \|v_{n_k}\|)^2} = 0,$$

which yields

$$\lim_{k\to\infty} \frac{\|x_{n_k} - Ux_{n_k}\|^2 + \|y_{n_k} - Ty_{n_k}\|^2 + \|Ax_{n_k} - By_{n_k}\|^2}{\|u_{n_k}\| + \|v_{n_k}\|} = 0. \tag{4.16}$$

Since

$$\begin{aligned}
\tau_{n_k} =& \gamma_{n_k} \frac{\|x_{n_k} - Ux_{n_k}\|^2 + \|y_{n_k} - Ty_{n_k}\|^2 + \|Ax_{n_k} - By_{n_k}\|^2}{\|u_{n_k}\|^2 + \|v_{n_k}\|^2} \\
\leq & \frac{4(\|x_{n_k} - Ux_{n_k}\|^2 + \|y_{n_k} - Ty_{n_k}\|^2 + \|Ax_{n_k} - By_{n_k}\|^2)}{(\|u_{n_k}\| + \|v_{n_k}\|)^2},
\end{aligned}$$

we have

$$\begin{aligned}
\|x_{n_k} - \tilde{u}_{n_k}\| + \|y_{n_k} - \tilde{v}_{n_k}\| =& \tau_{n_k}(\|u_{n_k}\| + \|v_{n_k}\|) \\
\leq & \frac{4(\|x_{n_k} - Ux_{n_k}\|^2 + \|y_{n_k} - Ty_{n_k}\|^2 + \|Ax_{n_k} - By_{n_k}\|^2)}{\|u_{n_k}\| + \|v_{n_k}\|}.
\end{aligned} \tag{4.17}$$

It follows from (4.16) and (4.17) that

$$\lim_{k\to\infty} (\|x_{n_k} - \tilde{u}_{n_k}\| + \|y_{n_k} - \tilde{v}_{n_k}\|) = 0. \tag{4.18}$$

To get (4.13), as $\lim_{k\to\infty}(1 - (1 - \alpha_{n_k})\rho^2) = 1 - \rho^2$ and

$$\lim_{k\to\infty} \alpha_{n_k}(\|f_1(x_{n_k}) - x^*\|^2 + \|f_2(y_{n_k}) - y^*\|^2) = 0,$$

we only need to verify

$$\limsup_{k\to\infty}(\langle f_1(x^*)-x^*,\tilde{u}_{n_k}-x^*\rangle+\langle f_2(y^*)-y^*,\tilde{v}_{n_k}-y^*\rangle)\le 0.$$

We can take subsequence $\{(x_{n_{k_l}},y_{n_{k_l}})\}$ of $\{(x_{n_k},y_{n_k})\}$ such that $(x_{n_{k_l}},y_{n_{k_l}})\rightharpoonup(\widetilde{x},\widetilde{y})$ as $l\to\infty$ and

$$\begin{aligned}&\limsup_{k\to\infty}(\langle f_1(x^*)-x^*,\tilde{u}_{n_k}-x^*\rangle+\langle f_2(y^*)-y^*,\tilde{v}_{n_k}-y^*\rangle)\\ =&\lim_{l\to\infty}(\langle f_1(x^*)-x^*,\tilde{u}_{n_{k_l}}-x^*\rangle+\langle f_2(y^*)-y^*,\tilde{v}_{n_{k_l}}-y^*\rangle)\\ =&\lim_{l\to\infty}(\langle f_1(x^*)-x^*,x_{n_{k_l}}-x^*\rangle+\langle f_2(y^*)-y^*,y_{n_{k_l}}-y^*\rangle)\\ =&\langle f_1(x^*)-x^*,\widetilde{x}-x^*\rangle+\langle f_2(y^*)-y^*,\widetilde{y}-y^*\rangle.\end{aligned}\tag{4.19}$$

Since $\omega_w(x_{n_k},y_{n_k})\subset\Gamma$ and (x^*,y^*) is the solution of variational inequality problem (4.4), we obtain from (4.19) that

$$\limsup_{k\to\infty}(\langle f_1(x^*)-x^*,\tilde{u}_{n_k}-x^*\rangle+\langle f_2(y^*)-y^*,\tilde{v}_{n_k}-y^*\rangle)\le 0.$$

It follows from Lemma 2.6 that

$$\lim_{n\to\infty}(\|x_n-x^*\|^2+\|y_n-y^*\|^2)=0,$$

which implies that $x_n\to x^*$ and $y_n\to y^*$. So the sequence $\{(x_n,y_n)\}$ strongly converges to the solution (x^*,y^*) of (1.1), which solves the variational inequality problem (4.4). □

Now, we apply Algorithm 4.1 to show the strong convergence result for the SEP (1.2). We assume that solution set of the SEP (1.2) is nonempty and its solution set is denoted by Ω.

Algorithm 4.3. Let $f_1:H_1\to H_1$ and $f_2:H_2\to H_2$ be two contractions with constants ρ_1, $\rho_2\in[0,\frac{1}{\sqrt{2}})$, and let $\{\alpha_n\}\subseteq[0,1]$ be a sequence of real numbers. Choose a positive sequence $\{\gamma_k\}_{k=1}^{\infty}\subset(0,2)$ and select the starting points $(x_0,y_0)\in H_1\times H_2$ arbitrarily. Assume that the nth iterate (x_n,y_n) has been constructed. If

$$\begin{cases}x_n-P_Cx_n+A^*(Ax_n-By_n)=0,\\ y_n-P_Qy_n-B^*(Ax_n-By_n)=0,\end{cases}$$

then stop; otherwise we calculate the (n+1)th iterate (x_{n+1},y_{n+1}) via the formula:

$$\begin{cases}u_n=x_n-P_Cx_n+A^*(Ax_n-By_n),\\ x_{n+1}=\alpha_nf_1(x_n)+(1-\alpha_n)(x_n-\tau_nu_n),\\ v_n=y_n-P_Qy_n-B^*(Ax_n-By_n),\\ y_{n+1}=\alpha_nf_2(y_n)+(1-\alpha_n)(y_n-\tau_nv_n),\end{cases}\tag{4.20}$$

where the step-sizes τ_n are chosen by (3.13).

Corollary 4.4. *Let $\{(x_n,y_n)\}$ be the sequence generated by Algorithm 4.3. Assume that*

$$0<\liminf_{n\to\infty}\gamma_n\le\limsup_{n\to\infty}\gamma_n<2,$$

$\lim_{n\to\infty}\alpha_n = 0$ *and* $\Sigma_{n=0}^{\infty}\alpha_n = +\infty$. *Then the sequence* $\{(x_n, y_n)\}$ *converges strongly to a solution* (x^*, y^*) *of the SEP (1.2), which solves the variational inequality problem:*

$$\begin{cases} \langle (I-f_1)x^*, x-x^*\rangle \ge 0, \\ \langle (I-f_2)y^*, y-y^*\rangle \ge 0, \end{cases} \quad \forall (x,y)\in\Omega.$$

5. NUMERICAL EXPERIMENTS

In this section, we provide some numerical experiments and show the performance of the proposed self-adaptive iterative Algorithm 3.1 and Algorithm 3.5 for solving the SECFP (1.1) and the SEP (1.2), respectively. All the codes are written in MATLAB and are performed on a personal Lenovo computer with Pentium(R) Dual-Core CPU @ 2.4GHz and RAM 2.00GB.

We denote the vector with all elements 0 by e_0, and the vector with all elements 1 by e_1 in what follows. In the numerical results listed in the following tables, "Iter." denote the number of iterations.

Example 5.1. The SEP (1.2) with $A = (a_{ij})_{J\times N} \in \mathbb{R}^{J\times N}$ and $B = (b_{ij})_{J\times M} \in \mathbb{R}^{J\times M}$, where $a_{ij} \in (0,1)$ and $b_{ij} \in (0,1)$ generated randomly. Take

$$C = \{x \in \mathbb{R}^N \,|\, \|x\| \le 0.25\},$$
$$Q = \{y \in \mathbb{R}^M \,|\, e_0 \le y \le L\},$$

where e_0 and L are the boundary of the box Q, and L is also generated randomly satisfying $e_1 \le L \le 2e_1$.

In Algorithm 3.6, we take $x_0 = 10e_1 \in \mathbb{R}^N$ and $y_0 = -10e_1 \in \mathbb{R}^M$ as initial points. In the implementation, we take $p(x) < \varepsilon = 10^{-4}$ as the stopping criterion, where

$$p(x) = \|x - P_C x\| + \|y - P_Q y\| + \|Ax - By\|.$$

We tried different step-sizes γ_n in different dimensional Euclidean space for solving this Example. For comparison, the same random values were taken in each test. Table 1 shows numerical results of Algorithm 3.6 for solving the Example with different step-sizes γ_n and dimensions. From Table 1, we can observe that Algorithm 3.6 behaved better with $\gamma_n \equiv 0.9$ than $\gamma_n \equiv 1$.

Example 5.2. Define operators $U : \mathbb{R}^2 \to \mathbb{R}^2$ and $T : \mathbb{R}^2 \to \mathbb{R}^2$ as follows:

$$U(x) = \frac{1}{3}x, \quad T(x) = (z_1, z_2)^T,$$

where $x = (x_1, x_2)^T \in \mathbb{R}^2$ and

$$z_i = \begin{cases} x_i, & x_i < 0, \\ 0, & x_i \ge 0, \end{cases} \quad (1 \le i \le 2).$$

Note that $F(U) = \{(0,0)^T\}$,

$$F(T) = \{(x_1, x_2)^T \in \mathbb{R}^2 : x_i \le 0,\ 1 \le i \le 2\},$$

and U and T are firmly quasi-nonexpansive operators. Indeed, $T(x) = (\frac{1}{2}I + \frac{1}{2}T_1)(x)$ for $x \in \mathbb{R}^2$, where $T_1(x) = (u_1, u_2)^T$ and

$$u_i = \begin{cases} x_i, & x_i < 0, \\ -x_i, & x_i \ge 0, \end{cases} \quad (1 \le i \le 2).$$

TABLE 1. Numerical results with different γ_n and dimensions.

		$N=10,\ M=10$				$N=10,\ M=20$			
γ_n		$J=10$	$J=30$	$J=40$	$J=50$	$J=10$	$J=30$	$J=40$	$J=50$
0.1	Iter.	486	610	650	588	899	1646	979	1024
0.2	Iter.	351	460	516	404	690	1125	854	817
0.3	Iter.	267	416	396	351	590	730	541	587
0.4	Iter.	269	266	318	307	448	611	384	374
0.5	Iter.	204	227	243	207	457	428	359	362
0.6	Iter.	178	196	189	253	318	370	354	330
0.7	Iter.	113	165	179	152	248	443	278	254
0.8	Iter.	131	174	146	148	202	390	288	235
0.9	Iter.	109	160	117	96	179	266	213	234
1.0	Iter.	616	998	960	618	1879	795	2068	662
1.1	Iter.	757	1149	1155	1027	1923	4081	2581	2357
1.2	Iter.	761	1155	1159	1033	1579	4113	2597	2371
1.3	Iter.	767	1161	1167	1039	1585	4149	2615	2387
1.4	Iter.	771	1169	1173	1045	1591	4185	2635	2405
1.5	Iter.	777	1177	1181	1053	1597	4233	2661	2425
1.6	Iter.	785	1189	1193	1063	1601	4295	2697	2453
1.7	Iter.	793	1201	1205	1075	1597	4381	2743	2489
1.8	Iter.	801	1217	1223	1093	1537	4521	2823	2541
1.9	Iter.	785	1225	1249	1123	1447	4819	3007	2653

Obviously, $F(T_1) = F(T)$ and T_1 is quasi-nonexpansive. So, T is firmly quasi-nonexpansive. Let

$$A = \begin{pmatrix} 2 & 5 \\ 1 & 2 \end{pmatrix}, B = \begin{pmatrix} 3 & 7 \\ 2 & 1 \end{pmatrix}.$$

We apply the proposed Algorithm 3.1 to the SECFP (1.1). In the implementation, we take $p(x) < \varepsilon = 10^{-4}$ as the stopping criterion, where

$$p(x) = \|x - Ux\| + \|y - Ty\| + \|Ax - By\|.$$

We take different initial points and step-sizes γ_n. Table 2 shows numerical results of Algorithm 3.1 for solving this Example.

Remark 5.3. In numerical experiment, it is revealed that the sequence generated by our proposed Algorithm 3.1 converges more quickly when we choose suitable step-size factors. Especially, the sequence converges more quickly for $\gamma_n = 0.9$ for the two examples. But, the sequence converges more slowly when $\gamma_n > 1$.

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TABLE 2. Numerical results for Algorithm 3.1 with different initial points and γ_n

γ_n		$x_0=(1,0)^T$ $y_0=(0,1)^T$	$x_0=(10,-10)^T$ $y_0=(20,-20)^T$	$x_0=(-1,10)^T$ $y_0=(-8,10)^T$
0.1	Iter.	380	536	457
0.2	Iter.	283	401	363
0.3	Iter.	237	351	325
0.4	Iter.	206	333	281
0.5	Iter.	210	289	245
0.6	Iter.	107	196	154
0.7	Iter.	171	213	207
0.8	Iter.	179	233	227
0.9	Iter.	77	164	110
1.0	Iter.	553	456	555
1.1	Iter.	667	978	810
1.2	Iter.	671	1002	814
1.3	Iter.	677	1020	820
1.4	Iter.	681	1040	824
1.5	Iter.	687	1068	830
1.6	Iter.	695	1108	838
1.7	Iter.	705	1166	856
1.8	Iter.	719	1254	924
1.9	Iter.	759	1416	1182

REFERENCES

[1] A. Moudafi, Alternating CQ-algorithm for convex feasibiity and split fixed-point problems, J. Nonlinear Convex Anal. 15 (2014), 809-818.

[2] H. Attouch, J. Bolte, P. Redont, A. Soubeyran, Alternating proximal algorithms for weakly coupled minimization problems. Applications to dynamical games and PDEs. J. Convex Anal. 15 (2008), 485-506.

[3] Y. Censor, T. Elfving, A multiprojection algorithm using Bregman projections in a product space, Numer. Algor. 8 (1994), 221-239.

[4] C. Byrne, Iterative oblique projection onto convex subsets and the split feasibility problem, Inverse Probl. 18 (2002), 441-453.

[5] C. Byrne, A unified treatment of some iterative algorithms in signal processing and image reconstruction, Inverse Probl. 20 (2004), 103-120.

[6] Y. Censor, A. Gibali, S. Reich, Algorithms for the split variational inequality problem, Numer. Algor. 59 (2012), 301-323.

[7] C. Byrne, Y. Censor, A. Gibali, S. Reich, The split common null point problem, J. Nonlinear Convex Anal. 13 (2012), 759-775.

[8] G. López, V. Martín-Márquez, F. Wang, H.K. Xu, Solving the split feasibility problem without prior knowledge of matrix norms, Inverse Probl. 27 (2012), 085004.

[9] X. Qin, A. Petrusel, J.C. Yao, CQ iterative algorithms for fixed points of nonexpansive mappings and split feasibility problems in Hilbert spaces, J. Nonlinear Convex Anal. 19 (2018), 157-165.

[10] L. Liu, A hybrid steepest descent method for solving split feasibility problems involving nonexpansive mappings, J. Nonlinear Convex Anal. 20 (2019), 471-488.

[11] Q.L. Dong, Y.C. Tang, Y.J. Cho, Th.M. Rassias, "Optimal" choice of the step length of the projection and contraction methods for solving the split feasibility problem, J. Glob. Optim. 71 (2018), 341-360.

[12] S. He, H. Tian, H.K. Xu, The selective projection method for convex feasibility and split feasibility problems, J. Nonlinear Convex Anal. 19 (2018), 1199-1215.

[13] H.K. Xu, Iterative methods for the split feasibility problem in infinite-dimensional Hilbert spaces, Inverse Probl. 26 (2010), 105018.

[14] X. Qin, J.C. Yao, A viscosity iterative method for a split feasibility problem, J. Nonlinear Convex Anal. 20 (2019), 1497-1506.

[15] X. Qin, S.Y. Cho, L. Wang, Strong convergence of an iterative algorithm involving nonlinear mappings of nonexpansive and accretive type, Optimization, 67 (2018), 1377-1388.

[16] Y. Censor, A. Segal, The split common fixed point problem for directed operators, J. Convex Anal. 16 (2009), 587-600.

[17] F. Wang, A new iterativemethod for the split common fixed point problem in Hilbert spaces, Optimization, 66 (2017), 407-415.

[18] A. Moudafi, E. Al-Shemas, Simultaneous iterative methods for split equality problems and application, Trans. Math. Program. Appl. 1 (2013), 1-11.

[19] S. Chang, J.C. Yao, C.F. Wen, L. Zhao, On the split equality fixed point problem of quasi-pseudo-contractive mappings without a priori knowledge of operator norms with applications, J. Optim. Theory Appl. (2020), https://doi.org/10.1007/s10957-020-01651-8.

[20] Q.L. Dong, D. Jiang, Simultaneous and semi-alternating projection algorithms for solving split equality problems, J. Inequal. Appl. 2018 (2018), 4.

[21] L.Y. Shi, Q.H. Ansari, J.C. Yao, C.F. Wen, Linear convergence of gradient projection algorithm for split equality problems, Optimization, 67 (2018), 2347-2358.

[22] J. Zhao, H. Zong, Solving the multiple-set split equality common fixed-point problem of firmly quasi-nonexpansive operators, J. Inequal. Appl. 2018 (2018), 83.

[23] J. Zhao, Y. Jia, H. Zhang, General alternative regularization methods for split equality common fixed-point problem, Optimization, 67 (2018), 619-635.

[24] J. Zhao, S. Wang, Mixed iterative algorithms for the multiple-set split equality common fixed-point problems without prior knowledge of operator norms, Optimization, 65 (2016), 1069-1083.

[25] J. Zhao, Solving split equality fixed-point problem of quasi-nonexpansive mappings without prior knowledge of operators norms, Optimization, 64 (2015), 2619-2630.

[26] J. Zhao, D. Hou, A self-adaptive iterative algorithm for the split common fixed point problems, Numer. Algor. 82 (2019), 1047-1063.

[27] G. Stampacchia, Formes bilineaires coercivites sur les ensembles convexes, C. R. Acad. Sci. Paris, 258 (1964), 4413-4416.

[28] J.L. Lions, G. Stampacchia, Variational inequalities, Commun. Pure Appl. Math. 20 (1967), 493-512.

[29] H.H. Bauschke, J.M. Borwein, On projection algorithms for solving convex feasibility problems, SIAM Rev. 38 (1996), 367-426.

[30] S. He, C. Yang, Solving the variational inequality problem defined on intersection of finite level sets, Abst. Appl. Anal. 2013 (2013), Article ID 942315.

[31] A. Moudafi, Viscosity approximation methods for fixed points problems, J. Math. Anal. Appl. 241 (2000), 46-55.