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# A NEW PROJECTION METHOD WITH A DYNAMIC STEPSIZE FOR THE SPLIT EQUALITY PROBLEM

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**Abstract.** In this paper, we introduce a new projection method with a dynamic stepsize for solving the split equality problem in Hilbert spaces. The weak convergence of the proposed method is established under the standard conditions. Two preliminary experiments are presented to illustrate the advantage of our method by comparing with other existing methods.
**Keywords.** Split equality problem; Weak convergence; Dynamic stepsize; Projection method.

## 1. Introduction

Let $H_1$, $H_2$ and $H_3$ be real Hilbert spaces, and let $C \subseteq H_1$, $Q \subseteq H_2$ be two nonempty closed convex sets. In this paper, we consider the split equality problem (SEP), which was first introduced by Moudafi [1]. It consists of finding $x, y$ with the property

$$x \in C,\ y \in Q, \text{ such that } Ax = By, \tag{1.1}$$

where $A : H_1 \to H_3$ and $B : H_2 \to H_3$ are two bounded linear operators. If $H_3 = H_2$ and $B = I$, the SEP is reduced to the split feasibility problem (SFP), which was first studied by Censor and Elfving [2] . The SFP has wide applications in the real world, such as, image reconstruction problems and gene regulatory network inference (see, e.g., [3, 4]).

Throughout this paper, we always assume that (1.1) is consistent (i.e., it has a solution) and its solution set is denoted by $\Gamma$, i.e.,

$$\Gamma = \{x \in C, y \in Q : Ax = By\}.$$

Most of current methods for investigating solutions of the SEP (1.1) are projection-based methods. We refer to [1, 5, 6, 7, 8, 9, 10, 11, 12, 13] for the convergence analysis of various

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projection methods. The first efficient method for the SEP is the alternating CQ algorithm proposed by Moudafi and Al-Shemas [14] as follows:

$$\begin{cases} x_{k+1} = P_C(x_k - \lambda_k A^*(Ax_k - By_k)), \\ y_{k+1} = P_Q(y_k + \lambda_k B^*(Ax_{k+1} - By_k)), \end{cases} \tag{1.2}$$

where $\lambda_k \in (\varepsilon, \min(\frac{1}{\|A\|^2}, \frac{1}{\|B\|^2} - \varepsilon)$, and $P_C$ and $P_Q$ are the metric projections onto $C$ and $Q$, respectively.

In 2018, Byrne and Moudafi [15] introduced the classical projection gradient algorithm, which is also called as the simultaneous iterative method:

$$\begin{cases} x_{k+1} = P_C(x_k - \lambda_k A^*(Ax_k - By_k)), \\ y_{k+1} = P_Q(y_k + \lambda_k B^*(Ax_k - By_k)), \end{cases} \tag{1.3}$$

where $\lambda_k \in (\varepsilon, 2/(\lambda_A + \lambda_B) - \varepsilon)$, $\lambda_A$ and $\lambda_B$ are the operator (matrix) norms $\|A\|$ and $\|B\|$ (or the largest eigenvalues of $A^*A$ and $B^*B$), respectively. Shi et al. [16] firstly presented the linear convergence rate of the gradient projection algorithm and the relaxed gradient projection algorithm provided that the SEP (1.1) satisfies the bounded linear regularity.

Very recently, Tan, Hu and Fang [17] investigated the gradient projection algorithm by using the technique of the projection dynamical system and showed that the trajectory of the dynamical system converges weakly to a solution of the approximate split equality problem as time variable goes to the infinity.

To determine stepsize $\lambda_k$ in the above algorithms, one needs to calculate (or estimate) the operator norms $\|A\|$ and $\|B\|$. In general, it is difficult or even impossible. In order to overcome this, the projection gradient algorithm with linesearch and the semi-alternating projection algorithms with linesearch were presented in [18] and [19], respectively. However, as it is common for all linesearch procedures, each inner iteration requires an evaluation of two gradients and two projections, which in the general case can eliminate all advantages of it. In 2015, Dong, He and Zhao [20] introduced a dynamical stepsize. In the same year, Vuong, Strodiot and Nguyen [21] further improved it and proposed a larger dynamic stepsize $\lambda_k$ as follows:

$$\lambda_k := \gamma_k \frac{2\|Ax_k - By_k\|^2}{\|A^*(Ax_k - By_k)\|^2 + \|B^*(Ax_k - By_k)\|^2}, \tag{1.4}$$

where $0 < \gamma_k < 1$. Note that the choice of the stepsize $\lambda_k$ in (1.4) is independent of the norms $\|A\|$ and $\|B\|$.

Inspired by the recent results on the split feasibility problem in [22], we propose a new projection method with a dynamic stepsize for the SEP and show its weak convergence under the standard conditions.

The structure of the article is as follows. In Section 2, we present some lemmas, which will be used in our main results. In Section 3, we present a new projection method with a dynamic stepsize and prove its weak convergence. In Section 4, the last section, some numerical results are provided, which illustrate the advantages of the proposed projection algorithm.

## 2. PRELIMINARIES

In this section, we present some definitions and lemmas which will be used in the main results.

In the sequel, we use the notations:

(1) $\rightharpoonup$ for weak convergence and $\to$ for strong convergence;
(2) $\omega_w(x_k) = \{x : \exists x_{k_l} \rightharpoonup x\}$ for the weak $\omega$-limit set of $\{x_k\}_{k\in\mathbb{N}}$.

The following trivial identity will be used for proving our main results:

$$\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x-y\|^2, \tag{2.1}$$

for all $\alpha \in \mathbb{R}$ and $(x,y) \in H \times H$.

Let $H$ be a real Hilbert space and let $K$ be a closed and convex nonempty subset of $H$. For a point $x \in H$, the classic metric projection of $x$ onto $K$, denoted by $P_K$, is defined by

$$P_K(x) := \operatorname{argmin}\{\|x-y\| : y \in K\}.$$

The following properties are characterizations of projections, which play a significant role in this paper.

Let $K$ be a closed convex nonempty subset of a real Hilbert space $H$. Given $x \in H$ and $z \in K$. Then $z = P_K x$ if and only if there holds the relation: $\langle x-z, y-z\rangle \leq 0$, for all $y \in K$. For any $x \in H$ and $z \in K$, the following inequalities hold

$$\|P_K(x) - z\|^2 \leq \|x-z\|^2 - \|P_K(x) - x\|^2,$$

and

$$\langle x - P_K x, x - z\rangle \geq \|x - P_K x\|^2.$$

Recall that a mapping $T : H \to H$ is said to be nonexpansive if

$$\|T(x) - T(y)\| \leq \|x-y\|, \ \forall x,y \in H.$$

The set of fixed points of $T$ is denoted by $Fix(T)$ in this paper. The class of nonexpansive mappings is essential in nonlinear analysis and optimization theory. For the existence of fixed points of the class of nonexpansive mappings, we refer to [23]. A trivial example is the projection operator.

**Lemma 2.1.** [24, Lemma 2.47] *Let $K$ be a closed and convex nonempty subset of a Hilbert space $H$ and let $T : K \to H$ be a nonexpansive mapping. Let $\{x_k\}_{k\in\mathbb{N}}$ be a sequence in $K$ and $x \in H$ such that $x_k \rightharpoonup x$ and $Tx - x_k \to 0$ as $k \to +\infty$. Then $x \in Fix(T)$.*

**Lemma 2.2.** [24, Theorem 4.27] *Let $K$ be a nonempty, closed and convex subset of a Hilbert space $H$. Let $\{x_k\}_{k\in\mathbb{N}}$ be a bounded sequence which satisfies the following properties:*

(i) *every sequential weak cluster point of $\{x_k\}_{k\in\mathbb{N}}$ lies in $K$;*
(ii) $\lim_{k\to\infty} \|x_k - x\|$ *exists for every $x \in K$.*

*Then $\{x_k\}_{k\in\mathbb{N}}$ converges weakly to a point in $K$.*

## 3. Main results

In this section, we present a new projection method and prove its weak convergence. Define a function $F : H_1 \times H_2 \to H_1$ by

$$F(x,y) = A^*(Ax - By), \tag{3.1}$$

and a function $G : H_1 \times H_2 \to H_2$ by

$$G(x,y) = B^*(By - Ax). \tag{3.2}$$

Next, we introduce the following iterative algorithm with a dynamic stepsize.

**Algorithm 1** A projection method

**Step 0.** Input $k := 0$, $(x_0, y_0) \in H_1 \times H_2$ and $\alpha \in (0,2)$.
**Step 1.** Given current iterate $(x_k, y_k)$, generate $(x_{k+1}, y_{k+1})$ by

$$\begin{cases} u_k = P_C(x_k - 2\lambda_k F(x_k, y_k)), \\ v_k = P_Q(y_k - 2\lambda_k G(x_k, y_k)), \end{cases} \tag{3.3}$$

and

$$\begin{cases} x_{k+1} := x_k + \alpha(u_k + \lambda_k F(x_k, y_k) - x_k), \\ y_{k+1} := y_k + \alpha(v_k + \lambda_k G(x_k, y_k) - y_k), \end{cases} \tag{3.4}$$

where

$$\lambda_k := \gamma \frac{\|Ax_k - By_k\|^2}{\|A^*(Ax_k - By_k)\|^2 + \|B^*(Ax_k - By_k)\|^2}, \tag{3.5}$$

and $\gamma \in (0,1)$.
**Step 2.** If $Ax_k = By_k$, $u_k = x_k$ and $v_k = y_k$, then terminate and $(x_k, y_k)$ is a solution. Otherwise, set $k := k+1$ and go to Step 1.

For simplicity, (3.3) and (3.4) can be rewritten as following:

$$\begin{aligned} x_{k+1} =& x_k + \alpha(u_k + \lambda_k(F(x_k, y_k) - x_k) \\ =&(1 - \frac{\alpha}{2})x_k + \frac{\alpha}{2}(2\lambda_k F(x_k, y_k) - x_k) + \frac{\alpha}{2} 2P_C(x_k - 2\lambda_k F(x_k, y_k)) \\ =&(1 - \frac{\alpha}{2})x_k + \frac{\alpha}{2}(2P_C(w_k) - w_k), \end{aligned} \tag{3.6}$$

where

$$w_k = x_k - 2\lambda_k F(x_k, y_k).$$

Similarly, we have

$$y_{k+1} =(1 - \frac{\alpha}{2})y_k + \frac{\alpha}{2}(2P_Q(q_k) - q_k), \tag{3.7}$$

where

$$q_k = y_k - 2\lambda_k G(x_k, y_k).$$

The following lemma shows that there is the lower bound for the stepsize $\lambda_k$.

**Lemma 3.1.** [8, Lemma 3.1] *For $\lambda_k$ defined in (3.5), it holds*

$$\lambda_k \geq \frac{\gamma}{\|A\|^2 + \|B\|^2}.$$

**Lemma 3.2.** *Let $\{(x_k, y_k)\}_{k \in \mathbb{N}}$ be generated by Algorithm 1. Then, for each $(x^*, y^*) \in \Gamma$,*

(i) *if $\varphi_k = \|x_k - x^*\|^2 + \|y_k - y^*\|^2$, then*

$$\begin{aligned} \varphi_{k+1} \leq& \varphi_k - 2\alpha\lambda_k(1-\gamma)\|Ax_k - By_k\|^2 \\ &- \frac{\alpha}{2}(1 - \frac{\alpha}{2})(\|2P_C(w_k) - w_k - x_k\|^2 + \|2P_Q(q_k) - q_k - y_k\|^2); \end{aligned} \tag{3.8}$$

(ii) *$\{\varphi_k\}_{k \in \mathbb{N}}$ converges.*

*Proof.* (i) From (2.1) and (3.6), we have

$$\begin{aligned}\|x_{k+1}-x^*\|^2 =&(1-\frac{\alpha}{2})\|x_k-x^*\|^2+\frac{\alpha}{2}\|2P_C(w_k)-w_k-x^*\|^2\\&-\frac{\alpha}{2}(1-\frac{\alpha}{2})\|2P_C(w_k)-w_k-x_k\|^2.\end{aligned} \tag{3.9}$$

Fixing $(x^*,y^*)\in\Gamma$, we have

$$\langle w_k-P_C(w_k),w_k-x^*\rangle \geq \|w_k-P_C(w_k)\|^2. \tag{3.10}$$

By taking into account the formula $\|a+b\|^2=\|a\|^2+\|b\|^2+2\langle a,b\rangle$ and (3.10), we obtain

$$\begin{aligned}\|2P_C(w_k)-w_k-x^*\|^2 =&\|2(P_C(w_k)-w_k)+(w_k-x^*)\|^2\\=&4(\|P_C(w_k)-w_k\|^2-\langle w_k-P_C(w_k),w_k-x^*\rangle)+\|w_k-x^*\|^2\\\leq&\|w_k-x^*\|^2.\end{aligned} \tag{3.11}$$

Using the definition of $w_k$, we get

$$\begin{aligned}\|w_k-x^*\|^2 =&\|x_k-2\lambda_k F(x_k,y_k)-x^*\|^2\\=&\|x_k-x^*\|^2+4\lambda_k^2\|F(x_k,y_k)\|^2-4\lambda_k\langle F(x_k,y_k),x_k-x^*\rangle.\end{aligned} \tag{3.12}$$

By combining (3.9), (3.11) and (3.12), we have

$$\begin{aligned}\|x_{k+1}-x^*\|^2 \leq&\|x_k-x^*\|^2+2\alpha\lambda_k^2\|F(x_k,y_k)\|^2-2\alpha\lambda_k\langle F(x_k,y_k),x_k-x^*\rangle\\&-\frac{\alpha}{2}(1-\frac{\alpha}{2})\|2P_C(w_k)-w_k-x_k\|^2.\end{aligned} \tag{3.13}$$

Similarly, we have

$$\begin{aligned}\|y_{k+1}-y^*\|^2 \leq&\|y_k-y^*\|^2+2\alpha\lambda_k^2\|G(x_k,y_k)\|^2-2\alpha\lambda_k\langle G(x_k,y_k),y_k-y^*\rangle\\&-\frac{\alpha}{2}(1-\frac{\alpha}{2})\|2P_Q(q_k)-q_k-y_k\|^2.\end{aligned} \tag{3.14}$$

Adding (3.13) and (3.14), we obtain

$$\begin{aligned}&\|x_{k+1}-x^*\|^2+\|y_{k+1}-y^*\|^2\\\leq&\|x_k-x^*\|^2+\|y_k-y^*\|^2+2\alpha\lambda_k^2(\|F(x_k,y_k)\|^2+\|G(x_k,y_k)\|^2)\\&-2\alpha\lambda_k(\langle F(x_k,y_k),x_k-x^*\rangle+\langle G(x_k,y_k),y_k-y^*\rangle)\\&-\frac{\alpha}{2}(1-\frac{\alpha}{2})(\|2P_C(w_k)-w_k-x_k\|^2+\|2P_Q(q_k)-q_k-y_k\|^2).\end{aligned} \tag{3.15}$$

It holds

$$-2\langle F(x_k,y_k),x_k-x^*\rangle=-\|Ax_k-Ax^*\|^2-\|Ax_k-By_k\|^2+\|Ax^*-By_k\|^2,$$

$$-2\langle G(x_k,y_k),y_k-y^*\rangle=-\|By_k-By^*\|^2-\|Ax_k-By_k\|^2+\|Ax_k-By^*\|^2.$$

In view of $Ax^*=By^*$, we have

$$-2(\langle F(x_k,y_k),x_k-x^*\rangle+\langle G(x_k,y_k),y_k-y^*\rangle)=-2\|Ax_k-By_k\|^2.$$

By using the definition of $\lambda_k$ and (3.15), we get

$$\begin{aligned}\varphi_{k+1} \leq&\varphi_k-2\alpha\lambda_k(1-\gamma)\|Ax_k-By_k\|^2\\&-\frac{\alpha}{2}(1-\frac{\alpha}{2})(\|2P_C(w_k)-w_k-x_k\|^2+\|2P_Q(q_k)-q_k-y_k\|^2).\end{aligned}$$

(ii) Using $\alpha \in (0,2)$ and $\gamma \in (0,1)$, we conclude from (3.8) that

$$\varphi_{k+1} \leq \varphi_k.$$

Hence, $\lim_{k\to\infty} \varphi_k$ exists due ot $\varphi_k \geq 0, \forall k \in \mathbb{N}$. □

**Lemma 3.3.** *Let $\{(x_k, y_k)\}_{k\in\mathbb{N}}$ be generated by Algorithm 1. Then, there hold*

$$\lim_{k\to\infty} \|Ax_k - Bv_k\| = 0, \tag{3.16}$$

*and*

$$\lim_{k\to\infty} \|P_C(x_k) - x_k\| = 0 \quad \text{and} \quad \lim_{k\to\infty} \|P_Q(y_k) - y_k\| = 0. \tag{3.17}$$

*Proof.* By using Lemma 3.2, we know

$$\begin{aligned} &\|x_{k+1} - x^*\|^2 + \|y_{k+1} - y^*\|^2 \\ \leq &\|x_k - x^*\|^2 + \|y_k - y^*\|^2 - 2\alpha\lambda_k(1-\gamma)\|Ax_k - By_k\|^2 \\ &- \frac{\alpha}{2}(1 - \frac{\alpha}{2})(\|2P_C(w_k) - w_k - x_k\|^2 + \|2P_Q(q_k) - q_k - y_k\|^2), \end{aligned}$$

and then

$$\begin{aligned} \lim_{k\to\infty} \Big\{ &2\alpha\lambda_k(1-\gamma)\|Ax_k - By_k\|^2 \\ &+ \frac{\alpha}{2}(1 - \frac{\alpha}{2})(\|2P_C(w_k) - w_k - x_k\|^2 + \|2P_Q(q_k) - q_k - y_k\|^2) \Big\} = 0. \end{aligned}$$

Using $\alpha \in (0,2)$ and $\gamma \in (0,1)$ again, we get

$$\lim_{k\to\infty} \lambda_k \|Ax_k - By_k\|^2 = 0, \tag{3.18}$$

$$\lim_{k\to\infty} \|2P_C(w_k) - w_k - x_k\| = 0 \quad \text{and} \quad \lim_{k\to\infty} \|2P_Q(q_k) - q_k - y_k\| = 0. \tag{3.19}$$

By use of (3.18) and Lemma 3.1, we get

$$\lim_{k\to\infty} \|Ax_k - By_k\| = 0.$$

Furthermore, we have

$$\lim_{k\to\infty} \lambda_k^2 (\|A^*(Ax_k - By_k)\|^2 + \|B^*(Ax_k - By_k)\|^2) = 0,$$

which leads to

$$\lim_{k\to\infty} \lambda_k \|A^*(Ax_k - By_k)\| = 0 \quad \text{and} \quad \lim_{k\to\infty} \lambda_k \|B^*(Ax_k - By_k)\| = 0. \tag{3.20}$$

Using the definition of $w_k$ and $F(x_k, y_k)$, we have

$$\|w_k - x_k\| = 2\lambda_k \|A^*(Ax_k - By_k)\|, \tag{3.21}$$

which together with (3.20) yields that

$$\lim_{k\to\infty} \|w_k - x_k\| = 0. \tag{3.22}$$

Similarly, we have

$$\lim_{k\to\infty} \|q_k - y_k\| = \lim_{k\to\infty} 2\lambda_k \|B^*(By_k - Ax_k)\| = 0. \tag{3.23}$$

Note that

$$\|P_C(w_k) - w_k\| \leq \frac{1}{2}(\|2P_C(w_k) - w_k - x_k\| + \|x_k - w_k\|).$$

Using the first formula of (3.19) and (3.22), we have

$$\lim_{k\to\infty} \|P_C(w_k) - w_k\| = 0. \tag{3.24}$$

Similarly, we obtain

$$\lim_{k\to\infty} \|P_Q(q_k) - q_k\| = 0. \tag{3.25}$$

Thanks to the nonexpansivity $P_C$, we get

$$\begin{aligned} \|P_C(x_k) - x_k\| \leq & \|P_C(x_k) - P_C(w_k)\| + \|P_C(w_k) - w_k\| + \|x_k - w_k\| \\ \leq & 2\|x_k - w_k\| + \|P_C(w_k) - w_k\|. \end{aligned}$$

Similarly, we have

$$\|P_Q(y_k) - y_k\| \leq 2\|y_k - q_k\| + \|P_Q(q_k) - q_k\|.$$

From (3.22) and (3.24), the first formula of (3.17) follows. By use of (3.23) and (3.25), we get the second formula of (3.17) immediately. □

**Theorem 3.4.** *Let $\{(x_k, y_k)\}_{k\in\mathbb{N}}$ be generated by Algorithm 1, then the sequence converges to a solution of the SEP* (1.1) *weakly.*

*Proof.* Let $z_k = (x_k, y_k)$. To show the weak convergence of $\{z_k\}_{k\in\mathbb{N}}$, it suffices to show that $\omega_w(z_k) \subseteq \Gamma$. To see this, fix $\hat{z} \in \omega_w(z_k)$ and let $\{z_{k_l}\}_{l\in\mathbb{N}}$ be a subsequence of $\{z_k\}_{k\in\mathbb{N}}$ converging weakly to $\hat{z}$. It is easy to see that $\hat{z} \in C \times Q$ by using (3.17) and Lemma 2.1. From (3.16) and the lower semicontinuity of norms, we obtain that

$$0 \leq \|A\hat{x} - B\hat{y}\| \leq \liminf_{l\to\infty} \|Ax_{k_l} - By_{k_l}\| \leq \lim_{k\to\infty} \|Ax_k - By_k\| = 0,$$

which yields $\|A\hat{x} - B\hat{y}\| = 0$, i.e., $A\hat{x} = B\hat{y}$. Therefore, $\hat{z} = (\hat{x}, \hat{y}) \in \Gamma$ and $\omega_w(z_k) \subseteq \Gamma$. The proof is complete. □

In Algorithm 1, we assume that projections $P_C$ and $P_Q$ are easily calculated. However, in some cases, it is impossible or needs too much work to calculate the projections. To deal with this situation, we consider a general case of the SEP (1.1), where $C$ and $Q$ are given by level sets of convex functions. From now on, we assume that $c : H_1 \to \mathbb{R}$ and $q : H_2 \to \mathbb{R}$ are convex functions, and $C$ and $Q$ are given, respectively, by

$$C = \{x \in \mathscr{H}_1 : c(x) \leq 0\}, \quad \text{and} \quad Q = \{y \in \mathscr{H}_2 : q(y) \leq 0\}.$$

Furthermore, we assume that $\partial c$ and $\partial q$ are bounded operators (i.e., bounded on any bounded set).

Given $(x_k, y_k)$, define the sets $C_k$ and $Q_k$ by the following half-spaces:

$$C_k = \{x \in H_1 : c(x_k) + \langle \xi_k, x - x_k \rangle \leq 0\},$$

where $\xi_k \in \partial c(x_k)$, and

$$Q_k = \{y \in H_2 : q(Ax_k) + \langle \eta_k, y - Ax_k \rangle \leq 0\},$$

where $\eta_k \in \partial q(Ax_k)$.

From the definition of the subgradient, it is clear that $C \subseteq C_k$ and $Q \subseteq Q_k$. The projections onto $C_k$ and $Q_k$ are easy to calculate since $C_k$ and $Q_k$ are half-spaces (see [25, 26]).

Below, we introduce a relaxed projection method.

**Algorithm 2** A relaxed projection method

**Step 0.** Input $k := 0$, $(x_0, y_0) \in H_1 \times H_2$ and $\alpha \in (0,2)$.
**Step 1.** Given current iterate $(x_k, y_k)$, generate $(x_{k+1}, y_{k+1})$ by

$$\begin{cases} u_k = P_{C_k}(x_k - 2\lambda_k F(x_k, y_k)), \\ v_k = P_{Q_k}(y_k - 2\lambda_k G(x_k, y_k)), \end{cases}$$

and

$$\begin{cases} x_{k+1} := x_k + \alpha(u_k + \lambda_k F(x_k, y_k) - x_k), \\ y_{k+1} := y_k + \alpha(v_k + \lambda_k G(x_k, y_k) - y_k), \end{cases}$$

where

$$\lambda_k := \gamma \frac{\|Ax_k - By_k\|^2}{\|A^*(Ax_k - By_k)\|^2 + \|B^*(Ax_k - By_k)\|^2},$$

and $\gamma \in (0,1)$.
**Step 2.** If $Ax_k = By_k$, $u_k = x_k$ and $v_k = y_k$, then terminate and $(x_k, y_k)$ is a solution. Otherwise, set $k := k+1$ and go to Step 1.

Following the proof in Theorem 3.4 and [20, Theorem 4.1], we can obtain the weak convergence of Algorithm 2 immediately.

**Theorem 3.5.** *Let $\{(x_k, y_k)\}_{k\in\mathbb{N}}$ be generated by Algorithm 2, then the sequence $\{(x_k, y_k)\}_{k\in\mathbb{N}}$ weakly converges to a solution of the SEP (1.1).*

## 4. Preliminary numerical results

In this section, we present two numerical examples to demonstrate the effectiveness of our algorithms.

We denote the vector with all elements 0 by $e_0$, and the vector with all elements 1 by $e_1$ in what follows. In the numerical results listed in the following table, 'Iter.' and 'Sec.' denote the number of iterations and the cpu time in seconds, respectively.

**Example 4.1.** [16] Let $H_1 = \mathbb{R}^2, H_2 = \mathbb{R}$ and $H_3 = \mathbb{R}^3$. Consider the SEP (1.1) with $C = H_1, Q = H_2$, and $A : H_1 \to H_3, B : H_2 \to H_3$ defined by

$$A(x,y) = (x,y,0) \quad \text{and} \quad B(z) = (z,0,0), \quad \text{for all } x,y,z \in \mathbb{R},$$

respectively.

Shi et al. [16] showed that the solution set of the SEP (1.1) is $\Gamma = \{(x,0,x) : x \in \mathbb{R}\}$. We take $x_0 = (23,600)^T, y_0 = 820$ as the initial value and then the exact solution of the SEP (1.1) is $x^* = (421.5,0)^T, y^* = 421.5$. Take $D_k = \|x_k - x^*\| + \|y_k - y^*\| < \varepsilon$ as the stopping criterion. We first tested different values of $\gamma$ and $\alpha$, and compared the corresponding iteration numbers. The value of $\varepsilon$ is taken as $10^{-4}$. From Figure 1, it is observed that the optimal parameters are $\gamma = 0.5$ and $\alpha = 1.91$.

Next we compare Algorithm 1 and Algorithm 1 in [21]. Take $\gamma = 0.5$ and $\alpha = 1.91$ in Algorithm 1 and take $\gamma_k = 0.25$ for Algorithm 1 in [21].

It is concluded from Figure 2 that Algorithm 1 needs less iterations than Algorithm 1 of [21] under the same errors.

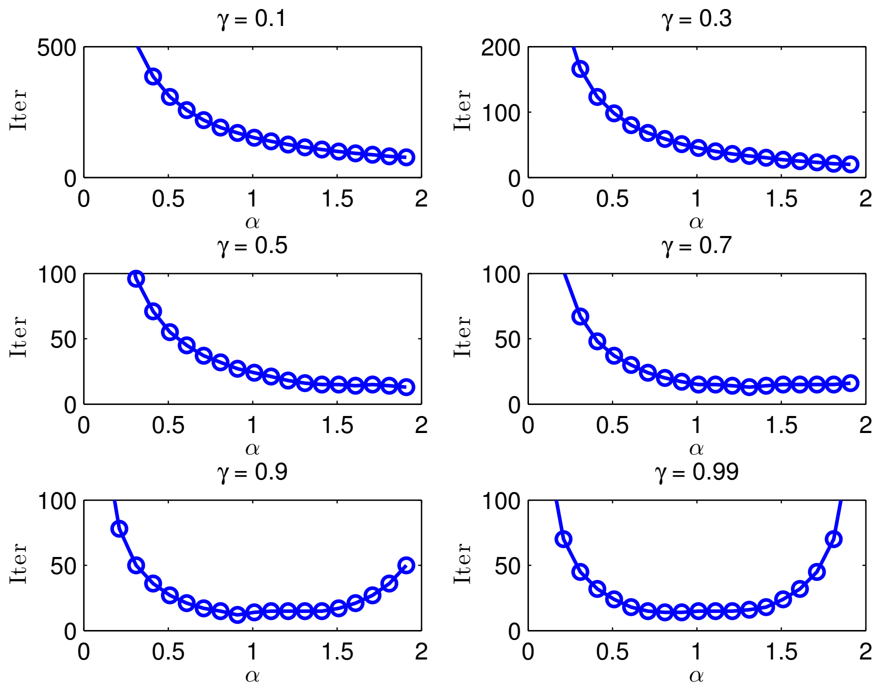


FIGURE 1. The relationship between $\alpha$ and the number of iterations when $\gamma$ is fixed for Example 4.1

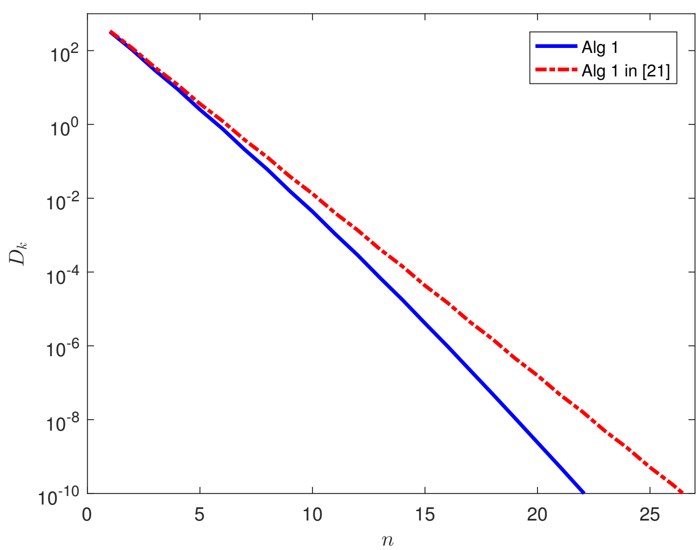


FIGURE 2. Comparison of two algorithms for Example 4.1

**Example 4.2.** The SEP (1.1) with $A = (a_{ij})_{J\times N}$, $B = (b_{ij})_{J\times M}$, $C = \{x \in \mathbb{R}^N | \|x\| \leq 0.25\}$, $Q = \{y \in \mathbb{R}^M | e_0 \leq y \leq u\}$, where $a_{ij} \in [0,1]$, $b_{ij} \in [0,1]$ and $u \in [e_1, 2e_1]$ are all generated randomly.

In the implementations, we take $\|Ax_k - By_k\| < \varepsilon = 10^{-3}$ as the stopping criterion. Select the initial value $x_0 = 10e_1$, $y_0 = -10e_1$ for two algorithms.

TABLE 1. Computational results for Example 4.2 with different dimensions.

| $(N,M)$ | $J$ | | 20 | 40 | 60 | 80 | 100 |
|---|---|---|---|---|---|---|---|
| $(50,50)$ | Algorithm 1 | Iter. | 49 | 137 | 690 | 3148 | 693 |
| | | Sec. | 0.009 | 0.024 | 0.125 | 0.663 | 0.143 |
| $(50,50)$ | Algorithm 1.1 in [21] | Iter. | 276 | 271 | 2189 | 3922 | 1334 |
| | | Sec. | 0.029 | 0.031 | 0.237 | 0.523 | 0.201 |
| $(80,100)$ | Algorithm 1 | Iter. | 42 | 95 | 154 | 276 | 460 |
| | | Sec. | 0.008 | 0.011 | 0.042 | 0.086 | 0.166 |
| $(80,100)$ | Algorithm 1.1 in [21] | Iter. | 99 | 137 | 219 | 601 | 842 |
| | | Sec. | 0.012 | 0.026 | 0.046 | 0.114 | 0.19 |
| $(200,150)$ | Algorithm 1 | Iter. | 34 | 65 | 100 | 122 | 220 |
| | | Sec. | 0.009 | 0.017 | 0.031 | 0.035 | 0.12 |
| $(200,150)$ | Algorithm 1.1 in [21] | Iter. | 119 | 76 | 103 | 223 | 299 |
| | | Sec. | 0.021 | 0.023 | 0.04 | 0.071 | 0.13 |

We compared Algorithm 1 and Algorithm 1 in [21]. For comparison, the same random values are taken in each test for two algorithms. We take $\gamma = 0.8, \alpha = 1.2$ for Algorithm 1, and $\gamma_k = 0.2$ for Algorithm 1 in [21]. We compared two algorithms for different problem sizes, and reported the results in Table 1, which illustrates that Algorithm 1 behaves better than Algorithm 1 in [21] from the iteration number and CPU time.

## 5. THE CONCLUSION

In this paper, we introduced new projection methods to solve the split equality problem and showed their weak convergence under the standard conditions. It is not easy to give the linear convergence of the proposed methods since $x_k$ and $y_k$ may be not in $C$ and $Q$, respectively. To employ the method presented in [6, 16] to obtain the linear convergence, one has to impose projections onto $C$ and $Q$ for $x_{k+1}$ and $y_{k+1}$ in (3.4), respectively, which obviously add the computational cost. Therefore, it is an open problem if the proposed methods have a linear convergence.

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