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THE SINGULARITIES OF ATTRACTIVE AND REPULSIVE TYPE TO FOURTH-ORDER DIFFERENTIAL EQUATIONS WITH TIME-DEPENDENT DEVIATING ARGUMENT

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Abstract. The paper investigates the existence of positive periodic solutions for a kind of fourth-order singular differential equation with time-dependent deviating argument. Based on the coincidence degree theory, we prove that there is at least one positive periodic solution to the quation. Two examples are given to support the applications of main theorems.

Keywords. Positive periodic solution; Fourth-order differential equation; Singularities of attractive and repulsive type.

1. Introduction

The Liénard equation [1]

$$x'' + f(x)x' + g(x) = 0, \tag{1.1}$$

has been intensively studied during the first half of 20th century as it can be used to model oscillating circuits or simple pendulums. For example, Van der Pol oscillator

$$x'' - \mu(1 - x^2)x' + x = 0,$$

is a Liénard equation. Recently, the existence of periodic solutions for Liénard equation was extensively studied (see [2]-[14]). In 2002, Wang [4] studed the following Liénard equation

$$x''(t) + f(x(t))x'(t) + g(x(t)) = p(t), \tag{1.2}$$

by using of the Poincaré-Birkhoff fixed point theorem. Wang obtained that the existence and the multiplicity of 2π-periodic solutions for (1.2). In 2005, Cheung and Ren [7] investigated a kind of p-Laplacian Liénard equation, that is,

$$(\phi_p(x'(t)))' + f(x(t))x'(t) + g(x(t - \delta(t))) = e(t),$$

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where $\phi_p : \mathbb{R} \to \mathbb{R}$ is given by $\phi_p(s) = |s|^{p-2}s$, $p > 1$ is a constant, $e \in C(\mathbb{R},\mathbb{R})$ is a T-periodic solution, and $\int_0^T e(t)dt = 0$. Their results are based on the celebrated coincidence degree method. In 2008, with the aid of the coincidence degree theory, Liu [9] established the existence and the uniqueness of periodic solutions for the following p-Laplacian Liénard equation

$$(\phi_p(x'(t)))' + f(x(t))x'(t) + g(x(t)) = e(t).$$

Indeed, most of early works concentrated on second-order Liénard equations without singularity. For recent results on singular Liénard equations, we refer to [15, 16, 17, 18]. By using the coincidence degree theory, Wang [17] investigated the existence of positive T-periodic solutions for a Liénard equation with a singularity of repulsive type and a deviating argument as follows

$$x''(t) + f(x(t))x'(t) + g(t, x(t-\delta)) = 0, \tag{1.3}$$

where δ is a constant and $\delta \in [0,T)$, and g is unbounded as $x \to 0^+$. In 2016, Xin and Cheng [18] investigated the existence of periodic solutions of a p-Laplacian Liénard equation with singularity

$$(\phi_p(x'(t)))' + f(x(t))x'(t) + g(t, x(t-\delta)) = 0, \tag{1.4}$$

where g has a singularity at $x = 0$. They proved that equation (1.4) has at least one positive T-periodic solution.

In this paper, inspired by the results in [7, 16, 17, 18], we consider the following fourth-order p-Laplacian singular Rayleigh equation with time-dependent deviating argument

$$(\phi_p(x''(t)))'' + f(x(t))x'(t) + g(t, x(t-\delta(t))) = e(t), \tag{1.5}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a continuous function, $e : \mathbb{R} \to \mathbb{R}$ is continuous periodic function with $e(t+T) \equiv e(t)$, $\delta \in C^1(\mathbb{R},\mathbb{R})$ is a T-periodic function, $g : \mathbb{R} \times (0,+\infty) \to \mathbb{R}$ is an L^2-Carathéodory function defined on $\mathbb{R}^2$, and $g(t,\cdot) = g(t+T,\cdot)$. Equation (1.5) is singularity of attractive type (resp. repulsive type) if $g(t,x) \to +\infty$ (resp. $g(t,x) \to -\infty$) as $x \to 0^+$ for $t \in \mathbb{R}$. Based on the coincidence degree theory, we prove that equation (1.5) has at least one positive T-periodic solution.

Remark 1.1. From the equations (1.3) and (1.4) in [17, 18], the nonlinear function g has a deviating argument (i.e., δ is a positive constant and $0 \le \delta < T$). In our paper, the nonlinear function g satisfies time-dependent deviating argument. For example, let $\delta(t) = \frac{1}{4}\cos 2t$. It is easy to see that estimating a lower bound of positive T-periodic solutions for equation (1.5) is more difficult than the estimation for equations (1.3) and (1.4). From [7, 9, 17, 18], the condition composed on $e(t)$ is $\int_0^T e(t)dt = 0$ or $e(t) \equiv 0$. It is redundant in this paper. Indeed, letting $e(t) = e^{\sin^2 2t}$, we have $\int_0^T e^{\sin^2 2t} \neq 0$. This shows that our results are more general.

2. Preliminaries

Lemma 2.1. *(Gaines and Mawhin [19]) Let X and Y be two Banach spaces, and let $L : D(L) \subset X \to Y$ be a Fredholm operator with index zero. Let $\Omega \subset X$ be an open bounded set and $N : \overline{\Omega} \to Y$ be L-compact on $\overline{\Omega}$. Assume that the following conditions hold:*

(1) $Lx \neq \lambda Nx, \forall\, x \in \partial\Omega \cap D(L),\ \lambda \in (0,1)$;
(2) $Nx \notin Im\, L, \forall\, x \in \partial\Omega \cap Ker\, L$;
(3) $\deg\{JQN, \Omega \cap Ker\, L, 0\} \neq 0$, where $J : Im\, Q \to Ker\, L$ is an isomorphism.

Then the equation $Lx = Nx$ has a solution in $\overline{\Omega} \cap D(L)$.

Lemma 2.2. [20] *If $\omega \in C^1(\mathbb{R}, \mathbb{R})$ and $\omega(0) = \omega(T) = 0$, then*

$$\left(\int_0^T |\omega(t)|^p dt\right)^{\frac{1}{p}} \leq \left(\frac{T}{\pi_p}\right)\left(\int_0^T |\omega'(t)|^p dt\right)^{\frac{1}{p}},$$

where $1 \leq p < \infty$, $\pi_p = 2\int_0^{(p-1)/p} \frac{ds}{(1-\frac{s^p}{p-1})^{1/p}} = \frac{2\pi(p-1)^{1/p}}{p\sin(\pi/p)}$.

In order to apply the topological degree theorem to study the existence of positive periodic solutions for (1.5), we rewrite (1.5) in the form:

$$\begin{cases} x_1''(t) = (\phi_q(x_2(t)), \\ x_2''(t) = -f(x_1(t))x_1'(t) - g(t, x_1(t-\delta(t))) + e(t), \end{cases} \tag{2.1}$$

where $\frac{1}{p} + \frac{1}{q} = 1$. Clearly, if $x(t) = (x_1(t), x_2(t))^\top$ is an T-periodic solution to (2.1), then $x_1(t)$ must be an T-periodic solution to (1.5). Thus, the problem of finding an T-periodic solution to (1.5) is reduced to the problem of finding solutions to (2.1).

Let

$$X := \{x = (x_1(t), x_2(t)) \in C^2(\mathbb{R}, \mathbb{R}^2) : x(t+T) - x(t) \equiv 0\}$$

be a space with norm $\|x\| := \max\{\|x_1\|, \|x_2\|\}$, and

$$Y := \{x = (x_1(t), x_2(t)) \in C^2(\mathbb{R}, \mathbb{R}^2) : x(t+T) - x(t) \equiv 0\}$$

a space with norm $\|x\|_\infty := \max\{\|x\|, \|x'\|\}$. Clearly, both X and Y are Banach spaces. We also define

$$L : D(L) \subset X \to Y, \text{ by } \quad (Lx)(t) = \begin{pmatrix} x_1''(t) \\ x_2''(t) \end{pmatrix}, \tag{2.2}$$

where

$$D(L) = \{x = (x_1, x_2)^\top \in C^2(\mathbb{R}, \mathbb{R}^2) : \ x(t+T) - x(t) \equiv 0, t \in \mathbb{R}\},$$

and a nonlinear operator $N : X \to Y$ by

$$(Nx)(t) = \begin{pmatrix} \phi_q(x_2(t)) \\ -f(x_1(t))x_1'(t) - g(t, x_1(t-\delta(t))) + e(t) \end{pmatrix}. \tag{2.3}$$

Then (2.1) can be converted to the abstract equation $Lx = Nx$.

From the definition of L, one can easily see that

$$\text{Ker } L \cong \mathbb{R}^2, \qquad \text{Im } L = \left\{ y \in Y : \int_0^T \begin{pmatrix} y_1(s) \\ y_2(s) \end{pmatrix} ds = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}.$$

So, L is a Fredholm operator with index zero. Let $P : X \to \text{Ker } L$ and $Q : Y \to \text{Im } Q \subset \mathbb{R}^2$ be defined by

$$Px := \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix}; \qquad Qy := \frac{1}{T}\int_0^T \begin{pmatrix} y_1(s) \\ y_2(s) \end{pmatrix} ds.$$

Then Im P=Ker L, and Ker Q=Im L. Let K denote the inverse of $L|_{\text{Ker}_p \cap D(L)}$. It is easy to see that Ker L =Im$Q = \mathbb{R}^2$ and

$$[Ky](t) = \text{col}\left(\int_0^T G_1(t,s)y_1(s)ds, \quad \int_0^T G_2(t,s)y_2(s)ds\right),$$

where

$$G_i(t,s)=\begin{cases}\frac{-s(T-t)}{T}, & 0\le s\le t\le T,\\ \frac{-t(T-s)}{T}, & 0\le t<s\le T,\end{cases}\qquad i=1,2. \tag{2.4}$$

3. MAIN RESULTS

In this section, we study that existence of positive T-periodic solutions of (1.5) with the singularity of repulsive type.

Theorem 3.1. *Assume that the following conditions hold:*

(H_1) there exist two positive constants D_1, D_2 with $D_1<D_2$ such that $g(t,x)-e(t)<0$ for all $(t,x)\in[0,T]\times(0,D_1)$, and $g(t,x)-e(t)>0$ for all $(t,x)\in[0,T]\times(D_2,+\infty)$.

(H_2) there exist positive constants a, b such that

$$g(t,x)\le ax^{p-1}+b,\ \ \text{for all } (t,x)\in[0,T]\times(0,+\infty).$$

(H_3) $g(t,x)=g_0(x)+g_1(t,x)$, where $g_0\in C((0,\infty);\mathbb{R})$ and $g_1:[0,T]\times[0,\infty)\to\mathbb{R}$ is an L^2-Carathéodory function.

(H_4) (singularity of repulsive type)

$$\int_0^1 g_0(x)dx=-\infty.$$

Then equation (1.5) has at least one positive T-periodic solution if

$$0<\frac{aT}{1-\delta'}\left(\frac{T}{\pi_p}\right)^{2p-1}<1,$$

where $\delta'=:\max_{t\in[0,T]}|\delta(t)|$.

Proof. Consider the operator equation

$$Lx=\lambda Nx,\quad \lambda\in(0,1),$$

where L and N are defined by equations (2.2) and (2.3). Set

$$\Omega_1=\{x:Lx=\lambda Nx,\lambda\in(0,1)\}.$$

If $x(t)=(x_1(t),x_2(t))^\top\in\Omega_1$, then

$$\begin{cases}x_1''(t)=\lambda\phi_q(x_2(t)),\\ x_2''(t)=-\lambda f(x_1(t))x_1'(t)-\lambda g(t,x_1(t-\delta(t)))+\lambda e(t).\end{cases} \tag{3.1}$$

Setting $x_2(t)=\frac{1}{\lambda^{p-1}}(\phi_p(x_1)''(t))$, we have

$$\phi_p(x_1''(t))''(t)+\lambda^p f(x_1(t))x_1'(t)+\lambda^p g(t,x_1(t-\sigma))=\lambda^p e(t). \tag{3.2}$$

Integrating both side of equation (3.2) over $[0,T]$, we have

$$\int_0^T(g(t,x_1(t-\delta(t)))-e(t))dt=0. \tag{3.3}$$

From condition (H_1), there exist two point ξ, $\eta\in(0,T)$ such that

$$x_1(\xi-\delta(\xi))\ge D_1,\qquad x_1(\eta)\le D_2. \tag{3.4}$$

From equation (3.4), we have

$$\begin{aligned} x(t) =&\frac{1}{2}(x(t)+x(t-T)) \\ =&\frac{1}{2}\left(x(\eta)+\int_{\eta}^{t}x'(s)ds+x(\eta)-\int_{t-T}^{\eta}x'(s)ds\right) \\ =&x(\eta)+\frac{1}{2}\int_{t-T}^{t}x'(s)ds \\ \leq&D_2+\frac{1}{2}\int_0^T|x'(t)|dt. \end{aligned} \tag{3.5}$$

Multiplying both sides of equation (3.2) by $x_1(t)$ and integrating over the interval $[0,T]$, we get

$$\begin{aligned} &\int_0^T\phi_p(x_1''(t))''x_1(t)dt+\lambda^p\int_0^T f(x_1(t))x_1'(t)x_1(t)dt+\lambda^p\int_0^T g(t,x_1(t-\delta(t)))x_1(t)dt \\ &=\lambda^p\int_0^T e(t)x_1(t)dt. \end{aligned} \tag{3.6}$$

Substituting $\int_0^T\phi_p(x_1''(t))''x_1(t)dt=\int_0^T|x_1''(t)|^pdt$ and $\int_0^T f(x_1(t))x_1'(t)x_1(t)dt=0$ into (3.6), we get

$$\begin{aligned} \int_0^T|x_1''(t)|^pdt =&-\lambda^p\int_0^T g(t,x_1(t-\delta(t)))x_1(t)dt+\lambda^p\int_0^T e(t)x_1(t)dt \\ \leq&\int_0^T|g(t,x_1(t-\delta(t)))||x_1(t)|dt+\int_0^T|e(t)||x_1(t)|dt \\ \leq&\|x_1\|\int_0^T|g(t,x_1(t-\delta(t)))|dt+\|e\|\|x_1\|. \end{aligned} \tag{3.7}$$

where $e(t):=\max_{t\in[0,T]}|e(t)|$. Using condition (H_2) and equation (3.3), we obtain that

$$\begin{aligned} \int_0^T|g(t,x_1(t-\delta))|dt =&\int_{g(t,x_1)\geq 0}g(t,x_1(t-\delta(t)))dt-\int_{g_1(t,x_1)\leq 0}g(t,x_1(t-\delta(t)))dt \\ =&2\int_{g(t,x_1)\geq 0}g(t,x_1(t-\delta(t)))dt-\int_0^T e(t)dt \\ \leq&2a\int_0^T x_1(t-\delta(t))^{p-1}dt+2bT+\int_0^T|e(t)|dt \\ =&2a\int_0^T|x_1(t-\delta(t))|^{p-1}dt+2bT+\|e\|T \\ \leq&\frac{2a}{1-\delta'}\int_0^T|x_1(t)|^{p-1}dt+2bT+\|e\|T. \end{aligned} \tag{3.8}$$

In view of

$$\frac{aT}{1-\delta'}\left(\frac{T}{\pi_p}\right)^{2p-1}>0,$$

we have $\delta' < 1$. Substituting (3.5) and (3.8) into (3.7), we find from the Hölder inequality that

$$\begin{aligned}
&\int_0^T |x_1(t)|^p dt \\
&\leq \frac{2a}{1-\delta'}\left(D_2+\frac{1}{2}\int_0^T |x_1'(t)|dt\right)\int_0^T |x_1(t)|^{p-1}dt+(2bT+2\|e\|T)\left(D_2+\frac{1}{2}\int_0^T |x_1'(t)|dt\right) \\
&\leq \frac{aT}{1-\delta'}\left(\int_0^T |x_1'(t)|^p dt\right)^{\frac{1}{p}}\left(\int_0^T |x_1(t)|^p dt\right)^{\frac{p-1}{p}}+\frac{2aD_2T^{\frac{1}{p}}}{1-\delta'}\left(\int_0^T |x_1(t)|^p dt\right)^{\frac{p-1}{p}} \\
&\quad +\frac{G_1T^{\frac{1}{q}}}{2}\left(\int_0^T |x_1'(t)|^p dt\right)^{\frac{1}{p}}+G_1D_2,
\end{aligned} \tag{3.9}$$

where $G_1 = 2(b+\|e\|)T$. Let $\omega(t) = x_1(t+\eta)-x_1(\eta)$, where $x_1(\eta) \leq D_2$. Then $\omega(0) = \omega(T) = 0$. From Lemma 2.2 and the Minkowski's inequality [21], we have

$$\begin{aligned}
\left(\int_0^T |x_1(t)|^p dt\right)^{\frac{1}{p}} &= \left(\int_0^T |\omega(t)+x_1(\eta)|^p dt\right)^{\frac{1}{p}} \\
&\leq \left(\int_0^T |\omega(t)|^p dt\right)^{\frac{1}{p}}+\left(\int_0^T |x_1(\eta)|^p dt\right)^{\frac{1}{p}} \\
&\leq \left(\frac{T}{\pi_p}\right)\left(\int_0^T |\omega'(t)|^p dt\right)^{\frac{1}{p}}+D_2T^{\frac{1}{p}} \\
&= \left(\frac{T}{\pi_p}\right)\left(\int_0^T |x_1'(t)|^p dt\right)^{\frac{1}{p}}+D_2T^{\frac{1}{p}}.
\end{aligned} \tag{3.10}$$

On the other hand, in view of $x_1(0) = x_1(T)$, there exists a point $t_1 \in (0,T)$ such that $x_1'(t_1) = 0$. Letting $\omega_*(t) = x_1'(t+t_1)$, we see that $\omega_*(0) = \omega_*(T) = 0$ for $\omega_* \in C^1(\mathbb{R},\mathbb{R})$. From Lemma 2.2, we have that

$$\begin{aligned}
\left(\int_0^T |x_1'(t)|^p dt\right)^{\frac{1}{p}} &= \left(\int_0^T |\omega_*(t)|^p dt\right)^{\frac{1}{p}} \\
&\leq \left(\frac{T}{\pi_p}\right)\left(\int_0^T |\omega_*'(t)|^p dt\right)^{\frac{1}{p}} \\
&= \left(\frac{T}{\pi_p}\right)\left(\int_0^T |x_1''(t)|^p dt\right)^{\frac{1}{p}}.
\end{aligned} \tag{3.11}$$

Substituting (3.11) into (3.10), we have

$$\left(\int_0^T |x_1(t)|^p dt\right)^{\frac{1}{p}} \leq \left(\frac{T}{\pi_p}\right)^2\left(\int_0^T |x_1''(t)|^p dt\right)^{\frac{1}{p}}+D_2T^{\frac{1}{p}}. \tag{3.12}$$

Substituting (3.11) and (3.12) into (3.9), we have

$$\begin{aligned}
&\int_0^T |(x_1)''(t)|^p dt\\
\le& \frac{aT}{1-\delta'}\left(\frac{T}{\pi_p}\right)\left(\int_0^T |x_1''(t)|^p dt\right)^{\frac{1}{p}}\left(\left(\frac{T}{\pi_p}\right)^2\left(\int_0^T |x_1''(t)|^p dt\right)^{\frac{1}{p}}+D_2T^{\frac{1}{p}}\right)^{p-1}\\
&+\frac{2aD_2T^{\frac{1}{p}}}{1-\delta'}\left(\left(\frac{T}{\pi_p}\right)^2\left(\int_0^T |x_1''(t)|^p dt\right)^{\frac{1}{p}}+D_2T^{\frac{1}{p}}\right)^{p-1}\\
&+\frac{G_1T^{\frac{1}{q}}}{2}\left(\frac{T}{\pi_p}\right)\left(\int_0^T |x_1''(t)|^p dt\right)^{\frac{1}{p}}+G_1D_2\\
=&\frac{aT}{1-\delta'}\left(\frac{T}{\pi_p}\right)^{2p-1}\left(1+\frac{D_2T^{\frac{1}{p}}}{\left(\frac{T}{\pi_p}\right)^2\left(\int_0^T |x_1''(t)|^p dt\right)^{\frac{1}{p}}}\right)^{p-1}\int_0^T |x_1''(t)|^p dt\\
&+\frac{2aD_2T^{\frac{1}{p}}}{1-\delta'}\left(\frac{T}{\pi_p}\right)^{2p-2}\left(1+\frac{D_2T^{\frac{1}{p}}}{\left(\frac{T}{\pi_p}\right)^2\left(\int_0^T |x_1''(t)|^p dt\right)^{\frac{1}{p}}}\right)^{p-1}\left(\int_0^T |x_1''(t)|^p dt\right)^{\frac{p-1}{p}}\\
&+\frac{G_1T^{\frac{1}{q}}}{2}\left(\frac{T}{\pi_p}\right)\left(\int_0^T |x_1''(t)|^p dt\right)^{\frac{1}{p}}+G_1D_2.
\end{aligned}\tag{3.13}$$

Next, we introduce a classical inequality. There exists a $k(p)>0$, which is dependent on p only, such that

$$(1+x)^p \le 1+(1+p)x, \qquad \text{for} \quad x\in[0,k(p)]. \tag{3.14}$$

Next, we consider the following two cases.

Case 1. If

$$\frac{D_2T^{\frac{1}{p}}}{\left(\frac{T}{\pi_p}\right)^2\left(\int_0^T |x_1''(t)|^p dt\right)^{\frac{1}{p}}} > k(p),$$

then

$$\left(\int_0^T |x_1''(t)|^p dt\right)^{\frac{1}{p}} < \frac{D_2T^{\frac{1}{p}}}{k(p)}\left(\frac{T}{\pi_p}\right)^{-2}.$$

From equations (3.5) and (3.11), we deduce

$$\begin{aligned}
x(t) \le& D_2+\frac{1}{2}T^{\frac{1}{q}}\left(\int_0^T |x'(t)|^p dt\right)^{\frac{1}{p}}\\
\le& D_2+\frac{1}{2}T^{\frac{1}{q}}\left(\frac{T}{\pi_p}\right)\left(\int_0^T |x''(t)|^p dt\right)^{\frac{1}{p}}\\
\le& D_2+\frac{TD_2}{2k(p)}\left(\frac{T}{\pi_p}\right)^{-1} := M_1.
\end{aligned}\tag{3.15}$$

Case 2. If

$$\frac{D_2 T^{\frac{1}{p}}}{\left(\frac{T}{\pi_p}\right)^2 \left(\int_0^T |x_1''(t)|^p dt\right)^{\frac{1}{p}}} < k(p),$$

we obtain from equations (3.13) and (3.14) that

$$\begin{aligned}
&\int_0^T |(x_1)''(t)|^p dt \\
&\leq \frac{aT}{1-\delta'}\left(\frac{T}{\pi_p}\right)^{2p-1}\left(1+\frac{D_2 T^{\frac{1}{p}} p}{\left(\frac{T}{\pi_p}\right)^2 \left(\int_0^T |x_1''(t)|^p dt\right)^{\frac{1}{p}}}\right)\int_0^T |x_1''(t)|^p dt \\
&\quad + \frac{2aD_2 T^{\frac{1}{p}}}{1-\delta'}\left(\frac{T}{\pi_p}\right)^{2p-2}\left(1+\frac{D_2 T^{\frac{1}{p}} p}{\left(\frac{T}{\pi_p}\right)^2 \left(\int_0^T |x_1''(t)|^p dt\right)^{\frac{1}{p}}}\right)\left(\int_0^T |x_1''(t)|^p dt\right)^{\frac{p-1}{p}} \\
&\quad + \frac{G_1 T^{\frac{1}{q}}}{2}\left(\frac{T}{\pi_p}\right)\left(\int_0^T |x_1''(t)|^p dt\right)^{\frac{1}{p}} + G_1 D_2 \\
&= \frac{aT}{1-\delta'}\left(\frac{T}{\pi_p}\right)^{2p-1}\int_0^T |x_1''(t)|^p dt + \frac{aD_2 T^{1+\frac{1}{p}}}{1-\delta'}\left(Tp+2\left(\frac{T}{\pi}\right)\right)\left(\frac{T}{\pi_p}\right)^{2p-2} \\
&\cdot\left(\int_0^T |x_1''(t)|^p dt\right)^{\frac{p-1}{p}} + \frac{2aD_2^2 T^{\frac{2}{p}} p}{1-\delta'}\left(\frac{T}{\pi_p}\right)^{2p-4}\left(\int_0^T |x_1''(t)|^p dt\right)^{\frac{p-2}{p}} \\
&\quad + \frac{G_1 T^{\frac{1}{q}}}{2}\left(\frac{T}{\pi_p}\right)\left(\int_0^T |x_1''(t)|^p dt\right)^{\frac{1}{p}} + G_1 D_2.
\end{aligned} \tag{3.16}$$

Since

$$\frac{aT}{1-\delta'}\left(\frac{T}{\pi_p}\right)^{2p-1} < 1,$$

it is easy to see that there exists a positive constant M_1' such that

$$\int_0^T |x_1''(t)|^p dt \leq M_1'.$$

From equation (3.5) and Lemma 2.2, we see that

$$\begin{aligned}
x_1(t) \leq& D_2 + \frac{1}{2}\int_0^T |x_1'(t)| dt \\
\leq& D_2 + \frac{1}{2}T^{\frac{1}{q}}\left(\frac{T}{\pi_p}\right)\left(\int_0^T |x_1''(t)|^p dt\right)^{\frac{1}{p}} \\
\leq& D_2 + \frac{1}{2}T^{\frac{1}{q}}\left(\frac{T}{\pi_p}\right)(M_1')^{\frac{1}{p}} := M_1.
\end{aligned} \tag{3.17}$$

From equation (3.5), we obtain

$$\|x_1'\| \leq x_1'(t_1) + \frac{1}{2}\int_0^T |x_1''(t)|dt \leq \frac{1}{2}T^{\frac{1}{q}}\left(\int_0^T |x_1''(t)|^d t\right)^{\frac{1}{p}} \leq \frac{1}{2}T^{\frac{1}{q}}M_1'^{\frac{1}{p}} := M_2, \tag{3.18}$$

due to $x_1'(t_1) = 0$. Form $x_2(0) = x_2(T)$, we know that there exists a point $t_2 \in (0,T)$ such that $x_2'(t_2) = 0$. From equations (3.8), (3.17) and (3.18), we deduce

$$\begin{aligned}
\|x_2'\| \leq & \frac{1}{2}\int_0^T |x_2''(t)|dt \\
\leq & \frac{\lambda}{2}\left(\int_0^T |f(x_1(t))||x_1'(t)|dt + \int_0^T |g(t,x_1(t-\sigma))|dt + \int_0^T |e(t)|dt\right) \\
\leq & \frac{\lambda}{2}(\|f_{M_1}\|M_2T + 2aTM_1^{p-1} + 2bT + 2T\|e\|) := \lambda M_3,
\end{aligned} \tag{3.19}$$

where $\|f_{M_1}\| = \max_{|x_1|\leq M_1} |f(x_1)|$. It follows that $\int_0^T x_2(t)dt = \int_0^T \phi_p(x_1''(t))dt = 0$, which implies that there is a point $t_3 \in (0,T)$ such that $x_2(t_3) = 0$. Hence,

$$\|x_2\| \leq \frac{1}{2}\int_0^T |x_2'(t)|dt \leq \lambda TM_3 := \lambda M_4. \tag{3.20}$$

On the other hand, it follow from equation (3.2) and condition (H_3) that

$$(\phi_p(x_1''(t)))'' + \lambda^p f(x_1(t))x_1'(t) + \lambda^p(g_0(x_1(t-\delta(t))) + g_1(t,x_1(t-\delta(t))) = \lambda^p e(t). \tag{3.21}$$

Let $\xi \in [0,T]$ be as in equation (3.4), for any $t \in [\xi,T]$. Multiplying both sides of equation (3.21) by $x_1'(t-\delta)(1-\delta'(t))$ and integrating on $[\xi,t]$, we arrive at

$$\begin{aligned}
&\lambda^p \int_{x_1(\xi-\delta(\xi))}^{x_1(t-\delta(t))} g_0(u)du \\
&= \lambda^p \int_\xi^t g_0(x_1(s-\delta(s)))x_1'(s-\delta)(1-\delta'(s))ds \\
&= -\int_\xi^t (\phi_p(x_1''(s)))''x_1'(s-\delta)(1-\delta'(s))ds - \lambda^p \int_\xi^t f(x_1(s))x_1'(s-\delta)(1-\delta'(s))ds \\
&\quad - \lambda^p \int_\xi^t g_1(s,x_1(s-\delta(s)))x_1'(s-\delta)(1-\delta'(s))ds \\
&\quad + \lambda^p \int_\xi^t e(s)x_1'(s-\delta)(1-\delta'(s))ds.
\end{aligned} \tag{3.22}$$

Furthermore, by use of (3.16), (3.17), (3.18) and (3.19), we have

$$\begin{aligned}
&\lambda^p \left| \int_{x_1(\xi-\delta(\xi))}^{x_1(t-\delta(t))} g_0(u)du \right| \\
&= \Big| -\int_\xi^t (\phi_p(x_1''(s)))'' x_1'(s-\delta)(1-\delta'(s))ds - \lambda^p \int_\xi^t f(x_1(s))x_1'(s-\delta)(1-\delta'(s))ds \\
&-\lambda^p \int_\xi^t g_1(s,x_1(s-\delta(s)))x_1'(s-\delta)(1-\delta'(s))ds + \lambda^p \int_\xi^t e(s)x_1'(s-\delta)(1-\delta'(s))ds \Big| \\
&\le (1+\delta')\|x_1'\| \int_0^T |\phi_p(x_1''(s)))''|ds + \lambda^p(1+\delta')\|x_1'\| \int_0^T |f(x_1(s))|ds \qquad (3.23) \\
&\quad + \lambda^p(1+\delta')\|x_1'\| \int_0^T |g_1(s,x_1(s-\delta(s)))|ds + \lambda^p(1+\delta')\|x_1'\| \int_0^T |e(s)|ds \\
&\le \lambda^p(1+\delta')M_2 \left(\int_0^T |f(x'(s)))|ds + \int_0^T |g(s,x_1(s))|ds + \int_0^T |e(s)|ds \right) \\
&\quad + \lambda^p(1+\delta')(M_2T\|f_{M_1}\| + M_2\|g_{M_1}\|T + M_2\|e\|T) \\
&\le \lambda^p(1+\delta')M_2(M_3 + \|f_{M_1}\|T + \|g_{M_1}\|T + \|e\|T).
\end{aligned}$$

where $g_{M_1} = \max_{0\le x\le M_1} |g_1(t,x)| \in L^2(0,T)$ is as in condition (H_3). From singular condition (H_4), we know that there exists a positive constant M_5 such that

$$x_1(t-\delta(t)) \ge M_5, \qquad \forall\ t \in [\xi, T]. \tag{3.24}$$

The case $t \in [0,\xi]$ (i.e., $x_1(t-\delta(t)) \in [-\delta(0), \xi-\delta(\xi)]$) can be treated similarly. From equations (3.16), (3.17), (3.18), (3.19) and (3.24), we let

$$\Omega = \{x = (x_1,x_2)^\top : E_1 \le x_1(t) \le E_2,\ \|x_1'\| \le E_3,\ \|x_2\| \le E_4 \text{ and } \|x_2'\| \le E_5,\ \forall\, t \in [0,T]\},$$

where $0 < E_1 < \min(M_5,\ D_1)$, $E_2 > \max(M_1,\ D_2)$, $E_3 > M_2$, $E_4 > M_4$ and $E_5 > M_3$, and $\Omega_2 = \{x : x \in \partial\Omega \cap Ker\ L\}$. Then, $\forall\, x \in \partial\Omega \cap Ker\ L$,

$$QNx = \frac{1}{T} \int_0^T \begin{pmatrix} \phi_q(x_2(t)) \\ -f(x_1)x_1'(t) - g(t,x_1) + e(t) \end{pmatrix} dt.$$

If $QNx = 0$, then $x_2(t) = 0,\ x_1 = E_2$ or E_1. If $x_1(t) = E_2$, then

$$0 = \int_0^T (g(t,E_2) - e(t))dt.$$

From condition (H_2), we have $x_1(t) \le D_2 \le E_2$, which yields a contradiction. Similarly, we have the case that $x_1 = E_1$. We also have $QNx \neq 0$, i.e., $\forall\ x \in \partial\Omega \cap Ker\ L,\ x \notin Im\ L$. Hence, assumptions (1) and (2) of Lemma 2.1 are satisfied. Define the isomorphism $J : Im\ Q \to Ker\ L$ as follows

$$J(x_1,x_2)^\top = (x_2, -x_1)^\top.$$

Letting $H(\mu,x) = -\mu x + (1-\mu)JQNx,\ (\mu,x) \in [0,1] \times \Omega$, we have, $\forall\ (\mu,x) \in (0,1) \times (\partial\Omega \cap Ker\ L)$,

$$H(\mu,x) = \begin{pmatrix} -\mu x_1 - \frac{1-\mu}{T} \int_0^T (g(t,x_1) - e(t))dt \\ -\mu x_2 - (1-\mu)\phi_q(x_2) \end{pmatrix}.$$

From condition (H_2), we get $x^\top H(\mu,x)\neq 0$, $\forall\ (\mu,x)\in(0,1)\times(\partial\Omega\cap Ker\ L)$. Hence

$$\begin{aligned}\deg\{JQN,\Omega\cap Ker\ L,0\}&=\deg\{H(0,x),\Omega\cap Ker\ L,0\}\\&=\deg\{H(1,x),\Omega\cap Ker\ L,0\}\\&=\deg\{I,\Omega\cap Ker\ L,0\}\neq 0.\end{aligned}$$

So assumption (3) of Lemma 2.1 is satisfied. Hence, we conclude that equation $Lx=Nx$ has a solution $x=(x_1,x_2)^\top$ on $\bar{\Omega}\cap D(L)$, i.e., equation (2.1) has a T-periodic solution $x_1(t)$. □

Remark 3.2. If equation (1.5) satisfies the singularity of attractive type, i.e., $\int_0^1 g_0(x)dx=+\infty$. Obviously, the attractive condition, and (H_1), (H_2) and (H_4) are contradictive. Therefore, the above method and conditions cannot be applicable to the existence of positive periodic solutions of equation (1.5) with the singularity of attractive type. Next, we will give another conditions to prove the existence of positive T-periodic solutions.

Theorem 3.3. *Let condition (H_3) hold. In addition, assume that the following conditions hold:*
(H_5) there exist two positive constants D_3, D_4 with $D_3<D_4$ such that $g(t,x)-e(t)>0$ for all $(t,x)\in[0,T]\times(0,D_3)$, and $g(t,x)-e(t)<0$ for all $(t,x)\in[0,T]\times(D_4,+\infty)$;
(H_6) there exist positive constants a', b' such that

$$-g(t,x)\leq a'x^{p-1}+b',\qquad \textit{for all}\qquad (t,x)\in[0,T]\times(0,+\infty).$$

(H_7) (singularity of attractive type)

$$\int_0^1 g_0(x)dx=+\infty.$$

Then equation (1.5) has at least one positive T-periodic solution if

$$0<\frac{a'T}{1-\delta'}\left(\frac{T}{\pi_p}\right)^{2p-1}<1.$$

Proof. We follow the same strategy as in the proof of Theorem 3.1. From equation (3.3) and condition (H_5), we know that there are points $\tau,\ \nu\in(0,T)$ such that

$$x_1(\tau-\delta(\tau))\geq D_3,\qquad x_1(\nu)\leq D_4. \tag{3.25}$$

Next, we consider $\int_0^T|g(t,x_1(t-\delta(t)))|dt$. From equation (3.8) and condition (H_6), we obtain

$$\begin{aligned}\int_0^T|g(t,x_1(t-\delta(t)))|dt&=\int_{g(t,x_1)\geq 0}g(t,x_1(t-\delta(t)))dt-\int_{g(t,x_1)\leq 0}g(t,x_1(t-\delta(t)))dt\\&=-2\int_{g(t,x_1)\leq 0}g(t,x_1(t-\delta(t)))dt+\int_0^T e(t)dt\\&\leq 2a'\int_0^T x_1^{p-1}(t-\delta(t))dt+2b'T+\int_0^T|e(t)|dt\\&=2a'\int_0^T|x_1(t-\delta(t))|^{p-1}dt+2b'T+\|e\|T\\&\leq\frac{2a'}{1-\delta'}\int_0^T|x_1(t)|^{p-1}dt+2b'T+\|e\|T,\end{aligned}$$

The proof left is as same as Theorem 3.1. This completes the proof. □

4. EXAMPLES

In this section, we present two examples to illustrate Theorems 3.1 and 3.3.

Example 4.1. Consider the fourth-order Liénard equation with the singularity of repulsive type and time-dependent deviating argument:

$$(\phi_p(x''(t)))''+f(x(t))x'(t)+\frac{1}{20\pi}(\sin 2t+3)x\left(t-\frac{1}{4}\cos 2t\right)-\frac{1}{x^{\kappa}\left(t-\frac{1}{4}\cos 2t\right)}=\cos 2t \tag{4.1}$$

where $\kappa\geq 1$and $p=4$, f is a continuous function.

It is clear that $T=\pi$, $g(t,x)=\frac{1}{20\pi}(\sin 2t+3)x-\frac{1}{x^{\kappa}}$, $\delta(t)=\frac{1}{4}\cos 2t$, $e(t)=\cos 2t$, $a=\frac{1}{5\pi}$, $\delta'=\frac{1}{2}$, $\pi_4=\frac{2\pi(p-1)^{\frac{1}{p}}}{p\sin(\pi/p)}=\frac{2\pi(4-1)^{\frac{1}{4}}}{4\cdot\frac{\sqrt{2}}{2}}=\pi\times\left(\frac{3}{4}\right)^{\frac{1}{4}}$. It is obvious that $(H_1)-(H_4)$ hold. Now we consider

$$\frac{aT}{1-\delta'}\left(\frac{T}{\pi_p}\right)^{2p-1}=\frac{2}{5\pi}\times\pi\times\left(\frac{\pi}{\pi\times\left(\frac{3}{4}\right)^{\frac{1}{4}}}\right)^{7}=\frac{2}{5}\times\left(\frac{4}{3}\right)^{\frac{7}{4}}\approx 0.6618<1$$

By use of Theorem 3.1, we know (4.1) has at least one positive π-periodic solution.

Example 4.2. Consider the fourth-order Liénard equation with the singularity of attractive type:

$$x^{(4)}(t)-f(x(t)x'(t)-\frac{1}{3\pi}(\cos 4t+5)x\left(t-\frac{1}{8}\sin^2 2t\right)+\frac{5}{x^{\kappa'}\left(t-\frac{1}{8}\sin^2 2t\right)}=e^{\cos^2(2t)}, \tag{4.2}$$

where $\kappa'\geq 1$, $p=2$, and f is a continuous function. It is clear that $T=\frac{\pi}{2}$, $g(t,x)=-\frac{1}{3\pi}(\cos 4t+5)x+\frac{5}{x^{\kappa'}}$, $\delta=\frac{1}{8}\sin^2 2t$, $e(t)=e^{\cos^2(2t)}$, $\delta'=\frac{1}{2}$, $\pi_2=\frac{2\pi(p-1)^{\frac{1}{p}}}{p\sin(\pi/p)}=\frac{2\pi(2-1)^{\frac{1}{2}}}{4\times\frac{1}{2}}=\pi$, and $a'=\frac{2}{\pi}$. It is obvious that (H_3), $(H_5)-(H_7)$ hold. Now we consider

$$\frac{a'T}{1-\delta'}\left(\frac{T}{\pi_p}\right)^{2p-1}=\frac{4}{\pi}\times\frac{\pi}{2}\times\left(\frac{1}{2}\right)^{3}=\frac{1}{4}<1.$$

From Theorem 3.3, we know that (4.2) has at least one positive $\frac{\pi}{2}$-periodic solution.

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