J. Nonlinear Funct. Anal. 2020 (2020), Article ID 44 https://doi.org/10.23952/jnfa.2020.44

Journal of Nonlinear Functional Analysis
Available online at http://jnfa.mathres.org

A SPLITTING METHOD FOR A FAMILY OF EQUILIBRIUM AND INCLUSION PROBLEMS

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Abstract. The propose of this paper is to introduce a hybrid forward-backward iterative method for studying common solutions of a family of equilibrium problem and a family of monotone mappings. The strong convergence of the hybrid forward-backward iterative method is obtained in the framework of real Hilbert spaces.

Keywords. Forward-backward iterative method; Fixed point; Monotone mapping; Nonexpansive mapping; Variational inequality.

1. Introduction

Let H be a real Hilbert space with norm $\|\cdot\|$ and inner produce $\langle\cdot,\cdot\rangle$. Let C be a convex and closed nonempty set in H. For any vector x in space H, there exists a unique vector in C, denoted by P_Cx, such that

$$P_Cx := \operatorname{argmin}\{\|x-y\|, y\in C\}.$$

P_Cx is the nearest vector in C to x and P_C is called the metric or nearest point projection of H onto C. One knows that P_C has the following elegant properties:

- $\langle x-P_Cx, y-P_Cx\rangle \le 0, \forall y\in C$;
- $\|P_Cx-P_Cy\|^2 \le \langle P_Cx-P_Cy, x-y\rangle, \forall y\in H$;
- $\|x-P_C(x)\|^2 \le \|x-y\|^2 - \|y-P_C(x)\|^2, \forall y\in C$.

If subset C is a half-space $H_{u,v}=\{x:\langle u,x\rangle\le v\}$, we have the closed projection formula computed by

$$P_{H_{u,v}}(x) = x-\max\{[\langle u,x\rangle - v]/\|u\|^2, 0\}u.$$

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Received August 18, 2020; Accepted October 14, 2020.

The projection of x onto the intersection of a hyperplane and a box

$$C = H_{u,v} \cap \text{Box}[a,b] = \left\{ x \in \mathbb{R}^n : u^\top x = v, a \le x \le b \right\}$$

is computed by

$$P_C(x) = P_{\text{Box}[a,b]}(x - \mu^* u),$$

with μ^* being a solution of the equation $\varphi(\mu) = u^\top P_{\text{Box}[a,b]}(x - \mu u) - v$.

Let $B : C \times C \to \mathbb{R}$ be a real bifunction. In this paper, one always assume that bifunction B satisfies the following restrictions

(R1) $B(x,x) = 0$, for all $x \in C$;

(R2) $B(x,y) + B(y,x) \le 0$, for all $x,y \in C$;

(R3) $\limsup_{t\downarrow 0} B(tz + (1-t)x, y) \le B(x,y)$, for each $x,y,z \in C$;

(R4) $y \mapsto B(x,y)$ is convex and weakly lower semi-continuous for each $x \in C$.

In this paper, we focus on the equilibrium problem with the bifunction in the sense of Blum and Oettli [1]

$$\text{finding } x^* \in C \text{ such that } B(x^*, x) \ge 0, \quad \forall x \in C. \tag{1.1}$$

This equilibrium problem, which was first considered by Fan [2], is quite general. Indeed, it includes saddle problems, complementary problems, fixed point problems, and variational inequality problems. On the other hand, it also finds a number of real applications in machine learning, economic and finance, traffic network, medical imaging etc; see, e.g., [3, 4, 5, 6, 7, 8] and the references therein. From Blum and Oettli [1], one can define a resolvent mapping and transfer the solution problem of the equilibrium problem into a fixed point problem of the resolvent mapping. Recently, a number of iterative methods have been suggested and analyzed for the equilibrium problem in Hilbert and Banach spaces; see, e.g., [9, 10, 11, 12, 13, 14] and the references therein.

Let $T : C \to H$ be a mapping defined by $B(x,y) = \langle Tx, y - x \rangle$ for all $x,y \in C$. We have that x^* is solution of (1.1) if and only if x^* is a solution to the variational inequality, which consist of finding $x \in C$ such that

$$\langle Tx, y - x \rangle \ge 0, \quad \forall y \in C. \tag{1.2}$$

Furthermore, one defines a mapping S by $S := P_C(I - \lambda T)$, where λ is a positive real constant. It is clear that x^* is a fixed point of mapping S, that is, $Sx^* = x^*$ if and only if it is a solution to variational inequality (1.3). Thus, fixed point methods are applicable to the variational inequality (and also the equilibrium problem). Recently, a number of fixed point iterative methods have been suggested for the variational inequality. Various convergence theorems of weak and strong were established in Hilbert and Banach spaces; see, e.g., [15, 16, 17, 18, 19, 20] and the references therein.

Let $A : C \to H$ be a single-valued mapping. Recall that A is said to be monotone if and only if

$$\langle Ax - Ay, x - y \rangle \ge 0, \quad \forall x,y \in C.$$

A is said to be ν-inverse-strongly monotone if and only if there exists a constant $\nu > 0$ such that

$$\langle Ax - Ay, x - y \rangle \ge \nu \|Ax - Ay\|^2, \quad \forall x,y \in C.$$

It is easy to see that inverse-strongly monotone mappings are monotone and (Lipschitz) continuous, that is, maximally monotone.

Let $M : H \to 2^H$ be multi-valued mapping. One employs $Graph(M) = \{(x,y) : y \in Mx\}$ to denote the graph of M, $Dom(M) = \{x \in H : Mx \neq \emptyset\}$ to denote its domain, and $Ran(M) = \{Bx : x \in Dom(M)\}$ to denote the range. Mapping M is said to be monotone if and only if $\langle x_1 - x_2, y_1 - y_2\rangle \geq 0$ for all $x_1 \in Dom(M)$, $x_2 \in Dom(M)$, $y_1 \in Mx_1$ and $y_2 \in Mx_2$. Mapping M is said to be maximal if its graph $Graph(M)$ is not contained in the graph of any other monotone mapping properly. From the theory of monotone mappings, one can define a resolvent operator $Res_r^M := (I + rM)^{-1}$, where r denotes a positive real number for a maximally monotone mapping M. One knows that a zero point x^* to mapping M if and only if it is a fixed point to Res_r^M, that is, $Res_r^M x^* = x^*$.

Let S be a mapping on H. S is said to be firmly nonexpansive if and only if

$$\|Sx - Sy\|^2 \leq \langle x - y, Sx - Sy\rangle,$$

$\forall x, y \in H$. It is known that metric projections, the resolvents of maximal monotone mappings, and the resolvents of the equilibrium problem (see below) are firmly nonexpansive. Furthermore, S is said to be nonexpansive if and only if

$$\|Sx - Sy\| \leq \|x - y\|, \quad \forall x, y \in H.$$

It is clear that the class of firmly nonexpansive mappings is a subclass of nonexpansive mappings. One knows that every nonexpansive mapping defined on the convex, closed and bounded subset of C of a Hilbert space H has a nonempty fixed point set and the set is convex and closed. For recent fixed point methods for dealing with zero points of maximally monotone mappings in Hilbert spaces, one refers to [21, 22, 23, 24, 25] and the references therein.

Next, we turn our attention to another important problem of maximal monotone mappings. Recently, finding a zero point of the sum of maximally monotone operators is now under spotlight. In this paper, we consider the case of the sum of two monotone operators

$$0 \in (A + M)(x), \tag{1.3}$$

where A and M are two monotone operators. There are number of problems can be remodelled as the inclusion problem, for example, the initial value problem of the evolution equation

$$\begin{cases} 0 \in (A + M)u + \frac{\partial u}{\partial t}, \\ u_0 = u(0). \end{cases}$$

Indeed, (1.3) provides a general framework for a number of problems in nonlinear equations and engineering optimization. Therefore, solving problem (1.3) is necessary and important from the viewpoint of theory and applications. Iterative methods are popular to investigate problem (1.3), in particular, the splitting methods (forward-backward splitting, Douglas–Rachford splitting and Peaceman–Rachford splitting). A splitting method means an method that involves individual operators instead of the sum of them. In the setting of infinite dimensional spaces, the methods of Douglas–Rachford splitting and Peaceman–Rachford splitting are not desirable. The method of forward-backward splitting recently has received much attention and various convergent theorems were established; see, e.g., [26, 27, 28, 29] and the references therein.

In this paper, we focus on the following convex feasibility problem, which consists of finding a common point in the convex sets $\cap_{m=1}^N (Sol(B_m) \cap (A + M)^{-1}(0))$, where N is some positive integer. We investigate a forward-backward splitting method and analyze its convergence analysis in the framework of Hilbert spaces. Our strong convergence theorem does not require any compact assumption imposed on the involved mappings and sets. To show our main convergence theorem, we need the following definitions and lemmas.

Lemma 1.1. [26] *Let H be a Hilbert space and let C be a convex, closed nonempty subset of H. Let $M : H \rightrightarrows H$ be a maximally set-valued monotone operator and let $A : C \to H$ be a single-valued inverse-strongly monotone mapping. Then $(M+A)^{-1}(0) = Fix(Res^M_\mu(I-\mu A))$, the fixed point set of $Res^M_\mu(I-\mu A)$.*

Lemma 1.2. [1]*Let H be a Hilbert space and let C be a convex, closed nonempty subset of H. Let $B : C \times C \to \mathbb{R}$ be a bifunction, which satisfies restrictions* (R1)-(R4). *For any $x \in H$ and $r > 0$, there is $z \in C$ such that*

$$\eta B(z,y) + \langle y-z, z-x\rangle \geq 0, \quad \forall y \in C.$$

Further, if we define

$$Res^B_\eta x := \Big\{ z \in C : \eta B(z,y) + \langle y-z, z-x\rangle \geq 0,\ \forall y \in C \Big\}$$

for any $\eta > 0$ and $x \in H$, then $Fix(Res^B_\eta) = Sol(B)$ is convex and closed, where $Fix(Res^B_\eta)$ denotes the set of fixed points of mapping Res^B_η, and Res^B_η is single-valued firmly nonexpansive.

Lemma 1.3. [30] *Let H be a Hilbert space and let C be a convex, closed nonempty subset of H. Let $T : C \to C$ be a nonexpansive mapping. Then the fixed point set of T, $Fix(T)$ is convex and closed.*

2. Main results

Theorem 2.1. *Let C be a convex, closed nonempty subset of a real Hilbert space H. Let N be some positive integer. Let $B_m : C \times C \to \mathbb{R}$ be a bifunction with restrictions (R1)-(R4) for each $1 \leq m \leq N$. Let $A_m : C \to H$ be a ν_m-inverse-strongly monotone mapping and let M_m be a maximally monotone mapping on H for each $1 \leq m \leq N$. Let $\{\alpha_n\}$ be a real sequence in $[a,1)$, where a is some real constant in $(0,1)$. Let $\{\eta_{n,m}\}$ be a real sequence with $\liminf_{n\to\infty} \eta_{n,m} > 0$ for each $1 \leq m \leq N$. Let $\{\mu_{n,m}\}$ be a real sequence in $[b,c]$, where $0 < b$ and $c \leq 2\nu_m$ for each $1 \leq m \leq N$. Let $\{\varphi_{n,m}\}$ be real sequence in $(d,1)$, with d is a real in $(0,1)$ for each $1 \leq m \leq N$ and $\sum_{m=1}^N \varphi_{n,m} = 1$. Let $\{x_n\}$ be a vector sequence defined by*

$$\begin{cases} x_1 \in H, \\ C_1 = H, \\ \eta_{n,m} B_m(y_{n,m}, y) + \langle y - y_{n,m}, y_{n,m} - x_n\rangle \geq 0, \quad \forall y \in C, \\ z_n = (1-\alpha_n)x_n + \alpha_n \sum_{m=1}^N \varphi_{n,m} Res^{M_m}_{\mu_{n,m}}(I-\mu_{n,m}A_m)y_{n,m}, \\ C_{n+1} = \{\xi \in C_n : \|x_n - \xi\| \geq \|z_n - \xi\|\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \quad n \geq 0. \end{cases}$$

If $\Omega := \cap_{m=1}^N (Sol(B_m) \cap (A_m+M_m)^{-1}(0)) \neq \emptyset$, then the sequence $\{x_n\}$ generated above converges to $P_\Omega x_1$ in norm.

Proof. From Lemma 1.2, one asserts that $\cap_{m=1}^N Sol(B_m)$ is convex and closed. Since each M_m is maximally monotone, one has that $Res^{M_m}_{\mu_{n,m}}$ is nonexpansive for each m and each n. On the other hand, one has

$$\begin{aligned}&\|(I-\mu_{n,m}A_m)x-(I-\mu_{n,m}A_m)y\|^2\\&=\|x-y\|^2-2\mu_{n,m}\langle x-y,A_mx-A_my\rangle+\mu_{n,m}^2\|A_mx-A_my\|^2\\&\le\|x-y\|^2-\mu_{n,m}(2\nu_m-\mu_{n,m})\|A_mx-A_my\|^2\\&\le\|x-y\|^2.\end{aligned}$$

It yields that $I-\mu_{n,m}A_m$ is nonexpansive for each m and each n. From Lemmas 1.1 and 1.3, one asserts that $\cap_{m=1}^N(A_m+M_m)^{-1}(0)$ is convex and closed. This proves that the metric projection on Ω is well defined.

Now, we show that C_n is a convex and closed. From the construction of C_n, we conclude that C_n is closed. Note that $C_1=H$ is a convex set. We now suppose that C_j is a convex set. Observe that C_{j+1} can be re-written as

$$C_{j+1}=\{\xi\in C_j:2\langle z_j-x_j,\xi\rangle\le\|z_j\|^2-\|x_j\|^2\}.$$

Let ξ_1 and ξ_2 be two vectors in C_{j+1}. Let $\bar{\xi}=\nu\xi_1+(1-\nu)\xi_2$, where ν is a real in $(0,1)$. From the construction of C_{j+1}, one easily sees that $2\langle z_j-x_j,\bar{\xi}\rangle\le\|z_j\|^2-\|x_j\|^2$. This shows that $\bar{\xi}\in C_{j+1}$. This proves the convexity of C_n. So, the metric projection onto C_n is well defined.

Now, we are in a position to show that Ω lines in C_n. It is clear that $\Omega\subset C_1$. Next, we suppose $\Omega\subset C_j$. For the same positive integer j, we aim to show $\Omega\subset C_{j+1}$. For any $p\in\Omega$, we conclude from Lemmas 1.1 and 1.2, we have

$$\begin{aligned}\|z_j-p\|&\le\alpha_j\|\sum_{m=1}^N\varphi_{j,m}Res_{\mu_{j,m}}^{M_m}(I-\mu_{j,m}A_m)y_{j,m}-p\|+(1-\alpha_j)\|x_j-p\|\\&\le\alpha_j\sum_{m=1}^N\varphi_{j,m}\|Res_{\mu_{j,m}}^{M_m}(I-\mu_{j,m}A_m)y_{j,m}-p\|+(1-\alpha_j)\|x_j-p\|\\&\le\alpha_j\sum_{m=1}^N\varphi_{j,m}\|Res_{\eta_{j,m}}^{B_m}x_j-p\|+(1-\alpha_j)\|x_j-p\|\\&\le\|x_j-p\|,\end{aligned}$$

that is, $p\in C_{j+1}$. This yields that $\Omega\subset C_n$ for each n. Next, we turn our attention to the boundedness of $\{x_n\}$. For any $p\in\Omega$, borrowing the fact that $x_n=P_{C_n}x_1$, we arrive at $\|p-x_1\|\ge\|x_n-x_1\|$. In particular, we have $\|P_\Omega x_1-x_1\|\ge\|x_n-x_1\|\ge0$. This proves that $\{x_n\}$ is a bounded sequence. From the known fact that every bounded sequences in Hilbert spaces has s weakly convergent subsequence, we denote a subsequence $\{x_{n_j}\}$ of $\{x_n\}$, which converges to x^* weakly. An element equality in Hilbert spaces yields that

$$\begin{aligned}&\|x_{n+1}-x_n\|^2\\&=\|x_n-x_1\|^2+2\langle x_n+x_1-x_{n+1}-x_n,x_n-x_1\rangle+\|x_{n+1}-x_1\|^2\\&=\|x_{n+1}-x_1\|^2+2\langle x_n-x_{n+1},x_n-x_1\rangle-\|x_n-x_1\|^2.\end{aligned}$$

From $x_{n+1}\in C_n$ and $x_n=P_{C_n}x_1$, we arrive at

$$\|x_{n+1}-x_n\|^2\le\|x_{n+1}-x_1\|^2-\|x_n-x_1\|^2,$$

On the other hand,

$$
\begin{aligned}
0 &\le \langle x_1 - x_n, x_n - x_{n+1}\rangle \\
&= \langle x_1 - x_n, x_n - x_1 + x_1 - x_{n+1}\rangle \\
&\le -\|x_1 - x_n\|^2 + \|x_1 - x_n\|\|x_1 - x_{n+1}\|.
\end{aligned}
$$

It follows that $\|x_1 - x_n\| \le \|x_1 - x_{n+1}\|$. So, $\lim_{n\to\infty}\|x_n - x_1\|$ exists. This further implies $\lim_{n\to\infty}\|x_n - x_{n+1}\| = 0$. Thanks to the fact that $x_n \in C_n$ for each n, we assert that $\lim_{n\to\infty}\|x_{n+1} - z_n\| = 0$. Due to

$$\|x_n - z_n\| \le \|x_n - x_{n+1}\| + \|x_{n+1} - z_n\|,$$

we have $\lim_{n\to\infty}\|x_n - z_n\| = 0$. Put

$$\kappa_n = \sum_{m=1}^{N} \varphi_{n,m} Res_{\mu_{n,m}}^{M_m}(I - \mu_{n,m}A_m)y_{n,m}$$

and fix $p \in \Omega$. Since both $Res_{\mu_{n,m}}^{M_m}$ and $I - \mu_{n,m}A_m$ are nonexpansive, we have

$$
\begin{aligned}
\|z_n - p\|^2 &\le \alpha_n\|\kappa_n - p\|^2 + (1-\alpha_n)\|x_n - p\|^2 \\
&\le \alpha_n \sum_{m=1}^{N} \varphi_{n,m}\|Res_{\mu_{n,m}}^{M_m}(I - \mu_{n,m}A_m)y_{n,m} - p\|^2 + (1-\alpha_n)\|x_n - p\|^2 \\
&\le \alpha_n \sum_{m=1}^{N} \varphi_{n,m}\|y_{n,m} - p\|^2 + (1-\alpha_n)\|x_n - p\|^2.
\end{aligned}
$$

From Lemma 1.2, we have

$$
\begin{aligned}
2\|y_{n,m} - p\|^2 &= 2\|Res_{\eta_{n,m}}^{B_m}x_n - Res_{\eta_{n,m}}^{B_m}p\|^2 \\
&\le 2\langle, x_n - p, Res_{\eta_{n,m}}^{B_m}x_n - Res_{\eta_{n,m}}^{B_m}p\rangle \\
&= 2\langle x_n - p, y_{n,m} - p\rangle \\
&= \|y_{n,m} - p\|^2 + \|x_n - p\|^2 - \|y_{n,m} - x_n\|^2.
\end{aligned}
$$

Thus,

$$\|y_{n,m} - p\|^2 \le \|x_n - p\|^2 - \|y_{n,m} - x_n\|^2,$$

and then

$$
\begin{aligned}
\|z_n - p\|^2 &\le \alpha_n \sum_{m=1}^{N} \varphi_{n,m}(\|x_n - p\|^2 - \|y_{n,m} - x_n\|^2) + (1-\alpha_n)\|x_n - p\|^2 \\
&\le \|x_n - p\|^2 - \alpha_n \sum_{m=1}^{N} \varphi_{n,m}\|y_{n,m} - x_n\|^2.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\alpha_n \sum_{m=1}^{N} \varphi_{n,m}\|y_{n,m} - x_n\|^2 &\le \|x_n - p\|^2 - \|z_n - p\|^2 \\
&\le (\|x_n - p\| + \|z_n - p\|)\|x_n - z_n\|.
\end{aligned}
$$

By use of the restrictions on $\{\alpha_n\}$ and $\{\varphi_{n,m}\}$, we find that $\lim_{n\to\infty}\|y_{n,m} - x_n\| = 0$ for each $1 \le m \le N$. For any $y \in C$, we have

$$\eta_{n,m}B_m(y_{n,m}, y) + \langle y - y_{n,m}, y_{n,m} - x_n\rangle \ge 0.$$

By use of the monotonicity of B_m, (R2), we obtain that

$$\frac{1}{\eta_{n,m}}\langle y-y_{n,m}, y_{n,m}-x_n\rangle \geq B_m(y, y_{n,m})$$

Note that $\{y_{n_j,m}\}$, which is a subsequence of $\{y_{n,m}\}$ converges to x^* weakly. By replacing n by n_j, we find from (R4) that $B_m(y,x^*)\leq 0$, $\forall y\in C$. It follows that

$$B_m((1-\zeta)y+\zeta x^*, x^*)\leq 0,$$

where ζ is a real number in $(0,1)$, and then

$$\begin{aligned}
&B_m((1-\zeta)y+\zeta x^*,(1-\zeta)y+\zeta x^*)\\
&\leq \zeta B_m((1-\zeta)y+\zeta x^*,x^*)+(1-\zeta)B_m((1-\zeta)y+\zeta x^*,y)\\
&\leq (1-\zeta)B_m((1-\zeta)y+\zeta x^*,y).
\end{aligned}$$

From (R1), we get that $B_m((1-\zeta)y+\zeta x^*,y)\geq 0$, $\forall y\in C$. Letting $c\uparrow 1$ and using restriction (R3), we see that $B_m(x^*,y)\geq 0$, $\forall y\in C$. This prove that $x^*\in Sol(B_m)$, for each m.

Next, we prove that $x^*\in (A_m+M_m)^{-1}(0)$ for each m. Setting $t_{n,m}=(I+\mu_{n,m}M_m)^{-1}(I-\mu_{n,m}A_m)y_{n,m}$, we have

$$M_m t_{n,m}\ni \frac{y_{n,m}-t_{n,m}}{\mu_{n,m}}-A_m y_{n,m}.$$

Since, for each $1\leq m\leq N$, B_m is maximally monotone, for any $(\psi_m,\phi_m)\in M_m$, we have

$$\langle t_{n,m}-\psi_m, \frac{y_{n,m}-t_{n,m}}{\mu_{n,m}}-A_m y_{n,m}-\phi_m\rangle\geq 0.$$

It is not hard to see that $y_{n,m}-t_{n,m}\to 0$ as $n\to\infty$ for each m. Indeed, since each A_m is inverse-strongly monotone, we have

$$\begin{aligned}
&\|(I-\mu_{n,m}A_m)y_{n,m}-(I-\mu_{n,m}A_m)p\|^2\\
&=\mu_{n,m}^2\|A_m y_{n,m}-A_m p\|^2-2\mu_{n,m}\langle y_{n,m}-p, A_m y_{n,m}-A_m p\rangle+\|y_{n,m}-p\|^2\\
&\leq \|y_{n,m}-p\|^2-\mu_{n,m}(2\nu_m-\mu_{n,m})\|A_m y_{n,m}-A_m p\|^2\\
&\leq \|x_n-p\|^2-\mu_{n,m}(2\nu_m-\mu_{n,m})\|A_m y_{n,m}-A_m p\|^2, \quad \forall 1\leq m\leq N.
\end{aligned}$$

This implies from the convexity of $\|\cdot\|^2$ that

$$\begin{aligned}
&\|z_n-p\|^2\\
&\leq \alpha_n\|\kappa_n-p\|^2+(1-\alpha_n)\|x_n-p\|^2\\
&\leq \alpha_n\sum_{m=1}^N \varphi_{n,m}\|Res_{\mu_{n,m}}^{M_m}(I-\mu_{n,m}A)y_{n,m}-p\|^2+(1-\alpha_n)\|x_n-p\|^2\\
&\leq \alpha_n\sum_{m=1}^N \varphi_{n,m}\|(I-\mu_{n,m}A)y_{n,m}-(I-\mu_{n,m}A)y_{n,m}p\|^2+(1-\alpha_n)\|x_n-p\|^2\\
&\leq \alpha_n\sum_{m=1}^N \varphi_{n,m}(\|x_n-p\|^2-\mu_{n,m}(2\nu_m-\mu_{n,m})\|A_m y_{n,m}-A_m p\|^2)+(1-\alpha_n)\|x_n-p\|^2\\
&\leq \|x_n-p\|^2-\alpha_n\sum_{m=1}^N \varphi_{n,m}(2\nu_m-\mu_{n,m})\mu_{n,m}\|A_m y_{n,m}-A_m p\|^2.
\end{aligned}$$

Thus

$$\begin{aligned}&\mu_{n,m}\varphi_{n,m}(2\nu_m-\mu_{n,m})\alpha_n\|A_my_{n,m}-A_mp\|^2\\&\leq\|x_n-p\|^2-\|z_n-p\|^2\\&\leq(\|x_n-p\|+\|z_n-p\|)\|x_n-z_n\|,\quad\forall 1\leq m\leq N.\end{aligned}$$

From the boundedness of $\{x_n\}$ and $\{z_n\}$, and conditions imposed on the real sequences, we have $A_my_{n,m}-A_mp\to 0$, for each $1\leq m\leq N$, as $n\to\infty$. By use of the firm nonexpansivity, we have

$$\begin{aligned}&2\|t_{n,m}-p\|^2\\&\leq 2\langle(I-\mu_{n,m}A_m)y_{n,m}-(I-\mu_{n,m}A_m)p,Res_{\mu_n}^{M_m}(I-\mu_{n,m}A_m)y_{n,m}-p\rangle\\&=\|t_{n,m}-p\|^2+\|(I-\mu_{n,m}A_m)y_{n,m}-(I-\mu_{n,m}A_m)p\|^2\\&\quad-\|(t_{n,m}-p)-(I-\mu_{n,m}A_m)y_{n,m}+(I-\mu_{n,m}A_m)p\|^2\\&\leq\|t_{n,m}-p\|^2+\|y_{n,m}-p\|^2-\|t_{n,m}+\mu_{n,m}(A_my_{n,m}-A_mp)-y_{n,m}\|^2\\&\leq\|t_{n,m}-p\|^2+\|y_{n,m}-p\|^2-\|t_{n,m}-y_{n,m}\|^2-\mu_{n,m}^2\|A_my_{n,m}-A_mp\|^2\\&\quad+2\mu_{n,m}\|t_{n,m}-y_{n,m}\|\|A_my_{n,m}-A_mp\|\\&\leq\|t_{n,m}-p\|^2+\|y_{n,m}-p\|^2-\|t_{n,m}-y_{n,m}\|^2\\&\quad+2\mu_{n,m}\|t_{n,m}-y_{n,m}\|\|A_my_{n,m}-A_mp\|.\end{aligned}$$

Hence,

$$\begin{aligned}\|z_n-p\|^2&\leq\alpha_n\|\kappa_n-p\|^2+(1-\alpha_n)\|x_n-p\|^2\\&\leq\alpha_n\sum_{m=1}^N\varphi_{n,m}\|t_{n,m}-p\|^2+(1-\alpha_n)\|x_n-p\|^2\\&\leq\alpha_n\sum_{m=1}^N\varphi_{n,m}(\|y_{n,m}-p\|^2-\|t_{n,m}-y_{n,m}\|^2\\&\quad+2\mu_{n,m}\|t_{n,m}-y_{n,m}\|\|A_my_{n,m}-A_mp\|)+(1-\alpha_n)\|x_n-p\|^2\\&\leq\alpha_n\sum_{m=1}^N\varphi_{n,m}(\|x_n-p\|^2-\|t_{n,m}-y_{n,m}\|^2\\&\quad+2\mu_{n,m}\|t_{n,m}-y_{n,m}\|\|A_my_{n,m}-A_mp\|)+(1-\alpha_n)\|x_n-p\|^2\\&\leq\|x_n-p\|^2-\alpha_n\sum_{m=1}^N\varphi_{n,m}\|t_{n,m}-y_{n,m}\|^2\\&\quad+2\mu_{n,m}\alpha_n\sum_{m=1}^N\varphi_{n,m}\|t_{n,m}-y_{n,m}\|\|A_my_{n,m}-A_mp\|,\end{aligned}$$

and then

$$\begin{aligned}\alpha_n\varphi_{n,m}\|t_{n,m}-y_{n,m}\|^2 &\le \|x_n-p\|^2-\|z_n-p\|^2\\ &\quad +2\mu_{n,m}\alpha_n\sum_{m=1}^{N}\varphi_{n,m}\|t_{n,m}-y_{n,m}\|\|A_my_{n,m}-A_mp\|\\ &\le (\|x_n-p\|+\|z_n-p\|)\|x_n-z_n\|\\ &\quad +2\mu_{n,m}\alpha_n\sum_{m=1}^{N}\varphi_{n,m}\|t_{n,m}-y_{n,m}\|\|A_my_{n,m}-A_mp\|.\end{aligned}$$

This prove that $t_{n,m}-y_{n,m}\to 0$, for each $1\le m\le N$, as $n\to\infty$. This indicates that $\{t_{n_j,m}\}$, which is the subsequence of $\{t_{n,m}\}$, converges to x^* for each m. Since the graph of maximally monotone mappings is weakly-strongly closed, we confirm that

$$\langle x^*-\psi_m, -A_mx^*-\phi_m\rangle \ge 0.$$

This indicates

$$x^*\in(A_m+B_m)^{-1}(0).$$

From above, we prove that $x^*\in\Omega$. Since norms are weakly lower semi-continuous, we have

$$\begin{aligned}\|x_1-P_\Phi x_1\| &\le \|x_1-x^*\|\\ &\le \liminf_{j\to\infty}\|x_1-x_{n_j}\| \le \limsup_{j\to\infty}\|x_1-x_{n_j}\|\\ &\le \|x_1-P_\Phi x_1\|,\end{aligned}$$

which yields that

$$\|x_1-x^*\| = \lim_{j\to\infty}\|x_1-x_{n_j}\| = \|x_1-P_\Phi x_1\|.$$

In view of the fact that H is a Hilbert space, one concludes that $\{x_{n_j}\}$ converges to $P_\Phi x_1$ in norm. Therefore, $\{x_n\}$ converges to $p_\Phi x_1$ in norm. The proof is completed. □

Corollary 2.2. *Let C be a convex, closed nonempty subset of a real Hilbert space H. Let $B: C\times C\to\mathbb{R}$ be a bifunction with restrictions (R1)-(R4). Let $A: C\to H$ be a ν-inverse-strongly monotone mapping and let M be a maximally monotone mapping on H. Let $\{\alpha_n\}$ be a real sequence in $[a,1)$, where a is some real constant in $(0,1)$. Let $\{\eta_n\}$ be a real sequence with $\liminf_{n\to\infty}\eta_n>0$. Let $\{\mu_n\}$ be a real sequence in $[b,c]$, where $0<b$ and $c\le 2\nu$. Let $\{x_n\}$ be a vector sequence defined by*

$$\begin{cases} x_1\in H,\\ C_1=H,\\ \eta_nB(y_n,y)+\langle y-y_n, y_n-x_n\rangle\ge 0, \quad \forall y\in C,\\ z_n=(1-\alpha_n)x_n+\alpha_nRes^M_{\mu_n}(I-\mu_nA)y_n,\\ C_{n+1}=\{\xi\in C_n: \|x_n-\xi\|\ge\|z_n-\xi\|\},\\ x_{n+1}=P_{C_{n+1}}x_1, \quad n\ge 0.\end{cases}$$

If $\Omega:=(Sol(B)\cap(A+M)^{-1}(0)\neq\emptyset$, then the sequence $\{x_n\}$ converges to $P_\Omega x_1$ in norm.

Finally, we give a result on variational inequality (1.3). From now on, we let $\tau : H \to (-\infty, +\infty]$ be a lower semicontinuous, proper, convex function. We use $\partial\tau$ to present the subdifferential of τ, which is defined by

$$\partial\tau(x) = \{y \in H : \langle z - x, y\rangle \leq \tau(z) - \tau(x), \quad z \in H\}, \quad \forall x \in H.$$

From the celebrated results [31, 32], we find that the subdifferential $\partial\tau$ is a maximally monotone mapping. In addition, $\tau(x) = \min_{y\in H} \tau(y)$ if and only if $0 \in \partial\tau(x)$. Define the indicator function I_C of set C by

$$I_C(x) = \begin{cases} 0, & x \in C, \\ +\infty, & x \notin C. \end{cases}$$

Since I_C is proper, convex, and lower semicontinuous, one finds that the subdifferential ∂I_C of I_C is set-valued maximally monotone. For the maximally monotone mapping, one can define its resolvent mapping $Res_\mu^{I_C}$, i.e., $Res_\mu^{I_C} := (I + \mu\partial I_C)^{-1}$. Letting $x = Res_\mu^{I_C} y$, one has

$$\begin{aligned} y \in x + \mu\partial I_C x &\Longleftrightarrow y \in x + \mu N_C x \\ &\Longleftrightarrow \langle z - x, y - x\rangle \leq 0, \forall z \in C \\ &\Longleftrightarrow x = P_C y, \end{aligned}$$

where $N_C x$ is the normal cone defined by $N_C x := \{e \in H : \langle e, z - x\rangle, \forall z \in C\}$.

Corollary 2.3. *Let C be a convex, closed nonempty subset of a real Hilbert space H. Let N be some positive integer. Let $B_m : C \times C \to \mathbb{R}$ be a bifunction with restrictions (R1)-(R4) for each $1 \leq m \leq N$. Let $A_m : C \to H$ be a ν_m-inverse-strongly monotone mapping for each $1 \leq m \leq N$. Let $\{\alpha_n\}$ be a real sequence in $[a, 1)$, where a is some real constant in $(0, 1)$. Let $\{\eta_{n,m}\}$ be a real sequence with $\liminf_{n\to\infty} \eta_{n,m} > 0$ for each $1 \leq m \leq N$. Let $\{\mu_{n,m}\}$ be a real sequence in $[b, c]$, where $0 < b$ and $c \leq 2\nu_m$ for each $1 \leq m \leq N$. Let $\{\varphi_{n,m}\}$ be real sequence in $(d, 1)$, with d is a real in $(0, 1)$ for each $1 \leq m \leq N$ and $\sum_{m=1}^N \varphi_{n,m} = 1$. Let $\{x_n\}$ be a vector sequence defined by*

$$\begin{cases} x_1 \in H, \\ C_1 = H, \\ \eta_{n,m} B_m(y_{n,m}, y) + \langle y - y_{n,m}, y_{n,m} - x_n\rangle \geq 0, \quad \forall y \in C, \\ z_n = (1 - \alpha_n)x_n + \alpha_n \sum_{m=1}^N \varphi_{n,m} P_C(I - \mu_{n,m} A_m) y_{n,m}, \\ C_{n+1} = \{\xi \in C_n : \|x_n - \xi\| \geq \|z_n - \xi\|\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \quad n \geq 0. \end{cases}$$

If $\Omega := \cap_{m=1}^N (Sol(B_m) \cap VI(C, A_m)) \neq \emptyset$, then the sequence $\{x_n\}$ converges to $P_\Omega x_1$ in norm.

Funding

The first author was supported by the Science and Technology Foundation of Hebei Agricultural University under grant LG201612.

Acknowledgement

The authors are thankful to the referee for the useful suggestions and comments.

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