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EXISTENCE RESULTS AND THE ULAM STABILITY FOR FRACTIONAL DIFFERENTIAL EQUATIONS WITH HYBRID PROPORTIONAL-CAPUTO DERIVATIVES

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Abstract. In this paper, we study the Ulam-Hyers and the generalized Ulam-Hyers-Rassias stability for linear fractional differential equations with hybrid proportional-Caputo derivatives using the Laplace transform method. The existence and uniqueness of solutions for nonlinear fractional differential equations with hybrid proportional-Caputo derivatives are established by means of Schaefer's fixed point theorem, Banach's fixed point theorem and the generalized Gronwall's inequality. Two examples are also given to illustrate the main results.

Keywords. Ulam stability; Hybrid proportional-Caputo derivatives; Laplace transform method; Schaefer's fixed point theorem.

1. Introduction

Fractional differential equations have been studied extensively in the literature because of their applications in various fields of engineering and science; see, for example, the monographs [1, 2, 3, 4, 5], and the references therein.

Recently, Khalil et al. [6] introduce a new definition of the fractional derivative, called the conformable fractional derivative, with an obstacle that it does not tend to the original function as the order α tends to zero. The new definition is under the spotlight and many authors established various useful results based on the definition; see, for example, [7, 8, 9, 10, 11].

In control theory, a proportional derivative controller for controller output u at time t with two tuning parameters has the algorithm $u(t) = \kappa_P \mathscr{E}(t) + \kappa_d \frac{d}{dt}\mathscr{E}(t)$, where κ_P and κ_d are the proportional control parameter and the derivative control parameter, respectively. The function $\mathscr{E}$ is the error between the state variable and the process variable. This control law enables Ding et al. [12] to present the control of complex networks models.

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Inspired by the above concept of the proportional derivative controller, Anderson and Ulness [13] introduced a new way to define the proportional (conformable) derivative of order α by

$$ {}^{P}_{0}D^{\alpha}_{t}g(t)=k_1(\alpha,t)g(t)+k_0(\alpha,t)g'(t), $$

where g is differentiable function and $k_0,k_1:[0,1]\times\mathbb{R}\to[0,\infty)$ are continuous functions of the variable t and the parameter $\alpha\in[0,1]$. They satisfy the following conditions, for all $t\in\mathbb{R}$,

$$ \lim_{\alpha\to0^+}k_0(\alpha,t)=0,\ \lim_{\alpha\to1^-}k_0(\alpha,t)=1,\ k_0(\alpha,t)\neq0,\ \alpha\in(0,1], \tag{1.1} $$

$$ \lim_{\alpha\to0^+}k_1(\alpha,t)=1,\ \lim_{\alpha\to1^-}k_1(\alpha,t)=0,\ k_1(\alpha,t)\neq0,\ \alpha\in[0,1). \tag{1.2} $$

This newly defined local derivative tends to the original function as the order α tends to zero and hence it improves the conformable derivatives. In [14], Jarad, Abdeljawad and Alzabut discussed a special case of the proportional derivatives when $k_1(\alpha,t)=1-\alpha$ and $k_0(\alpha,t)=\alpha$. Very recently, Baleanu, Fernandez and Akgül [15] introduced two new hybrid fractional operators, denoted by ${}^{PC}_{0}D^{\alpha}_{t}$ and ${}^{CPC}_{0}D^{\alpha}_{t}$ by combining the proportional and Caputo operators in a new way. The latter is considered as a particular case where the functions k_0 and k_1 are constants with respect to t, depending only on α and it can be expressed as a linear combination of the Caputo fractional derivative and the Riemann-Liouville fractional integral (see Definition 2.1).

On the other hand, the stability problem of functional equations was originally raised in 1940 by Ulam. The problem posed by Ulam was the following: under what conditions there exist an additive mapping near an approximately additive mapping? (for more details, one refers to [16]). The first answer to the Ulam question was given in 1941 by Hyers in the case of Banach spaces (see [17]). Thereafter, this type of stability is called the Ulam-Hyers stability. In 1978, Rassias [18] obtained a remarkable generalization of the Ulam-Hyers stability of mappings by considering variables. As a matter of fact, the Ulam-Hyers stability and the Ulam-Hyers-Rassias stability have been taken up by a number of authors and the study of this area has grown to be one of the central subject in the mathematical analysis. For more details on the recent advances on the Ulam-Hyers stability and Ulam-Hyers-Rassias stability of differential equations, we refer to [19, 20, 21, 22, 23, 24].

Motivated by the results in [15], in the frame of the Laplace transform method, we study the Ulam-Hyers and the generalized Ulam-Hyers-Rassias stability of the following linear fractional differential equation with hybrid proportional-Caputo derivative:

$$ {}^{CPC}_{0}D^{\alpha}_{t}x(t)=g(t),\ t\in[0,T],\ T<\infty,\ 0<\alpha<1. \tag{1.3} $$

We also study the existence and uniqueness of solutions of the following nonlinear fractional differential equation with hybrid proportional-Caputo derivatives

$$ \begin{cases} {}^{PC}_{0}D^{\alpha}_{t}x(t)=f(t,x(t)),\ t\in[0,T],\ T<\infty,\ 0<\alpha<1,\\ x(0)=x_0\in\mathbb{R}, \end{cases} \tag{1.4} $$

where ${}^{CPC}_{0}D^{\alpha}_{t}, {}^{PC}_{0}D^{\alpha}_{t}$ denote the hybrid proportional-Caputo fractional derivatives of order α (see Definition 2.1), $g:[0,T]\to\mathbb{R}$ and $f:[0,T]\times\mathbb{R}\to\mathbb{R}$ are given functions will be specified later.

Up to the author's knowledge, there is no result on the existence of solutions as well as the Ulam stability results for fractional differential equations with hybrid proportional-Caputo fractional derivatives, which are highlights of this paper.

2. Preliminaries

Let $C([0,T],\mathbb{R})$ be the Banach space of all real-valued continuous functions from J into $\mathbb{R}$ equipped by the norm $\|x\| = \sup_{t\in[0,T]} |x(t)|$.

First, we recall from [25] the definition of the Caputo fractional derivative of order $\alpha \in (0,1)$ of a differentiable function $g(t)$:

$$ {}_0^C D_t^\alpha g(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} g'(\tau)\, d\tau, \tag{2.1} $$

and the Riemann-Liouville fractional integral [1]:

$$ {}_0^{RL} I_t^\beta g(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-\tau)^{1-\beta} g(\tau)\, d\tau, \tag{2.2} $$

for $\beta > 0$, where $g(t)$ is an integrable function. Clearly, (2.1) and (2.2) yield:

$$ {}_0^C D_t^\alpha g(t) = {}_0^{RL} I_t^{1-\alpha} g'(t). $$

Next, we present the definitions, the properties and the lemmas of the new hybrid proportional-Caputo fractional derivatives.

Definition 2.1. [15] The proportional-Caputo hybrid fractional derivative of order $\alpha \in (0,1)$ of a differentiable function $g(t)$ can be defined in one of two possible ways. The following general way:

$$ \begin{aligned} {}_0^{PC} D_t^\alpha g(t) &= \frac{1}{\Gamma(1-\alpha)} \int_0^t \big(k_1(\alpha,\tau)g(t) + k_0(\alpha,\tau)g'(t)\big)(t-\tau)^{-\alpha}\, d\tau \\ &= {}_0^{RL} I_t^{1-\alpha} \big(k_1(\alpha,\tau)g(t) + k_0(\alpha,\tau)g'(t)\big) \\ &= \big(k_1(\alpha,\tau)g(t) + k_0(\alpha,\tau)g'(t)\big) * \left(\frac{t^{-\alpha}}{\Gamma(1-\alpha)}\right), \end{aligned} \tag{2.3} $$

or the following simpler expression:

$$ \begin{aligned} {}_0^{CPC} D_t^\alpha g(t) &= \frac{1}{\Gamma(1-\alpha)} \int_0^t \big(k_1(\alpha)g(t) + k_0(\alpha)g'(t)\big)(t-\tau)^{-\alpha}\, d\tau \\ &= k_1(\alpha)\, {}_0^{RL} I_t^{1-\alpha} g(t) + k_0(\alpha)\, {}_0^C D_t^\alpha g(t). \end{aligned} \tag{2.4} $$

The latter is a simple linear combination of the Riemann-Liouville integral and the Caputo derivative. In both of these formulae, the function space domain is given by requiring that g is differentiable and both g and g' are locally L^1 functions on the positive reals.

Definition 2.2. [15] The inverse operators to the fractional PC and CPC derivatives (2.3)-(2.4) are given by

$$ {}_0^{PC} I_t^\alpha g(t) = \int_0^t \exp\left(-\int_u^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)} ds\right) \frac{{}_0^{RL} D_u^{1-\alpha} g(u)}{k_0(\alpha,u)}\, du, \tag{2.5} $$

$$ {}_0^{CPC} I_t^\alpha g(t) = \frac{1}{k_0(\alpha)} \int_0^t \exp\left(-\frac{k_1(\alpha)}{k_0(\alpha)}(t-u)\right) {}_0^{RL} D_u^{1-\alpha} g(u)\, du, \tag{2.6} $$

where ${}^{RL}_{0}D^{1-\alpha}_{u}$ denotes the Riemann-Liouville fractional derivative of order $1-\alpha$ given by

$$ {}^{RL}_{0}D^{1-\alpha}_{u} g(u) = \frac{1}{\Gamma(\alpha)} \frac{d}{du} \int_0^u (u-s)^{\alpha-1} g(s)\, ds. \tag{2.7}$$

For more details, we refer the reader to [1].

Lemma 2.3. [15] *The following inversion relations:*

$$ {}^{PC}_{0}D^{\alpha}_{t}\, {}^{PC}_{0}I^{\alpha}_{t} g(t) = g(t) - \frac{t^{-\alpha}}{\Gamma(1-\alpha)} \lim_{t\to 0} {}^{RL}_{0}I^{\alpha}_{t} g(t), \tag{2.8}$$

$$ {}^{PC}_{0}I^{\alpha}_{t}\, {}^{PC}_{0}D^{\alpha}_{t} g(t) = g(t) - \exp\left(-\int_0^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)} ds\right) g(0) \tag{2.9}$$

and

$$ {}^{CPC}_{0}D^{\alpha}_{t}\, {}^{CPC}_{0}I^{\alpha}_{t} g(t) = g(t) - \frac{t^{-\alpha}}{\Gamma(1-\alpha)} \lim_{t\to 0} {}^{RL}_{0}I^{\alpha}_{t} g(t), \tag{2.10}$$

$$ {}^{CPC}_{0}I^{\alpha}_{t}\, {}^{CPC}_{0}D^{\alpha}_{t} g(t) = g(t) - \exp\left(-\frac{k_1(\alpha,s)}{k_0(\alpha,s)} t\right) g(0) \tag{2.11}$$

are satisfied.

Lemma 2.4. [15] *The PC and CPC operators are non-local and singular and the Laplace transform of the CPC operator is given as follows:*

$$ \mathscr{L}\{{}^{CPC}_{0}D^{\alpha}_{t} x(t)\}(s) = \left[\frac{k_1(\alpha)}{s} + k_0(\alpha)\right] s^{\alpha} \mathscr{L}\{x(t)\}(s) - k_0(\alpha) s^{\alpha-1} x(0). \tag{2.12}$$

Remark 2.5. [15] In the limiting cases $\alpha \to 0$ and $\alpha \to 1$, we recover the following special cases:

$$ \begin{aligned} \lim_{\alpha\to 0} {}^{PC}_{0}D^{\alpha}_{t} g(t) &= \lim_{\alpha\to 0} {}^{CPC}_{0}D^{\alpha}_{t} g(t) = \int_0^t g(\tau)\, d\tau, \\ \lim_{\alpha\to 1} {}^{PC}_{0}D^{\alpha}_{t} g(t) &= \lim_{\alpha\to 1} {}^{CPC}_{0}D^{\alpha}_{t} g(t) = g(t). \end{aligned} $$

Theorem 2.6. *(Schaefer's fixed point theorem [26]) Let E be a Banach space. Let $\mathscr{P} : E \to E$ be a completely continuous operator and let*

$$ \mathscr{S}(\mathscr{P}) = \{z \in E : z = \mu \mathscr{P} z,\ \mu \in (0,1)\} $$

be a bounded set. Then $\mathscr{P}$ has a fixed point in E.

3. Ulam stability Results

In this section, we study the Ulam-Hyers and the generalized Ulam-Hyers-Rassias stability of (1.3).

Definition 3.1. Let $0<\alpha<1$ and let $g:[0,T]\to\mathbb{R}$ be a continuous function. Then, (1.3) is Ulam-Hyers stable if there exist $\Omega>0$ and $\varepsilon>0$ such that, for each solution $x\in C([0,T],\mathbb{R})$ of (1.3),

$$ \left|{}^{CPC}_{0}D^{\alpha}_{t} x(t) - g(t)\right| \le \varepsilon,\ \forall t \in [0,T], \tag{3.1}$$

and there exists a solution $y \in C([0,T],\mathbb{R})$ of (1.3) with

$$ \left|x(t)-y(t)\right| \le \Omega\varepsilon,\ \forall t\in[0,T]. $$

Definition 3.2. Let $0<\alpha<1$, $g:[0,T]\to\mathbb{R}$ be a continuous function and $\phi:[0,T]\to\mathbb{R}_+$ be continuous function. Then, (1.3) is generalized Ulam-Hyers-Rassias stable if there exist $C_{g,\phi}>0$ such that, for each solution $x\in C([0,T],\mathbb{R})$ of (1.3),

$$\left|{}^{CPC}_{0}D^{\alpha}_{t}x(t)-g(t)\right|\leq\phi(t),\ \forall t\in[0,T], \tag{3.2}$$

and there exists a solution $y\in C([0,T],\mathbb{R})$ of (1.3) with

$$\left|x(t)-y(t)\right|\leq C_{g,\phi}\ \phi(t),\ \forall t\in[0,T].$$

Theorem 3.3. *Let $0<\alpha<1$ and $g(t)$ be a given real function on $[0,T]$. If a function $x:[0,T]\to\mathbb{R}$ satisfies the inequality*

$$\left|{}^{CPC}_{0}D^{\alpha}_{t}x(t)-g(t)\right|\leq\varepsilon,\ \forall t\in[0,T], \tag{3.3}$$

for each $t\in[0,T]$ and $\varepsilon>0$, then there exists a solution $x^*:[0,T]\to\mathbb{R}$ of (1.3) such that

$$\left|x(t)-x^*(t)\right|\leq\frac{1}{|k_0(\alpha)|}\left(\frac{T^{\alpha-1}}{|A\Gamma(\alpha-2)|}+|B|T\max\left\{1,e^{-AT}\right\}\right)\varepsilon, \tag{3.4}$$

where $A=\frac{k_0(\alpha)}{k_1(\alpha)}$ and $B=\left(\frac{k_1(\alpha)}{k_0(\alpha)}\right)^{1-\alpha}$.

Proof. Let

$$h(t)={}^{CPC}_{0}D^{\alpha}_{t}x(t)-g(t),\ t\in[0,T]. \tag{3.5}$$

Taking the Laplace transform of (3.5) by Lemma 2.4, we have

$$\begin{aligned}H(s) &= \left[\frac{k_1(\alpha)}{s}+k_0(\alpha)\right]s^{\alpha}X(s)-k_0(\alpha)s^{\alpha-1}x(0)-G(s)\\ &= k_0(\alpha)\big(s+A\big)s^{\alpha-1}X(s)-k_0(\alpha)s^{\alpha-1}x(0)-G(s),\end{aligned} \tag{3.6}$$

where $H(s)=\mathscr{L}\{h(t)\}$, $X(s)=\mathscr{L}\{x(t)\}$ and $G(s)=\mathscr{L}\{g(t)\}$ denote the Laplace transforms of the functions h, x and g respectively. From (3.6), we get

$$\begin{aligned}X(s) &= \frac{1}{s+A}x(0)+\frac{1}{k_0(\alpha)s^{\alpha-1}\big(s+A\big)}\Big[G(s)+H(s)\Big]\\ &= \frac{1}{s+A}x(0)+\frac{1}{k_0(\alpha)}\Big(\frac{1}{As^{\alpha-1}}+\frac{B}{s+A}\Big)\Big[G(s)+H(s)\Big].\end{aligned} \tag{3.7}$$

Set

$$x^*(t)=x(0)e^{-At}+\frac{1}{k_0(\alpha)}\left(\frac{1}{A\Gamma(\alpha-2)}\int_0^t t^{\alpha-2}g(t-\tau)\,d\tau+B\int_0^t e^{-At}g(t-\tau)\,d\tau\right). \tag{3.8}$$

Taking the Laplace transform of (3.8), we get

$$X^*(s)=\frac{1}{s+A}x(0)+\frac{1}{k_0(\alpha)s^{\alpha-1}\big(s+A\big)}G(s). \tag{3.9}$$

Note that

$$\mathscr{L}\{{}^{CPC}_{0}D^{\alpha}_{t}x^*(t)\}(s)=k_0(\alpha)\big(s+A\big)s^{\alpha-1}X^*(s)-k_0(\alpha)s^{\alpha-1}x(0). \tag{3.10}$$

Substituting (3.9) into (3.10), we obtain

$$\mathscr{L}\{{}^{CPC}_{0}D^{\alpha}_{t}x^*(t)\}(s)=G(s)=\mathscr{L}\{g(t)\}(s),$$

which implies that $x^*(t)$ is a solution of Equation (1.3) since $\mathscr{L}$ is one-to-one. From (3.7) and (3.9), we have

$$X(s)-X^*(s)=\frac{1}{k_0(\alpha)}\Big(\frac{1}{As^{\alpha-1}}+\frac{B}{s+A}\Big)H(s),$$

which implies that

$$\begin{aligned} x(t)-x^*(t) &= \mathscr{L}^{-1}\{X(s)-X^*(s)\} \\ &= \frac{1}{k_0(\alpha)}\mathscr{L}^{-1}\Big\{\Big(\frac{1}{As^{\alpha-1}}+\frac{B}{s+A}\Big)H(s)\Big\} \\ &= \frac{1}{k_0(\alpha)}\left(\frac{1}{A\Gamma(\alpha-2)}t^{\alpha-2}*h(t)+Be^{-At}*h(t)\right). \end{aligned}$$

Therefore,

$$\begin{aligned} |x(t)-x^*(t)| &= \frac{1}{|k_0(\alpha)|}\left(\frac{1}{|A\Gamma(\alpha-2)|}\Big|t^{\alpha-2}*h(t)\Big|+|B|\Big|e^{-At}*h(t)\Big|\right) \\ &\le \frac{1}{|k_0(\alpha)|}\left(\frac{1}{|A\Gamma(\alpha-2)|}\int_0^t\Big|t^{\alpha-2}\Big|\Big|h(t-\tau)\Big|\,d\tau+|B|\int_0^t\Big|e^{-At}\Big|\Big|h(t-\tau)\Big|\,d\tau\right) \\ &\le \frac{1}{|k_0(\alpha)|}\left(\frac{1}{|A\Gamma(\alpha-2)|}\varepsilon\int_0^t\Big|t^{\alpha-2}\Big|\,d\tau+|B|\varepsilon\int_0^t\Big|e^{-At}\Big|\,d\tau\right) \\ &\le \frac{1}{|k_0(\alpha)|}\left(\frac{T^{\alpha-1}}{|A\Gamma(\alpha-2)|}+|B|T\max\left\{1,e^{-AT}\right\}\right)\varepsilon. \end{aligned}$$

Hence, (1.3) is Ulam-Hyers stable with the constant

$$\Omega=\frac{1}{|k_0(\alpha)|}\left(\frac{T^{\alpha-1}}{|A\Gamma(\alpha-2)|}+|B|T\max\left\{1,e^{-AT}\right\}\right).$$

□

Remark 3.4. Let $0<\alpha<1$ and g be a given real function on $[0,T]$. If a function $x:[0,T]\to\mathbb{R}$ satisfies the inequality

$$\left|{}^{CPC}_{\ 0}D^{\alpha}_{t}x(t)-g(t)\right|\le\phi(t), \tag{3.11}$$

then

$$\left|h(t)\right|\le\phi(t),$$

for each $t\in[0,T]$ and some function $\phi(t)>0$, where h is defined in (3.5). In view of Theorem 3.3, there exists a solution $x^*:[0,T]\to\mathbb{R}$ of (1.3) such that

$$x(t)-x^*(t)=\frac{1}{k_0(\alpha)}\left(\frac{1}{A\Gamma(\alpha-2)}t^{\alpha-2}*h(t)+Be^{-At}*h(t)\right),$$

and

$$
\begin{aligned}
|x(t)-x^*(t)| &\leq \frac{1}{|k_0(\alpha)|}\left(\frac{1}{|A\Gamma(\alpha-2)|}|t^{\alpha-2}*h(t)|+|B||e^{-At}*h(t)|\right)\\
&\leq \frac{1}{|k_0(\alpha)|}\left(\frac{1}{|A\Gamma(\alpha-2)|}\Big|t^{\alpha-2}\Big|\int_0^t|h(t-\tau)|\,d\tau\right.\\
&\quad \left.+|B|\max\{1,e^{-AT}\}\int_0^t\Big|h(t-\tau)\Big|\,d\tau\right)\\
&\leq \frac{1}{|k_0(\alpha)|}\left(\frac{T^{\alpha-2}}{|A\Gamma(\alpha-2)|}\Big|h(t)\Big|+|B|\max\left\{1,e^{-AT}\right\}|h(t)|\right)\\
&\leq \frac{1}{|k_0(\alpha)|}\left(\frac{T^{\alpha-2}}{|A\Gamma(\alpha-2)|}+|B|\max\left\{1,e^{-AT}\right\}\right)\phi(t),
\end{aligned}
$$

provided that $\int_0^t \Big|h(\tau)\Big|\,d\tau \leq h(t)$, for any $t\in[0,T]$. Thus, (1.3) is generalized Ulam-Hyers-Rassias stable with respect to ϕ on $[0,T]$.

4. Existence and uniqueness results

In this section, we study the existence and uniqueness of solutions for (1.4).

The following auxiliary lemma concerns the linear issue of (1.4).

Lemma 4.1. *Let $0<\alpha<1$ and $\sigma\in C([0,T],\mathbb{R})$. Then the solution of the following linear fractional differential equation*

$$
\begin{cases}
{}^{PC}_{\ 0}D_t^{\alpha}x(t)=\sigma(t),\ ,\ t\in[0,T],\\
x(0)=x_0,
\end{cases} \tag{4.1}
$$

is equivalent to the Volterra integral equation:

$$
\begin{aligned}
x(t) &= \exp\left(-\int_0^t\frac{k_1(\alpha,s)}{k_0(\alpha,s)}ds\right)x_0\\
&\quad +\frac{1}{\Gamma(\alpha-1)}\int_0^t\int_0^u\exp\left(-\int_u^t\frac{k_1(\alpha,s)}{k_0(\alpha,s)}ds\right)\frac{(u-\tau)^{\alpha-2}}{k_0(\alpha,u)}\sigma(\tau)\,d\tau\,du.
\end{aligned} \tag{4.2}
$$

Proof. Applying the operator ${}^{PC}_{\ 0}I_t^{\alpha}(\cdot)$ on both sides of (4.1), we get

$$
{}^{PC}_{\ 0}I_t^{\alpha}\,{}^{PC}_{\ 0}D_t^{\alpha}x(t)={}^{PC}_{\ 0}I_t^{\alpha}\sigma(t).
$$

Using (2.5) and (2.7) together with Lemma 2.3, we get

$$
x(t)-\exp\left(-\int_0^t\frac{k_1(\alpha,s)}{k_0(\alpha,s)}ds\right)x(0)=\int_0^t\exp\left(-\int_u^t\frac{k_1(\alpha,s)}{k_0(\alpha,s)}ds\right)\frac{{}^{RL}_{\ 0}D_u^{1-\alpha}\sigma(u)}{k_0(\alpha,u)}\,du.
$$

In view of the following elementary relation between the Riemann-Liouville fractional derivative and fractional integral:

$$
\begin{aligned}
{}^{RL}_{\ 0}D_u^{1-\alpha}h(u) &= {}^{RL}_{\ 0}I_u^{\alpha-1}\sigma(u)\\
&= \frac{1}{\Gamma(\alpha-1)}\int_0^u(u-\tau)^{\alpha-2}\sigma(\tau)\,d\tau,
\end{aligned}
$$

we get the integral equation (4.2). The converse follows by direct computation. This completes the proof □

We consider the following assumptions.

($\mathbf{A1}$) The function $f:[0,T]\times\mathbb{R}\to\mathbb{R}$ is continuous.
($\mathbf{A2}$) There exists a constant $L>0$ such that

$$|f(t,x)-f(t,y)|\leq L|x-y|, \text{ for all } t\in[0,T],\ x,y\in\mathbb{R}.$$

Theorem 4.2. *If the assumptions* $(\mathbf{A1})-(\mathbf{A2})$ *are satisfied, then (1.4) has a unique solution on* $[0,T]$ *provided that*

$$\frac{LM_kT^{\alpha}}{\Gamma(\alpha+1)}<1, \tag{4.3}$$

where $M_k=\sup_{t\in[0,T]}\frac{1}{|k_0(\alpha,t)|}$.

Proof. We consider the operator $\mathscr{P}:C([0,T],\mathbb{R})\to C([0,T],\mathbb{R})$ defined by

$$\begin{aligned}(\mathscr{P}x)(t) &= \exp\left(-\int_0^t\frac{k_1(\alpha,s)}{k_0(\alpha,s)}ds\right)x_0+\frac{1}{\Gamma(\alpha-1)}\int_0^t\int_0^u\exp\left(-\int_u^t\frac{k_1(\alpha,s)}{k_0(\alpha,s)}ds\right)\\ &\quad\times\frac{(u-\tau)^{\alpha-2}}{k_0(\alpha,u)}f(\tau,x(\tau))\,d\tau\,du.\end{aligned} \tag{4.4}$$

By (4.2), finding a solution of (1.4) in $C([0,T],\mathbb{R})$ is equivalent to finding a fixed point of the operator $\mathscr{P}$. For any $x,y\in C([0,T],\mathbb{R})$ and each $t\in[0,T]$, we have

$$\begin{aligned}&|(\mathscr{P}x)(t)-(\mathscr{P}y)(t)|\\ &=\left|\frac{1}{\Gamma(\alpha-1)}\int_0^t\int_0^u exp\left(-\int_u^t\frac{k_1(\alpha,s)}{k_0(\alpha,s)}ds\right)\frac{(u-\tau)^{\alpha-2}}{k_0(\alpha,u)}\big(f(\tau,x(\tau))-f(\tau,y(\tau))\big)\,d\tau\,du\right|\\ &\leq\frac{1}{\Gamma(\alpha-1)}\int_0^t\int_0^u\frac{(u-\tau)^{\alpha-2}}{|k_0(\alpha,u)|}\big|f(\tau,x(\tau))-f(\tau,y(\tau))\big|\,d\tau\,du\\ &\leq\frac{LM_k}{\Gamma(\alpha-1)}\int_0^t\int_0^u(u-\tau)^{\alpha-2}|x(\tau)-y(\tau)|\,d\tau\,du\\ &\leq\frac{LM_kT^{\alpha}}{\Gamma(\alpha+1)}\|x-y\|.\end{aligned}$$

In view of (4.3) and Banach's fixed point, we deduct that $\mathscr{P}$ is a contraction. Hence, (1.4) has a unique solution on $[0,T]$. This completes the proof. □

Next, we show that the existence of solutions for (1.4) via the Schaefer's fixed point theorem.

Theorem 4.3. *If the assumptions* $(\mathbf{A1})-(\mathbf{A2})$ *are satisfied, then (1.4) has at least one solution on* $[0,T]$.

Proof. We consider $\mathscr{P}$ as in (4.4). The proof will be given in several steps.
Step 1. $\mathscr{P}$ is continuous.

Let $\{x_n\}$ be a sequence such that $x_n \to x$ in $C([0,T],\mathbb{R})$. For each $t \in [0,T]$, we get

$$
\begin{aligned}
&|\mathscr{P}x_n(t) - \mathscr{P}x(t)| \\
&= \left| \frac{1}{\Gamma(\alpha-1)} \int_0^t \int_0^u exp\left(-\int_u^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)}ds\right) \frac{(u-\tau)^{\alpha-2}}{k_0(\alpha,u)} \big(f(\tau,x_n(\tau)) - f(\tau,x(\tau))\big)\, d\tau\, du \right| \\
&\leq \frac{M_k}{\Gamma(\alpha-1)} \int_0^t \int_0^u (u-\tau)^{\alpha-2} \|(f(\cdot,x_n(\cdot)) - f(\cdot,x(\cdot))\|\, d\tau\, du \\
&\leq \frac{M_k T^\alpha}{\Gamma(\alpha+1)} \|(f(\cdot,x_n(\cdot)) - f(\cdot,x(\cdot))\|.
\end{aligned}
$$

Therefore, the continuity of f implies that $\|\mathscr{P}x_n - \mathscr{P}x\| \to 0$, as $n \to \infty$. Hence $\mathscr{P}$ is continuous.

Step 2. $\mathscr{P}$ maps bounded sets into bounded sets of $C([0,T],\mathbb{R})$.

Indeed, we prove that for all $r > 0$, there exists a $\rho > 0$ such that, for every $x \in \mathscr{B}_r = \{x \in C([0,T],\mathbb{R}) : \|x\| \leq r)\}$, $\|\mathscr{P}x\| \leq \rho$.

In fact, for any $t \in [0,T]$ and set $M_f = \sup_{t\in[0,T]} |f(t,0)|$, we have

$$
\begin{aligned}
|\mathscr{P}x| \leq\; & \left| \exp\left(-\int_0^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)}ds\right) x_0 \right| + \left| \frac{1}{\Gamma(\alpha-1)} \int_0^t \int_0^u \exp\left(-\int_u^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)}ds\right) \right. \\
& \left. \times \frac{(u-\tau)^{\alpha-2}}{k_0(\alpha,u)} ((f(\tau,x(\tau)) - f(\tau,0)) + f(\tau,0))\, d\tau\, du \right| \\
\leq\; & |x_0| + \frac{1}{\Gamma(\alpha-1)} \int_0^t \int_0^u \frac{(u-\tau)^{\alpha-2}}{|k_0(\alpha,u)|} (|f(\tau,x(\tau)) - f(\tau,0)| + |f(\tau,0)|)\, d\tau\, du \\
\leq\; & |x_0| + \frac{M_k}{\Gamma(\alpha-1)} \int_0^t \int_0^u (u-\tau)^{\alpha-2} (L|x(\tau)| + |f(\tau,0)|)\, d\tau\, du \\
\leq\; & |x_0| + \frac{M_k}{\Gamma(\alpha-1)} \int_0^t \int_0^u (u-\tau)^{\alpha-2} (L\|x\| + M_f)\, d\tau\, du \\
\leq\; & |x_0| + \frac{M_k T^\alpha}{\Gamma(\alpha+1)} \big(Lr + M_f\big),
\end{aligned}
$$

which implies that

$$
\|\mathscr{P}x\| \leq |x_0| + \frac{M_k T^\alpha}{\Gamma(\alpha+1)} \big(Lr + M_f\big) := \rho.
$$

Step 3. $\mathscr{P}$ maps bounded sets into equicontinuous sets of $C([0,T],\mathbb{R})$.

For each $t_1, t_2 \in [0,T],\ t_1 < t_2$ and $x \in \mathscr{B}_r$, we have

$$
\begin{aligned}
&|(\mathscr{P}x)(t_2)-(\mathscr{P}x)(t_1)|\\
&\leq \left|exp\left(-\int_0^{t_2}\frac{k_1(\alpha,s)}{k_0(\alpha,s)}ds\right)x_0-exp\left(-\int_0^{t_1}\frac{k_1(\alpha,s)}{k_0(\alpha,s)}ds\right)x_0\right|\\
&+\frac{1}{\Gamma(\alpha-1)}\left|\int_0^{t_2}\int_0^{u}\left[\left(-\int_0^{t_2}\frac{k_1(\alpha,s)}{k_0(\alpha,s)}ds\right)-exp\left(-\int_0^{t_1}\frac{k_1(\alpha,s)}{k_0(\alpha,s)}ds\right)\right]\right.\\
&\left.\times\frac{(u-\tau)^{\alpha-2}}{k_0(\alpha,u)}(f(\tau,x(\tau))\,d\tau\,du\right|\\
&+\frac{1}{\Gamma(\alpha-1)}\left|\int_{t_1}^{t_2}\int_0^{u}exp\left(-\int_0^{t_1}\frac{k_1(\alpha,s)}{k_0(\alpha,s)}ds\right)\frac{(u-\tau)^{\alpha-2}}{k_0(\alpha,u)}(f(\tau,x(\tau))\,d\tau\,du\right|\\
&=\left|\frac{k_1(\alpha,\xi)}{k_0(\alpha,\xi)}x_0 exp\left(-\int_0^{\xi}\frac{k_1(\alpha,s)}{k_0(\alpha,s)}ds\right)(t_2-t_1)\right|\\
&+\frac{1}{\Gamma(\alpha-1)}\left|\int_0^{t_2}\int_0^{u}\frac{k_1(\alpha,\xi)}{k_0(\alpha,\xi)}exp\left(-\int_0^{\xi}\frac{k_1(\alpha,s)}{k_0(\alpha,s)}ds\right)(t_2-t_1)\frac{(u-\tau)^{\alpha-2}}{k_0(\alpha,u)}(f(\tau,x(\tau))\,d\tau\,du\right|\\
&+\frac{1}{\Gamma(\alpha-1)}\left|\int_{t_1}^{t_2}\int_0^{u}exp\left(-\int_0^{t_1}\frac{k_1(\alpha,s)}{k_0(\alpha,s)}ds\right)\frac{(u-\tau)^{\alpha-2}}{k_0(\alpha,u)}(f(\tau,x(\tau))\,d\tau\,du\right|\\
&\leq\left[\left|\frac{k_1(\alpha,\xi)}{k_0(\alpha,\xi)}x_0\right|+\frac{\bar{f}}{\Gamma(\alpha+1)}\left|\frac{k_1(\alpha,\xi)}{k_0^2(\alpha,\xi)}\right|t_2\right](t_2-t_1)+\frac{\bar{f}}{\Gamma(\alpha+1)}\left|\frac{1}{k_0(\alpha,\xi)}\right|(t_2^{\alpha}-t_1^{\alpha}),
\end{aligned}
$$

where $\bar{f}=\sup_{t\in[0,T]\times\mathscr{B}_r}|f(t,x(t)|,\ \xi\in(t_1,t_2)$.

As $t_1\to t_2$, the right hand side of the above inequality tends to zero independently of $x\in\mathscr{B}_r$. Thus, $\mathscr{P}$ is equicontinuous. As a consequence of Step 1-Step 3 with the Arzelà-Ascoli theorem, we deduce that $\mathscr{P}$ is completely continuous.

Step 4. The priori bounds.

We show that the set $\mathscr{S}(\mathscr{P})=\{x\in C([0,T],\mathbb{R}):x=\mu\mathscr{P}x,\ \mu\in(0,1)\}$ is bounded. Let $x\in\mathscr{S}(\mathscr{P})$. Then, $x=\mu\mathscr{P}x$ for some $\mu\in(0,1)$. For each $t\in[0,T]$, we have

$$
\begin{aligned}
|x(t)|&\leq|x_0|+\frac{1}{\Gamma(\alpha-1)}\int_0^t\int_0^u\frac{(u-\tau)^{\alpha-2}}{|k_0(\alpha,u)|}(|f(\tau,x(\tau))-f(\tau,0)|+|f(\tau,0)|)\,d\tau\,du\\
&\leq|x_0|+\frac{M_k}{\Gamma(\alpha-1)}\int_0^t\int_0^u(u-\tau)^{\alpha-2}(L|x(\tau)|+|f(\tau,0)|)\,d\tau\,du\\
&\leq|x_0|+\left[\frac{M_kL}{\Gamma(\alpha-1)}\int_0^t\int_0^u(u-\tau)^{\alpha-2}|x(\tau)|\,d\tau\,du+\frac{M_kM_f}{\Gamma(\alpha-1)}\int_0^t\int_0^u(u-\tau)^{\alpha-2}\,d\tau\,du\right]\\
&\leq|x_0|+\left[\frac{M_kL}{\Gamma(\alpha-1)}\int_0^t\int_\tau^t(u-\tau)^{\alpha-2}|x(\tau)|\,du\,d\tau+\frac{M_kM_fT^{\alpha}}{\Gamma(\alpha+1)}\right],
\end{aligned}
$$

which implies that

$$|x(t)|\leq a+\frac{b}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}|x(\tau)|\,d\tau,$$

where $a = |x_0| + \frac{M_k M_f T^\alpha}{\Gamma(\alpha+1)}$ and $b = M_k L$.

In view of the generalized Gronwall's inequality (see Diethelm and Ford [27]), one has

$$|x(t)| \leq a\mathbb{E}_\alpha(bt^\alpha) < \infty,$$

where $\mathbb{E}_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha+1)}$ is the one-parameter Mittag-Leffler function. Then the set $\mathscr{S}(\mathscr{P})$ is bounded. Hence, Schaefer's fixed point theorem (Theorem 2.6) guarantees that $\mathscr{P}$ has a fixed point, which is a solution of (1.4). This completes the proof. □

5. Examples

In this section, we consider two fractional differential equations with hybrid proportional-Caputo derivatives.

Example 5.1. Consider

$$ {}^{CPC}_{\;\;\,0}D_t^{\frac{1}{2}} x(t) = \frac{t}{\sqrt{\pi}} + \frac{1}{10},\; t \in [0,1]. \tag{5.1}$$

Here $\alpha = \frac{1}{2}, T = 1, k_0(\alpha) = \alpha = \frac{1}{2}, k_1(\alpha) = 1-\alpha = \frac{1}{2}$ and $g(t) = \frac{t}{\sqrt{\pi}} + \frac{1}{10}$. Let $x(t) = \sqrt{t}$. From (2.4), we get

$$\begin{aligned} {}^{CPC}_{\;\;\,0}D_t^{\frac{1}{2}} \sqrt{t} &= \frac{1}{\Gamma\left(\frac{1}{2}\right)} \int_0^t \left(\frac{1}{2}\sqrt{t} + \frac{1}{4\sqrt{t}}\right)(t-\tau)^{-\frac{1}{2}}\, d\tau \\ &= \frac{1}{\sqrt{\pi}}\left(t + \frac{1}{2}\right). \end{aligned}$$

Choose $\varepsilon = \frac{1}{2}$. Observe that $x(t)$ satisfies:

$$\begin{aligned} \left|{}^{CPC}_{\;\;\,0}D_t^{\alpha} x(t) - g(t)\right| &= \left|\frac{1}{\sqrt{\pi}}\left(t + \frac{1}{2}\right) - \frac{t}{\sqrt{\pi}} - \frac{1}{10}\right| \\ &= \left|\frac{1}{2\sqrt{\pi}} - \frac{1}{10}\right| \leq \frac{1}{2\sqrt{\pi}} < \frac{1}{2}. \end{aligned}$$

Since $x_1(0) = 0$, $A = \frac{k_0(\alpha)}{k_1(\alpha)} = 1$ and $B = \left(\frac{k_1(\alpha)}{k_0(\alpha)}\right)^{1-\alpha} = 1$, we have that (3.8) gives an exact solution $x^*(t)$ of (1.3) as:

$$\begin{aligned} x^*(t) &= 2\left(\frac{3}{4\sqrt{\pi}} \int_0^t t^{-\frac{3}{2}} \left(\frac{(t-\tau)}{\sqrt{\pi}} + \frac{1}{10}\right) d\tau + \int_0^t e^{-t}\left(\frac{(t-\tau)}{\sqrt{\pi}} + \frac{1}{10}\right) d\tau\right) \\ &= \frac{3}{2\sqrt{\pi}}\left(\frac{1}{10\sqrt{t}} + \frac{1}{2\sqrt{\pi}}\sqrt{t}\right) + 2e^{-t}\left(\frac{1}{10}t + \frac{1}{2\sqrt{\pi}}t^2\right). \end{aligned}$$

By (3.4), we conclude that

$$\begin{aligned}
|x(t)-x^*(t)| &= \left|\sqrt{t}-\frac{3}{2\sqrt{\pi}}\left(\frac{1}{10\sqrt{t}}+\frac{1}{2\sqrt{\pi}}\sqrt{t}\right)-2e^{-t}\left(\frac{1}{10}t+\frac{1}{2\sqrt{\pi}}t^2\right)\right| \\
&= \left|(1-\frac{3}{4\pi})\sqrt{t}-\frac{3}{20\sqrt{\pi}}\frac{1}{\sqrt{t}}-2e^{-t}\left(\frac{1}{10}t+\frac{1}{2\sqrt{\pi}}t^2\right)\right| \\
&\leq \left|\sqrt{t}-\frac{3}{20\sqrt{\pi}}\frac{1}{\sqrt{t}}\right| \\
&\leq \left|t-\frac{3}{20\sqrt{\pi}}\frac{1}{\sqrt{t}}\right| \\
&\leq t+\frac{3}{4\sqrt{\pi}}\frac{1}{\sqrt{t}}=\left(\frac{3}{2\sqrt{\pi}}t^{-\frac{1}{2}}+2t\right)\frac{1}{2}=\Omega\varepsilon,
\end{aligned}$$

where $\Gamma\left(-\frac{3}{2}\right)=\frac{4\sqrt{\pi}}{3}$ is calculated from the well-known formula

$$\Gamma\left(-n+\frac{1}{2}\right)=\frac{(-1)^n\,2^n}{1\cdot 3\cdot 5\cdots(2n-1)}\sqrt{\pi},\ n=2.$$

Thus, By Theorem 3.3, we have

$$\Omega=\frac{1}{|k_0(\alpha)|}\left(\frac{T^{\alpha-1}}{|A\Gamma(\alpha-2)|}+|B|T\max\left\{1,e^{-AT}\right\}\right)=\frac{3}{2\sqrt{\pi}}T^{-\frac{1}{2}}+2T,\ \varepsilon=\frac{1}{2}.$$

Hence, (5.1) is Ulam-Hyers stable.

Example 5.2. Consider

$$\begin{cases} {}^{PC}_{0}D_t^{\frac{1}{3}}x(t)=\frac{e^{-2t}}{9+t}\frac{|x(t)|}{1+|x(t)|},\ t\in[0,1], \\ x(0)=0. \end{cases} \tag{5.2}$$

Set $\alpha=\frac{1}{3}, T=1$ and $f(t,x)=\frac{e^{-2t}}{9+t}\frac{|x|}{1+|x|}$. For any $t\in[0,1]$ and $x,y\in\mathbb{R}$,

$$\begin{aligned}
|f(t,x)-f(t,y)| &= \frac{e^{-2t}}{9+t}\left|\frac{|x|}{(1+|x|)}-\frac{|y|}{(1+|y|)}\right| \\
&\leq \frac{e^{-2t}}{9+t}\frac{|x-y|}{(1+|x|)(1+|y|)} \\
&\leq \frac{e^{-2t}}{9+t}|x-y| \\
&\leq \frac{1}{9}|x-y|.
\end{aligned}$$

Thus, $(A2)$ holds with $L=\frac{1}{9}$. If we choose $k_0(\alpha,t)=\alpha t^{1-\alpha}$ and $k_1(\alpha,t)=(1-\alpha)t^{\alpha}$, then

$$M_k=\sup_{t\in[0,1]}\frac{1}{|k_0(\alpha,t)|}=\sup_{t\in[0,1]}\frac{1}{|\alpha t^{1-\alpha}|}=\frac{1}{\alpha}=3.$$

Thus the condition (4.3) of Theorem 4.2 yields

$$\frac{LM_kT^{\alpha}}{\Gamma(\alpha+1)}=\frac{3}{9\times\Gamma\left(\frac{4}{3}\right)}=0.3732821739<1.$$

Hence, Theorem 4.2 admits a unique solution of (5.2) on $[0,1]$.

6. The conclusion

By using the Laplace transform method, the Ulam-Hyers and the generalized Ulam-Hyers-Rassias stability of linear fractional differential equations with hybrid proportional-Caputo derivatives were investigated. Further, based on a fixed point approach together with a generalized Gronwall's inequality, the existence and uniqueness theorems of solution were established. Finally, two examples were inserted to illustrate the applicability of the theoretical results. In the first example, we are able to prove that the solution of the considered linear fractional differential equation with hybrid proportional-Caputo derivative is Ulam-Hyers stable, which supports the outcomes of Theorem 3.3. As for the second example, the uniqueness result stated in Theorem 4.2 was verified.

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