J. Nonlinear Funct. Anal. 2020 (2020), Article ID 46 https://doi.org/10.23952/jnfa.2020.46

MATHRES

Journal of Nonlinear Functional Analysis

Available online at http://jnfa.mathres.org

INFINITELY MANY FAST HOMOCLINIC SOLUTIONS FOR DIFFERENT CLASSES OF DAMPED VIBRATION SYSTEMS

MOHSEN TIMOUMI

Department of Mathematics, Monastir University, Monastir, Tunisia

Abstract. In this paper, we study the existence and multiplicity of fast homoclinic orbits for the class of damped vibration systems $\ddot{u}(t)+(q(t)I_N+B)\dot{u}(t)+\frac{1}{2}q(t)Bu(t)-L(t)u(t)+\nabla W(t,u(t))=0$, $\forall t\in\mathbb{R}$, where $L(t)$ is not required to be either uniformly positive definite or coercive, and $W(t,x)$ is of subquadratic or superquadratic growth as $|x|\to\infty$, or satisfies only local conditions near the origin (i.e., it can be subquadratic, superquadratic or asymptotically quadratic at infinity). To the best of our knowledge, there is no result concerning the existence and multiplicity of homoclinic orbits for the system with the conditions.

Keywords. Damped vibration systems; Fast homoclinic solutions; Variational methods; Critical points; Local conditions.

1. Introduction

Consider the following damped vibration system

$$(\mathscr{DV}) \qquad \ddot{u}(t)+(q(t)I_N+B)\dot{u}(t)+\frac{1}{2}q(t)Bu(t)-L(t)u(t)+\nabla W(t,u(t))=0,\ \forall t\in\mathbb{R},$$

where $q\in C(\mathbb{R},\mathbb{R})$, I_N is the $N\times N$ identity matrix, B is an antisymmetric $N\times N$ constant matrix, $L\in C(\mathbb{R},\mathbb{R}^{N^2})$ is a symmetric matrix-valued function unnecessary coercive or positive definite and $W:\mathbb{R}\times\mathbb{R}^N\longrightarrow\mathbb{R}$ is a continuous function, differentiable in the second variable with continuous derivative $\nabla W(t,x)=\frac{\partial W}{\partial x}(t,x)$. If $q(t)=0$ for all $t\in\mathbb{R}$ and $B=0$, then $(\mathscr{DV})$ is just the following second order Hamiltonian system

$$(\mathscr{HS}). \qquad \ddot{u}(t)-L(t)u(t)+\nabla W(t,u(t))=0,\ \forall t\in\mathbb{R}.$$

As a special case of the dynamical systems, Hamiltonian systems play an important role in practical problems concerning relativistic mechanics, gaz dynamics, nuclear physics, fluid mechanics and others. A solution u of $(\mathscr{HS})$ is called homoclinic (to 0) if $u(t)\longrightarrow 0$ and $\dot{u}(t)\longrightarrow 0$ as

E-mail address: m_timoumi@yahoo.com.

Received September 19, 2020; Accepted October 22, 2020.

$|t| \to \infty$. u is nontrivial if $u \neq 0$. In the past three decades, the existence and multiplicity of homoclinic solutions for system $(\mathscr{HS})$ have been studied via critical point theory and variational methods. The main difficulty in dealing with system $(\mathscr{HS})$ arises from the fact that $H^1(\mathbb{R})$ is not compactly embedded in $L^p(\mathbb{R})$ for $p \in [2,\infty]$. To overcome this difficulty, many authors have considered the periodic case; see [1, 2, 3, 4] and the references cited therein. In this case, the existence of homoclinic solutions can be obtained by going to the limit of periodic solutions of approximating problems. If $L(t)$ and $W(t,x)$ are neither autonomous in t nor periodic in t, the existence of homoclinic solutions of $(\mathscr{HS})$ is quite different from the ones just described because of the lack of compactness of the Sobolev embedding; see [5, 6, 7, 8] and the references cited therein. In this last case, to study the existence of homoclinic solutions, one needs some conditions on L to recover the compactness of the Sobolev embedding. As far as the cases $q(t) \neq 0$ or $B \neq 0$ are concerned, there are only a few authors who studied homoclinic orbits for $(\mathscr{DV})$. We refer to [9, 10, 11] for $q(t) = 0$ for all $t \in \mathbb{R}$ and $B \neq 0$, and [12, 13, 14, 15, 16] for $q(t) \neq 0$ and $B = 0$. It is worthy of pointing out that some coercive assumptions on L are often needed to obtain the existence of homoclinic solutions of $(\mathscr{DV})$. For the general case $q(t) \neq 0$ and $B \neq 0$, to our best knowledge, there are only two authors [17, 18] concerning the existence and multiplicity of homoclinic solutions for $(\mathscr{DV})$. In [17], the author obtained the existence of at least one ground state homoclinic orbit for $(\mathscr{DV})$ when L satisfies the Ding's coercive condition [7] and W satisfies a kind of superquadratic condition at infinity by using a variant generalized weak linking theorem. Recently, the authors in [18] obtained the existence of infinitely many fast homoclinic solutions (see Definition 2.1) for $(\mathscr{DV})$ when W is locally defined and superquadratic near the origin by using the following conditions on q and L:

(Q_γ) there exists a constant $\gamma > 1$ such that

$$\|q\|_\infty < \infty,\ Q(t) = \int_0^t q(s)ds \longrightarrow +\infty \ as\ |t| \longrightarrow \infty \ and\ \int_{|t|\geq 1} e^{Q(t)}\,|t|^{-\gamma}dt < +\infty;$$

(L_1) there exists a constant $l_0 \geq 0$ such that

$$l(t) = \inf_{|\xi|=1} L(t)\xi\cdot\xi \geq -l_0,\ \forall t \in \mathbb{R};$$

$$(L_{\gamma,Q}) \qquad meas_Q\left(\left\{t \in \mathbb{R}/\,|t|^{-\gamma}L(t) < bI_N\right\}\right) < +\infty,\ \forall b > 0,$$

where $meas_Q$ denotes the Lebesgue's measure on $\mathbb{R}$ with density $e^{Q(t)}$ and γ is defined in (Q_γ). Here, for two $N\times N$ symmetric matrices M_1 and M_2, we say that $M_1 < M_2$ if

$$\min_{x\in\mathbb{R}^N,|x|=1} (M_1 - M_2)x\cdot x < 0$$

and $M_1 \geq M_2$ if $M_1 < M_2$ does not hold.

In this paper, we focus on the study of infinitely many fast nontrivial homoclinic orbits for $(\mathscr{DV})$ when $W(t,x)$ is of subquadratic or superquadratic growth as $|x| \longrightarrow \infty$, or satisfies only locally subquadratic conditions near the origin (i.e., it can be subquadratic, superquadratic or asymptotically quadratic at infinity). The remainder of this article consists of four sections. After presenting some preliminaries in Section 2, we establish, in Sections 3 and 4, the existence of fast homoclinic orbits, respectively, for the subquadratic and superquadratic growth conditions at infinity. The last Section, Section 5, is devoted to the case where the nonlinearity still only satisfies locally conditions near the origin.

2. Preliminaries

In order to introduce the concept of fast homoclinic solutions for $(\mathscr{DV})$ conveniently, we first describe some properties of the weighted Sobolev space E on which the certain variational functional associated with $(\mathscr{DV})$ is defined and the fast homoclinic solutions of $(\mathscr{DV})$ are the critical points of such functional. We use $L^2_Q(\mathbb{R})$ to denote the Hilbert space of measurable functions from $\mathbb{R}$ into $\mathbb{R}^N$ under the inner product

$$< u, v >_{L^2_Q} = \int_{\mathbb{R}} e^{Q(t)} u(t) \cdot v(t) dt$$

and the induced norm

$$\|u\|_{L^2_Q} = (\int_{\mathbb{R}} e^{Q(t)} |u(t)|^2 dt)^{\frac{1}{2}}.$$

Similarly, $L^s_Q(\mathbb{R})$ $(1 \leq s < \infty)$ denotes the Banach space of functions on $\mathbb{R}$ with values in $\mathbb{R}^N$ under the norm

$$\|u\|_{L^s_Q} = (\int_{\mathbb{R}} e^{Q(t)} |u(t)|^s dt)^{\frac{1}{s}}$$

and $L^\infty_Q(\mathbb{R})$ denotes the Banach space of functions on $\mathbb{R}$ with values in $\mathbb{R}^N$ under the norm

$$\|u\|_{L^\infty_Q} = ess\sup\left\{e^{\frac{Q(t)}{2}} |u(t)| / t \in \mathbb{R}\right\}.$$

Let A be the self-adjoint extension of the operator $-\frac{d^2}{dt^2} + L(t)$ with the domain $\mathscr{D}(A) \subset L^2_Q(\mathbb{R})$. Let $\{\mathscr{E}(\lambda) / -\infty < \lambda < \infty\}$ denote the resolution of A, and $U = I - \mathscr{E}(0) - \mathscr{E}(-0)$. It is well known that U commutes with A, $|A|$ and $|A|^{\frac{1}{2}}$, and $A = |A|U$ is the polar decomposition of A. Set $E = \mathscr{D}(|A|^{\frac{1}{2}})$ and define on E the inner product by

$$< u, v >_0 = < |A|^{\frac{1}{2}} u, |A|^{\frac{1}{2}} v >_{L^2_Q} + < u, v >_{L^2_Q}$$

and the corresponding norm by

$$\|u\|_0 = < u, u >_0^{\frac{1}{2}}:$$

Definition 2.1. A solution u of $(\mathscr{DV})$ is called a fast homoclinic solution if $u \in E$.

Lemma 2.2. [18] *Suppose that L and q satisfy (Q_γ), (L_1) and $(L_{\gamma,Q})$. Then E is compactly embedded in $L^p(\mathbb{R})$ for any $p \in [1, \infty]$.*

Define an operator $K : E \longrightarrow E$ by

$$< Ku, v >_0 = \int_{\mathbb{R}} e^{Q(t)} [B\dot{u} \cdot v + \frac{1}{2} qBu \cdot v] dt + \int_{\mathbb{R}} e^{Q(t)} (I_N - L(t)) u \cdot v dt$$

for all $u, v \in E$. Then it is easy to check that K is a bounded self-adjoint linear operator. By the classical spectral theory, we know that the spectrum $\sigma(I-K)$ of $I-K$ consists of eigenvalues numbered in $\lambda_1 \leq \lambda_2 \leq ... \longrightarrow +\infty$, and a corresponding system of eigenfunctions $(e_j)_{j\in\mathbb{N}}$ $((I-K)e_j = \lambda_j e_j)$ forms an orthonormal basis in $L^2_Q(\mathbb{R})$. Here, I denotes the identity operator. Let $k^- = card\{k / \lambda_k < 0\}$, $k^0 = card\{k / \lambda_k = 0\}$ and $\bar{k} = k^- + k^0$, where $card(S)$ denotes the number of elements of set S. Set $E^- = span\{e_1, ..., e_{k^-}\}$, $E^0 = ker(I-K)$ and

$E^+ = \overline{span\left\{e_{\bar{k}+1}, \dots\right\}}$. Then one has $E = E^- \oplus E^0 \oplus E^+$. Besides $E^- \oplus E^0$ is finite dimensional since K is compact. Furthermore, we introduce on E the equivalent new inner product

$$< u, v >=< (I-K)u^+, v^+ >_0 - < (I-K)u^-, v^- >_0 + < u^0, v^0 >_{L^2_Q}$$

for $u = u^- + u^0 + u^+, v = v^- + v^0 + v^+ \in E^- \oplus E^0 \oplus E^+$ and the corresponding norm $\|.\| =< .,. >^{\frac{1}{2}}$. In the following, the inner product $< .,. >$ and the norm $\|.\|$ will always be used. By Lemma 2.2, we know that E is compactly embedded in $L^p_Q(\mathbb{R})$ for all $p \in [1, \infty]$ and as a consequence, for all $p \in [1, \infty]$, there exists a constant $\eta_p > 0$ such that

$$\|u\|_{L^p_Q} \leq \eta_p \|u\|, \ \forall u \in E. \tag{2.1}$$

By use of the definition of $< .,. >$, E^- and E^+, we have

$$< (I-K)u, u >= \pm \|u\|^2, \ \forall u \in E^\pm. \tag{2.2}$$

3. SUBQUADRATIC CASE

In this section, we are interested in the existence of infinitely many fast homoclinic solutions for system $(\mathscr{DV})$ when the potential $W(t,x)$ is subquadratic at infinity with respect to x. More precisely, we make the following assumptions.

(W_1) $W(t,x) \geq 0$ for all $(t,x) \in \mathbb{R} \times \mathbb{R}^N$ and there exist constants $0 < \mu < 2$ and $R > 0$ such that

$$\nabla W(t,x) \cdot x \leq \mu W(t,x), \ \forall t \in \mathbb{R}, \ |x| \geq R,$$

and

$$\nabla W(t,x) \cdot x \leq 2W(t,x), \ \forall t \in \mathbb{R}, \ |x| \leq R;$$

(W_2) $\lim_{|x| \longrightarrow 0} \frac{W(t,x)}{|x|^2} = +\infty$ uniformly for $t \in \mathbb{R}$;

(W_3) $W(t,0) = 0, \forall t \in \mathbb{R}$ and there exists a constant $c > 0$ such that

$$|\nabla W(t,x)| \leq c\,|x|, \ \forall t \in \mathbb{R}, \ |x| \leq R,$$

where R is the constant introduced in (W_1);

(W_4) $\liminf_{|x| \longrightarrow \infty} \frac{W(t,x)}{|x|} \geq a$, where a is a positive constant.

Our main result in this Section reads as follows.

Theorem 3.1. *Suppose that (Q_γ), (L_1), $(L_{\gamma,Q})$, $(W_1) - (W_4)$ hold and $W(t,x)$ is even in x for all $t \in \mathbb{R}$. Then $(\mathscr{DV})$ possesses infinitely many nontrivial fast homoclinic solutions.*

Example 3.2. Let $W(t,x) = a(t)\,|x|^\mu$, where $0 < \inf_{t \in \mathbb{R}} a(t) \leq \sup_{t \in \mathbb{R}} a(t) < \infty$ for all $t \in \mathbb{R}$ and $\mu \in [1,2[$. It is easy to check that W satisfies conditions $(W_1) - (W_4)$.

Proof of Theorem 3.1. In the following, c_n, $n \in \mathbb{N}$, denotes some various positive constants. For system $(\mathscr{DV})$, we associate the following functional defined on the space E introduced in Section 2 by

$$f(u) = \frac{1}{2} \int_{\mathbb{R}} e^{Q(t)}[|\dot{u}(t)|^2 + L(t)u(t) \cdot u(t) + B(t)u(t) \cdot \dot{u}(t)]dt - \int_{\mathbb{R}} e^{Q(t)} W(t,u(t))dt.$$

From (2.2), we find that f can be rewritten as

$$f(u)=\frac{1}{2}(\|u^+\|^2-\|u^-\|^2)-g(u),\ u=u^-+u^0+u^+\in E,$$

where

$$g(u)=\int_{\mathbb{R}}e^{Q(t)}W(t,u(t))dt,\ u\in E.$$

It is well known that under assumptions of Theorem 3.1 the functional f is continuously differentiable on E and its critical points on E are exactly the fast homoclinic solutions of the system $(\mathscr{DV})$. Moreover g' is compact and for all $u,v\in E$

$$\begin{aligned}f'(u)v=&\int_{\mathbb{R}}e^{Q(t)}[\dot{u}(t)\cdot\dot{v}(t)+L(t)u(t)\cdot v(t)-B\dot{u}(t)\cdot v(t)-\frac{1}{2}q(t)Bu(t)\cdot v(t)]dt\\&-\int_{\mathbb{R}}e^{Q(t)}\nabla W(t,u(t))\cdot v(t)dt.\end{aligned}$$

To prove Theorem 3.1, the following Variant Fountain Theorem developed by Zou [19] will be needed. Let X be a Banach space with the norm $\|.\|$ and $X=\overline{\oplus_{m\in\mathbb{N}}X_m}$ with $dimX_m<\infty$ for any $m\in\mathbb{N}$. Set

$$Y_k=\oplus_{m=1}^{k}X_m,\ Z_k=\overline{\oplus_{m=k}^{\infty}X_m}.$$

Consider a family of functionals $f_\lambda\in C^1(X,\mathbb{R})$ defined by

$$f_\lambda(u)=A(u)-\lambda B(u),\ u\in X,\ \lambda\in[1,2].$$

Lemma 3.3. [19] *Assume that the functionals f_λ defined previously, satisfy*
(T_1) *f_λ maps bounded sets into bounded sets uniformly for all $\lambda\in[1,2]$ and*

$$f_\lambda(-u)=f_\lambda(u),\ \forall(\lambda,u)\in[1,2]\times X;$$

(T_2) *$B(u)\geq 0$ for all $u\in X$ and $B(u)\longrightarrow+\infty$ as $\|u\|\longrightarrow\infty$ on any finite dimensional subspace of X;*
(T_3) *There exist $\rho_k>r_k>0$ such that*

$$a_k(\lambda)=\inf_{u\in Z_k,\|u\|=\rho_k}f_\lambda(u)\geq 0>b_k(\lambda)=\max_{u\in Y_k,\|u\|=r_k}f_\lambda(u)$$

for all $\lambda\in[1,2]$ and

$$d_k(\lambda)=\inf_{u\in Z_k,\|u\|\leq\rho_k}f_\lambda(u)\longrightarrow 0\ as\ k\longrightarrow\infty\ uniformly\ for\ \lambda\in[1,2].$$

Then there exist $\lambda_n\longrightarrow 1$, $u_{\lambda_n}\in Y_n$ such that

$$(f_{\lambda_n|Y_n})'(u_{\lambda_n})=0,\ f_{\lambda_n}(u_{\lambda_n})\longrightarrow c_k\in[d_k(2),b_k(1)].$$

Particularly, if (u_{λ_n}) has a convergent subsequence for every k, then f_1 has infinitely many nontrivial critical points $u_k\in X\setminus\{0\}$ satisfying $f_1(u_k)\longrightarrow 0^-$ as $k\longrightarrow\infty$.

For $m\in\mathbb{N}$, if $X_m=\mathbb{R}e_m$, where (e_m) is the sequence defined in Section 2, then Y_k and Z_k are defined as above. In order to apply the above Variant Fountain Theorem, we introduce the family of functionals

$$f_\lambda(u)=A(u)-\lambda B(u),\ (\lambda,u)\in[1,2]\times E,$$

where

$$A(u) = \frac{1}{2}\left\|u^+\right\|^2,\ B(u) = \frac{1}{2}\left\|u^-\right\|^2 + \int_{\mathbb{R}} e^{Q(t)} W(t,u(t))dt,$$

for $u = u^- + u^0 + u^+ \in E = E^- \oplus E^0 \oplus E^+$. Now, for $|x| \leq R$, we have from (W_3) and the Mean Value Theorem that

$$W(t,x) = \int_0^1 \nabla W(t,sx)\cdot x ds \leq \frac{c}{2}|x|^2 \leq \frac{cR}{2}|x|. \tag{3.1}$$

For $|x| \geq R$, set

$$\varphi(\xi) = W(t, \frac{Rx}{\xi|x|})\xi^\mu,\ \xi \in]0,1].$$

By (W_1), it holds

$$\begin{aligned}\varphi'(\xi) &= -\nabla W(t, \frac{Rx}{\xi|x|})\cdot\frac{Rx}{\xi^2|x|}\xi^\mu + \mu W(t,\frac{Rx}{\xi|x|})\xi^{\mu-1}\\ &= \xi^{\mu-1}[-\nabla W(t,\frac{Rx}{\xi|x|})\cdot\frac{Rx}{\xi|x|} + \mu W(t,\frac{Rx}{\xi|x|})] \geq 0.\end{aligned}$$

So φ is nondecreasing in $]0,1]$, and since $\xi = \frac{R}{|x|} \leq 1$, then

$$W(t,x)(\frac{R}{|x|})^\mu \leq W(t,\frac{Rx}{|x|}),$$

which together with (3.1) yields

$$W(t,x)(\frac{R}{|x|})^\mu \leq \frac{c}{2}R^2$$

and then

$$W(t,x) \leq \frac{c}{2}R^{2-\mu}|x|^\mu.$$

In view of (3.1), we have

$$W(t,x) \leq c_1(|x| + |x|^\mu),\ \forall (t,x) \in \mathbb{R}\times\mathbb{R}^N. \tag{3.2}$$

It follows from (2.1) that, for any $\lambda \in [1,2]$ and $u \in E$,

$$|f_\lambda(u)| \leq \frac{1}{2}\|u\|^2 + 2c_1\int_{\mathbb{R}} e^{Q(t)}(|u| + |u|^\mu)dt \leq \frac{1}{2}\|u\|^2 + 2c_1(\eta_1\|u\| + \eta_\mu\|u\|^\mu)$$

which implies that f_λ maps bounded sets into bounded sets uniformly for $\lambda \in [1,2]$. Note that $W(t,-x) = W(t,x)$. Hence, we have $f_\lambda(-u) = f_\lambda(u)$ for all $(\lambda,u) \in [1,2]\times E$. Thus, the condition (T_1) of Lemma 3.3 holds.

Lemma 3.4. *Assume that (Q_γ), (L_1), $(L_{\gamma,Q})$, (W_1) and (W_4) hold. Then $B(u) \geq 0$ for all $u \in E$ and $B(u) \longrightarrow +\infty$ as $\|u\| \longrightarrow \infty$ on any finite-dimensional subspace of E.*

Proof. By assumption (W_1), it is clear that $B(u) \geq 0$ for all $u \in E$. We claim that, for any finite-dimensional subspace F of E, there exists a constant $\varepsilon_0 > 0$ such that

$$meas_Q(\{t \in \mathbb{R}/ |u(t)| \geq \varepsilon_0\|u\|\}) \geq \varepsilon_0,\ \forall u \in F\setminus\{0\}. \tag{3.3}$$

If not, for any $k \in \mathbb{N}$, there exists $u_k \in F\setminus\{0\}$ such that

$$meas_Q(\left\{t \in \mathbb{R}/ |u_k(t)| \geq \frac{1}{k}\|u_k\|\right\}) < \frac{1}{k}.$$

Letting $v_k = \frac{u_k}{\|u_k\|} \in F$, we have $\|v_k\| = 1$ and

$$meas(\left\{t \in \mathbb{R}/\,|v_k(t)| \geq \frac{1}{k}\right\}) < \frac{1}{k},\ \forall k \in \mathbb{N}. \tag{3.4}$$

Since F is finite dimensional, by taking a subsequence if necessary, we can assume that $v_k \longrightarrow v_0$ in F for some $v_0 \in F$, $\|v_0\| = 1$. Recalling that any two norms on F are equivalent, one has

$$\int_{\mathbb{R}} e^{Q(t)}\,|v_k(t) - v_0(t)|\,dt \longrightarrow 0 \ as\ k \longrightarrow \infty. \tag{3.5}$$

Since $\|v_0\| = 1$, then $\|v_0\|_{L^\infty_Q} = \sup_{t\in\mathbb{R}} e^{\frac{Q(t)}{2}}\,|v_0(t)| > 0$. Hence there exists a constant $\sigma_0 > 0$ such that

$$meas_Q(\{t \in \mathbb{R}/\,|v_0(t)| \geq \sigma_0\}) \geq \sigma_0. \tag{3.6}$$

For any $k \in \mathbb{N}$, let

$$\Omega_k = \left\{t \in \mathbb{R}/\,|v_k(t)| < \frac{1}{k}\right\},\ \Omega_0 = \{t \in \mathbb{R}/\,|v_0(t)| \geq \sigma_0\}.$$

From (3.4) and (3.6), for any $k \in \mathbb{N}$ large enough, it holds

$$meas_Q(\Omega_0 \cap \Omega_k) = meas_Q(\Omega_0 \setminus \Omega_k^c) \geq meas_Q(\Omega_0) - meas_Q(\Omega_k^c) \geq \sigma_0 - \frac{1}{k} \geq \frac{\sigma_0}{2}.$$

Then, for k large enough,

$$\begin{aligned}\int_{\mathbb{R}} e^{Q(t)}\,|v_k(t) - v_0(t)|\,dt &\geq \int_{\Omega_0\cap\Omega_k} e^{Q(t)}\,|v_k(t) - v_0(t)|\,dt \\ &\geq \int_{\Omega_0\cap\Omega_k} e^{Q(t)}(|v_0(t)| - |v_k(t)|)dt \\ &\geq (\sigma_0 - \frac{1}{k})meas_Q(\Omega_0 \cap \Omega_k) \geq c_0\frac{\sigma_0^2}{4} > 0\end{aligned}$$

which contradicts (3.5). Therefore (3.3) holds. For the ε_0 given in (3.3), we denote

$$\Omega_u = \{t \in \mathbb{R}/\,|u(t)| \geq \varepsilon_0\,\|u\|\},\ \forall u \in F \setminus \{0\}.$$

It follows from (3.3) that

$$meas_Q(\Omega_u) \geq \varepsilon_0,\ \forall u \in F \setminus \{0\}.$$

From (W_4), there exists a constant $R_1 > R$ such that

$$W(t,x) \geq a\frac{|x|}{2},\ \forall t \in \mathbb{R},\ |x| \geq R_1. \tag{3.7}$$

Letting $u \in F$ be such that $\|u\| \geq \frac{R_1}{\varepsilon_0}$, we have

$$|u(t)| \geq \varepsilon_0\,\|u\| \geq R_1,\ \forall t \in \Omega_u,$$

which together with (3.7) yields, for all $u \in F$, $\|u\| \geq \frac{R_1}{\varepsilon_0}$,

$$\begin{aligned}B(u) = \frac{1}{2}\left\|u^-\right\|^2 + \int_{\mathbb{R}} e^{Q(t)}W(t,u)dt &\geq \int_{\Omega_u} e^{Q(t)}W(t,u)dt \\ \geq c_0\int_{\Omega_u} e^{Q(t)}a\frac{|u|}{2}dt &\geq c_0\frac{a\varepsilon_0}{2}\,\|u\|\,meas_Q(\Omega_u) \geq c_0\frac{a\varepsilon_0^2}{2}\,\|u\|.\end{aligned}$$

This implies that $B(u) \longrightarrow +\infty$ as $\|u\| \longrightarrow \infty$ on any finite-dimensional subspace F of E. The proof of Lemma 3.4 is completed. □

Lemma 3.5. *Suppose that (Q_γ), (L_1) and $(L_{\gamma,Q})$ hold. Then, for any $p \in [2,\infty]$,*

$$l_p(k) = \sup_{u\in Z_k, \|u\|=1} \|u\|_{L^p_Q} \longrightarrow 0 \text{ as } k \longrightarrow \infty.$$

Proof. It is clear that $0 < l_p(k+1) \leq l_p(k)$. Hence $l_p(k) \longrightarrow \bar{l}_p$ as $k \longrightarrow \infty$. For every $k \geq 1$, there exists $u_k \in Z_k$ such that $\|u_k\| = 1$ and $\|u_k\|_{L^p_Q} > \frac{1}{2} l_p(k)$. For any $v \in E$, let $v = \sum_{i=1}^{\infty} v_i e_i$. By the Cauchy-Schwartz inequality, one has

$$\begin{aligned} |< u_k, v >| &= \left| < u_k, \sum_{i=1}^{\infty} v_i e_i > \right| = \left| < u_k, \sum_{i=k+1}^{\infty} v_i e_i > \right| \\ &\leq \|u_k\| \left\| \sum_{i=k+1}^{\infty} v_i e_i \right\| \leq \sum_{i=k+1}^{\infty} |v_i| \, \|e_i\| \longrightarrow 0 \text{ as } k \longrightarrow \infty, \end{aligned}$$

which implies that $u_k \rightharpoonup 0$. Without loss of generality, Lemma 2.2 implies that $u_k \longrightarrow 0$ in $L^2_Q(\mathbb{R})$. Thus, $\bar{l}_p = 0$. The proof of Lemma 3.5 is completed. □

Lemma 3.6. *Assume that (Q_γ), (L_1), $(L_{\gamma,Q})$, (W_2) and (W_3) are satisfied. Then there exist a constant $k_0 \in \mathbb{N}$ and two sequences $0 < r_k < \rho_k \longrightarrow 0$ as $k \longrightarrow \infty$ such that*

$$a_k(\lambda) = \inf_{u\in Z_k, \|u\|=\rho_k} f_\lambda(u) > 0, \ \forall k \geq k_0, \ \forall \lambda \in [1,2], \tag{3.8}$$

$$b_k(\lambda) = \max_{u\in Y_k, \|u\|=r_k} f_\lambda(u) < 0, \ \forall k \geq k_0, \ \forall \lambda \in [1,2], \tag{3.9}$$

$$c_k(\lambda) = \inf_{u\in Z_k, \|u\|\leq\rho_k} f_\lambda(u) \longrightarrow 0 \text{ as } k \longrightarrow \infty \text{ uniformly for} \lambda \in [1,2]. \tag{3.10}$$

Proof. For any $u \in E$ with $\|u\| \leq \frac{R\sqrt{c_0}}{\eta_\infty}$, we have that (2.1) implies

$$\|u\|_{L^\infty} \leq \frac{1}{\sqrt{c_0}} \|u\|_{L^\infty_Q} \leq R. \tag{3.11}$$

Note that $Z_k \subset E^+$ for any $k \geq k^+ + 1$ with k^+ is defined in Section 2. It follows from (3.1) and (3.11) that, for all $k \geq k^+ + 1$ and $\|u\| \leq \frac{R\sqrt{c_0}}{\eta_\infty}$,

$$\begin{aligned} f_\lambda(u) &\geq \frac{1}{2}\|u\|^2 - 2\int_{\mathbb{R}} e^{Q(t)} W(t,u)dt \geq \frac{1}{2}\|u\|^2 - 2\int_{\mathbb{R}} e^{Q(t)} \frac{cR}{2} |u|\, dt \\ &\geq \frac{1}{2}\|u\|^2 - cR\|u\|_{L^1_Q}, \ \forall \lambda \in [1,2], \ \forall u \in Z_k. \end{aligned} \tag{3.12}$$

Let

$$M_k = l_1(k) = \sup_{u\in Z_k\setminus\{0\}} \frac{\|u\|_{L^1_Q}}{\|u\|}, \ \forall k \in \mathbb{N}. \tag{3.13}$$

It follows that

$$M_k \longrightarrow 0 \text{ as } k \longrightarrow \infty. \tag{3.14}$$

Combining (3.12) and (3.13) yields

$$f_\lambda(u) \geq \frac{1}{2}\|u\|^2 - cRM_k\|u\|, \tag{3.15}$$

for all $k \geq \bar{k}+1$ and $u \in Z_k$ with $\|u\| \leq \frac{R\sqrt{c_0}}{\eta_\infty}$. For any $k \geq \bar{k}+1$, let $\rho_k = 4cRM_k$. It follows from (3.14) that there exists an integer $k_0 \geq \bar{k}+1$ such that

$$\rho_k \leq \frac{R\sqrt{c_0}}{\eta_\infty},\ \forall k \geq k_0. \tag{3.16}$$

It follows from (3.15) and (3.16) that, for all $k \geq k_0$,

$$a_k(\lambda) = \inf_{u\in Z_k, \|u\|=\rho_k} f_\lambda(u) \geq \frac{1}{2}\rho_k^2 - cRM_k\rho_k = \frac{1}{4}\rho_k^2 > 0.$$

Hence (3.8) is satisfied. Now, for any $k \geq k_0$, we see that (3.15) implies that, for any $u \in Z_k$ with $\|u\| \leq \rho_k$,

$$f_\lambda(u) \geq -cRM_k\rho_k.$$

Since $f_\lambda(0) = 0$, we deduce that

$$0 \geq \inf_{u\in Z_k, \|u\|\leq\rho_k} f_\lambda(u) \geq -cRM_k\rho_k,\ \forall k \geq k_0,$$

which implies that (3.10) is satisfied. It remains to prove (3.9). Since the two norms $\|.\|_{L^2_Q}$ and $\|.\|$ are equivalent in finite-dimensional space Y_k, then, for any $k \in \mathbb{N}$, there exists a constant $d_k > 0$ such that

$$\|u\|_{L^2_Q} \geq d_k \|u\|,\ \forall u \in Y_k. \tag{3.17}$$

By (W_2), for any $k \in \mathbb{N}$, there exists a constant $\varepsilon_k > 0$ such that

$$W(t,x) \geq \frac{|x|^2}{d_k^2},\ \forall |x| \leq \varepsilon_k. \tag{3.18}$$

For any $k \in \mathbb{N}$ and $u \in E$ with $\|u\| \leq \frac{\varepsilon_k\sqrt{c_0}}{\eta_\infty}$, one has from (2.1)

$$\|u\|_{L^\infty} \leq \frac{1}{\sqrt{c_0}}\|u\|_{L^\infty_Q} \leq \varepsilon_k.$$

Consequently, for all $k \in \mathbb{N}$ and $u \in Y_k$ with $\|u\| \leq \frac{\varepsilon_k\sqrt{c_0}}{\eta_\infty}$, it follows from (3.17) and (3.18) that, for all $\lambda \in [1,2]$,

$$\begin{aligned} f_\lambda(u) &\leq \frac{1}{2}\|u^+\|^2 - \int_{\mathbb{R}} e^{Q(t)}W(t,u)dt \\ &\leq \frac{1}{2}\|u^+\|^2 - \int_{\mathbb{R}} e^{Q(t)}\frac{|u|^2}{d_k^2}dt \leq -\frac{1}{2}\|u\|^2, \forall \lambda \in [1,2]. \end{aligned} \tag{3.19}$$

For any $k \in \mathbb{N}$, choose $0 < r_k < \min\left\{\rho_k, \frac{\varepsilon_k\sqrt{c_0}}{\eta_\infty}\right\}$. Then it implies

$$b_k(\lambda) = \max_{u\in Y_k, \|u\|=r_k} f_\lambda(u) \leq -\frac{r_k^2}{2} < 0,\ \forall k \in \mathbb{N}.$$

The proof of Lemma 3.6 is completed. □

It follows from above that there exists a positive integer k_0 such that, for all $k \geq k_0$, all the conditions of Lemma 3.3 are satisfied. Therefore, for all $k \geq k_0$, there exist sequences $0 < \lambda_j \longrightarrow 1$, $u_j \in Y_j$ such that

$$(f_{\lambda_j|Y_j})'(u_{\lambda_j}) = 0,\ f_{\lambda_j}(u_{\lambda_j}) \longrightarrow \theta_k \in [d_k(2), b_k(1)]. \tag{3.20}$$

Lemma 3.7. *Under the assumptions of Theorem 3.1, the sequence (u_{λ_j}) is bounded in E.*

Proof. Set, for $j \in \mathbb{N}$,

$$\Lambda_j = \left\{ t \in \mathbb{R} / \left| u_{\lambda_j}(t) \right| \geq R_1 \right\},$$

where R_1 is defined in (3.7) and note by (W_1) that $\nabla W(t,x) \cdot x \leq 2W(t,x)$, $\forall (t,x) \in \mathbb{R} \times \mathbb{R}^N$. Hence, it holds from (W_1), (3.7) and (3.20) that

$$\begin{aligned}
-f_{\lambda_j}(u_{\lambda_j}) &= \frac{1}{2}(f_{\lambda_j|Y_j})'(u_{\lambda_j})u_{\lambda_j} - f_{\lambda_j}(u_{\lambda_j}) \\
&= \lambda_j \int_{\mathbb{R}} e^{Q(t)}[W(t,u_{\lambda_j}) - \frac{1}{2}\nabla W(t,u_{\lambda_j}) \cdot u_{\lambda_j}]dt \\
&= \lambda_j \int_{\Lambda_j} e^{Q(t)}[W(t,u_{\lambda_j}) - \frac{1}{2}\nabla W(t,u_{\lambda_j}) \cdot u_{\lambda_j}]dt \\
&+ \lambda_j \int_{\Lambda_j^c} e^{Q(t)}[W(t,u_{\lambda_j}) - \frac{1}{2}\nabla W(t,u_{\lambda_j}) \cdot u_{\lambda_j}]dt \\
&\geq \lambda_j \int_{\Lambda_j} e^{Q(t)}[W(t,u_{\lambda_j}) - \frac{1}{2}\nabla W(t,u_{\lambda_j}) \cdot u_{\lambda_j}]dt \\
&\geq \lambda_j \frac{2-\mu}{2} \int_{\Lambda_j} e^{Q(t)} W(t,u_{\lambda_j})dt \\
&\geq \lambda_j \frac{a\lambda_j(2-\mu)}{4} \int_{\Lambda_j} e^{Q(t)} \left| u_{\lambda_j} \right| dt, \ \forall j \in \mathbb{N}.
\end{aligned}$$

It follows from (3.20) that

$$\int_{\Lambda_j} e^{Q(t)} \left| u_{\lambda_j} \right| dt \leq c_2, \ \forall j \in \mathbb{N}. \tag{3.21}$$

For any $j \in \mathbb{N}$, let $\chi_j : \mathbb{R} \longrightarrow \mathbb{R}$ be the indicator of Λ_j, that is,

$$\chi_j(t) = \begin{cases} 1, \ if \ t \in \Lambda_j \\ 0, \ if \ t \in \Lambda_j^c. \end{cases}$$

From the definition of Λ_j and (3.21), we have

$$\left\| (1-\chi_j)u_{\lambda_j} \right\|_{L^\infty} \leq R_1 \ and \ \left\| \chi_j u_{\lambda_j} \right\|_{L^1_Q} \leq c_2, \ \forall j \in \mathbb{N}.$$

Therefore, from the equivalence of any two norms on finite-dimensional space $E^- \oplus E^0$, it holds that

$$\begin{aligned}
\left\| u^-_{\lambda_j} + u^0_{\lambda_j} \right\|^2_{L^2_Q} &= < u^-_j + u^0_j, u_j >_{L^2_Q} \\
&= < u^-_{\lambda_j} + u^0_{\lambda_j}, (1-\chi_j)u_{\lambda_j} >_{L^2_Q} + < u^-_{\lambda_j} + u^0_{\lambda_j}, \chi_j u_{\lambda_j} >_{L^2_Q} \\
&\leq \left\| u^-_{\lambda_j} + u^0_{\lambda_j} \right\|_{L^1_Q} \left\| (1-\chi_j)u_{\lambda_j} \right\|_{L^\infty} + \left\| u^-_{\lambda_j} + u^0_{\lambda_j} \right\|_{L^\infty} \left\| \chi_j u_{\lambda_j} \right\|_{L^1_Q} \\
&\leq \left(c_3 \left\| (1-\chi_j)u_{\lambda_j} \right\|_{L^\infty} + c_4 \left\| \chi_j u_{\lambda_j} \right\|_{L^1_Q} \right) \left\| u^-_{\lambda_j} + u^0_{\lambda_j} \right\|_{L^2_Q} \\
&\leq (c_3 R_1 + c_4 c_2) \left\| u^-_{\lambda_j} + u^0_{\lambda_j} \right\|_{L^2_Q}, \ \forall j \in \mathbb{N}.
\end{aligned}$$

Consequently

$$\left\|u_{\lambda_j}^- + u_{\lambda_j}^0\right\|_{L_Q^2} \le c_3 R_1 + c_4 c_2,\ \forall j \in \mathbb{N},$$

which together with the fact that $E^- \oplus E^0$ is of finite-dimensional implies that

$$\left\|u_{\lambda_j}^- + u_{\lambda_j}^0\right\| \le c_5,\ \forall j \in \mathbb{N}. \tag{3.22}$$

From

$$\left\|u_{\lambda_j}^+\right\|^2 = 2f_{\lambda_j}(u_{\lambda_j}) + \lambda_j \left\|u_{\lambda_j}^-\right\|^2 + 2\lambda_j \int_{\mathbb{R}} e^{Q(t)} W(t, u_{\lambda_j}) dt,\ \forall j \in \mathbb{N},$$

we deduce from (2.1), (3.2), (3.10) and (3.22) that

$$\begin{aligned}
\left\|u_{\lambda_j}\right\|^2 &= \left\|u_{\lambda_j}^- + u_{\lambda_j}^0\right\|^2 + \left\|u_{\lambda_j}^+\right\|^2 \\
&= \left\|u_{\lambda_j}^- + u_{\lambda_j}^0\right\|^2 + 2f_{\lambda_j}(u_{\lambda_j}) + \lambda_j \left\|u_{\lambda_j}^-\right\|^2 + 2\lambda_j \int_{\mathbb{R}} e^{Q(t)} W(t, u_{\lambda_j}) dt \\
&\le c_6 + c_7(\left\|u_{\lambda_j}\right\|_{L_Q^1} + \left\|u_{\lambda_j}\right\|_{L_Q^\mu}^\mu) \le c_6 + c_8\left(\left\|u_{\lambda_j}\right\| + \left\|u_{\lambda_j}\right\|^\mu\right),\ \forall j \in \mathbb{N}.
\end{aligned}$$

Since $\mu < 2$, this implies that (u_{λ_j}) is bounded in E. The proof of Lemma 3.7 is completed. □

It remains to prove that (u_{λ_j}) has a strongly convergent subsequence in E. Since $E^- \oplus E^0$ is finite-dimensional, we can assume by Lemma 3.7 that

$$u_{\lambda_j}^- \longrightarrow u^-,\ u_{\lambda_j}^0 \longrightarrow u^0,\ u_{\lambda_j}^+ \rightharpoonup u^+ \text{ and } u_{\lambda_j} \rightharpoonup u \text{ as } j \longrightarrow \infty \tag{3.23}$$

for some $u = u^- + u^0 + u^+ \in E^- \oplus E^0 \oplus E^+$. In virtue of the Riez Representation Theorem, $(f_{\lambda_j|Y_j})' : Y_j \longrightarrow Y_j^*$ and $g' : E \longrightarrow E^*$ can be viewed as $(f_{\lambda_j|Y_j})' : Y_j \longrightarrow Y_j$ and $g' : E \longrightarrow E$, where Y_j^* and E^* are the dual spaces of Y_j and E, respectively. Letting $P_j : E \longrightarrow Y_j$ be the orthogonal projection for all $j \in \mathbb{N}$, we have

$$0 = (f_{\lambda_j|Y_j})'(u_{\lambda_j}) = e^{Q(t)}[u_{\lambda_j}^+ - \lambda_j u_{\lambda_j}^-] - \lambda_j P_j g'(u_{\lambda_j}),\ \forall j \in \mathbb{N},$$

that is,

$$u_{\lambda_j}^+ = \lambda_j [u_{\lambda_j}^- + e^{Q(t)} P_j g'(u_{\lambda_j})],\ \forall j \in \mathbb{N}. \tag{3.24}$$

Since $g' : E \longrightarrow E$ is compact, then, without loss of generality, (3.23) implies that the right-hand side of (3.24) converges strongly in E. So, $u_{\lambda_j}^+ \longrightarrow u^+$ in E. This together with (3.23) implies $u_{\lambda_j} \longrightarrow u$ in E. It follows from Lemma 3.3 that $f_1 = f$ possesses infinitely many nontrivial critical points, which implies that $(\mathscr{DV})$ has infinitely many nontrivial fast homoclinic orbits. The proof of Theorem 3.1 is completed.

4. SUPERQUADRATIC CASE

In this section, we are interested in the existence of infinitely many fast homoclinic orbits of $(\mathscr{DV})$ when the potential $W(t,x)$ is superquadratic at infinity with respect to x. More precisely, we make the following assumptions:

(W_1) $$\frac{W(t,x)}{|x|^2} \longrightarrow +\infty \text{ as } |x| \longrightarrow \infty, \text{ uniformly in } t \in \mathbb{R};$$

(W_2) $$\nabla W(t,x) \cdot x \ge 2W(t,x) \ge 0,\ \forall (t,x) \in \mathbb{R} \times \mathbb{R}^N;$$

(W_3) $$\frac{|\nabla W(t,x)|}{|x|} \longrightarrow 0 \ as \ |x| \longrightarrow 0, \ uniformly \ in \ t \in \mathbb{R};$$

(W_4) there exist constants $\alpha > 0$ and $a > 0$ such that

$$|\nabla W(t,x)| \leq a(|x|^{\alpha} + 1), \ \forall (t,x) \in \mathbb{R} \times \mathbb{R}^N;$$

(W_5) there exist constants $\beta \geq \alpha$, $\beta > 1$, $b > 0$ and $r > 0$ such that

$$\nabla W(t,x) \cdot x - 2W(t,x) \geq b\,|x|^{\beta}, \ \forall t \in \mathbb{R}, \ \forall |x| \geq r.$$

Our main results in this section read as follows.

Theorem 4.1. *Assume that* (Q_γ), (L_1), $(L_{\gamma,Q})$ *and* $(W_1) - (W_5)$ *hold. Then the damped vibration system* $(\mathscr{DV})$ *possesses at least one nontrivial fast homoclinic solution.*

Theorem 4.2. *Assume that* (Q_γ), (L_1), $(L_{\gamma,Q})$ *and* $(W_1) - (W_5)$ *hold and* $W(t,x)$ *is even in* $x \in \mathbb{R}^N$. *Then* $(\mathscr{DV})$ *has infinitely many distinct fast homoclinic solutions.*

Example 4.3. Let

$$W(t,x) = |x|^2 \, ln(1 + |x|^2).$$

A straightforward computation shows that W satisfies our Theorems 4.1 and 4.2.

Proof of Theorem 4.1. Now we are going to establish the corresponding variational framework to obtain fast homoclinic solutions for $(\mathscr{DV})$. To this end, define the functional $f : E \longrightarrow \mathbb{R}$ as in Section 3. It is well known that the functional f is continuously differentiable on E and its critical points on E are exactly the fast homoclinic solutions of the system $(\mathscr{DV})$. For the existence and multiplicity of critical points of f, we appeal to the following abstract critical lemmas. Let E be a Banach space and $f \in C^1(E,\mathbb{R})$. As usual, we say that f satisfies the Palais-Smale condition ((PS) for short) if any sequence $(u_k) \subset E$ for which $(f(u_k))$ is bounded and $f'(u_k) \longrightarrow 0$ as $k \longrightarrow \infty$ possesses a convergent subsequence.

Lemma 4.4. *(*[20, Generalized Mountain Pass Theorem]*). Let* E *be an infinite dimensional Banach space such that* $E = V \oplus X$, *where* V *is finite dimensional. If* $f \in C^1(E,\mathbb{R})$ *and the following conditions hold*
(f_1) f *satisfies the* (PS) *condition;*
(f_2) *there are constants* $\rho, \delta > 0$ *such that*

$$f_{|\partial B_\rho \cap X} \geq \delta;$$

where $\partial B_\rho = \{u \in E : \|u\| = \rho\}$*;*
(f_3) *there are constants* $r > \rho$, $M > 0$ *and* $e \in X$ *with* $\|e\| = 1$ *such that*

$$f_{|\partial\Lambda} \leq 0 \ and \ f_{|\Lambda} \leq M,$$

where

$$\Lambda = (B_r \cap V) \oplus \{se : 0 \leq s \leq r\}.$$

Then f *has a critical point* u *with* $f(u) \geq \delta$.

Lemma 4.5. *([20, Symmetric Mountain Pass Theorem]). Let E be an infinite dimensional Banach space such that $E = V \oplus X$, where V is finite dimensional. If $f \in C^1(E,\mathbb{R})$ is even and satisfies $f(0) = 0$, (f_1), (f_2) and*
$(f_3^{'})$ for each finite dimensional subspace $\tilde{E} \subset E$, there is an $R = R(\tilde{E}) > 0$ such that $f \leq 0$ on $\tilde{E} \setminus B_R$.
Then f possesses an unbounded sequence of critical values.

In the following, c_n, $n \in \mathbb{N}$, denotes some various constants.

Lemma 4.6. *If (Q_γ), (L_1), $(L_{\gamma,Q})$, (W_2), (W_4) and (W_5) hold, then f satisfies the (PS) condition.*

Proof. Let $(u_k) \subset E$ be a (PS) sequence. That is, there exists a constant $M > 0$ such that

$$|f(u_k)| \leq M, \ \forall k \in \mathbb{N} \text{ and } f'(u_k) \longrightarrow 0, \text{ as } k \longrightarrow \infty.$$

We claim that (u_k) is bounded. If not, passing to a subsequence if necessary, we may assume that $\|u_k\| \longrightarrow \infty$ as $k \longrightarrow \infty$. By (W_2) and (W_5), we have

$$\begin{aligned} 2f(u_k) - f'(u_k)u_k &= \int_{\mathbb{R}} e^{Q(t)}[\nabla W(t,u_k) \cdot u_k - 2W(t,u_k)]dt \\ &\geq b \int_{\{t\in\mathbb{R}:|u_k(t)|\geq r\}} e^{Q(t)} |u_k(t)|^\beta \, dt \end{aligned} \tag{4.1}$$

for all positive integer k, which implies that

$$\frac{1}{\|u_k\|} \int_{\{t\in\mathbb{R}:|u_k(t)|\geq r\}} e^{Q(t)} |u_k(t)|^\beta \, dt \longrightarrow 0 \tag{4.2}$$

as $k \longrightarrow \infty$. Let

$$v_k(t) = \begin{cases} u_k(t), \ if \ |u_k(t)| \leq r, \\ 0, \ if \ |u_k(t)| > r, \end{cases} \tag{4.3}$$

and

$$w_k(t) = u_k(t) - v_k(t) \tag{4.4}$$

for all positive integer k and all $t \in \mathbb{R}$. From (4.1) and (4.4), we get

$$c_1(1 + \|u_k\|) \geq b \, \|w_k\|^\beta_{L^\beta_Q} \tag{4.5}$$

for all positive integer k. It follows from Hölder's inequality, (4.3), (4.4) and the equivalence of the norms on the finite dimensional subspace $E^- \oplus E^0$ that

$$\begin{aligned} \|u_k^- + u_k^0\|^2_{L^2_Q} &= < u_k^- + u_k^0, u_k >_{L^2_Q} \\ &= < u_k^- + u_k^0, v_k >_{L^2_Q} + < u_k^- + u_k^0, w_k >_{L^2_Q} \\ &\leq \left\|u_k^- + u_k^0\right\|_{L^1_Q} \|v_k\|_{L^\infty} + \left\|u_k^- + u_k^0\right\|_{L^{\beta'}_Q} \|w_k\|_{L^\beta_Q} \\ &\leq c_2 \left\|u_k^- + u_k^0\right\|_{L^2_Q} (1 + \|w_k\|_{L^\beta_Q}) \end{aligned} \tag{4.6}$$

for all positive integer k, where $\beta' = \frac{\beta}{\beta-1}$ $(\beta > 1)$ is the Hölder's conjugate of β. From the equivalence of the norms on the finite dimensional subspace $E^- \oplus E^0$, (4.5) and (4.6), we obtain

$$\begin{aligned}\|u_k^- + u_k^0\| &\leq c_3 \|u_k^- + u_k^0\|_{L_Q^2} \leq c_4(1 + \|w_k\|_{L_Q^\beta}) \\ &\leq c_5(1 + \|u_k\|^{\frac{1}{\beta}})\end{aligned}$$

for all positive integer k, which implies that

$$\frac{\|u_k^- + u_k^0\|}{\|u_k\|} \longrightarrow 0 \tag{4.7}$$

as $k \longrightarrow \infty$. It follows from (W_4) that

$$\begin{aligned}f'(u_k)u_k^+ &= \|u_k^+\|^2 - \int_{\mathbb{R}} e^{Q(t)} \nabla W(t, u_k) \cdot u_k^+ dt \\ &\geq \|u_k^+\|^2 - \int_{\mathbb{R}} e^{Q(t)} |\nabla W(t, u_k)| \, |u_k^+| \, dt \\ &\geq \|u_k^+\|^2 - a \int_{\mathbb{R}} e^{Q(t)} |u_k|^\alpha \, |u_k^+| \, dt - a \int_{\mathbb{R}} e^{Q(t)} |u_k^+| \, dt \\ &\geq \|u_k^+\|^2 - \frac{a}{\sqrt{c_0}} \|u_k^+\|_{L_Q^\infty} \int_{\{t \in \mathbb{R}: |u_k(t)| \geq r\}} e^{Q(t)} |u_k|^\alpha dt \\ &\quad - ar^\alpha \int_{\{t \in \mathbb{R}: |u_k(t)| < r\}} e^{Q(t)} |u_k^+| \, dt - a \int_{\mathbb{R}} e^{Q(t)} |u_k^+| \, dt \\ &\geq \|u_k^+\|^2 - \frac{a}{\sqrt{c_0}} \|u_k^+\|_{L_Q^\infty} r^{\alpha-\beta} \int_{\{t \in \mathbb{R}: |u_k(t)| \geq r\}} e^{Q(t)} |u_k|^\beta dt \\ &\quad - ar^\alpha \|u_k^+\|_{L_Q^1} - a \|u_k^+\|_{L_Q^1} \\ &\geq \|u_k^+\|^2 - \frac{a\eta_\infty}{\sqrt{c_0}} \|u_k^+\| r^{\alpha-\beta} \int_{\{t \in \mathbb{R}: |u_k(t)| \geq r\}} e^{Q(t)} |u_k|^\beta dt \\ &\quad - ar^\alpha \eta_1 \|u_k^+\| - a\eta_1 \|u_k^+\|\end{aligned}$$

which together with (4.2) implies

$$\frac{\|u_k^+\|}{\|u_k\|} \longrightarrow 0 \tag{4.8}$$

as $k \longrightarrow \infty$. It follows from (4.7) and (4.8) that

$$1 = \frac{\|u_k\|}{\|u_k\|} \leq \frac{\|u_k^- + u_k^0\| + \|u_k^+\|}{\|u_k\|} \longrightarrow 0$$

as $k \longrightarrow \infty$, which is a contradiction. Hence (u_k) must be bounded. Moreover, we have

$$\|u_k^+ - u^+\|^2 = (f'(u_k) - f'(u))(u_k^+ - u^+) + (g'(u_k) - g'(u))(u_k^+ - u^+).$$

Going to a subsequence if necessary, we may assume, by using Lemma 2.2, that $u_k \rightharpoonup u$ weakly in E and

$$u_k \longrightarrow u \text{ in both } L_Q^2(\mathbb{R}) \text{ and } L_Q^\infty(\mathbb{R}) \text{ as } k \longrightarrow \infty. \tag{4.9}$$

Since g' is continuous, we deduce that $g'(u_k) \longrightarrow g'(u)$. Therefore, $u_k^+ \longrightarrow u^+$ in E. From (4.9) and the equivalence of the norms on the finite dimensional subspace $E^- \oplus E^0$, we obtain that

$u_k^0 \longrightarrow u^0$ and $u_k^- \longrightarrow u^-$ in E as $k \longrightarrow \infty$. Hence (u_k) has a convergent subsequence, which shows that the (PS) condition holds. The proof of Lemma 4.5 is achieved. □

Lemma 4.7. *Assume that* (Q_γ), (L_1), $(L_{\gamma,Q})$, (W_2) *and* (W_3) *are satisfied. Then there are constants* $\rho > 0$ *and* $\delta > 0$ *such that* $f_{|S} \geq \delta$, *where* $S = \{u \in E^+ : \|u\| = \rho\}$.

Proof. By (W_3), for all $\varepsilon > 0$, there exists $\nu > 0$ such that

$$|\nabla W(t,x)| \leq \varepsilon |x|, \ \forall t \in \mathbb{R}, \ \forall |x| \leq \nu,$$

which with (W_2) and the Mean Value Theorem gives

$$W(t,x) = \int_0^1 \nabla W(t,sx) \cdot x ds \leq \frac{\varepsilon}{2} |x|^2, \ \forall t \in \mathbb{R}, \ \forall |x| \leq \nu.$$

Choose $\varepsilon = (2\eta_2^2)^{-1}$ and take $\rho = \frac{\nu\sqrt{c_0}}{\eta_\infty}$, $\delta = \frac{\rho^2}{4}$. By Lemma 2.2, we get

$$\begin{aligned} f(u) &= \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}} e^{Q(t)} W(t,u(t))dt \geq \frac{1}{2}\|u\|^2 - \frac{\varepsilon}{2}\int_{\mathbb{R}} e^{Q(t)} |u(t)|^2 dt \\ &\geq \frac{1}{2}\|u\|^2 - \frac{\varepsilon}{2}\eta_2^2 \|u\|^2 = \frac{1}{4}\|u\|^2 = \frac{\rho^2}{4} = \delta \end{aligned}$$

for all $u \in S$. The proof of Lemma 4.7 is completed. □

Lemma 4.8. *Assume that* (Q_γ), (L_1), $(L_{\gamma,Q})$, (W_1), (W_2) *and* (W_5) *are satisfied. Let* $e \in E^+$ *with* $\|e\| = 1$. *Then there exist* $r_1, r_2 > 0$ *such that* $f(u) \leq 0$, $\forall u \in \partial\Lambda$, *where* $\Lambda = \{se : 0 \leq s \leq r_1\} \oplus \left\{u \in E^- \oplus E^0 : \|u\| \leq r_2\right\}$.

Proof. Let $e \in E^+$ with $\|e\| = 1$ and $F = span\{e\} \oplus E^- \oplus E^0$. By the proof of Lemma 3.4, there exists a constant $\varepsilon_0 > 0$ such that

$$meas_Q(\{t \in \mathbb{R} : |u(t)| \geq \varepsilon_0 \|u\|\}) \geq \varepsilon_0, \ \forall u \in F \setminus \{0\}. \tag{4.10}$$

For $u = u^- + u^0 + u^+ \in F$, let

$$\Omega_u = \{t \in \mathbb{R} / |u(t)| \geq \varepsilon_0 \|u\|\}.$$

By (W_1), for $d = \frac{1}{2\varepsilon_0^3} > 0$, there exists $R_1 > 0$ such that

$$W(t,x) \geq d|x|^2, \ \forall |x| \geq R_1, \ \forall t \in \mathbb{R}.$$

Hence,

$$W(t,u(t)) \geq d|u(t)|^2 \geq d\varepsilon_0^2 \|u\|^2 \tag{4.11}$$

for all $u \in F$ with $\|u\| \geq \frac{R_1}{\varepsilon_0}$ and $t \in \Omega_u$. It follows from (W_2), (4.10) and (4.11) that

$$\begin{aligned} f(u) &= \frac{1}{2}\|u^+\|^2 - \frac{1}{2}\|u^-\|^2 - \int_{\mathbb{R}} e^{Q(t)} W(t,u(t))dt \\ &\leq \frac{1}{2}\|u^+\|^2 - \int_{\Omega_u} e^{Q(t)} W(t,u(t))dt \\ &\leq \frac{1}{2}\|u^+\|^2 - d\varepsilon_0^2 \|u\|^2 meas_Q(\Omega_u) \\ &\leq \frac{1}{2}\|u^+\|^2 - d\varepsilon_0^3 \|u\|^2 \\ &= \frac{1}{2}\|u^+\|^2 - \frac{1}{2}\|u\|^2 \leq 0, \end{aligned} \tag{4.12}$$

for all $u \in F$ with $\|u\| \geq \frac{R_1}{\varepsilon_0}$. Let $r_1 > 0$ and denote

$$\Lambda = \{se/0 \leq s \leq r_1\} \oplus \left\{u \in E^- \oplus E^0 : \|u\| \leq r_1\right\}.$$

Then $\partial\Lambda = \Lambda_1 \cup \Lambda_2 \cup \Lambda_3$, where

$$\Lambda_1 = \left\{u \in E^- \oplus E^0 : \|u\| \leq r_1\right\},$$

$$\Lambda_2 = r_1 e + \left\{u \in E^- \oplus E^0 / \|u\| \leq r_1\right\},$$

$$\Lambda_3 = \{se/0 \leq s \leq r_1\} \oplus \left\{u \in E^- \oplus E^0 / \|u\| = r_1\right\}.$$

It follows from (4.12) that

$$f(u) \leq 0,\ \forall u \in \Lambda_2 \cup \Lambda_3$$

for all $r_1 \geq \frac{R}{\varepsilon_0}$. From (W_2), we have

$$f(u) \leq 0,\ \forall u \in E^- \oplus E^0,$$

which implies that $f(u) \leq 0$, $\forall u \in \Lambda_1$. Hence, $f(u) \leq 0$, $\forall u \in \partial\Lambda$, for all $r_1 > max\left\{\rho, \frac{R_1}{\varepsilon_0}\right\}$, where ρ is defined in Lemma 4.7. This completes the proof. □

From Lemma 4.4, we find that f has a critical point u satisfying $f(u) \geq \delta > 0$ where δ is given in Lemma 4.8. Since $f(0) = 0$, then u is nontrivial and $(\mathscr{DV})$ possesses a nontrivial fast homoclinic solution. The proof of Theorem 4.1 is achieved.

Proof of Theorem 4.2. We have $f(0) = 0$ and since $W(t,x)$ is even with respect to the second variable, then f is even. The assumptions (f_1) and (f_2) are proved above. Let us prove $(f_3^{'})$. Letting $\widetilde{E} \subset E$ be a finite dimensional subspace of E, we have that there exists $m \geq 1$ such that $\widetilde{E} \subset E^- \oplus E^0 \oplus span\{w_1, ..., w_m\} = X^m$, where $w_k = e_{n^- + n^0 + k}$, $k \geq 1$. Replacing the subspace $F = span\{e\} \oplus E^- \oplus E^0$, introduced in the proof of Lemma 4.8, and following the same steps, we obtain $R_m > 0$ such that

$$f(u) \leq \frac{1}{2}\left\|u^+\right\|^2 - \frac{1}{2}\|u\|^2 \leq 0,\ \forall u \in X^m,\ \|u\| \geq R_m.$$

Hence $(f_3^{'})$ is verified. Therefore, it follows from Lemma 4.5 that f possesses an unbounded sequence of critical points. Hence $(\mathscr{DV})$ possesses infinitely may fast homoclinic solutions.

5. Local conditions

In this section, we are interested in a general case where the potential $W(t,x)$ satisfies only locally conditions near the origin with respect to x and do not satisfy any additional hypotheses at infinity. More precisely, we present the following assumptions:

(W_1) there exist constants $r, c > 0$ and $\nu \in]0,1[$ such that

$$|\nabla W(t,x)| \leq c\,|x|^{\nu},\ \forall t \in \mathbb{R},\ |x| \leq r;$$

(W_2) there exists $\rho \in]0,r]$ such that

$$W(t,-x) = W(t,x),\ and\ W(t,x) \geq 0,\ \forall t \in \mathbb{R}\ |x| \leq \rho;$$

$$(W_3) \qquad \lim_{|x|\to 0} \frac{|W(t,x)|}{|x|^2} = +\infty,\ uniformly\ for\ all\ t \in \mathbb{R}.$$

Our main result in this Section reads as follows.

Theorem 5.1. *Assume that* (Q_γ), (L_1), $(L_{\gamma,Q})$ *and* $(W_1)-(W_3)$ *are satisfied. Then* $(\mathscr{DV})$ *possesses infinitely many nontrivial fast homoclinic orbits* (u_k) *such that*

$$f(u_k)=\frac{1}{2}\int_{\mathbb{R}} e^{Q(t)}[|\dot{u}_k(t)|^2+L(t)u_k(t)\cdot u_k(t)+Bu_k(t)\cdot\dot{u}_k(t)]dt-\int_{\mathbb{R}} e^{Q(t)}W(t,u_k(t))dt\longrightarrow 0$$

as $k\longrightarrow\infty$.

In the following, we give some examples which satisfy our assumptions.

Example 5.2. (The subquadratic case at infinity). Let $W(t,x)=h(t)\,|x|^\theta\,ln(1+|x|^2)$, where $(t,x)\in\mathbb{R}\times\mathbb{R}^N$, $\theta\in]1,2[$ and $h\in C(\mathbb{R},\mathbb{R})$ with $0<\inf_{t\in\mathbb{R}}h(t)\leq\sup_{t\in\mathbb{R}}h(t)<\infty$. It is easy to see that $W(t,x)$ satisfies the conditions $(W_1)-(W_3)$ and the subquadratic condition at infinity, i.e., $\lim_{|x|\longrightarrow\infty}\frac{W(t,x)}{|x|^2}=0$.

Example 5.3. (The superquadratic case at infinity). Let $W(t,x)=h(t)(|x|^\nu+|x|^\theta\,ln(1+|x|^2))$, where $(t,x)\in\mathbb{R}\times\mathbb{R}^N$, $\theta\in]1,2[$, $\nu\in]2,\infty[$ and $h\in C(\mathbb{R},\mathbb{R})$ with $0<\inf_{t\in\mathbb{R}}h(t)\leq\sup_{t\in\mathbb{R}}h(t)<\infty$. It is easy to see that $W(t,x)$ satisfies the conditions $(W_1)-(W_3)$ and the superquadratic condition at infinity, i.e., $\lim_{|x|\longrightarrow\infty}\frac{W(t,x)}{|x|^2}=+\infty$.

Example 5.4. (The asymptotically quadratic case at infinity). Let $W(t,x)=\frac{1}{2}S(t)x\cdot x+|x|^\theta\,ln(1+|x|^2)$, where $S:\mathbb{R}\longrightarrow\mathbb{R}^{N^2}$ is a bounded symmetric $N\times N$ matrix-valued function and $\theta\in]1,2[$. It is clear that $W(t,x)$ is asymptotically quadratic at infinity with respect to x and satisfies the conditions $(W_1)-(W_3)$.

Proof of Theorem 5.1. Consider the continuously differentiable functional $f:E\longrightarrow\mathbb{R}$ introduced in Section 3 whose critical points on E are the fast homoclinic solutions of the system $(\mathscr{DV})$. We shall use the following Variant Symmetric Mountain Pass Lemma due to Kajikiya [21] to prove our result. We first recall the notion of genus. Let E be a Banach space and let A be a subset of E. A is said to be symmetric if $u\in A$ implies $-u\in A$. For a closed symmetric set A, which does not contain the origin, we define the genus $\gamma(A)$ of A by the smallest integer k for which there exists an odd continuous mapping from A to $\mathbb{R}^k\setminus\{0\}$. If such a k does not exist, then we define $\gamma(A)=+\infty$. Moreover, we set $\gamma(\phi)=0$. Let

$$\Gamma_k=\{A\subset E/A \text{ is a closed symmetric subset},\ 0\notin A,\ \gamma(A)\geq k\}.$$

The properties of genus used in the proof of Theorem 5.1 are summarized as follows.

Lemma 5.5. [21] *Let A and B be closed symmetric subsets of E that do not contain the origin. Then the following hold.*
a) If $A\subset B$, then $\gamma(A)\leq\gamma(B)$.
b) The $n-$dimensional sphere S^n has a genus of $n+1$ by the Borsuk-Ulam theorem.

Lemma 5.6. [21] *Let E be an infinite-dimensional Banach space and let $f\in C^1(E,\mathbb{R})$ satisfy the following*
(f_1) *$f(0)=0$, f is even and bounded from below and f satisfies the $(PS)-$condition;*

(f_2) *For each $k \in \mathbb{N}$, there exists $A_k \subset \Gamma_k$ such that $\sup_{u \in A_k} f(u) < 0$. Then f possesses a sequence of critical points (u_k) such that*

$$f(u_k) \leq 0,\ u_k \neq 0,\ \forall k \in \mathbb{N} \text{ and } \lim_{k \longrightarrow \infty} u_k = 0.$$

Now, let $\theta \in C^1(\mathbb{R}^+, \mathbb{R}^+)$ satisfying

$$\left\{ \begin{array}{l} \theta(s) = 1 \ for\ s \in [0, \frac{\rho\sqrt{c_0}}{2\eta_\infty}],\ \theta(s) = 0 \ for\ s \geq \frac{\rho\sqrt{c_0}}{\eta_\infty}, \\ \theta'(s) < 0 \ for\ \frac{\rho\sqrt{c_0}}{2\eta_\infty} < s < \frac{\rho\sqrt{c_0}}{\eta_\infty}, \end{array} \right. \tag{5.1}$$

where ρ is defined in (W_2). Consider the new functional h defined on E by

$$h(u) = \frac{1}{2}\|u\|^2 - \theta(\|u\|)\Big(\left\|u^-\right\|^2 + \frac{1}{2}\left\|u^0\right\|^2 + \int_{\mathbb{R}} e^{Q(t)} W(t,u)dt\Big).$$

Remark 5.7. It is clear that $h \in C^1(E, \mathbb{R})$, $h(u) = f(u)$ for all $\|u\| \leq \frac{\rho\sqrt{c_0}}{2\eta_\infty}$ and thus critical points of h satisfying $\|u\| \leq \frac{\rho\sqrt{c_0}}{2\eta_\infty}$ are exactly critical points of f. Consequently, to prove Theorem 5.1, we will apply Lemma 5.6 to the functional h instead of f.

Lemma 5.8. *Assume that (Q_γ), (L_1), $(L_{\gamma,Q})$, (W_1) and (W_2) are satisfied. Then h satisfies the (PS)-condition.*

Proof. Let (u_n) be a Palais-Smale sequence, that is, $(h(u_n))$ is bounded and $h'(u_n) \longrightarrow 0$ as $n \longrightarrow \infty$. If $u \in E$ with $\|u\| \geq \frac{\rho\sqrt{c_0}}{\eta_\infty}$, then we find from the definition of θ and h that $h(u) = \frac{1}{2}\|u\|^2$, which implies that

$$h(u) \longrightarrow +\infty \text{ as } \|u\| \longrightarrow \infty. \tag{5.1}$$

Since $(h(u_n))$ is bounded, then (5.1) implies that (u_n) is bounded. Thus, passing to a subsequence if necessary, we can assume by Lemma 2.2 that $u_n \rightharpoonup u = u^- + u^0 + u^+$, $u_n^+ \rightharpoonup u^+$ and $u_n^+ \longrightarrow u^+$ in $L^1_Q(\mathbb{R})$. On the other hand, if $\|u_n\| \geq \frac{\rho\sqrt{c_0}}{\eta_\infty}$, we have $h'(u_n)u_n = \|u_n\|^2 \geq \frac{\rho^2 c_0}{\eta_\infty^2}$ contradicting the fact that $h'(u_n) \longrightarrow 0$ as $n \longrightarrow \infty$. Hence, we can assume that $\|u_n\| \leq \frac{\rho\sqrt{c_0}}{\eta_\infty}$ for all $n \in \mathbb{N}$, which togeher with (2.2) implies that

$$|u_n(t)| \leq \|u_n\|_{L^\infty} \leq \frac{1}{\sqrt{c_0}}\|u_n\|_{L^\infty_Q} \leq \frac{\eta_\infty}{\sqrt{c_0}}\|u_n\| \leq \rho.$$

This, together with (W_1), implies that

$$\begin{aligned} \left|\int_{\mathbb{R}} e^{Q(t)} \nabla W(t,u_n)\cdot(u_n^+ - u^+)dt\right| &\leq \int_{\mathbb{R}} e^{Q(t)} \left|\nabla W(t,u_n)\right|\left|u_n^+ - u^+\right| dt \\ &\leq c\|u_n\|^\nu_{L^\infty} \int_{\mathbb{R}} e^{Q(t)} \left|u_n^+ - u^+\right| dt \\ &\leq c\rho^\nu \int_{\mathbb{R}} e^{Q(t)} \left|u_n^+ - u^+\right| dt \longrightarrow 0 \end{aligned} \tag{5.2}$$

as $n \longrightarrow \infty$. Now, we have

$$\begin{aligned} h'(u_n)(u_n^+ - u^+) = &< u_n, u_n^+ - u^+ > - \theta'(\|u_n\|) < \frac{u_n}{\|u_n\|}, u_n^+ - u^+ > \Big(\frac{1}{2}\left\|u_n^0\right\|^2 + \left\|u_n^-\right\|^2 \\ &+ \int_{\mathbb{R}} e^{Q(t)} W(t,u_n)dt\Big) - \theta(\|u_n\|)\int_{\mathbb{R}} e^{Q(t)} \nabla W(t,u_n)\cdot(u_n^+ - u^+)dt, \end{aligned}$$

which together with (5.2) and the fact that $h'(u_n) \longrightarrow 0$ implies

$$\Big[1 - \frac{\theta'(\|u_n\|)}{\|u_n\|}\Big(\frac{1}{2}\left\|u_n^0\right\|^2 + \left\|u_n^-\right\|^2 + \int_{\mathbb{R}} e^{Q(t)}W(t,u_n)dt\Big)\Big] < u_n, u_n^+ - u^+ > \longrightarrow 0 \tag{5.3}$$

as $n \longrightarrow \infty$. Since $\|u_n\| \leq \frac{\rho\sqrt{c_0}}{\eta_\infty}$, then $|u_n(t)| \leq \rho$ and $W(t,u_n(t)) \geq 0$ for all $n \in \mathbb{N}$ and $t \in \mathbb{R}$ by (W_2). Hence, the definition of θ implies

$$1 - \frac{\theta'(\|u_n\|)}{\|u_n\|}\Big(\frac{1}{2}\left\|u_n^0\right\|^2 + \left\|u_n^-\right\|^2 + \int_{\mathbb{R}} e^{Q(t)}W(t,u_n)dt\Big) \geq 1,\ \forall n \in \mathbb{N}.$$

It follows from (5.3) that

$$< u_n, u_n^+ - u^+ > \longrightarrow 0 \ as\ n \longrightarrow \infty.$$

By virtue of $u_n^+ \rightharpoonup u^+$, we have $\|u_n^+\| \longrightarrow \|u^+\|$ and then $u_n^+ \longrightarrow u^+$. Noting that E^- and E^0 are finite dimensional subspaces, we have $u_n^- \longrightarrow u^-$ and $u_n^0 \longrightarrow u^0$. Therefore $u_n \longrightarrow u$ in E and h satisfies the $(PS)-$condition. □

Now, the definitions of h and θ imply that $h(u) = \frac{1}{2}\|u\|^2 = h(-u)$ for all $\|u\| \geq \frac{\rho\sqrt{c_0}}{\eta_\infty}$. If $\|u\| \leq \frac{\rho\sqrt{c_0}}{\eta_\infty}$, we have as above $|u(t)| \leq \rho$ for all $t \in \mathbb{R}$, which together with (W_2) implies $W(t,-u(t)) = W(t,u(t))$ for all $t \in \mathbb{R}$ and $h(-u) = h(u)$. Thus h is even in E. We claim that h is bounded from below. If not, there exists a sequence (u_n) such that

$$h(u_n) \longrightarrow -\infty \ as\ n \longrightarrow \infty. \tag{5.4}$$

By (W_1), (W_2) and the definitions of h and θ, it is easy to verify that h maps bounded sets into bounded sets. It follows from (5.4) that $\|u_n\| \longrightarrow \infty$ as $n \longrightarrow \infty$. Thus, (5.1) implies that $h(u_n) \longrightarrow +\infty$ as $n \longrightarrow \infty$, which contradicts (5.4). Hence the condition (f_1) of Lemma 5.6 is verified.

Finally, we show that h satisfies the condition (f_2) of Lemma 5.6. For any positive integer k, let $E_k = \oplus_{m=1}^k X_m$, and $X_m = \mathbb{R}e_m$, where the sequence (e_m) is defined in Section 2. Since E_k is finite dimensional, there exists a positive constant β_k such that

$$\|u\| \leq \beta_k \|u\|_{L^2_Q},\ \forall u \in E_k. \tag{5.5}$$

By (W_3), there exists a constant $R > 0$ such that

$$W(t,x) \geq \beta_k^2 |x|^2,\ \forall t \in \mathbb{R},\ |x| \leq R. \tag{5.6}$$

Letting $u \in E$ be such that $\|u\| \leq \frac{R\sqrt{c_0}}{\eta_\infty}$, we know that $|u(t)| \leq R$ for all $t \in \mathbb{R}$. It follows from (5.6) that

$$W(t,u(t)) \geq \beta_k^2 |u(t)|^2,\ \forall t \in \mathbb{R}. \tag{5.7}$$

Therefore, by (5.5) and (5.7), for all $u \in E_k$ with $0 < \|u\| = r_k \leq \min\left\{\frac{\rho\sqrt{c_0}}{2\eta_\infty}, \frac{R\sqrt{c_0}}{\eta_\infty}\right\}$, we have

$$\begin{aligned} h(u) &= \frac{1}{2}\left\|u^+\right\|^2 - \frac{1}{2}\left\|u^-\right\|^2 - \int_{\mathbb{R}} e^{Q(t)}W(t,u)dt \\ &\leq \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}} e^{Q(t)}\beta_k^2 |u(t)|^2 dt \\ &\leq \frac{1}{2}\|u\|^2 - \|u\|^2 = -\frac{1}{2}\|u\|^2 = -\frac{1}{2}r_k^2, \end{aligned}$$

which implies

$$\{u \in E_k \setminus \{0\} \,/\, \|u\| = r_k\} \subset A_k, \tag{5.8}$$

where $A_k = \left\{u \in E \,/\, h(u) \leq -\frac{1}{2}\eta_k^2\right\}$. Thus, Lemma 5.5 and (5.8) imply

$$\gamma(A_k) \geq \gamma\Big(\{u \in E_k \setminus \{0\} \,/\, \|u\| = r_k\} \Big) \geq k.$$

Hence, by the definition of Γ_k, we have $A_k \subset \Gamma_k$. Moreover, the definition of A_k implies $\sup_{u \in A_k} h(u) \leq -\frac{1}{2}\eta_k^2 < 0$. All the conditions of Lemma 5.6 hold. This completes the proof.

REFERENCES

[1] S. Lu, X. Lv, P. Yan, Existence of homoclinics for a class of Hamiltonian systems, Nonlinear Anal. 72 (2010), 390-398.

[2] P.H. Rabinowitz, Homoclinic orbits for a class of Hamiltonian systems, Proc. Roy. Soc. Edinburgh 114 (1990), 33-38.

[3] M. Izydorek, J. Janczewska; Homoclinic solutions for a class of the second order Hamiltonian sysstems, J. Differential Equations, 219 (2005), 375-389.

[4] X. Lv, J. S. Lu, Jiang, Homoclinic solutions for a class of second order Hamiltonian systems, Nonlinear Anal. 13 (2012), 176-185.

[5] G. Chen, Homoclinic orbits for second order Hamiltonian systems with asymptotically linear terms at infinity, Adv. Difference Equations 2014 (2014), 114.

[6] G. Chen, Z. He, Infinitely many homoclinic solutions for a class of second order Hamiltonian systems, Adv. Difference Equations 2014 (2014), 161.

[7] Y. Ding, Existence and multiplicity results for homoclinic solutions to a class of Hamiltonian systems, Nonlinear Anal. 25 (1995), 1095-1113.

[8] L. Chu, Q. Zhang, Homoclinic solutions for a class of second order Hamiltonian systems with locally defined potentials, Nonlinear Anal. 75 (2012), 3188-3197.

[9] J, Sun, J.J. Nieto, M. Otero-Novoa, On homoclinic orbits for a class of damped vibration systems, Adv. Difference Equations 20112 (2012), 102.

[10] X. Wu, W. Zhang, Existence and multiplicity of homoclinic solutions for a class of damped vibration problems, Nonlinear Anal. 74 (2011), 4392-4398.

[11] G. Chen, Nonperiodic damped vibration systems with sublinear terms at infinity: Infinitely many homoclinic orbits, Nonlinear Anal. 92 (2013), 168-175.

[12] M. Timoumi, Infinitely many fast homoclinic orbits for a class of superquadratic damped vibration systems, J. Elliptic Parabolic Equ. 6 (2020), 451-471.

[13] F. Khelifi, M. Timoumi, Even homoclinic orbits for a class of damped vibration systems, Indigationes Math. 28 (2017), 1111-1125.

[14] Y. Li, Q. Zhang, Existence and multiplicity of fast homoclinic solutions for a class of nonlinear second order nonautonomous systems in a weighted Sobolev space, J. Functional Spaces 2015 (2015), Article ID 495040.

[15] R.P. Agarwal, P. Chen, X. Tang, Fast homoclinic solutions for a class of damped vibration problems, Appl. Math. Comput. 219 (2013), 6053-6065.

[16] P. Chen, X.H. Tang, Fast homoclinic solutions for a class of damped vibration problems with sub-quadratic potentials, Math. Nachr. 219 (2013), 6053-6065.

[17] M. Timoumi, On ground-state homoclinic orbits of a class of superquadratic damped vibration systems, Mediterr. J. Math. 15 (2018) 53.

[18] A.R. Chouikha, M. Timoumi, Infinitely many fast homoclinic solutions for damped vibration systems with locally defined potentials, Nonlinear Stud. 27 (2020), 1-19.

[19] W. Zou, Variant Fountain Theorems and their applications, Manuscripta Math. 104 (2001), 343-358.

[20] P.H. Rabinowitz, Minimax methods in critical point theory with applications to differential equations, in: CBMS Reg. Conf. Ser. in Math., vol. 65, American Mathematical Society, Providence, R.I, 1985.

[21] R. Kajikiya, A critical point theorem related to the symmetric mountain pass lemma and its applications to elliptic equations, J. Funct. Anal. 225 (2005), 352-370.