J. Nonlinear Funct. Anal. 2020 (2020), Article ID 42 https://doi.org/10.23952/jnfa.2020.42

MATHRES

Journal of Nonlinear Functional Analysis
Available online at http://jnfa.mathres.org

GENERAL DECAY AND BLOW-UP RESULTS OF SOLUTIONS FOR A VISCOELASTIC WAVE EQUATION WITH ROBIN-DIRICHLET CONDITIONS

LE THI PHUONG NGOC[1], LE MINH TRI[2,3], LE TRAN NGOC TRAN[2,3], NGUYEN THANH LONG[2,3,*]

[1]Department of Mathematics, University of Khanh Hoa, Nha Trang City, Vietnam
[2]Department of Mathematics and Computer Science, University of Science, Ho Chi Minh City, Vietnam
[3]Department of Mathematics, Vietnam National University, Ho Chi Minh City, Vietnam

Abstract. In this paper, we investigate the unique existence, general decay and blow-up of solutions of an initial-boundary value problem for a viscoelastic wave equation with Robin-Dirichlet conditions. The proof is based on Faedo-Galerkin method associated with a priori estimate, weak convergence and compactness techniques. A numerical example is also given to illustrate the decay property of the solutions.
Keywords. Nonlinear wave equation; Faedo-Galerkin method; Blow-up; Exponential decay; Lyapunov functional.

1. Introduction

In this paper, we are concerned with the following initial and boundary value problem

$$u_{tt} - \mu(t)u_{xx} + \int_0^t g(t-s)u_{xx}(x,s)ds + \lambda u_t = K(x,t)F(u) + f(x,t),\ 0 < x < 1,\ t > 0, \quad (1.1)$$

$$u_x(0,t) - u(0,t) = u(1,t) = 0, \quad (1.2)$$

$$u(x,0) = \tilde{u}_0(x),\ u_t(x,0) = \tilde{u}_1(x), \quad (1.3)$$

where $\lambda > 0$ is given constant, and μ, K, F, f, g, $\tilde{u}_0$ and $\tilde{u}_1$ are given functions with some conditions.

It is well known that the single viscoelastic wave equation of the form

$$u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u ds + h(u_t) = f(u),\ (x,t) \in \Omega \times \mathbb{R}_+,$$

with initial and boundary conditions, where $\Omega \subset \mathbb{R}^n$ is bounded domains with a smooth boundary $\partial\Omega$, has been extensively studied and many results concerning existence, nonexistence,

*Corresponding author.
E-mail addresses: ngoc1966@gmail.com (L.T.P. Ngoc), leminhtri.06.03.97@gmail.com (L.M. Tri), ngoctran711@gmail.com (L.T.N. Tran), longnt2@gmail.com (N.T. Long).
Received May 9, 2020; Accepted September 10, 2020.

exponential decay and blow-up in finite time have been proved (see [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15] and references therein).

In [1], Berrimi and Messaoudi considered the following problem

$$u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u ds = |u|^{\gamma} u, \ (x,t) \in \Omega \times \mathbb{R}_+,$$

where $\gamma > 0$ and g is a nonnegative and decaying function. They established a local existence theorem and proved that, for certain initial data and suitable conditions on g and γ, the solution is global with energy, which decays exponentially or polynomially depending on the rate of the decay of the relaxation function g. Later, these results were improved in [11], Messaoudi [11] established a general decay result.

In [15], Wang studied the nonlinear viscoelastic equation as follows

$$u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u ds + u_t = |u|^{p-1} u, \ (x,t) \in \Omega \times \mathbb{R}_+,$$

and established a blow-up result with positive energy under some appropriate assumptions on g and initial conditions.

In [10], Messaoudi considered the following problem

$$u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u ds + |u_t|^{m-2} u_t = |u|^{p} u, \ (x,t) \in \Omega \times \mathbb{R}_+,$$

with initial conditions and Dirichlet boundary conditions. For nonincreasing positive functions g and for $p > m$, Messaoudi proved that solutions with positive initial energy blow up in finite time.

In [8], Long et al. considered the following initial-boundary value problem

$$\begin{cases} u_{tt} - u_{xx} + \int_0^t k(t-s)u_{xx} ds + |u_t|^{q-2} u_t = f(x,t,u), \ (x,t) \in (0,1) \times (0,T), \\ u_x(0,t) = u(0,t), \, u_x(1,t) + \eta u(1,t) = g(t), \\ u(x,0) = u_0(x), \ u_t(x,0) = u_1(x), \end{cases}$$

where $\eta \geq 0$, $q \geq 2$ are given constants and u_0, u_1, g, k, f are given functions. The first result obtained in [8] is the unique existence of a weak solution $u(t)$. On the other hand, in case of $f(x,t,u) = -|u|^{p-2}u + F(x,t)$, the solution $u(t)$ is exponentially decay to zero as $t \to +\infty$.

In [6], Li and He considered the nonlinear viscoelastic wave equation with linear damping, nonlinear damping and source term as follows

$$u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u ds + u_t + |u_t|^{m-2} u_t = |u|^{p-2} u, \ (x,t) \in \Omega \times \mathbb{R}_+,$$

and established a general decay result with $m \geq 1$, $p > 2$ and $\Omega \subset \mathbb{R}^n$ is bounded domains with smooth boundary $\partial\Omega$. In [7], Li and He also investigated the nonlinear viscoelastic wave equation with strong damping of the form

$$u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u ds - \Delta u_t + u_t = |u|^{p-2} u, \ (x,t) \in \Omega \times \mathbb{R}_+,$$

for relaxation functions satisfying a relation of the form $g'(t) \leq -\xi(t) \ g(t), \ t \in \mathbb{R}_+$. They proved the local existence and global existence theorems and established a general decay rate estimate. On the other hand, the finite time blow-up result for some solutions with negative initial energy and positive initial energy was obtained.

In the absence of the viscoelastic term (that is, $g = 0$), the above equations reduce to the following form

$$u_{tt} - \Delta u + h(u_t) = f(u),\ (x,t) \in \Omega \times \mathbb{R}_+,$$

with the damping term $h(u_t)$ and the source term $f(u)$. The interaction between the damping term and the source term was first considered by Levine ([16, 17]) in the case $h(u_t) = au_t$ and a polynomial source term of the form $f(u) = b|u|^{p-2}u$. He showed that the solutions with negative initial energy blow up in finite time. The main tool used in [16, 17] is the "concavity method". This method was also used in [18], where Levine and Payne proved the global nonexistence of the heat equation as follows

$$\begin{cases} u_t - \Delta u = 0,\ (x,t) \in \Omega \times [0,T], \\ \dfrac{\partial u}{\partial n} = f(u),\ (x,t) \in \partial\Omega \times [0,T], \\ u(x,0) = u_0(x). \end{cases}$$

Recently, Long et al. [19] considered the following nonlinear heat equation with a viscoelastic term and Robin conditions

$$\begin{aligned} & u_t - \frac{\partial}{\partial x}[\mu_1(x,t)u_x] + \int_0^t g(t-s)\frac{\partial}{\partial x}[\mu_2(x,s)u_x(x,s)]\,ds \\ = \ & f(u) + f_1(x,t), (x,t) \in (0,1) \times (0,T) \end{aligned}$$

and obtained the results relative to existence, blow-up and decay of a weak solution. The main tools are the Faedo-Galerkin method and a modified energy functional together with the technique of the Lyapunov functional.

In this paper, motivated by the above works, we consider the viscoelastic wave equation (1.1)-(1.3) by applying and modifying the methods used in [10, 14, 19]. It consists of five sections. First, we present preliminaries and state two theorems of the unique existence in Section 2. In Section 3, we give a sufficient condition to guarantee the global existence and the exponential decay of weak solutions with the decaying relaxation function g. In Section 4, with $f \equiv 0$, we prove the blow-up property when the initial energy is negative, null and positive corresponding the variety conditions of the relaxation function $g(t)$. Here, we first state the blow-up result when the initial energy is negative based on defining a modified energy functional together with the technique of the Lyapunov functional. Next, we continue to show that the weak solution $u(t)$ blows up at finite time when the initial energy is null or positive by using the concavity method introduced by Levine-Payne [18]. Finally, we present numerical results.

2. Existence and uniqueness

Put $\Omega = (0,1)$, $Q_T = \Omega \times (0,T)$, $T > 0$. Let us denote the usual functions spaces used in this spaces by $C^m(\overline{\Omega})$, $W^{m,p} = W^{m,p}(\Omega)$, $L^P = L^p(\Omega)$ and $H^m = H^m(\Omega)$ for $1 \le p \le \infty$ and $m \in N$. Let $\langle \cdot,\cdot \rangle$ be either the scalar product in L^2 or the scalar product [20] between a Banach space and its dual. The notation $\|\cdot\|$ stands for the norm of L^2 and $\|\cdot\|_X$ is denoted for the norm of the Banach space X.

We call X' the dual space of X and denote by $L^p(0,T;X)$, $1 \le p \le \infty$, the Banach space of the function $u : (0,T) \to X$ measurable such that

$$\|u\|_{L^p(0,T;X)} = \left(\int_0^T \|u(t)\|_X^p\,dt \right)^{1/p} < \infty \text{ for } 1 \le p < \infty,$$

and

$$\|u\|_{L^\infty(0,T;X)} = \underset{0<t<T}{ess\sup} \|u(t)\|_X \text{ for } p=\infty.$$

Since the domain of interest is one-dimentional, we let $u(t)$, $u' = u_t$, $u'' = u_{tt}$, $\nabla u = u_x$ and $\Delta u = u_{xx}$ denote $u(x,t)$, $\frac{\partial u}{\partial t}$, $\frac{\partial^2 u}{\partial t^2}$, $\frac{\partial u}{\partial x}$ and $\frac{\partial^2 u}{\partial x^2}$, respectively.

We define

$$V = \left\{ v \in H^1 : v(1) = 0 \right\},$$

the closed subspace of H^1. On V, we shall use the equivalent norms $v \mapsto \|v\|_{H^1}$ and $v \mapsto \|v_x\|$. Precisely,

$$\|v_x\| \le \|v\|_{H^1} \le \sqrt{2}\,\|v_x\|, \ \forall v \in V. \tag{2.1}$$

We put

$$a(u,v) = \langle u_x, v_x \rangle + u(0)v(0), \ \forall u,\ v \in V, \tag{2.2}$$

and

$$\|v\|_a = \sqrt{a(v,v)}. \tag{2.3}$$

The following standard lemmas read as the imbedding H^1 into $C(\overline{\Omega})$, the equivalence between the norms $\|v\|_a$ and $\|v_x\|$ and the existence of the Hilbert orthonormal base of V associated with the binlinear form $a(\cdot,\cdot)$ defined by (2.2).

Lemma 2.1. *The imbedding $H^1 \hookrightarrow C(\overline{\Omega})$ is compact and the following inequality holds*

$$\|v\|_{C(\overline{\Omega})} \le \sqrt{2}\,\|v\|_{H^1}, \ \forall v \in H^1.$$

Lemma 2.2. *The symmetric bilinear form $a(\cdot,\cdot)$ defined by (2.2) is continuous on $V \times V$ and coercive on V, i.e,*

(i) $|a(u,v)| \le 2\,\|u_x\|\,\|v_x\|$ for all $u,\ v \in V$,

(ii) $a(v,v) \ge \|v_x\|^2$ for all $v \in V$.

Remark 2.3. We note that on V, three norms $\|v\|_{H^1}$, $\|v_x\|$ and $\|v\|_a$ are equivalent.

The weak formulation of initial-boundary value problem (1.1)-(1.3) can be given in the following manner:

Find $u \in W_T = \left\{ u \in L^\infty(0,T;H^2 \cap V) : u' \in L^\infty(0,T;V),\ u'' \in L^\infty(0,T;L^2) \right\}$ such that u satisfies the variational equation

$$\begin{aligned} &\langle u''(t),v\rangle + \mu(t)a(u(t),v) + \lambda\langle u'(t),v\rangle - \textstyle\int_0^t g(t-s)a(u(s),v)ds \\ &= \langle K(t)F(u(t)),v\rangle + \langle f(t),v\rangle,\ \forall v\in V, \end{aligned} \tag{2.4}$$

together with the initial conditions

$$u(0) = \tilde{u}_0,\ u'(0) = \tilde{u}_1. \tag{2.5}$$

We make the following assumptions

$(\mathbf{H_0})$ $\lambda > 0$;

$(\mathbf{H}_1)$ $\tilde{u}_0 \in V \cap H^2$ and $\tilde{u}_1 \in V$;

$(\mathbf{H}_2)$ $\mu \in H^2(0,T)$ and $\mu(t) \ge \mu_0 > 0,\ \forall t \in [0,T]$;

$(\mathbf{H}_3)$ $g \in W^{2,1}(0,T)$;

$(\mathbf{H_4})$ $K \in C^1(\overline{Q_T})$;

$(\mathbf{H_5})$ $F \in C^1(\mathbb{R})$;

$(\mathbf{H}_6)$ $f, f_t \in L^2(Q_T)$.

Using the standard Faedo-Galerkin method, which is introduced by Lions in [21], we can prove the following theorem, which implies that problem (1.1)-(1.3) has a unique weak solution.

Theorem 2.4. *Let* $(\mathbf{H_0}) - (\mathbf{H}_6)$ *hold for* $T > 0$. *Then, there exist* $\tilde{T} \in (0,T]$ *and a unique solution of the problem (2.4)-(2.5) such that* $u \in W_{\tilde{T}}$.

Combining Theorem 2.4 and using the standard arguments of density, we obtain the following theorem. This theorem indicates that there exists a unique weak solution to (1.1)-(1.3) when $\tilde{u}_0$ and $\tilde{u}_1$ belong to larger spaces.

Theorem 2.5. *Let* $(\mathbf{H_0})$, $(\mathbf{H}_2) - (\mathbf{H}_6)$ *hold for* $T > 0$, $\tilde{u}_0 \in V$ *and* $\tilde{u}_1 \in L^2$. *Then, there exist* $\tilde{T} \in (0,T]$ *and a unique solution of the problem (2.4)-(2.5such that*

$$u \in C^1([0,\tilde{T}];V) \cap C([0,\tilde{T}];L^2). \tag{2.6}$$

3. Global existence and decay of solutions

To state the theorems relative to the global existence and the general decay, we use assumptions on the given datum as follows

$(\bar{\mathbf{H}}_2)$ $\mu \in C^2(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+)$ and $\inf_{t \geq 0} \mu(t) = \mu_0 > 0$;

$(\bar{\mathbf{H}}_3)$ $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a C^1 function satisfying the following conditions

(i) $\mu_0 - \int_0^{+\infty} g(s)ds = L > 0$,

(ii) There exists a positive differentiable function $\xi(t)$ such that

$$g'(t) \leq -\xi(t)g(t) < 0, \ \forall t \geq 0.$$

Furthermore, $\xi(t)$ satisfies

$$\xi'(t) \leq 0, \ \forall t \geq 0, \ \int_0^\infty \xi(s)ds = +\infty;$$

$(\bar{\mathbf{H}}_4)$ $K \in C^1([0,1] \times \mathbb{R}_+)$ and K satisfies two conditions below

(i) $0 < K_1 \leq K(x,t) \leq K_2, \forall (x,t) \in [0,1] \times \mathbb{R}_+$,

(ii) $K_t(x,t) \geq 0, \forall (x,t) \in [0,1] \times \mathbb{R}_+$;

$(\bar{\mathbf{H}}_5)$ the source term $F(u)$ satisfies the following conditions

(i) there exist positive constants d_j, p_j with $p_j > 2, \forall j = \overline{1,N}$ such that

$$0 \leq uF(u) \leq \sum_{j=1}^N d_j |u|^{p_j}, \ \forall u \in \mathbb{R};$$

(ii) there exist two positive constants p and q with $q > p > 2$ such that

$$p\int_0^u F(z)dz \leq uF(u) \leq q\int_0^u F(z)dz, \ \forall u \in \mathbb{R};$$

$(\bar{\mathbf{H}}_6)$ $f \in L^2((0,1) \times \mathbb{R}_+)$ and there exist positive constants C_*, γ_* such that

$$\|f(t)\|^2 \leq C_* e^{-\gamma_* t}, \ \forall t \geq 0.$$

Remark 3.1. There are several functions satisfying the assumption $(\bar{\mathbf{H}}_4)$, such as

$$K(x,t)=\phi(x)(2-e^{-t}) \text{ or } K(x,t)=\phi(x)\left[4-\left(sin^2t+2sintcost+2\right)e^{-t}\right],$$

for any $\phi\in C^1([0,1],\mathbb{R}_+)$. The examples of function g, which satisfy $(\bar{\mathbf{H}}_3)$, can be found in [11].

Remark 3.2. We represent examples for the functions F satisfying $(\bar{\mathbf{H}}_5)$, that is,

$$F(u)=|u|^{p-2}u \text{ or } F(u)=|u|^{p-2}u\ln^r\left(e+u^2\right).$$

Remark 3.3. In this paper, we shall achieve the decay result stated in Theorem 3.10 below without using the assumption $|\xi'(t)/\xi(t)|\le k$ as in [7], to estimate the Lyapunov functional.

Remark 3.4. Since $\xi'(t)\le 0$, then $\xi(t)\le\xi(0)$ for all $t\ge 0$.

Let us present the following functionals

$$\begin{aligned} E(t) &= \frac{1}{2}\left\|u'(t)\right\|^2+J(t), \quad (3.1)\\ J(t) &= \frac{1}{2}\left(\mu(t)-\int_0^t g(s)ds\right)\|u(t)\|_a^2+\frac{1}{2}(g\diamond u)(t)-\int_0^1 K(x,t)\int_0^{u(x,t)}F(z)dzdx,\\ I(t) &= (g\diamond u)(t)+\left(\mu(t)-\int_0^t g(s)ds\right)\|u(t)\|_a^2-p\int_0^1 K(x,t)\int_0^{u(x,t)}F(z)dzdx, \end{aligned}$$

where

$$(g\diamond u)(t)=\int_0^t g(t-s)\|u(t)-u(s)\|_a^2ds. \quad (3.2)$$

Lemma 3.5. *Suppose that* $(\tilde{u}_0,\tilde{u}_1)\in V\times L^2$ *and the assumptions* $(\bar{\mathbf{H}}_2)-(\bar{\mathbf{H}}_6)$ *hold. If* u *is the solution of (2.4)-(2.5), then the energy functional* $E(t)$ *satisfies*

$$E'(t)\le-\left(\lambda-\frac{\varepsilon_0}{2}\right)\left\|u'(t)\right\|^2-\frac{1}{2}\xi(t)(g\diamond u)(t)+\frac{1}{2\varepsilon_0}\|f(t)\|^2\le\frac{1}{2\varepsilon_0}\|f(t)\|^2,\ \forall t\in[0,\tilde{T}], \quad (3.3)$$

for all $0<\varepsilon_0<2\lambda$.

Proof. Multiplying (1.1) by $u'(t)$ and intergrating over Ω, we get

$$\begin{aligned} E'(t) &= -\lambda\left\|u'(t)\right\|^2+\mu'(t)\|u(t)\|_a^2+\frac{1}{2}(g'\diamond u)(t)-\frac{1}{2}g(t)\|u(t)\|_a^2 \quad (3.4)\\ &\quad -\int_0^1 K_t(x,t)dx\int_0^{u(x,t)}F(z)dz+\langle f(t),u'(t)\rangle\\ &\le -\lambda\left\|u'(t)\right\|^2-\frac{1}{2}\xi(t)(g\diamond u)(t)+\langle f(t),u'(t)\rangle. \end{aligned}$$

Applying the Young's inequality for product

$$ab\le\frac{\varepsilon a^2}{2}+\frac{b^2}{2\varepsilon},$$

we verify that there exists a positive constant ε_0 such that

$$\langle f(t),u'(t)\rangle\le\frac{\varepsilon_0}{2}\left\|u'(t)\right\|^2+\frac{1}{2\varepsilon_0}\|f(t)\|^2. \quad (3.5)$$

Combining (3.4) and (3.5), Lemma 3.5 is proved. □

Lemma 3.6. *Assume the assumptions* $(\bar{\mathbf{H}}_2)-(\bar{\mathbf{H}}_6)$ *hold. Additionally, suppose that* $(\tilde{u}_0,\tilde{u}_1)\in V\times L^2$ *such that*

$$\begin{cases} I(0)>0, \\ \eta = L-\frac{K_2}{p}\sum_{j=1}^{N} d_j R_*^{p_j-2}>0, \\ \|\mu\|_{L^\infty(\mathbb{R}_+)}<\frac{p}{q}\left(\mu_0-\int_0^\infty g(s)ds\right), \end{cases} \tag{3.6}$$

where $R_*=\sqrt{\frac{2p}{L(p-2)}\left(E(0)+\frac{1}{2\varepsilon_0}\int_0^\infty \|f(t)\|^2 dt\right)}$. *Then, we deduce that* $I(t)>0,\ \forall t\geq 0.$

Proof. By use of the continuity of $I(t)$ and $I(0)>0$, we have that there exists $T_1>0$ such that $I(t)>0$ for all $t\in[0,T_1]$. We will prove that $I(t)>0$, for all $t\geq 0$. Under the definition of the functional $J(t)$, we have

$$\begin{aligned} J(t) &= \frac{p-2}{2p}\left[(g\diamond u)(t)+\left(\mu(t)-\int_0^t g(s)ds\right)\|u(t)\|_a^2\right]+\frac{1}{p}I(t) \\ &\geq \frac{p-2}{2p}L\|u(t)\|_a^2,\ \forall t\in[0,T_1]. \end{aligned} \tag{3.7}$$

Hence,

$$\begin{aligned} \frac{p}{(p-2)L}\left\|u'(t)\right\|^2+\|u(t)\|_a^2 &= \frac{2p}{(p-2)L}\left[\frac{1}{2}\left\|u'(t)\right\|^2+\frac{p-2}{2p}L\|u(t)\|_a^2\right] \\ &\leq \frac{2p}{(p-2)L}\left[\frac{1}{2}\left\|u'(t)\right\|^2+J(t)\right]=\frac{2p}{(p-2)L}E(t) \\ &\leq \frac{2p}{L(p-2)}\left(E(0)+\frac{1}{2\varepsilon_0}\int_0^\infty\|f(t)\|^2dt\right)=R_*^2,\ \forall t\in[0,T_1]. \end{aligned} \tag{3.8}$$

Using $(\bar{\mathbf{H}}_5)_{(ii)}$, we obtain the following inequality

$$\begin{aligned} \int_0^1 K(x,t)dx\int_0^{u(x,t)}F(z)dz &\leq \frac{1}{p}\int_0^1 K(x,t)u(x,t)F(u(x,t))dx \\ &\leq \frac{K_2}{p}\sum_{j=1}^N d_j\int_0^1|u(x,t)|^{p_j}dx \\ &\leq \frac{K_2}{p}\sum_{j=1}^N d_j\|u(t)\|_a^{p_j} \\ &\leq \frac{K_2}{p}\sum_{j=1}^N d_jR_*^{p_j-2}\|u(t)\|_a^2. \end{aligned} \tag{3.9}$$

Therefore,

$$I(t)\geq(g\diamond u)(t)+\eta\|u(t)\|_a^2,\ \forall t\in[0,T_1], \tag{3.10}$$

where

$$\eta=L-\frac{K_2}{p}\sum_{j=1}^N d_jR_*^{p_j-2}>0.$$

Denote $T_{\max} = \sup\{T > 0 : I(t) > 0, \forall t \in [0,T]\}$ and assume $T_{\max} < +\infty$. From the continuity of $I(t)$, we have $I(T_{\max}) \geq 0$.

We consider two cases for $I(T_{\max})$:

Case $I(T_{\max}) > 0$. Using the same arguments as above, we imply that there exists $\tilde{T} > T_{\max}$ such that $I(t) > 0$, $\forall t \in [0,\tilde{T}]$. This is a contradiction.

Case $I(T_{\max}) = 0$. By use of (3.10), we obtain $(g \diamond u)(T_{\max}) = \|u(T_{\max})\|_a = 0$, which leads to $\int_0^{T_{\max}} g(T_{\max}-s)\|u(s)\|_a^2 ds = 0$. From the continuity of function $s \longmapsto g(T_{\max}-s)\|u(s)\|_a^2 = 0$ on $[0,T_{\max}]$ and $g(T_{\max}-s) > 0$, $\forall s \in [0,T_{\max}]$, we imply $u(s) = 0$, $\forall s \in [0,T_{\max}]$. So, $I(0) = 0 < I(0)$. This is a contradiction.

Thus, in both cases as above, we have $T_{\max} = +\infty$, i.e, $I(t) > 0$, $\forall t \geq 0$. Lemma 3.6 is proved. □

Theorem 3.7. *Let $(\bar{\mathbf{H}}_2)-(\bar{\mathbf{H}}_6)$ hold. Suppose that $(\tilde{u}_0,\tilde{u}_1) \in V \times L^2$ such that (3.6) holds. Then the solution $u(t)$ of (2.4)-(2.5) is bounded and global.*

Since the proof is similar to that of Theorem 4.3 in [1], we omit it here.

In order to get the decay result, we establish the Lyapunov functional as follows

$$\mathscr{F}(t) = E(t) + \delta\psi(t), \tag{3.11}$$

where δ is a positive constant specified later and $\psi(t) = \langle u'(t),u(t)\rangle + \frac{\lambda}{2}\|u(t)\|^2$.

Lemma 3.8. *Assume that (3.6) is satisfied and set*

$$\tilde{E}(t) = \left\|u'(t)\right\|^2 + \|u(t)\|_a^2 + (g \diamond u)(t) + I(t). \tag{3.12}$$

Then, there exist positive constants α_1 and α_2 such that

$$\alpha_1\tilde{E}(t) \leq \mathscr{F}(t) \leq \alpha_2\tilde{E}(t),\ \forall t \geq 0.$$

Proof. It is easy to see from (3.11) that

$$\begin{aligned}
\mathscr{F}(t) &\leq \frac{1+\delta}{2}\left\|u'(t)\right\|^2 + \left[\frac{p-2}{2p}\left(\mu(t) - \int_0^t g(s)ds\right) + \frac{\delta(1+\lambda)}{2}\right]\|u(t)\|_a^2 \\
&\quad + \frac{p-2}{2p}(g \diamond u)(t) + \frac{1}{p}I(t) \\
&\leq \frac{1+\delta}{2}\left\|u'(t)\right\|^2 + \left[\frac{(p-2)\|\mu\|_{L^\infty(\mathbb{R}_+)}}{2p} + \frac{\delta(1+\lambda)}{2}\right]\|u(t)\|_a^2 \\
&\quad + \frac{p-2}{2p}(g \diamond u)(t) + \frac{1}{p}I(t) \\
&\leq \alpha_2\tilde{E}(t),
\end{aligned} \tag{3.13}$$

where

$$\alpha_2 = \max\left\{\frac{1+\delta}{2}, \frac{(p-2)\|\mu\|_{L^\infty(\mathbb{R}_+)}}{2p} + \frac{\delta(1+\lambda)}{2}\right\}.$$

With similar calculations, we have the following estimate

$$\mathscr{F}(t) \geq \frac{1-\delta}{2}\left\|u'(t)\right\|^2 + \frac{(p-2)L - \delta p}{2p}\|u(t)\|_a^2 + \frac{p-2}{2p}(g \diamond u)(t) + \frac{1}{p}I(t). \tag{3.14}$$

If $\delta < \min\left\{1, \dfrac{L(p-2)}{p}\right\}$, then

$$\mathscr{F}(t) \geq \alpha_1 \tilde{E}(t), \tag{3.15}$$

where

$$\alpha_1 = \min\left\{\frac{1-\delta}{2}, \frac{(p-2)L-\delta p}{2p}, \frac{1}{p}, \frac{p-2}{2p}\right\}.$$

This completes the proof of Lemma 3.8. □

Remark 3.9. It is obviously that the energy functional $E(t)$ is equivalent to $\tilde{E}(t)$ in the sense that there exist two positive constants $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ such that $\tilde{\alpha}_1\tilde{E}(t) \leq E(t) \leq \tilde{\alpha}_2\tilde{E}(t)$.

Theorem 3.10. *Assume that* $(\bar{\mathbf{H}}_2)-(\bar{\mathbf{H}}_6)$ *hold. If the initial conditions* $(\tilde{u}_0, \tilde{u}_1) \in V \times L^2$ *satisfy (3.6), then there exist positive constants* C *and* γ *such that*

$$\left\|u'(t)\right\|^2 + \|u(t)\|_a^2 \leq C\exp\left(-\gamma\int_0^t \xi(s)ds\right), \ \forall t \geq 0. \tag{3.16}$$

Proof. Multiplying (1.1) by $u(t)$ and intergrating with respect to x over Ω, we obtain

$$\begin{aligned} &\psi'(t) \\ = &\left\|u'(t)\right\|^2 - \left(\mu(t) - \int_0^t g(s)ds\right)\|u(t)\|_a^2 + \int_0^t g(t-s)a(u(s)-u(t),u(t))\,ds \\ &+\int_0^1 K(x,t)u(x,t)F(u(x,t))dx + \langle f(t),u(t)\rangle. \end{aligned} \tag{3.17}$$

Combining (3.17) and $(\bar{\mathbf{H}}_6)_{(ii)}$, we easily get the following estimate

$$\begin{aligned} \psi'(t) \leq\ & \left\|u'(t)\right\|^2 - \left[L - \frac{\varepsilon_1}{2}(\mu_0 - L + 1)\right]\|u(t)\|_a^2 + \frac{1}{2\varepsilon_1}(g\diamond u)(t) \\ &+q\int_0^1 K(x,t)dx\int_0^{u(x,t)} F(z)dz + \frac{1}{2\varepsilon_1}\|f(t)\|^2, \end{aligned} \tag{3.18}$$

where ε_1 is a positive constant. It follows from Lemma 3.5 and (3.18) that

$$\begin{aligned} \mathscr{F}'(t) \leq\ & -\left(\lambda - \frac{\varepsilon_0}{2} - \delta\right)\left\|u'(t)\right\|^2 - \delta\left[L - \frac{\varepsilon_1}{2}(\mu_0 - L + 1)\right]\|u(t)\|_a^2 + \frac{\delta}{2\varepsilon_1}(g\diamond u)(t) \\ &+\delta q\int_0^1 K(x,t)dx\int_0^{u(x,t)} F(z)dz + \frac{1}{2}\left(\frac{1}{\varepsilon_0} + \frac{\delta}{\varepsilon_1}\right)\|f(t)\|^2. \end{aligned} \tag{3.19}$$

For $\varepsilon_2 \in (0,1)$, we note that

$$\begin{aligned} &\delta q\int_0^1 K(x,t)dx\int_0^{u(x,t)} F(z)dz \\ = &\frac{-\delta q}{p}\left[\varepsilon_2 I(t) - (g\diamond u)(t) - \left(\mu(t) - \int_0^t g(s)ds\right)\|u(t)\|_a^2\right] - \frac{\delta q(1-\varepsilon_2)}{p}I(t) \\ \leq &\frac{-\delta q}{p}(g\diamond u)(t) - \frac{\delta q}{p}\left(\varepsilon_2\eta - \|\mu\|_{L^\infty(\mathbb{R}_+)}\right)\|u(t)\|_a^2 - \frac{\delta q(1-\varepsilon_2)}{p}I(t). \end{aligned} \tag{3.20}$$

We can respectively choose ε_2 and ε_1 as follows

$$\begin{aligned}\varepsilon_2 &< \min\left\{1, \frac{1}{\eta}\|\mu\|_{L^\infty(\mathbb{R}_+)}\right\},\\ \varepsilon_1 &< \frac{2}{\mu_0 - L + 1}\left[L + \frac{q}{p}\left(\varepsilon_2\eta - \|\mu\|_{L^\infty(\mathbb{R}_+)}\right)\right].\end{aligned}$$

If ε_1 and ε_2 are fixed, then the choice of δ satisfying $0 < \delta < \lambda - \frac{\varepsilon_0}{2}$ make

$$\begin{aligned}\beta_1 &= \lambda - \frac{\varepsilon_0}{2} - \delta > 0,\\ \beta_2 &= \delta\left[L - \frac{\varepsilon_1}{2}(\mu_0 - L + 1) + \frac{q}{p}\left(\varepsilon_2\eta - \|\mu\|_{L^\infty(\mathbb{R}_+)}\right)\right] > 0,\\ \beta_3 &= \frac{\delta q(1-\varepsilon_2)}{p} > 0.\end{aligned}$$

It follows from (3.19) and (3.20) that

$$\begin{aligned}&\mathscr{F}'(t)\\ \le\;& -\beta_1\|u'(t)\|^2 - \beta_2\|u(t)\|_a^2 - \beta_3 I(t) + \frac{\delta}{2\varepsilon_1}(g\diamond u)(t) + \frac{1}{2}\left(\frac{1}{\varepsilon_0} + \frac{\delta}{\varepsilon_1}\right)\|f(t)\|^2 \quad (3.21)\\ \le\;& -\beta\tilde{E}(t) + \left[\beta + \delta\left(\frac{q}{p} + \frac{1}{2\varepsilon_1}\right)\right](g\diamond u)(t) + \frac{1}{2}\left(\frac{1}{\varepsilon_0} + \frac{\delta}{\varepsilon_1}\right)\|f(t)\|^2\\ \le\;& -\frac{\beta}{\tilde{\alpha}_2}E(t) + \beta_4(g\diamond u)(t) + \frac{1}{2}\left(\frac{1}{\varepsilon_0} + \frac{\delta}{\varepsilon_1}\right)\|f(t)\|^2,\end{aligned}$$

where $\beta = \min\{\beta_1, \beta_2, \beta_3\}$, and $\beta_4 = \beta + \delta\left(\frac{q}{p} + \frac{1}{2\varepsilon_1}\right)$. Combining (3.21) and (3.3), we obtain

$$\begin{aligned}\xi(t)\mathscr{F}'(t) &\le -\frac{\beta}{\tilde{\alpha}_2}\xi(t)E(t) + \beta_4\xi(t)(g\diamond u)(t) + \frac{1}{2}\left(\frac{1}{\varepsilon_0} + \frac{\delta}{\varepsilon_1}\right)\xi(t)\|f(t)\|^2 \quad (3.22)\\ &\le -\frac{\beta}{\tilde{\alpha}_2}\xi(t)E(t) - 2\beta_4E'(t) + d_*e^{-\gamma_* t},\end{aligned}$$

where

$$d_* = C_*\left[\frac{1}{2}\left(\frac{1}{\varepsilon_0} + \frac{\delta}{\varepsilon_1}\right)\xi(0) + \frac{\beta_4}{\varepsilon_0}\right].$$

We consider the functional $\mathscr{L}(t) = \xi(t)\mathscr{F}(t) + 2\beta_4E(t)$ introduced by [6], which is equivalent to $E(t)$. From estimate (3.22), we find that there exists a positive constant $\gamma < \frac{\gamma_*}{\xi(0)}$ such that

$$\mathscr{L}'(t) \le -\gamma\xi(t)\mathscr{L}(t) + d_*e^{-\gamma_* t}. \tag{3.23}$$

Integrating (3.23) with regard to time variable leads to

$$\mathscr{L}(t) \le \left(\frac{d_*}{\gamma_* - \gamma\xi(0)} + \mathscr{L}(0)\right)\exp\left(-\gamma\int_0^t \xi(s)ds\right). \tag{3.24}$$

This completes the proof of Theorem 3.10. □

Remark 3.11. In [1, 6, 11], the functional

$$\chi(t)=\int_{\Omega}u'(x,t)\int_{0}^{t}g(t-s)\,(u(x,t)-u(x,s))\,dsdx,$$

was used to obtain general rates of decay. In our result, we establish the general rate decay without using this functional.

Remark 3.12. It can be obtained exponential decay for $\xi(t)\equiv a$, polynomial decay for $\xi(t)=a(1+t)^{-1}$, and logarit decay for $\xi(t)=\dfrac{a}{(1+t)\,(1+\ln(1+t))}$, where $a>0$ is a constant. For more details, we have

(i) In case $\xi(t)\equiv a:\exp\left(-\gamma\int_0^t\xi(s)ds\right)=\exp\left(-\gamma at\right)$ (exponential decay);

(ii) In case $\xi(t)=a(1+t)^{-1}:\exp\left(-\gamma\int_0^t\xi(s)ds\right)=\dfrac{1}{(1+t)^{\gamma a}}$ (polynomial decay);

(iii) In case $\xi(t)=\dfrac{a}{(1+t)\,(1+\ln(1+t))}:\exp\left(-\gamma\int_0^t\xi(s)ds\right)=\dfrac{1}{(\;1+\ln(1+t))^{\gamma a}}$ (logarit decay).

4. Blow-up at finite time

This section will prove that the weak solution to problem (1.1)-(1.3) blows up at finite time with $f(x,t)\equiv 0$, under some suitable conditions of source term $F(u)$ and relaxation function $g(t)$. We will consider the blow-up property in three cases of the initial energy, when the initial energy is negative, null or positive.

Let us state the blow-up result when the initial energy is negative. In this case, we suppose that source term $F(u)$ satisfies the following assumption.

$(\mathbf{F_1})$ F is in class C^1 and satisfies

(i) there exist two positive constants γ and $k>2$ such that

$$uF(u)\geq\gamma|u|^k,\;\forall u\in\mathbb{R};$$

(ii) there exist two positive constants $p>2$ and q such that

$$p\int_0^u F(z)dz\leq uF(u)\leq q\int_0^u F(z)dz,\;\forall u\in\mathbb{R}. \tag{4.1}$$

Theorem 4.1. *Assume that the assumptions* $(\overline{\mathbf{H}}_\mathbf{2})$, $(\overline{\mathbf{H}}_\mathbf{4})$ *and* $(\overline{\mathbf{H}}_\mathbf{5})$ *hold. Suppose that* $g:\mathbb{R}_+\to\mathbb{R}_+$ *such that* $g'(t)\leq 0$ *and satisfies*

$(\mathbf{G}_0^*)$ $0<\int_0^\infty g(s)ds<\dfrac{\mu_0(p-2)}{p-2+1/p}=\mu_0\left(1-\dfrac{1}{(p-1)^2}\right).$

Then, for any initial conditions $(\tilde{u}_0,\tilde{u}_1)\in V\times L^2$ *such that* $E(0)<0$, *the weak solution* $u(t)$ *of the problem (1.1)-(1.3) blows up at finite time.*

In order to prove Theorem 4.1, we establish a series of lemmas.

Lemma 4.2. *If* u *is the solution of (2.4)-(2.5), then the energy functional* $E(t)$ *is nonincreasing, i.e,* $E'(t)\leq 0$.

From the proof of Lemma 3.5, we can conclude the desired conclusion immediately.

Lemma 4.3. *There exists a positive constant C such that*

$$\|u(t)\|_{L^k}^s \le C\left(\|u(t)\|_a^2 + \int_0^1 K(x,t)\int_0^{u(x,t)} F(z)dzdx\right), \tag{4.2}$$

for $2 \le s \le k$.

Proof. We consider two cases for $\|u(t)\|_{L^k}$.

(i) Case 1: If $\|u(t)\|_{L^k} \le 1$, then $\|u(t)\|_{L^k}^s \le \|u(t)\|_{L^k}^2 \le \|u(t)\|_a^2$ due to the continuous embedding $V \hookrightarrow L^k$.

(ii) Case 2: We consider $\|u(t)\|_{L^k} > 1$. From (4.1), we have the following inequalities

$$\frac{1}{q}yF(y) \le \int_0^y F(z)dz \le \frac{1}{p}yF(y),\ \forall y \in \mathbb{R}.$$

Combining with the assumptions $(\bar{\mathbf{H}}_4)$ and $(\mathbf{F_1})$, we obtain

$$\begin{aligned}\|u(t)\|_{L^k}^s &\le \|u(t)\|_{L^k}^k = \int_0^1 |u(x,t)|^k dx \\ &\le \frac{1}{\gamma}\int_0^1 u(x,t)F(u(x,t))dx \\ &\le \frac{q}{K_1}\int_0^1 K(x,t)dx\int_0^{u(x,t)} F(z)dz.\end{aligned}$$

This completes the proof of this lemma. □

Proof of Theorem 4.1. Put $H(t) = -E(t)$. Then, it follows from Lemma 4.2 that

$$0 < H(0) \le H(t) \le \int_0^1 K(x,t)dx\int_0^{u(x,t)} F(z)dz. \tag{4.3}$$

To prove this theorem, we use the functional in [10] and [14] with a slight modification, that is,

$$L(t) = H^{1-\sigma}(t) + \psi(t), \tag{4.4}$$

where $0 < \sigma < \dfrac{k-2}{2k}$. Taking the derivative of (4.4) and using equation (1.1), we obtain

$$\begin{aligned}L'(t) &= (1-\sigma)H^{-\sigma}(t)H'(t) + \left\|u'(t)\right\|^2 - \left(\mu(t) - \int_0^t g(s)ds\right)\|u(t)\|_a^2 \\ &\quad + \int_0^t g(t-s)a\left(u(s)-u(t),u(t)\right)ds + \int_0^1 K(x,t)u(x,t)F(u(x,t))dx \\ &\ge \left\|u'(t)\right\|^2 - \left(\mu(t) - \int_0^t g(s)ds\right)\|u(t)\|_a^2 \\ &\quad -\frac{\theta}{2}(g\diamond u)(t) - \frac{1}{2\theta}\int_0^t g(s)ds\,\|u(t)\|_a^2 + p\int_0^1 K(x,t)dx\int_0^{u(x,t)} F(z)dz\end{aligned} \tag{4.5}$$

$$\begin{aligned}
&\geq \left\|u'(t)\right\|^2-\left(\mu(t)-\int_0^t g(s)ds\right)\|u(t)\|_a^2\\
&\quad+\theta\left[H(t)+\frac{1}{2}\left\|u'(t)\right\|^2+\frac{1}{2}\left(\mu(t)-\int_0^t g(s)ds\right)\|u(t)\|_a^2\right]\\
&\quad-\frac{1}{2\theta}\int_0^t g(s)ds\,\|u(t)\|_a^2+\left(p-\frac{\theta}{2}\right)\int_0^1 K(x,t)dx\int_0^{u(x,t)}F(z)dz\\
&\geq \theta H(t)+\left(1+\frac{\theta}{2}\right)\left\|u'(t)\right\|^2+\left[\left(\frac{\theta}{2}-1\right)\left(\mu_0-\int_0^\infty g(s)ds\right)-\frac{1}{2\theta}\int_0^\infty g(s)ds\right]\|u(t)\|_a^2\\
&\quad+(p-\theta)\int_0^1 K(x,t)dx\int_0^{u(x,t)}F(z)dz,
\end{aligned}$$

for all $\theta>0$. According to condition $(\mathbf{G}_0^*)$, we find that there exists a positive number $\theta_*\in(2,p)$ such that

$$0<\int_0^\infty g(s)ds<\mu_0\left(1-\frac{1}{(\theta_*-1)^2}\right)<\mu_0\left(1-\frac{1}{(p-1)^2}\right),$$

which leads to

$$\left[\left(\frac{\theta_*}{2}-1\right)\left(\mu_0-\int_0^\infty g(s)ds\right)-\frac{1}{2\theta_*}\int_0^\infty g(s)ds\right]\|u(t)\|_a^2>0.$$

By choosing $\theta=\theta_*$, it follows from (4.5) that there exists a positive constant d_0 such that

$$L'(t)\geq d_0\left(H(t)+\left\|u'(t)\right\|^2+\|u(t)\|_a^2+\int_0^1 K(x,t)dx\int_0^{u(x,t)}F(z)dz\right). \tag{4.6}$$

Using te Cauchy-Schwarz's inequality and the continuous imbedding $L^2\hookrightarrow L^k$, we get the estimate

$$\left|\langle u'(t),u(t)\rangle\right|\leq\left\|u'(t)\right\|\|u(t)\|_{L^k}, \tag{4.7}$$

which implies

$$\left|\langle u'(t),u(t)\rangle\right|^{\frac{1}{1-\sigma}}\leq\left\|u'(t)\right\|^{\frac{1}{1-\sigma}}\|u(t)\|_{L^k}^{\frac{1}{1-\sigma}}. \tag{4.8}$$

Applying Young's inequality for $\frac{1}{p_1}+\frac{1}{p_2}=1$, we find that

$$\left|\langle u'(t),u(t)\rangle\right|^{\frac{1}{1-\sigma}}\leq\frac{1}{p_1}\left\|u'(t)\right\|^{\frac{p_1}{1-\sigma}}+\frac{1}{p_2}\|u(t)\|_{L^k}^{\frac{p_2}{1-\sigma}}. \tag{4.9}$$

Taking $p_1=2(1-\sigma)$, we find that estimate (4.9) becomes

$$\left|\langle u'(t),u(t)\rangle\right|^{\frac{1}{1-\sigma}}\leq\frac{1}{p_1}\left\|u'(t)\right\|^2+\frac{1}{p_2}\|u(t)\|_{L^k}^{\frac{2}{1-2\sigma}}. \tag{4.10}$$

Combining (4.10) and applying Lemma 4.3 for $s=\frac{2}{1-2\sigma}\leq k$, $s=\frac{2}{1-\sigma}\leq k$, we investigate that there exists a positive constant d_1 such that

$$\begin{aligned}
&\left|\langle u'(t),u(t)\rangle\right|^{\frac{1}{1-\sigma}}+\|u(t)\|^{\frac{2}{1-\sigma}} &(4.11)\\
&\leq d_1\left(\left\|u'(t)\right\|^2+\|u(t)\|_a^2+\int_0^1 K(x,t)dx\int_0^{u(x,t)}F(z)dz\right).
\end{aligned}$$

Therefore, we have the following estimate

$$\begin{aligned} L^{\frac{1}{1-\sigma}}(t) &= \left(H^{1-\sigma}(t)+\langle u'(t),u(t)\rangle+\frac{\lambda}{2}\|u(t)\|^2\right)^{\frac{1}{1-\sigma}} \\ &\leq Const\left(H(t)+\left|\langle u'(t),u(t)\rangle\right|^{\frac{1}{1-\sigma}}+\|u(t)\|^{\frac{2}{1-\sigma}}\right) \\ &\leq Const\left(H(t)+\left\|u'(t)\right\|^2+\|u(t)\|_a^2+\int_0^1 K(x,t)dx\int_0^{u(x,t)}F(z)dz\right). \end{aligned} \tag{4.12}$$

Combining (4.12) and (4.6), we refer that there exists a positive constant D such that

$$L'(t) \geq DL^{\frac{1}{1-\sigma}}(t). \tag{4.13}$$

By integrating (4.13) with respect to t over $(0,t)$, we deduce that

$$L^{\frac{1}{1-\sigma}}(t) \geq \frac{1}{L^{\frac{-\sigma}{1-\sigma}}(0)-\frac{D\sigma t}{1-\sigma}}. \tag{4.14}$$

Therefore, the solution u blows up at finite time given by

$$\bar{T}=\frac{1-\sigma}{D\sigma}L^{\frac{-\sigma}{1-\sigma}}(0).$$

This completes the proof. □

In the following part, we prove that the weak solution $u(t)$ blows up at finite time when the initial energy is null or positive. In this process, we use the concavity method introduced by Levine-Payne [18].

Suppose that $g:\mathbb{R}_+\to\mathbb{R}_+$ such that $g'(t)\leq 0$ and the following additional assumptions hold

$(\mathbf{G}_0)$ $\int_0^\infty g(s)ds < \dfrac{(p-2)\mu_0}{2(p-2+1/p)}$;

$(\mathbf{G}_1)$ $\int_0^t v(\tau)\int_0^\tau e^{\frac{\lambda(\tau-s)}{2}}g(\tau-s)v(s)dsd\tau \geq 0$, $\forall v\in C^1(\mathbb{R}_+)$, $\forall t\geq 0$.

Remark 4.4. We demonstrate the function $g(t)$ satisfying $(\mathbf{G}_0)-(\mathbf{G}_1)$ by the following examples

$$g(t)=e^{-\alpha t},\text{ for }\alpha\geq\frac{\lambda}{2};\text{ or }g(t)=e^{-\frac{\lambda}{2}t}\cos^2(\alpha t),\text{ for }\alpha>0.$$

Moreover, we also assume that the source term $F(u)$ satisfying

$(\mathbf{F_1^*})$ $\int_0^u F(z)dz\geq 0$, $\forall u\in\mathbb{R}$;

$(\mathbf{F_2^*})$ There exists a positive constant p such that

$$p\int_0^u F(z)dz \leq uF(u),\ \forall u\in\mathbb{R}. \tag{4.15}$$

Remark 4.5. With the examples in Section 3, we can give more examples for the functions F satisfying conditions $(\mathbf{F_1^*})$ and $(\mathbf{F_2^*})$ as below

$F(u)=|u|^{p-2}u\left(\dfrac{e^u+e^{-u}}{2}\right)$; or $F(u)=|u|^{p-2}ue^{u^2}$.

Theorem 4.6. *Assume that* $\lambda > 0$, $p > 3$, $2\lambda^2 < (p-2)\mu_0$ *and* $(\bar{\mathbf{H}}_2)$, $(\bar{\mathbf{H}}_4)$, $(\mathbf{G}_0)$, $(\mathbf{F}_\mathbf{1}^*)$, $(\mathbf{F}_\mathbf{2}^*)$ *hold. If* $(\tilde{u}_0, \tilde{u}_1) \in V \times L^2$ *such that* $E(0) = 0$ *and* $\langle \tilde{u}_0, \tilde{u}_1 \rangle > 0$*, then the solution of the problem (1.1)-(1.3) blows up at finite time.*

Proof. It follows from the Lemma 4.2 that the derivative of energy functional is negative, which implies that $E(t) \le E(0)$. We define the auxiliary functional

$$G(t) = \|u(t)\|^2. \tag{4.16}$$

By direct computation, we achieve that

$$\begin{cases} G'(t) = 2\langle u'(t), u(t)\rangle, \\ G''(t) = 2\langle u''(t), u(t)\rangle + 2\left\|u'(t)\right\|^2. \end{cases} \tag{4.17}$$

Multiplying equation (1.1) to $u(t)$ and integrating over Ω, we get the following estimates

$$\begin{aligned} \frac{1}{2}G''(t) &= \left\|u'(t)\right\|^2 - \left(\mu(t) - \int_0^t g(s)ds\right)\|u(t)\|_a^2 - \lambda\langle u'(t), u(t)\rangle \\ &\quad + \int_0^t g(t-s)a(u(s)-u(t), u(t))ds + \int_0^1 K(x,t)u(x,t)F(u(x,t))dx \\ &\ge \left\|u'(t)\right\|^2 - \left(\mu(t) - \int_0^t g(s)ds\right)\|u(t)\|_a^2 - \frac{\lambda\varepsilon}{2}\|u'(t)\|^2 - \frac{\lambda}{2\varepsilon}\|u(t)\|_a^2 \\ &\quad - \frac{\tilde{\varepsilon}}{2}(g\diamond u)(t) - \frac{1}{2\tilde{\varepsilon}}\int_0^t g(s)ds\,\|u(t)\|_a^2 + p\int_0^1 K(x,t)dx\int_0^{u(x,t)} F(z)dz \\ &\ge \frac{1}{2}(2-\lambda\varepsilon)\left\|u'(t)\right\|^2 - \left[\left(\mu(t) - \int_0^t g(s)ds\right) + \frac{1}{2\tilde{\varepsilon}}\int_0^t g(s)ds + \frac{\lambda}{2\varepsilon}\right]\|u(t)\|_a^2 \\ &\quad + \frac{p}{2}\left\|u'(t)\right\|^2 + \frac{p}{2}\left(\mu(t) - \int_0^t g(s)ds\right)\|u(t)\|_a^2 + \frac{1}{2}(p-\tilde{\varepsilon})(g\diamond u)(t) - pE(t) \\ &\ge \frac{1}{2}(p+2-\lambda\varepsilon)\left\|u'(t)\right\|^2 + \frac{1}{2}(p-\tilde{\varepsilon})(g\diamond u)(t) \\ &\quad + \frac{1}{2}\left[(p-2)\left(\mu(t) - \int_0^t g(s)ds\right) - \frac{1}{\tilde{\varepsilon}}\int_0^t g(s)ds - \frac{\lambda}{\varepsilon}\right]\|u(t)\|_a^2 \\ &\ge \frac{1}{2}(p+2-\lambda\varepsilon)\left\|u'(t)\right\|^2 + \frac{1}{2}(p-\tilde{\varepsilon})(g\diamond u)(t) \\ &\quad + \frac{1}{2}\left[(p-2)\left(\mu_0 - \int_0^\infty g(s)ds\right) - \frac{1}{\tilde{\varepsilon}}\int_0^\infty g(s)ds - \frac{\lambda}{\varepsilon}\right]\|u(t)\|_a^2. \end{aligned} \tag{4.18}$$

Choose $\tilde{\varepsilon} = p$ and $\varepsilon = \dfrac{2\lambda}{\mu_0(p-2)}$. From the choices of ε, $\tilde{\varepsilon}$ and the assumptions, we obtain that

$$\begin{cases} p+2-\lambda\varepsilon > p+1 > 0, \\ (p-2)\left(\mu_0 - \int_0^\infty g(s)ds\right) - \dfrac{1}{\tilde{\varepsilon}}\int_0^\infty g(s)ds - \dfrac{\lambda}{\varepsilon} > 0, \\ p - \tilde{\varepsilon} = 0. \end{cases} \tag{4.19}$$

Combining (4.18) and (4.19), we refer that

$$G''(t) \ge (p+1)\left\|u'(t)\right\|^2. \tag{4.20}$$

Therefore,

$$G''(t)G(t)-\frac{p+1}{4}\left(G'(t)\right)^2 \geq (p+1)\left(\|u(t)\|^2\left\|u'(t)\right\|^2-\left|\langle u(t),u'(t)\rangle\right|^2\right)\geq 0. \tag{4.21}$$

Inequality (4.21) implies that

$$\frac{d}{dt}\left(\frac{G'(t)}{G^{(p+1)/4}(t)}\right)\geq 0, \tag{4.22}$$

which leads to

$$\frac{G'(t)}{G^{(p+1)/4}(t)}\geq \frac{G'(0)}{G^{(p+1)/4}(0)}. \tag{4.23}$$

Integrating (4.23) with respect to time variable from 0 to t, we obtain

$$G^{(p-3)/4}(t)\geq \frac{4G^{(p+1)/4}(0)}{4G(0)-(p-3)G'(0)t}. \tag{4.24}$$

Inequality (4.24) shows that the solution blows up at finite time

$$\bar{T}=\frac{4G(0)}{(p-3)G'(0)}=\frac{2\|\tilde{u}_0\|^2}{(p-3)\langle\tilde{u}_0,\tilde{u}_1\rangle}.$$

This completes the proof. □

Lemma 4.7. *Assume that $g(t)$ satisfies $(\mathbf{G}_0)$, $(\mathbf{G}_1)$ and h is a twice continuously differentiable satisfying*

$$\begin{cases} h''(t)+\lambda h'(t)>2\int_0^t g(t-s)a(u(s),u(t))ds,\ t\in(0,\tilde{T}),\\ h(0)>0,\ h'(0)>0, \end{cases} \tag{4.25}$$

where $u(t)$ is the corresponding solution to (1.1)-(1.3). Then, h is strictly increasing.

Proof. We consider the following ordinary differential equation

$$\begin{cases} \bar{h}''(t)+\lambda \bar{h}'(t)=2\int_0^t g(t-s)a(u(s),u(t))ds,\ t\in(0,\tilde{T}),\\ \bar{h}(0)=h(0),\ \bar{h}'(0)=0. \end{cases} \tag{4.26}$$

It is easy to see that (4.26) has unique solution $\bar{h}(t)$ determined on $[0,\tilde{T}]$. From (4.26), the derivative of $\bar{h}(t)$ can be written as follows

$$\begin{aligned} \bar{h}'(t)&=2\int_0^t e^{\lambda(\tau-t)}\int_0^\tau g(\tau-s)a(u(s),u(\tau))dsd\tau\\ &=2\int_0^t e^{\lambda(\tau-t)}\int_0^\tau g(\tau-s)\left[\int_0^1 u_x(x,s)u_x(x,\tau)dx+u(0,s)u(0,\tau)\right]dsd\tau\\ &=2e^{-\lambda t}\int_0^1\int_0^t\left(e^{\frac{\lambda\tau}{2}}u_x(x,\tau)\right)\int_0^\tau e^{\frac{\lambda(\tau-s)}{2}}g(\tau-s)\left(e^{\frac{\lambda s}{2}}u_x(x,s)\right)dsd\tau dx\\ &\quad+2e^{-\lambda t}\int_0^t\left(e^{\frac{\lambda\tau}{2}}u(0,\tau)\right)\int_0^\tau e^{\frac{\lambda(\tau-s)}{2}}g(\tau-s)\left(e^{\frac{\lambda s}{2}}u(0,s)\right)dsd\tau\geq 0, \end{aligned} \tag{4.27}$$

which implies that $\bar{h}(t)\geq\bar{h}(0)=h(0)$.

We next show that $h'(t)>\bar{h}'(t)$, for $t\geq 0$. It follows from (4.25) and (4.26) that

$$\begin{cases} \left(h''(t)-\bar{h}''(t)\right)+\lambda\left(h'(t)-\bar{h}'(t)\right)>0,\ t\in[0,\tilde{T}),\\ h(0)-\bar{h}(0)=0,\ h'(0)-\bar{h}'(0)>0. \end{cases} \tag{4.28}$$

Inequality (4.28) implies that

$$h'(t)-\bar{h}'(t)\geq e^{-\lambda t}\left(h'(0)-\bar{h}'(0)\right)>0,\ \forall t\in[0,\tilde{T}). \tag{4.29}$$

This shows that $h'(t) > 0$, i.e, $h(t)$ is strictly increasing in $[0, \tilde{T})$. □

Lemma 4.8. *Suppose that* $(\tilde{u}_0, \tilde{u}_1) \in V \times L^2$ *such that* $\langle \tilde{u}_0, \tilde{u}_1 \rangle > 0$. *If the weak solution* $u(t)$ *satisfies* $\tilde{I}(t) < 0$, $\forall t \in [0, \tilde{T})$, *where the functional* $\tilde{I}(t)$ *is defined as*

$$\tilde{I}(t) = \tilde{I}(u(t)) = \mu(t) \|u(t)\|_a^2 - \int_0^1 K(x,t)u(x,t)F(u(x,t))dx, \tag{4.30}$$

then $\|u(t)\|^2$ *is strictly increasing on* $[0, \tilde{T})$.

Proof. It follows from (4.18) that

$$\begin{aligned} \frac{1}{2}G''(t) &= \left\|u'(t)\right\|^2 - \mu(t)\|u(t)\|_a^2 - \lambda\langle u'(t), u(t)\rangle + \int_0^t g(t-s)a(u(s),u(t))ds \qquad (4.31)\\ &\quad + \int_0^1 K(x,t)u(x,t)F(u(x,t))dx \\ &= \left\|u'(t)\right\|^2 - \tilde{I}(t) - \lambda\langle u'(t), u(t)\rangle + \int_0^t g(t-s)a(u(s),u(t))ds \\ &> -\lambda\langle u'(t), u(t)\rangle + \int_0^t g(t-s)a(u(s),u(t))ds \\ &= -\frac{\lambda}{2}G'(t) + \int_0^t g(t-s)a(u(s),u(t))ds. \end{aligned}$$

Therefore, we obtain that

$$\begin{aligned} G''(t) + \lambda G'(t) &> 2\int_0^t g(t-s)a(u(s),u(t))ds, \qquad (4.32)\\ G(0) &= \|\tilde{u}_1\|^2 > 0, \\ G'(0) &= 2\langle \tilde{u}_0, \tilde{u}_1\rangle > 0. \end{aligned}$$

Applying Lemma 4.7, we conclude that $G(t)$ is strictly increasing. This completes the proof. □

Theorem 4.9. *Assume that* $p \geq 2 + \sqrt{2}$, $2\lambda^2 < (p-2)\mu_0$ *and* $(\bar{\mathbf{H}}_2)$, $(\bar{\mathbf{H}}_4)$, $(\mathbf{G}_0)$, $(\mathbf{G}_1)$, $(\mathbf{F}_1^*)$, $(\mathbf{F}_2^*)$ *hold. If the initial conditions* $(\tilde{u}_0, \tilde{u}_1) \in V \times L^2$ *satisfy the following conditions*

$$\begin{cases} E(0) > 0, \\ E(0) < \dfrac{1}{4p}\left[(p-2)\mu_0 - 2\left(p - 2 + \dfrac{1}{p}\right)\int_0^\infty g(s)ds\right]\|\tilde{u}_0\|^2, \\ \langle \tilde{u}_0, \tilde{u}_1\rangle > 0, \\ \tilde{I}(0) = \tilde{I}(\tilde{u}_0) < 0, \end{cases} \tag{4.33}$$

then the weak solution $u(t)$ *of the problem (1.1)-(1.3) blows up at finite time.*

Proof. First, we claim that

$$\tilde{I}(t) < 0, \ \forall t \in [0, \tilde{T}), \tag{4.34}$$

and

$$\|u(t)\|^2 > \frac{4pE(0)}{(p-2)\mu_0 - 2(p-2+1/p)\int_0^\infty g(s)ds}, \ \forall t \in [0, \tilde{T}). \tag{4.35}$$

Let us denote $t_{min} > 0$ as follow

$$t_{min} = \inf\left\{t \in (0, \tilde{T}] : \tilde{I}(t) = 0\right\}. \tag{4.36}$$

Assume that $t_{min} < \tilde{T}$. By the continuity of the functional $\tilde{I}(t)$, we see that $\tilde{I}(t) < 0$, $\forall t \in [0, t_{min})$ and $\tilde{I}(t_{min}) = 0$. Consequently, it follows from Lemma 4.8 that

$$\|u(t)\|^2 > \|\tilde{u}_0\|^2 > \frac{4pE(0)}{(p-2)\mu_0 - 2(p-2+1/p)\int_0^\infty g(s)ds}, \; \forall t \in [0, t_{min}). \tag{4.37}$$

Moreover, it is obvious that $\|u(t)\|^2$ is continuous on $[0, t_{min}]$. Hence, the following inequality is achieved

$$\|u(t_{min})\|^2 > \frac{4pE(0)}{(p-2)\mu_0 - 2(p-2+1/p)\int_0^\infty g(s)ds}. \tag{4.38}$$

On the other hand, it follows from Lemma 4.2 and $\tilde{I}(t_{min}) = 0$ that

$$\begin{aligned} E(0) \;\; \geq \;\; & \frac{1}{2}\left(\mu(t_{min}) - \int_0^{t_{min}} g(s)ds\right)\|u(t_{min})\|_a^2 \\ & + \frac{1}{2}(g \diamond u)(t_{min}) - \int_0^1 K(x, t_{min}) \int_0^{u(x,t_{min})} F(z)dzdx \\ \geq \;\; & \frac{1}{2}\left(\mu(t_{min}) - \int_0^{t_{min}} g(s)ds\right)\|u(t_{min})\|_a^2 + \frac{1}{2}(g \diamond u)(t_{min}) \\ & - \frac{1}{p}\int_0^1 K(x, t_{min}) u(x, t_{min}) F(u(x, t_{min}))dx \\ \geq \;\; & \left[\frac{p-2}{2p}\mu(t_{min}) - \frac{1}{2}\int_0^{t_{min}} g(s)ds\right]\|u(t_{min})\|_a^2 \\ \geq \;\; & \frac{1}{2p}\left[(p-2)\mu_0 - p\int_0^\infty g(s)ds\right]\|u(t_{min})\|^2 \\ \geq \;\; & \frac{1}{2p}\left[(p-2)\mu_0 - 2\left(p - 2 + \frac{1}{p}\right)\int_0^\infty g(s)ds\right]\|u(t_{min})\|^2. \end{aligned} \tag{4.39}$$

Therefore,

$$\|u(t_{min})\|^2 \leq \frac{4pE(0)}{(p-2)\mu_0 - 2(p-2+1/p)\int_0^\infty g(s)ds}. \tag{4.40}$$

There is a contradiction between (4.38) and (4.40). This leads to $t_{min} = \tilde{T}$. Additionally, this also employs that $\|u(t)\|^2$ is strictly increasing on $[0, \tilde{T})$. Next, we prove the blow-up result of the weak solution $u(t)$. We introduce the funtional

$$\tilde{G}(t) = \|u(t)\|^2 + \Lambda t^2, \tag{4.41}$$

where Λ is a positive constant which, is chosen later. Using the similar computation as in (4.19), we have the estimate

$$\begin{aligned} \frac{1}{2}\tilde{G}''(t) \;\; \geq \;\; & \frac{1}{2}(p + 2 - \lambda\varepsilon)\left\|u'(t)\right\|^2 + \frac{1}{2}(p - \tilde{\varepsilon})(g \diamond u)(t) \\ & + \frac{1}{2}\left[(p-2)\left(\mu_0 - \int_0^\infty g(s)ds\right) - \frac{1}{\tilde{\varepsilon}}\int_0^\infty g(s)ds - \frac{\lambda}{\varepsilon}\right]\|u(t)\|_a^2 + \Lambda - pE(0) \\ \geq \;\; & \frac{1}{2}(p + 2 - \lambda\varepsilon)\left\|u'(t)\right\|^2 + \frac{1}{2}(p - \tilde{\varepsilon})(g \diamond u)(t) \\ & + \frac{1}{2}\left[(p-2)\left(\mu_0 - \int_0^\infty g(s)ds\right) - \frac{1}{\tilde{\varepsilon}}\int_0^\infty g(s)ds - \frac{\lambda}{\varepsilon}\right]\|\tilde{u}_0\|^2 + \Lambda - pE(0). \end{aligned} \tag{4.42}$$

If we choose $\tilde{\varepsilon} = p$ and $\varepsilon = \dfrac{2\lambda}{\mu_0(p-2)}$, then

$$\frac{1}{2}\tilde{G}''(t) \geq \frac{p+1}{2}\left\|u'(t)\right\|^2 + \frac{1}{4}\left[(p-2)\mu_0 - 2\left(p-2+\frac{1}{p}\right)\int_0^\infty g(s)ds\right]\|\tilde{u}_0\|^2 + \Lambda - pE(0). \tag{4.43}$$

Thus, if Λ is small enough, then

$$p\Lambda \leq \frac{1}{4}\left[(p-2)\mu_0 - 2\left(p-2+\frac{1}{p}\right)\int_0^\infty g(s)ds\right]\|\tilde{u}_0\|^2 - pE(0). \tag{4.44}$$

From (4.43) and (4.44), we get

$$\tilde{G}''(t) \geq (p+1)\left(\left\|u'(t)\right\|^2 + \Lambda\right). \tag{4.45}$$

As a result, we obtain the following inequality

$$\begin{aligned} &\tilde{G}''(t)\tilde{G}(t) - \frac{p+1}{4}\left(\tilde{G}'(t)\right)^2 \\ \geq\ & (p+1)\left(\Lambda\left\|u(t)-tu'(t)\right\|^2 + \left\|u'(t)\right\|^2\|u(t)\|^2 - \left|\langle u'(t),u(t)\rangle\right|^2\right) \geq 0. \end{aligned} \tag{4.46}$$

Then, we conclude that the weak solution blows up at finite time $\bar{T} = \frac{2\|\tilde{u}_0\|^2}{(p-3)\langle\tilde{u}_0,\tilde{u}_1\rangle}$. Theorem 4.7 is completely proved. □

5. Numerical results

Consider the following problem:

$$\begin{cases} u_{tt} + \lambda u_t - \mu(t)u_{xx} + \int_0^t g(t-s)u_{xx}(x,s)ds = K(x,t)\,|u|^{p-2}u\ln^r(e+u^2) \\ \qquad\qquad + f(x,t),\ 0<x<1,\ 0<t<T, \\ u_x(0,t) - u(0,t) = u(1,t) = 0, \\ u(x,0) = \tilde{u}_0(x),\ u_t(x,0) = \tilde{u}_1(x), \end{cases} \tag{5.1}$$

where $p=6$, $r=\dfrac{5}{2}$, $\lambda=1$ are constants and the functions $\mu(t)$, $g(t)$, $K(x,t)$, f, $\tilde{u}_0$ and $\tilde{u}_1$ are defined by

$$\begin{cases} \mu(t) = 1+e^{-t},\ g(t) = e^{-3t},\ K(x,t) = (1+x)(2-e^{-t}), \\ f(x,t) = 2e^{-t}(1+e^{-t})^2 - \frac{1}{4}(1+x)(2-e^{-t})e^{-5t}w^5(x)\ln^{5/2}\left[e+e^{-2t}w^2(x)\right], \\ \tilde{u}_0(x) = w(x),\ \tilde{u}_1(x) = -w(x), \\ w(x) = -2x^2+x+1. \end{cases} \tag{5.2}$$

The exact solution of problem (5.1) with $\mu(t)$, $g(t)$, $K(x,t)$, f, $\tilde{u}_0$ and $\tilde{u}_1$ defined in (5.2) respectively is the function u_{ex} given by

$$u_{ex}(x,t) = e^{-t}w(x). \tag{5.3}$$

In order to solve problem (5.1) numerically, we consider the differential system for the unknowns $U_i(t) \equiv u(x_i,t)$, $V_i(t) = \frac{dU_i}{dt}(t)$, with $x_i = i\triangle x$, $\triangle x = \frac{1}{N+1}$, $i = \overline{0,N+1}$:

$$\begin{cases} \frac{dU_i}{dt}(t) = V_i(t),\, i = \overline{0,N}, \\ \frac{dV_0}{dt}(t) = -\lambda V_0(t) + \alpha(t)\left[-(1+\triangle x)U_0(t) + U_1(t)\right] \\ \qquad\qquad -\frac{1}{(\triangle x)^2}\int_0^t g(t-s)\left[-(1+\triangle x)U_0(s)+U_1(s)\right]ds + F_0\left(x_0,t,U_0(t)\right), \\ \frac{dV_i}{dt}(t) = -\lambda V_i(t) + \alpha(t)\left(U_{i-1}(t) - 2U_i(t) + U_{i+1}(t)\right) \\ \qquad\qquad -\frac{1}{(\triangle x)^2}\int_0^t g(t-s)\left(U_{i-1}(s) - 2U_i(s) + U_{i+1}(s)\right)ds \\ \qquad\qquad +F_i\left(x_i,t,U_i(t)\right),\, i = \overline{1,N-1}, \\ \frac{dV_N}{dt}(t) = -\lambda V_N(t) + \alpha(t)\left(U_{N-1}(t) - 2U_N(t)\right) \\ \qquad\qquad -\frac{1}{(\triangle x)^2}\int_0^t g(t-s)\left(U_{N-1}(s) - 2U_N(s)\right)ds + F_N\left(x_N,t,U_N(t)\right), \\ U_i(0) = \tilde{u}_0(x_i),\, V_i(0) = \tilde{u}_1(x_i),\, i = \overline{0,N}, \end{cases} \tag{5.4}$$

and

$$\begin{aligned} &F_i\left(x_i,t,U_i(t)\right) = K(x_i,t)\left|U_i(t)\right|^{p-2}U_i(t)\ln^r\left(e+U_i^2(t)\right) + f(x_i,t),\, i = \overline{0,N}, \\ &\alpha(t) = \frac{\mu(t)}{(\triangle x)^2} = (N+1)^2\mu(t). \end{aligned} \tag{5.5}$$

Then system (5.4) is equivalent to

$$\begin{cases} \frac{dX}{dt}(t) = A(t)X(t) - \frac{1}{(\triangle x)^2}\int_0^t g(t-s)BX(s)ds + F(t,X(t)), \\ X(0) = X_0, \end{cases} \tag{5.6}$$

where

$$\begin{cases} X(t) = (U_0(t),\cdots,U_N(t),V_0(t),\cdots,V_N(t))^T \in \mathbb{R}^{2N+2}, \\ F(t,X(t)) = (0,\cdots,0,F_0(t,X(t)),\cdots,F_N(t,X(t)))^T \in \mathbb{R}^{2N+2}, \\ F_i(t,X(t)) = K(x_i,t)\left|U_i(t)\right|^{p-2}U_i(t)\ln^r\left(e+U_i^2(t)\right) + f(x_i,t),\, i = \overline{0,N}, \\ X_0 = (\tilde{u}_0(x_0),\cdots,\tilde{u}_0(x_N),\tilde{u}_1(x_0),\cdots,\tilde{u}_1(x_N)\ )^T \in \mathbb{R}^{2N+2}, \end{cases} \tag{5.7}$$

$$A(t) = \begin{bmatrix} O & E \\ \alpha(t)\hat{B} & -\lambda E \end{bmatrix} \in \mathfrak{M}_{2N+2},\; B = \begin{bmatrix} O & O \\ \hat{B} & O \end{bmatrix} \in \mathfrak{M}_{2N+2}, \tag{5.8}$$

$$E=\begin{bmatrix}1 & 0 & \cdots & 0\\ 0 & 1 & \ddots & \vdots\\ \vdots & \ddots & \ddots & 0\\ 0 & \cdots & 0 & 1\end{bmatrix}\in\mathfrak{M}_{N+1},$$

$$\hat{B}=\begin{bmatrix}-b_0 & 1 & 0 & \cdots & \cdots & 0\\ 1 & -2 & 1 & 0 & \cdots & 0\\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots\\ \vdots & 0 & \ddots & \ddots & \ddots & 0\\ \vdots & \vdots & \ddots & 1 & -2 & 1\\ 0 & 0 & \cdots & 0 & 1 & -2\end{bmatrix}\in\mathfrak{M}_{N+1}, \tag{5.9}$$

where $b_0=1+\triangle x=\dfrac{N+2}{N+1}$.

The nonlinear differential system (5.6) is solved by using the following linear recursive scheme generated by the nonlinear terms $F_i\left(x_i,t,U_i(t)\right)=K(x_i,t)\left|U_i(t)\right|^{p-2}U_i(t)\ln^r\left(e+U_i^2(t)\right)+f(x_i,t):$

$$\begin{cases}\frac{dX^{(m)}}{dt}(t)=A(t)X^{(m)}(t)-\dfrac{1}{(\triangle x)^2}\int_0^t g(t-s)BX^{(m)}(s)ds+F^{(m)}(t),\\ X^{(m)}(0)=X_0,\end{cases} \tag{5.10}$$

where

$$\begin{cases}X^{(m)}(t)=\left(U_0^{(m)}(t),\cdots,U_N^{(m)}(t),V_0^{(m)}(t),\cdots,V_N^{(m)}(t)\right)^T\in\mathbb{R}^{2N+2},\\ F^{(m)}(t)\equiv F\left(t,X^{(m-1)}(t)\right)=\left(0,\cdots,0,F_0(t,X^{(m-1)}(t)),\cdots,F_N(t,X^{(m-1)}(t))\right)^T\\ \qquad =\left(F_1^{(m)}(t),\cdots,F_{2N+2}^{(m)}(t)\right)^T\in\mathbb{R}^{2N+2},\\ F_i\left(t,X^{(m-1)}(t)\right)=K(x_i,t)\left|U_i^{(m-1)}(t)\right|^{p-2}U_i^{(m-1)}(t)\ln^r\left(e+\left|U_i^{(m-1)}(t)\right|^2\right)\\ \qquad\qquad +f(x_i,t),\ i=\overline{0,N},\\ X_0=\left(\tilde{u}_0(x_0),\cdots,\tilde{u}_0(x_N),\tilde{u}_1(x_0),\cdots,\tilde{u}_1(x_N)\ \right)^T\in\mathbb{R}^{2N+2},\end{cases} \tag{5.11}$$

and

$$F_i^{(m)}(t)=\begin{cases}0,\ i=\overline{1,N+1},\\ F_{i-N-2}(t,X^{(m-1)}(t))\\ =K(x_{i-N-2},t)\left|U_{i-N-2}^{(m-1)}(t)\right|^{p-2}U_{i-N-2}^{(m-1)}(t)\ln^r\left(e+\left|U_{i-N-2}^{(m-1)}(t)\right|^2\right)\\ \qquad +f(x_{i-N-2},t),\ i=\overline{N+2,2N+2}.\end{cases} \tag{5.12}$$

In order to solve problem (5.10) numerically, we will approximate $\frac{dX^{(m)}}{dt}(t_j)$ as follows

$$\begin{aligned}&\frac{dX^{(m)}}{dt}(t_j)\approx\frac{X_{j+1}^{(m)}-X_j^{(m)}}{\Delta t},\\ &X_j^{(m)}=X^{(m)}(t_j),\ t_j=j\Delta t,\ \Delta t=\frac{T}{M},\ j=\overline{0,M}.\end{aligned} \tag{5.13}$$

Therefore

$$\begin{cases} \dfrac{X_{j+1}^{(m)}-X_j^{(m)}}{\Delta t} = A(t_j)X_j^{(m)} - \dfrac{1}{(\triangle x)^2}\sum_{\nu=1}^{j}\int_{t_{\nu-1}}^{t_\nu} g(t_j-s)BX^{(m)}(s)ds \\ \qquad\qquad +F^{(m)}(t_j),\ j=\overline{0,M-1}, \\ X_0^{(m)} = X_0. \end{cases} \tag{5.14}$$

On the other hand, we approximate $\sum_{\nu=1}^{j}\int_{t_{\nu-1}}^{t_\nu} g(t_j-s)BX^{(m)}(s)ds$ as follows

$$\begin{aligned} \sum_{\nu=1}^{j}\int_{t_{\nu-1}}^{t_\nu} g(t_j-s)BX^{(m)}(s)ds &\approx \sum_{\nu=1}^{j}\frac{\Delta t}{2}\left[g(t_j-t_{\nu-1})BX_{\nu-1}^{(m)}+g(t_j-t_\nu)BX_\nu^{(m)}\right] \\ &= \frac{\Delta t}{2}\sum_{\nu=1}^{j}B\left(g(t_{j-\nu+1})X_{\nu-1}^{(m)}+g(t_{j-\nu})X_\nu^{(m)}\right) \\ &= \frac{\Delta t}{2}\sum_{\nu=1}^{j}B\left(g_{j-\nu+1}X_{\nu-1}^{(m)}+g_{j-\nu}X_\nu^{(m)}\right) \\ &= \frac{\Delta t}{2}\left[g_jBX_0+g_0BX_j^{(m)}+2\sum_{\nu=1}^{j-1}g_{j-\nu}BX_\nu^{(m)}\right], \end{aligned} \tag{5.15}$$

where $g_j = g(t_j)$. Hence

$$\begin{cases} X_{j+1}^{(m)} = \left(\mathbb{I}+\Delta tA(t_j)-\dfrac{(\triangle t)^2}{2(\triangle x)^2}g_0B\right)X_j^{(m)} - \dfrac{(\triangle t)^2}{(\triangle x)^2}\sum_{\nu=1}^{j-1}g_{j-\nu}BX_\nu^{(m)} \\ \qquad\qquad -\dfrac{(\triangle t)^2}{2(\triangle x)^2}g_jBX_0+\Delta tF^{(m)}(t_j),\ j=\overline{0,M-1}, \\ X_0^{(m)} = X_0, \end{cases} \tag{5.16}$$

where $\mathbb{I}$ the identity matrix of size $2N+2$. We write (5.16) in the form

$$\begin{cases} X_1^{(m)} = (\,\mathbb{I}+\Delta tA(t_0))X_0+\Delta tF^{(m)}(0), \\ X_{j+1}^{(m)} = \mathscr{F}_j\left[X_1^{(m)},X_2^{(m)},\cdots,X_j^{(m)}\right],\ j=\overline{1,M-1}, \end{cases} \tag{5.17}$$

in which

$$\begin{cases} \mathscr{F}_j\left[X_1^{(m)},X_2^{(m)},\cdots,X_j^{(m)}\right] = \left(\mathbb{I}+\Delta tA(t_j)-\dfrac{(\triangle t)^2}{2(\triangle x)^2}g_0B\right)X_j^{(m)} \\ \qquad\qquad -\dfrac{(\triangle t)^2}{(\triangle x)^2}\sum_{\nu=1}^{j-1}g_{j-\nu}BX_\nu^{(m)} \\ \qquad\qquad -\dfrac{(\triangle t)^2}{2(\triangle x)^2}g_jBX_0+\Delta tF^{(m)}(t_j),\ j=\overline{1,M-1}, \\ F^{(m)}(0) = F(0,X_0) = (0,\cdots,0,F_0(0,X_0),\cdots,F_N(0,X_0)\,)^T \in \mathbb{R}^{2N+2}, \\ F_i(0,X_0) = K(x_i,0)\,|\tilde{u}_0(x_i)|^{p-2}\,\tilde{u}_0(x_i)\ln^r\left(e+|\tilde{u}_0(x_i)|^2\right)+f(x_i,0),\ i=\overline{0,N}. \end{cases} \tag{5.18}$$

For fixed N, M, we find $\left(X_1^{(m)}, X_2^{(m)}, \cdots, X_M^{(m)}\right)$ by (5.17), (5.18) such that

$$\max_{1\leq j\leq M}\left|X_j^{(m)}-X_j^{(m-1)}\right|_1<\varepsilon=10^{-3}, \tag{5.19}$$

where $|\cdot|_1$ is the norm in the space $\mathbb{R}^{2N+2}$ given as below

$$\left|X_j^{(m)}-X_j^{(m-1)}\right|_1=\sum_{i=0}^{N}\left(\left|U_i^{(m)}(t_j)-U_i^{(m-1)}(t_j)\right|+\left|V_i^{(m)}(t_j)-V_i^{(m-1)}(t_j)\right|\right), \tag{5.20}$$

with

$$X_j^{(m)}=X^{(m)}(t_j)=\left(U_0^{(m)}(t_j),\cdots,U_N^{(m)}(t_j),V_0^{(m)}(t_j),\cdots,V_N^{(m)}(t_j)\right)^T\in\mathbb{R}^{2N+2}. \tag{5.21}$$

If (5.19) holds, then $X_j^{(m)}$ is chosen as follows

$$X_j^{(m)}\equiv X_j=\left(U_0(t_j),\cdots,U_N(t_j),V_0(t_j),\cdots,V_N(t_j)\right)^T,$$

and the following error is obtained

$$E_{N,M}(u)=\max_{1\leq j\leq M}\max_{0\leq i\leq N}\left|u_{ex}\left(x_i,t_j\right)-U_i(t_j)\right|. \tag{5.22}$$

For the detail, we give the following errors $E_{N,M}(u)$ with respect to N, M

N	M	$E_{N,M}(u)$
8	110	0.0190
9	130	0.0092
10	150	0.0080
11	170	0.0070
12	200	0.0060
13	230	0.0052
14	260	0.0046
15	290	0.0041
20	490	0.0024
25	730	0.0016
30	1025	0.0012

(5.23)

Obviously, the errors is decreasing when M, N are increasing.

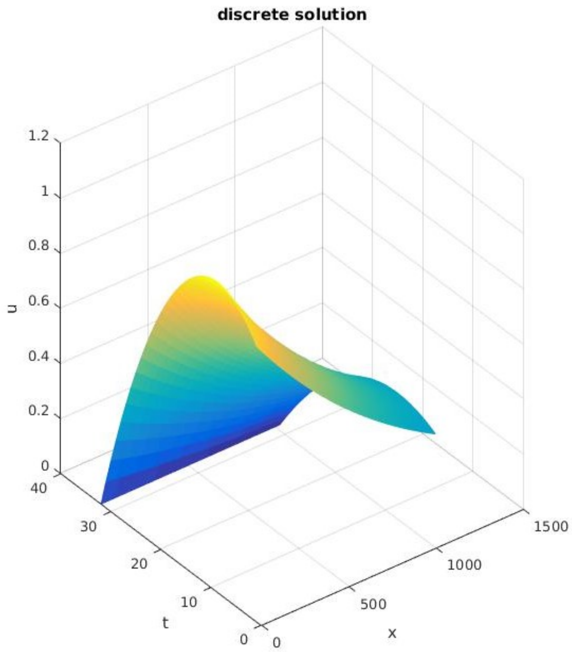

FIGURE 1. Approximated solution ($N = 30$, $M = 1025$)

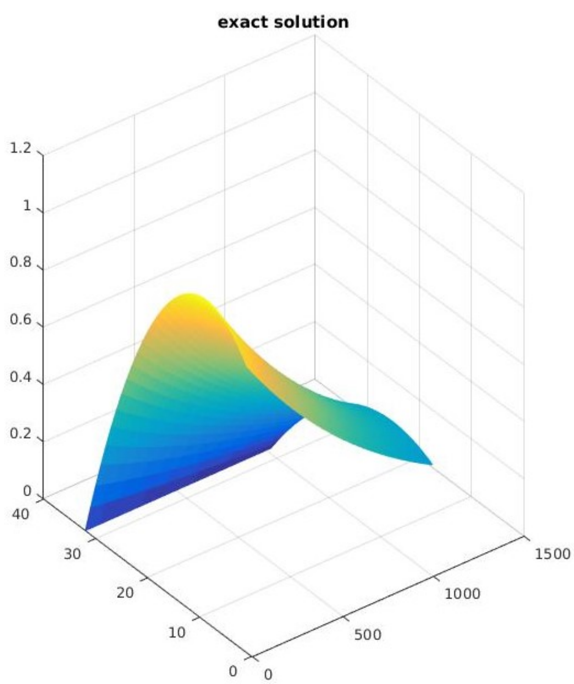

FIGURE 2. Exact solution

In Fig. 1, we have drawn the approximation solution $u_{dis}(x,t)$ of problem (5.1) with $T=1$, $p=6$, $r=\frac{5}{2}$, $\lambda=1$ and $\mu(t)$, $g(t)$, $K(x,t)$, f, $\tilde{u}_0$, $\tilde{u}_1$ given in (5.2), where $N=30$, $M=1025$, while Fig. 2 represents the corresponding exact solution $u_{ex}(x,t)$ given in (5.3). Therefore, in both cases, we notice the very good decay of these surfaces from $T=0$ to $T=1$.

Funding

This research is funded by Vietnam National University HoChiMinh City (VNU-HCM) under grant number B2020-18-01.

Acknowledgements

The authors wish to express their sincere thanks to the Editor and the referees for the suggestions, remarks and valuable comments.

REFERENCES

[1] S. Berrimi, S. A. Messaoudi, Existence and decay of solutions of a viscoelastic equation with a nonlinear source, Nonlinear Anal. 64 (2006), 2314-2331.
[2] S. Boulaaras, D. Ouchenane, General decay for a coupled Lamé system of nonlinear viscoelastic equations, Math. Meth. Appl. Sci. 43 (2020), 1717-1735.
[3] Boumaaza, S. Boulaaras, General decay for Kirchhoff type in viscoelasticity with not necessarily decreasing kernel, Math. Meth. Appl. Sci. 41 (2018), 6050-6069.
[4] L. Jie, L. Fei, Blow-up of solution for an integro-differential equation with arbitrary positive initial energy, Bound Value Probl. 2015 (2015), 96.
[5] M. Kafini, S. A. Messaoudi, A blow-up result in a Cauchy viscoelastic problem, Appl. Math. Lett. 21 (2008), 549-553.
[6] Q. Li, L. He, General decay of solutions to a viscoelastic wave equation with linear damping, nonlinear damping and source term, Appl. Anal. 99 (2020), 1248-1259.
[7] Q. Li, L. He, General decay and blow-up of solutions for a nonlinear viscoelastic wave equation with strong damping, Boundary Value Probl, 2018 (2018), 153.
[8] N. T. Long, Alain P. N. Dinh, L. X. Truong, Existence and decay of solutions of a nonlinear viscoelastic problem with a mixed nonhomogeneous condition, Numer. Funct. Anal. Optim. 29 (2008), 1363-1393.
[9] N. T. Long, L. T. P. Ngoc, On a nonlinear wave equation with boundary conditions of two-point type, J. Math. Anal. Appl. 385 (2012), 1070-1093.
[10] S. A. Messaoudi, Blow-up of positive-initial-energy solutions of a nonlinear viscoelastic hyperbolic equation, J. Math. Anal. Appl. 320 (2006), 902-915.
[11] S. A. Messaoudi, General decay of the solution energy in a viscoelastic equation with a nonlinear source, Nonlinear Anal. 69 (2008), 2589-2598.
[12] S. A. Messaoudi, B. Said-Houari, Global nonexistence of positive initial-energy solutions of a system of nonlinear viscoelastic wave equations with damping and source terms, J. Math. Anal. Appl. 365 (2010), 277-287.
[13] N. Mezouar, S. Boulaaras, Global existence and decay of solutions for a class of viscoelastic Kirchhoff equation, Bull. Malaysian Math. Sci. Soc. 43 (2020), 725-755.
[14] L. X. Truong, L. T. P. Ngoc, Alain P. N. Dinh, N. T. Long, Existence, blow-up and exponential decay estimates for a nonlinear wave equation with boundary conditions of two-point type, Nonlinear Anal. 74 (2011), 6933-6949.
[15] Y. Wang, A global nonexistence theorem for viscoelastic equations with arbitrary positive initial energy, Appl. Math. Lett. 22 (2009), 1394-1400.
[16] H. A. Levine, Instability and nonexistence of global solutions of nonlinear wave equation of the form $Pu_{tt}=Au+F(u)$, Trans. Amer. Math. Soc. 192 (1974), 1-21.
[17] H. A. Levine, Some additional remarks on the nonexistence of global solutions to nonlinear wave equations, SIAM J. Math. Anal. 5 (1974), 138-146.

[18] H. A. Levine, L. E. Payne, Some nonexistence theorems for initial-boundary value problems with nonlinear boundary constraints, Proc. Amer. Math. Soc. 46 (1974), 277-284.

[19] N. T. Long, N. V. Y, L. T. P. Ngoc, Exponential decay and blow-up results for nonlinear heat equation with viscoelastic term and Robin conditions, Annales Polonici Math. 119 (2017), 121-145.

[20] H. Brezis, Functional Analysis, Sobolev Spaces and Partial Differential Equations, Springer Science & Business Media, 2010.

[21] J. L. Lions, Quelques méthodes de résolution des problèmes aux limites nonlinéaires, Dunod; Gauthier-Villars, Paris, 1969.