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ON HIGH-ORDER GENERALIZED NEUTRAL SINGULAR DIFFERENTIAL EQUATIONS WITH TIME-DEPENDENT DEVIATING ARGUMENT

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Abstract. Based on the coincidence degree theory, we investigate the existence of a positive periodic solution of a kind of high-order p-Laplacian generalized neutral singular differential equation with time-dependent deviating argument.

Keywords. Positive periodic solution; High-order; Generalized neutral; Singular; Time-dependent deviating argument.

1. INTRODUCTION

The periodic solution problem for neutral differential equations attracted much attention during the past years, see [1]-[13]. In 2009, Ren and Cheng [13] discussed the following high-order neutral differential equation

$$(\phi_p(x(t)-cx(t-\tau))^{(l)})^{(n-l)} = F(t,x(t),x'(t),\cdots,x^{(l-1)}(t)), \tag{1.1}$$

where $p \geq 2$, $\phi_p(x) = |x|^{p-2}x$ for $x \neq 0$ and $\phi_p(0) = 0$. The existence of periodic solutions for equation (1.1) was obtained in the general case (i.e., $|c| \neq 1$ in [12]) and in the critical case (i.e. $|c| = 1$ in [13]), respectively. In 2015, Cheng and Ren [4] investigated a kind of fourth-order generalized neutral Liénard equation

$$(\phi_p(x(t)-c(t)x(t-\tau(t)))'')'' + f(x(t))x'(t) + g(t,x(t),x(t-\sigma(t)),x'(t)) = p(t), \tag{1.2}$$

where $c \in C(\mathbb{R},\mathbb{R})$ is a T-periodic function for some $T > 0$ and $c(t) \neq 1$. They obtained the existence of a periodic solution for equation (1.2) by using the coincidence degree theory. The conditions they presented to guarantee the existence of a periodic solution are efficient and beautiful.

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On the other hand, there are also some results on neutral differential equations with a singularity, see, e.g., [14, 15]. In 2015, Kong, Lu and Laing [14] studied the following second-order neutral singular Liénard equation

$$(x(t)-cx(t-\tau))''+f(x(t))x'(t)+g(t,x(t-\sigma))=p(t), \tag{1.3}$$

where c is a constant with $|c|<1$, $g\in C(\mathbb{R}\times(0,+\infty),\mathbb{R})$ is a T-periodic function about t and has a singularity at $x=0$. Based on the Mawhin's continuation theorem, they proved that equation (1.3) has at least one positive T-periodic solution. In 2017, Kong and Lu [15] investigated the following fourth-order neutral singular Liénard equation

$$(\phi_p(x(t)-cx(t-\delta))'')''+f(x(t))x'(t)+g^*(t,x(t-\sigma(t)))=p(t), \tag{1.4}$$

where c is a constant with $|c|<1$, f is a continuous function, $g^*(t,x(t-\sigma(t))=g_0^*(x(t))+g_1^*(t,x(t-\sigma(t)))$, $g_1^*\in C(\mathbb{R}\times\mathbb{R},\mathbb{R})$ is a T-periodic function about t, and $g_0^*(x)\in C((0,+\infty),\mathbb{R})$ can be singular at $x=0$. Using the coincidence degree theory, they obtained the existence of a positive T-periodic solution for equation (1.4).

In this paper, inspired by the results presented in [4, 12, 13, 14, 15], we discuss the existence of a positive periodic solution for the following high-order p-Laplacian generalized neutral singular differential equation with time-dependent deviating argument

$$(\phi_p(x(t)-c(t)x(t-\tau(t)))'')^{(n-2)}+f(t,x'(t))+g(t,x(t-\sigma(t)))=p(t), \tag{1.5}$$

where $f\in C(\mathbb{R}\times\mathbb{R},\mathbb{R})$ is a T-periodic function about t and $f(t,0)\equiv 0$, $c,\ \delta\in C^2(\mathbb{R},\mathbb{R})$ are T-periodic functions with $|c(t)|\neq 1$, for all $t\in[0,T]$, $\sigma\in C^1(\mathbb{R},\mathbb{R})$ is a T-periodic function and $\sigma':=\max_{t\in\mathbb{R}}|\sigma'(t)|<1$, $p\in C(\mathbb{R},\mathbb{R})$ is a T-periodic function with $\int_0^T p(t)dt=0$, $g(t,x)=g_0(x)+g_1(t,x)$, where $g_1\in C(\mathbb{R}\times\mathbb{R},\mathbb{R})$ is a T-periodic function about t, $g_0\in C((0,\infty);\mathbb{R})$ has a strong singularity of repulsive type at the origin, i.e.,

$$\lim_{x\to 0^+} g_0(x)=-\infty \text{ and } \lim_{x\to 0^+}\int_1^x g_0(s)ds=+\infty. \tag{1.6}$$

Remark 1.1. It is worth mentioning that the friction term $f(x)x'(t)$ of equations (1.3) and (1.4) in [14, 15] satisfies $\int_0^T f(x(t))x'(t)dt=0$, which is crucial to estimate the *priori bounds* of positive T-periodic solutions for equations (1.3) and (1.4). However, in this paper, the friction term $f(t,x')$ may not satisfy $\int_0^T f(t,x'(t))dt=0$. For example, let $f(t,x')=(\sin^2 2t+3)\cos x'(t)$. It is obvious that $\int_0^T(\sin^2 2t+3)\cos x'(t)dt\neq 0$. This implies that our methods to estimate *priori bounds* of a positive T-periodic solution for equation (1.5) is more difficult than equations (1.3) and (1.4).

Remark 1.2. In [14, 15], coefficient c of neutral operator $(A_1x)(t):=x(t)-cx(t-\tau)$ satisfies $|c|<1$. In this paper, variable coefficient $c(t)$ satisfies $|c(t)|<1$ or $|c(t)|>1$. In addition, the singular term g_0^* of equation (1.4) has not deviating argument (i.e. $\sigma\equiv 0$). The singular term g_0 of this paper satisfies time-dependent deviating argument. It is easy to verify that the result on estimating *lower bounds* of a positive periodic solution for equation (1.5) is more complex than equation (1.4). This shows that our result is more general.

2. Preliminaries

We first recall qualitative properties of the neutral operator $(Ax)(t):=x(t)-c(t)x(t-\tau(t))$ [16, 17] and the coincidence degree theory [18]

Lemma 2.1. [16, 17] *If $|c(t)| \neq 1$, then the operator $(Ax)(t)$ has a continuous inverse A^{-1} on the space*

$$C_T := \{x|x \in (\mathbb{R},\mathbb{R}),\ x(t+T) \equiv x(t),\ \forall\, t \in \mathbb{R}\},$$

and satisfies

(1) $\left|\left(A^{-1}x\right)(t)\right| \leq \frac{\|x\|}{1-c^*}$, *for* $c^* := \max_{t\in[0,T]}|c(t)| < 1$, $\forall\, x \in C_T$, *where* $\|x\| := \max_{t\in\mathbb{R}}|x(t)|$;

(2) $\left|\left(A^{-1}x\right)(t)\right| \leq \frac{\|x\|}{c_*-1}$, *for* $c_* := \min_{t\in[0,T]}|c(t)| > 1$, $\forall\, x \in C_T$.

Lemma 2.2. [18] *Suppose that X and Y are two Banach spaces, and $L : D(L) \subset X \to Y$ is a Fredholm operator with index zero. Let $\Omega \subset X$ be an open bounded set and $N : \overline{\Omega} \to Y$ be L-compact on $\overline{\Omega}$. Assume that the following conditions hold:*

(1) $Lx \neq \lambda Nx, \forall\, x \in \partial\Omega \cap D(L),\ \lambda \in (0,1)$;

(2) $Nx \notin Im\, L, \forall\, x \in \partial\Omega \cap Ker\, L$;

(3) $\deg\{JQN, \Omega \cap Ker\, L, 0\} \neq 0$, *where $J : Im\, Q \to Ker\, L$ is an isomorphism.*

Then the equation $Lx = Nx$ has a solution in $\overline{\Omega} \cap D(L)$.

In order to use the coincidence degree theory [18], we rewrite (1.5) in the form:

$$\begin{cases} (Ax_1)''(t) = \phi_q(x_2(t)), \\ x_2^{(n-2)}(t) = -f(t,x_1'(t)) - g(t,x_1(t-\sigma(t))) + p(t), \end{cases} \tag{2.1}$$

where $\frac{1}{p}+\frac{1}{q} = 1$. Clearly, if $x(t) = (x_1(t),x_2(t))^\top$ is periodic solution of equation (2.1), then $u_1(t)$ must be a periodic solution of equation (1.5). Therefore, the problem of finding a T-periodic solution of equation (1.5) is reduced to finding one for equation (2.1). Set

$$X := \{x(t) = (x_1(t),x_2(t))^\top \in C^n(\mathbb{R},\mathbb{R}^2) : x(t+T) \equiv x(t),\ t \in \mathbb{R}\}$$

with the norm $\|x\| := \max\{\|x_1\|, \|x_2\|\}$ and

$$Y := \{x(t) = (x_1(t),x_2(t))^\top \in C^1(\mathbb{R},\mathbb{R}^2) : x(t+T) \equiv x(t),\ t \in \mathbb{R}\}$$

with the norm $\|x\|_\infty = \max\{\|x\|, \|x'\|\}$. Clearly, X and Y are both Banach spaces. Define

$$L : D(L) \subset X \to Y, \text{by } (Lx)(t) = \begin{pmatrix} (Ax_1)''(t) \\ x_2^{(n-2)}(t) \end{pmatrix},$$

where

$$D(L) = \{x(t) = (x_1(t),x_2(t))^\top \in C^2(\mathbb{R},\mathbb{R}^2) :\ x(t+T) \equiv x(t), t \in \mathbb{R}\}.$$

Define a nonlinear operator $N : X \to Y$ as follows

$$(Nx)(t) = \begin{pmatrix} \phi_q(x_2(t)) \\ -f(t,x_1'(t)) - g(t,x_1(t-\sigma(t))) + p(t) \end{pmatrix}. \tag{2.2}$$

Then equation (2.1) can be converted to the abstract equation $Lx = Nx$. $\forall x \in KerL$, $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in KerL$, i.e. $\begin{cases} (x_1(t) - c(t)x_1(t-\tau(t)))'' = 0, \\ x_2^{n-2}(t) = 0, \end{cases}$

we have

$$\begin{cases} x_1(t) - c(t)x_1(t-\delta(t)) = a_1 t + a_0, \\ x_2(t) = b_{n-3}t^{n-3} + b_{n-4}t^{n-4} + \cdots + b_1 t + b_0, \end{cases}$$

where $a_1, a_0, b_{n-3}, b_{n-4}, \cdots, b_1, b_0 \in \mathbb{R}$ are constants. From $x_1(t) - c(t)x_1(t-\tau(t)) \in C_T$, $x_2(t) \in C_T$, we have $a_1 = 0,\ b_1 = b_2 = \cdots = b_{n-3} = 0$. Let $\phi(t) \neq 0$ be a solution of $x_1(t) - c(t)x_1(t-\tau(t)) = 1$. Then $KerL = x = \begin{pmatrix} a\phi(t),\ a \in \mathbb{R} \\ b,\ b \in \mathbb{R} \end{pmatrix}$. From the definition of L, one can easily see that

$$Ker\ L \cong \mathbb{R}^n,\ Im\ L = \left\{ y \in Y : \int_0^T \begin{pmatrix} y_1(s) \\ y_2(s) \end{pmatrix} ds = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}.$$

So L is a Fredholm operator with index zero. Let $P : X \to Ker\ L$ and $Q : Y \to Im\ Q \subset \mathbb{R}^2$ be defined by

$$Px = \begin{pmatrix} (Ax_1)(0) \\ x_2(0) \end{pmatrix};\ Qy = \frac{1}{T}\int_0^T \begin{pmatrix} y_1(s) \\ y_2(s) \end{pmatrix} ds.$$

Then $Im\ P$=$Ker\ L$, $Ker\ Q$=$Im\ L$. Letting K denote the inverse of $L|_{Ker\ p\ \cap D(L)}$, we have

$$[L_P^{-1}y](t) = \begin{pmatrix} (A^{-1}Gy_1)(t) \\ (Gy_2)(t) \end{pmatrix},$$

$$\begin{aligned} [Gy_1](t) &= (Ax_1)'(0)t + \int_0^t (t-s)y_1(s)ds, \\ [Gy_2](t) &= \sum_{i=1}^{n-3} \frac{1}{i!} b_i t^i + \frac{1}{(n-3)!}\int_0^t (t-s)^{n-3} y_2(s)ds, \end{aligned} \tag{2.3}$$

where $b_i := x_2^{(i)}(0)$ is defined by the following

$$E_1 Z = C,\ \text{where } E_1 = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ e_1 & 1 & 0 & \cdots & 0 & 0 \\ e_2 & e_1 & 1 & \cdots & 0 & 0 \\ \cdots & & & & & \\ e_{n-5} & e_{n-6} & e_{n-7} & \cdots & 1 & 0 \\ e_{n-4} & e_{n-5} & e_{n-6} & \cdots & e_1 & 0 \end{pmatrix}_{(n-3)\times(n-3)}.$$

$Z = (b_{n-3}, b_{n-4} \cdots, a_1)^\top$, $C = (c_1, c_2, \cdots, c_{n-5})^\top$, $c_i = -\frac{1}{i!T}\int_0^T (T-s)^i y_1(s)ds$ and $e_j = \frac{T^j}{(j+1)!}$, $j = 1, 2, \cdots, n-4$. Therefore, from (2.2) and (2.3), we can get that N is L-compact on $\bar{\Omega}$.

3. Main results

In this section, applying Lemma 2.2, we study the existence of a positive periodic solution to equation (1.5) in the cases that $c^* < 1$ and $c_* > 1$. First, we consider equation (1.5) in the case that $c^* < 1$.

Theorem 3.1. *Suppose that the condition $c^* < 1$ hold. Furthermore, assume that the following conditions hold:*

(H_1) *There exist a positive constant N such that*

$$|f(t,v)| \leq N,\ \textit{for all}\ (t,v) \in [0,T] \times \mathbb{R};$$

(H_2) *There exist two positive constants $d_1,\ d_2$ with $d_1 < d_2$ such that $g(t,x) < -N$ for all $(t,x) \in [0,T] \times (0,d_1)$, and $g(t,x) > N$ for all $(t,x) \in [0,T] \times (d_2,+\infty)$;*

(H_3) *There exist two positive constants a, b such that*

$$g(t,x) \le ax^{p-1} + b, \text{ for all } (t,x) \in [0,T] \times (0,+\infty).$$

Then, equation (1.5) has at least one positive T-periodic solution if the following one of condition hold:

$$0 < \frac{aT^{p+1}\left(\frac{T}{2\pi}\right)^{n+p-5}}{2^p(1-\sigma')\left(1-c^*-\left(\frac{T^2}{4\pi}c_2+Tc_1+Tc_1\tau_1+\frac{Tc^*\tau_2}{2}+c^*\tau_1^2+2c^*\tau_1\right)\right)^{p-1}} < 1,$$

where $\delta_i = \max_{t\in[0,\omega]}|\delta^{(i)}(t)|$, and $c_i = \max\limits_{t\in[0,\omega]}|c^{(i)}(t)|$, $i = 1,2$.

Proof. Consider the operator equation

$$Lx = \lambda Nx,\ \lambda \in (0,1),$$

where L and N are defined by equations (2.2) and (2.3). Set

$$\Omega_1 = \{x : Lx = \lambda Nx,\ \lambda \in (0,1)\}.$$

If $x(t) = (x_1(t), x_2(t))^\top \in \Omega_1$, then

$$\begin{cases} (Ax_1)''(t) = \lambda\phi_q(x_2(t)) \\ x_2^{(n-2)}(t) = -\lambda f(t,x_1'(t)) - \lambda g(t,x_1(t-\sigma(t))) + \lambda p(t). \end{cases} \tag{3.1}$$

Substituting $x_2(t) = \frac{1}{\lambda^{p-1}}\phi_p((Ax_1)''(t))$ into (3.1), we obtain

$$(\phi_p(Ax_1)''(t))^{(n-2)} + \lambda^p f(t,x_1'(t)) + \lambda^p g(t,x_1(t-\sigma(t))) = \lambda^p p(t). \tag{3.2}$$

Integrating of both sides of equation (3.2) over $[0,T]$, we get

$$\int_0^T (f(t,x_1'(t)) + g(t,x_1(t-\sigma(t)))dt = 0. \tag{3.3}$$

From equation (3.3) and condition (H_1), it is clear that

$$-NT \le \int_0^T g(t,x_1(t-\sigma(t))dt \le NT.$$

By condition (H_2), we have that there exist two points $\xi,\ \eta \in (0,T)$ such that

$$x_1(\xi-\sigma(\xi)) \ge d_1, \text{ and } x_1(\eta) \le d_2. \tag{3.4}$$

Then, from equation (3.4), we arrive at

$$\begin{aligned} \|x_1\| &= \max_{t\in[0,T]}|x_1(t)| = \max_{t\in[\eta,\eta+T]}|x_1(t)| \\ &= \max_{t\in[\eta,\eta+T]}\left|\frac{1}{2}(x_1(t)+x_1(t-T))\right| \\ &= \max_{t\in[\eta,\eta+T]}\left|\frac{1}{2}\left(\left(x_1(\eta)+\int_\eta^t x_1'(s)ds\right)+\left(x_1(\eta)-\int_{t-T}^\eta x_1'(s)ds\right)\right)\right| \\ &\le \max_{t\in[\eta,\eta+T]}\left\{d_2+\frac{1}{2}\left(\int_\eta^t|x_1'(s)|ds+\int_{t-T}^\eta|x_1'(s)|ds\right)\right\} \\ &\le d_2+\frac{1}{2}\int_0^T|x_1'(s)|ds. \end{aligned} \tag{3.5}$$

On the other hand, from $(Ax_1)(t) = x_1(t) - c(t)x_1(t-\tau(t))$, we obtain

$$\begin{aligned}(Ax_1)''(t) &= (x_1(t) - c(t)x_1(t-\tau(t)))'' \\ &= (x_1'(t) - c'(t)x_1(t-\tau(t)) - c(t)x_1'(t-\tau(t)) + c(t)x_1'(t-\tau(t))\tau'(t))' \\ &= x_1''(t) - c(t)x_1''(t-\tau(t)) - [c''(t)x_1(t-\tau(t)) \\ &\quad + (2c'(t) - 2c'(t)\tau'(t) - c(t)\tau''(t))x_1'(t-\tau(t)) \\ &\quad + (c(t)(\tau'(t))^2 - 2c(t)\tau'(t))x_1''(t-\tau(t))].\end{aligned} \tag{3.6}$$

Furthermore, the above equation implies

$$\begin{aligned}(Ax_1'')(t) = (Ax_1)''(t) &+ c''(t)x_1(t-\tau(t)) + (2c'(t) - 2c'(t)\tau'(t) - c(t)\tau''(t))u_1'(t-\tau(t)) \\ &+ (c(t)(\tau'(t))^2 - 2c(t)\tau'(t))u_1''(t-\tau(t)).\end{aligned}$$

In view of $c^* < 1$, using Lemma 2.1, it is clear that

$$\begin{aligned}\|x_1''\| &= \max_{t\in[0,T]} \left|A^{-1}Ax_1''(t)\right| \\ &\le \frac{\max_{t\in[0,T]} |Ax_1''(t)|}{1-c^*} \\ &\le \frac{\phi_q(\|x_2\|) + c_2\|x_1\| + (2c_1 + 2c_1\tau_1 + c^*\tau_2)\|x_1'\| + (c^*\tau_1^2 + 2c^*\tau_1)\|x_1''\|}{1-c^*}.\end{aligned} \tag{3.7}$$

From equation (3.5), applying the Hölder inequality and the Wirtinger inequality (see [19, Lemma 2.4]), we deduce

$$\begin{aligned}\|x_1\| &\le d_2 + \frac{1}{2}\int_0^T |u_1'(t)|dt \\ &\le d_2 + \frac{T^{\frac{1}{2}}}{2}\left(\int_0^T |x_1'(t)|^2 dt\right)^{\frac{1}{2}} \\ &\le d_2 + \frac{T^{\frac{3}{2}}}{4\pi}\left(\int_0^T |x_1''(t)|^2 dt\right)^{\frac{1}{2}} \\ &\le d_2 + \frac{T^2}{4\pi}\|x_1''\|.\end{aligned} \tag{3.8}$$

According to $x_1(0) = x_1(T)$, there exists a point $t_1 \in (0,T)$ such that $x_1'(t_1) = 0$. From equation (3.5), we get

$$\|x_1'\| \le x_1'(t_1) + \frac{1}{2}\int_0^T |x_1''(t)|dt \le \frac{T}{2}\|x_1''\|. \tag{3.9}$$

Substituting equations (3.8) and (3.9) into (3), we obtain

$$\begin{aligned}\|x_1''\| &\le \frac{\phi_q(\|x_2\|) + c_2\left(d_2 + \frac{T^2}{4\pi}\|x_1''\|\right) + \frac{T}{2}(2c_1 + 2c_1\tau_1 + c^*\tau_2)\|x_1''\| + (c^*\tau_1^2 + 2c^*\tau_1)\|x_1''\|}{1-c^*} \\ &\le \frac{\phi_q(\|x_2\|) + \left(\frac{T^2}{4\pi}c_2 + Tc_1 + Tc_1\tau_1 + \frac{T}{2}c^*\tau_2 + c^*\tau_1^2 + 2c^*\tau_1\right)\|x_1''\| + c_2 d_2}{1-c^*}.\end{aligned}$$

Since $1-c^*-\left(\frac{T^2}{4\pi}c_2+Tc_1+\frac{T}{2}c_1\tau_1+Tc^*\tau_2+c^*\tau_1^2+2c^*\tau_1\right)>0$, we get

$$\|x_1''\|\leq\frac{\phi_q(\|x_2\|)+c_2d_2}{1-c^*-\left(\frac{T^2}{4\pi}c_2+Tc_1+Tc_1\tau_1+\frac{T}{2}c^*\tau_2+c^*\tau_1^2+2c^*\tau_1\right)}. \tag{3.10}$$

In view of

$$\int_0^T(\phi_q(x_2(t))dt=\int_0^T(Ax_1(t))''(t)dt=0,$$

there exists a point $t_2\in(0,T)$ such that $x_2(t_2)=0$. Applying the Hölder inequality, the Wirtinger inequality and equation (3.5), it is easy to verify that

$$\begin{aligned}\|x_2\|\leq&\frac{1}{2}\int_0^T|x_2'(t)|dt\\\leq&\frac{\sqrt{T}}{2}\left(\int_0^T|x_2'(t)|^2\right)^{\frac{1}{2}}\\\leq&\frac{\sqrt{T}}{2}\left(\frac{T}{2\pi}\right)^{n-4}\left(\int_0^T|x_2^{(n-2)}(t)|^2dt\right)^{\frac{1}{2}}\\\leq&\frac{T}{2}\left(\frac{T}{2\pi}\right)^{n-4}\|x_2^{(n-2)}\|.\end{aligned} \tag{3.11}$$

Besides, from $x_2^{(n-4)}(0)=x_2^{(n-4)}(T)$, there exists a point $t_3\in(0,T)$ such that $x_2^{(n-3)}(t_3)=0$. From the second equation of (3.1), equation (3.5) and condition (H_1), we get

$$\begin{aligned}2\|x_2^{(n-2)}\|\leq&2\left(x_2^{n-3}(t_3)+\frac{1}{2}\int_0^T\left|x_2^{n-2}(t)\right|dt\right)\\=&\lambda\int_0^T|-f(t,x_1'(t))-g(t,x_1(t-\sigma(t)))+p(t)|dt\\\leq&\int_0^T|f(t,x_1'(t))|dt+\int_0^T|g(t,x_1(t-\sigma(t)))|dt+\int_0^T|p(t)|dt\\\leq&\int_0^T|g(t,x_1(t-\sigma(t)))|dt+NT+\|p\|T,\end{aligned} \tag{3.12}$$

where $\|p\|:=\max_{t\in[0,T]}|p(t)|$. From conditions (H_1), (H_3) and equation (3.3), we deduce

$$\begin{aligned}\int_0^T|g(t,x_1(t-\sigma(t)))|dt=&\int_{g(t,x_1)\geq0}g(t,x_1(t-\sigma(t)))dt-\int_{g(t,x_1)\leq0}g(t,x_1(t-\sigma(t)))dt\\=&2\int_{g(t,x_1)\geq0}g(t,x_1(t-\sigma(t)))dt+\int_0^Tf(t,x_1'(t))dt\\\leq&2\int_{g(t,x_1)\geq0}(ax_1^{p-1}(t-\sigma(t))+b)dt+\int_0^T|f(t,x_1'(t))|dt\\\leq&\frac{2a}{1-\sigma'}\int_0^T|x_1(t)|^{p-1}dt+2bT+NT,\end{aligned} \tag{3.13}$$

since $\sigma' < 1$. Substituting equations (3.8) and (3.13) into (3.12), we see that

$$\begin{aligned}
2\|x_2^{(n-2)}\| \leq & \frac{2a}{1-\sigma'}\int_0^T |x_1(t)|^{p-1}dt + 2bT + 2NT + \|p\|T \\
\leq & \frac{2Ta}{1-\sigma'}\|x_1\|^{p-1} + (2b+2N+\|p\|)T \\
\leq & \frac{2Ta}{1-\sigma'}\left(d_2 + \frac{T^2}{4\pi}\|x_1''\|\right)^{p-1} + N_1 \\
\leq & \frac{2Ta}{1-\sigma'}\left(\frac{T^2}{4\pi}\right)^{p-1}\left(1+\frac{4\pi d_2}{T^2\|x_1''\|}\right)^{p-1}\|x_1''\|^{p-1} + N_1,
\end{aligned} \tag{3.14}$$

where $N_1 := (2b+N+\|p\|)T$.

Next, we introduce a classical inequality. There exists a $\mu(p) > 0$, which is dependent on p only,

$$(1+x)^p \leq 1 + (1+p)x, \text{ for } x \in [0, \mu(p)].$$

Then, we consider the following two cases:

Case 1. If $\frac{4\pi d_2}{T^2\|x_1''\|} > \mu(p)$, then it is obvious that

$$\|x_1''\| < \frac{4\pi d_2}{T^2\mu(p)} := M_3'.$$

Case 2. If $\frac{4\pi d_2}{T^2\|x_1''\|} \leq \mu(p)$, substituting equations (3.8) and (3.9) into (3.14), we deduce

$$\begin{aligned}
\|x_2^{(n-2)}\| \leq & \frac{Ta}{1-\sigma'}\left(\frac{T^2}{4\pi}\right)^{p-1}\left(1+\frac{4\pi d_2(p-1)}{T^2\|x_1''\|}\right)\|x_1''\|^{p-1} + \frac{N_1}{2} \\
\leq & \frac{Ta}{1-\sigma'}\left(\frac{T^2}{4\pi}\right)^{p-1}\|x_1''\|^{p-1} + \frac{Tad_2(p-1)\left(\frac{T^2}{4\pi}\right)^{p-2}}{(1-\sigma')}\|x_1''\|^{p-2} + \frac{N_1}{2}.
\end{aligned} \tag{3.15}$$

Substituting (3.10) into (3.15), we arrive at

$$\begin{aligned}
\|x_2^{(n-2)}\| \leq & \frac{Ta\left(\frac{T^2}{4\pi}\right)^{p-1}\left(\phi_q(\|x_2\|)+c_2d_2\right)^{p-1}}{(1-\sigma')\left(1-c^*-\left(\frac{T^2}{4\pi}c_2+Tc_1+Tc_1\tau_1+\frac{Tc^*\tau_2}{2}+c^*\tau_1^2+2c^*\tau_1\right)\right)^{p-1}} \\
& + \frac{Tad_2(p-1)\left(\frac{T^2}{4\pi}\right)^{p-2}\left(\phi_q(\|x_2\|)+c_2d_2\right)^{p-2}}{(1-\sigma')\left(1-c^*-\left(\frac{T^2}{4\pi}c_2+Tc_1+Tc_1\tau_1+\frac{Tc^*\tau_2}{2}+c^*\tau_1^2+2c^*\tau_1\right)\right)^{p-2}} + \frac{N_1}{2}.
\end{aligned} \tag{3.16}$$

Combination of equations (3.16) and (3.11), we obtain

$$\begin{aligned}\|x_2\| \leq & \frac{T}{2}\left(\frac{T}{2\pi}\right)^{n-4}\|x_2^{(n-2)}\| \\ \leq & \frac{T}{2}\left(\frac{T}{2\pi}\right)^{n-4}\left[\frac{Ta\left(\frac{T^2}{4\pi}\right)^{p-1}\left(\phi_q(\|x_2\|)+c_2d_2\right)^{p-1}}{(1-\sigma')\left(1-c^*-\left(\frac{T^2}{4\pi}c_2+Tc_1+Tc_1\tau_1+\frac{Tc^*\tau_2}{2}+c^*\tau_1^2+2c^*\tau_1\right)\right)^{p-1}}\right. \\ & \left. +\frac{Tad_2(p-1)\left(\frac{T^2}{4\pi}\right)^{p-2}\left(\phi_q(\|x_2\|)+c_2d_2\right)^{p-2}}{(1-\sigma')\left(1-c^*-\left(\frac{T^2}{4\pi}c_2+Tc_1+Tc_1\tau_1+\frac{Tc^*\tau_2}{2}+c^*\tau_1^2+2c^*\tau_1\right)\right)^{p-2}}+\frac{N_1}{2}\right].\end{aligned}$$

Furthermore, the above equation implies

$$\begin{aligned}\|x_2\| \leq & \frac{T}{2}\left(\frac{T}{2\pi}\right)^{n-4}\left[\frac{Ta\left(\frac{T^2}{4\pi}\right)^{p-1}\left(\|x_2\|+(p-1)\|x_2\|^{2-q}c_2d_2+\cdots+(c_2d_2)^{p-1}\right)}{(1-\sigma')\left(1-c^*-\left(\frac{T^2}{4\pi}c_2+Tc_1+Tc_1\tau_1+\frac{Tc^*\tau_2}{2}+c^*\tau_1^2+2c^*\tau_1\right)\right)^{p-1}}\right. \\ & \left. +\frac{Tad_2(p-1)\left(\frac{T^2}{4\pi}\right)^{p-2}\left(\|x_2\|^{2-q}+\cdots+(c_2d_2)^{p-2}\right)}{(1-\sigma')\left(1-c^*-\left(\frac{T^2}{4\pi}c_2+Tc_1+Tc_1\tau_1+\frac{Tc^*\tau_2}{2}+c^*\tau_1^2+2c^*\tau_1\right)\right)^{p-2}}+\frac{N_1}{2}\right].\end{aligned}$$

Since $p \geq 2$ and

$$\frac{aT^{p+1}\left(\frac{T}{2\pi}\right)^{n+p-5}}{2^p(1-\sigma')\left(1-c^*-\left(\frac{T^2}{4\pi}c_2+Tc_1+Tc_1\tau_1+\frac{Tc^*\tau_2}{2}+c^*\tau_1^2+2c^*\tau_1\right)\right)^{p-1}}<1,$$

we have that there exists a positive constant M_3'' (independent of λ) such that

$$\|x_2\| \leq M_3''.$$

Take $M_3 := \max\{M_3', M_3''\}$. Therefore, we have

$$\|x_2\| \leq M_3. \tag{3.17}$$

Substituting equations (3.17) into (3.10), it is clear that

$$\|x_1''\| \leq \frac{M_3^{q-1}+c_2d_2}{1-c^*-\left(\frac{T^2}{4\pi}c_2+Tc_1+Tc_1\tau_1+\frac{T}{2}c^*\tau_2+c^*\tau_1^2+2c^*\tau_1\right)} := M_2^*.$$

From equation (3.8), it is easy to verify that

$$\|x_1\| \leq d_2+\frac{T^2}{4\pi}\|x_1''\| \leq d_2+\frac{T^2}{4\pi}M_2^* := M_1. \tag{3.18}$$

From equation (3.9), we see that

$$\|x_1'\| \leq \frac{T}{2}\|x_1''\| \leq \frac{T}{2}M_2^* := M_2. \tag{3.19}$$

On the other hand, from equation (3.16) and (3.17), we get

$$
\begin{aligned}
&\|x_2^{(n-2)}\| \\
&\leq \frac{Ta\left(\frac{T^2}{4\pi}\right)^{p-1}\left(M_3^{q-1}+c_2d_2\right)^{p-1}}{(1-\sigma')\left(1-c^*-\left(\frac{T^2}{4\pi}c_2+Tc_1+Tc_1\tau_1+\frac{Tc^*\tau_2}{2}+c^*\tau_1^2+2c^*\tau_1\right)\right)^{p-1}} \\
&+\frac{Tad_2(p-1)\left(\frac{T^2}{4\pi}\right)^{p-2}\left(M_3^{q-1}+c_2d_2\right)^{p-2}}{(1-\sigma')\left(1-c^*-\left(\frac{T^2}{4\pi}c_2+Tc_1+Tc_1\tau_1+\frac{Tc^*\tau_2}{2}+c^*\tau_1^2+2c^*\tau_1\right)\right)^{p-2}}+\frac{N_1}{2}:=M_4'.
\end{aligned}
\tag{3.20}
$$

From equation (3.11) and (3.20), we have

$$
\|x_2'\| \leq \frac{T}{2}\left(\frac{T}{2\pi}\right)^{n-5}\|x_2^{(n-2)}\| \leq \frac{T}{2}\left(\frac{T}{2\pi}\right)^{n-5} M_4' = M_4. \tag{3.21}
$$

Next, we claim that there exists a positive constant M_5 such that

$$
x_1(t) \geq M_5. \tag{3.22}
$$

In fact, equation (3.2) can be rewritten in the form

$$
(\phi_p(Ax_1)''(t))^{(n-2)}+\lambda^p f(t,x_1'(t))+\lambda^p(g_0(x_1(t-\sigma(t)))+g_1(t,x_1(t-\sigma(t)))=\lambda^p p(t). \tag{3.23}
$$

Let $\xi \in [0,T]$ be as in equation (3.4), for any $t \in [\xi,T]$. Multiplying both sides of equation (3.23) by $x_1'(t-\sigma(t))(1-\sigma'(t))$ and integrating on $[\xi,t]$, we deduce

$$
\begin{aligned}
\lambda^p\int_{x_1(\xi-\sigma(\xi))}^{x_1(t-\sigma(t))} g_0(v)dv =&\lambda^p\int_\xi^t g_0(x_1(s-\sigma(s)))x_1'(s-\sigma(s))(1-\sigma'(s))ds \\
=&-\int_\xi^t (\phi_p(Ax_1)''(s))^{(n-2)}x_1'(s-\sigma(s))(1-\sigma'(s))ds \\
&-\lambda^p\int_\xi^t f(t,x_1'(s))x_1'(s-\sigma(s))(1-\sigma'(s))ds \\
&-\lambda^p\int_\xi^t g_1(s,x_1(s-\sigma(s)))x_1'(s-\sigma(s))(1-\sigma'(s))ds \\
&+\lambda^p\int_\xi^t p(s)x_1'(s-\sigma(s))(1-\sigma'(s))ds.
\end{aligned}
\tag{3.24}
$$

By equations (3.2), (3.13), (3.18) and (3.19), we get

$$
\begin{aligned}
&\lambda^p \left| \int_{x_1(\xi-\sigma(\xi))}^{x_1(t-\sigma(t))} g_0(v)dv \right| \\
\leq &(1+\sigma')\|x_1'\| \int_0^T |(\phi_p(Ax_1)''(s))^{(n-2)}|ds + \lambda^p(1+\sigma')\|x_1'\| \int_0^T |f(s,x_1'(s))|ds \\
&+ \lambda^p(1+\sigma')\|x_1'\| \int_0^T |g_1(s,x_1(s-\sigma(s)))|ds + \lambda^p(1+\sigma')\|x_1'\| \int_0^T |p(s)|ds \\
\leq &\lambda^p(1+\sigma')M_2 \left(\int_0^T |f(s,x_1'(s)))|ds + \int_0^T |g(s,x_1(s-\sigma(s)))|ds + \int_0^T |p(s)|ds \right) \\
&+ \lambda^p(1+\sigma')(M_2NT + M_2\|g_{M_1}\|T + M_2\|p\|T) \\
\leq &\lambda^p(1+\sigma')M_2 \left(2NT + \frac{2aTM_1^{p-1}}{1-\sigma'} + 2Tb + \|p\|T \right) \\
&+ \lambda^p(1+\sigma')M_2(NT + \|g_{M_1}\|T + \|p\|T) \\
\leq &\lambda^p(1+\sigma')M_2T \left(3N + \frac{2aM_1^{p-1}}{1-\sigma'} + 2b + \|g_{M_1}\| + 2\|p\| \right),
\end{aligned}
\tag{3.25}
$$

where $g_{M_1} = \max_{0\leq x\leq M_1} |g_1(t,x)|$. From strong force condition (1.6), it is clear that there exists constant $M_5' > 0$ such that

$$x_1(t-\sigma(t)) \geq M_5', \ \forall\, t \in [\xi, T].$$

The case $t \in [0,\xi]$ (i.e. $u_1(t-\sigma(t) \in [-\sigma(0), \xi-\sigma(\xi)])$ can be treated similarly. Therefore, we obtain that equation (3.22) holds. Having in mind equations (3.17), (3.18), (3.19), (3.21) and (3.22), we define

$$\Omega_2 = \{x = (x_1,x_2)^\top \in X : E_5 \leq x_1(t) \leq E_1,\ \|x_1\| \leq E_2,\ \|x_2\| \leq E_3,\ \|x_2'\| \leq E_4,\ \forall\, t \in [0,T]\},$$

where $0 < E_5 < \min\{M_5,\ d_1\}$, $E_2 > \max\{M_1,\ d_2\}$, $E_j > M_j,\ j = 2,3,4$ and $\Omega = \{x : x \in \partial\Omega_2 \cap Ker\ L\}$. Then, $\forall x \in \partial\Omega \cap Ker\ L$,

$$QNx = \frac{1}{T} \int_0^T \begin{pmatrix} \phi_q(x_2(t)) \\ -f(t,x_1'(t)) - g(t,x_1(t-\sigma(t))) + p(t) \end{pmatrix} dt.$$

If $QNx = 0$, then $x_1 = E_1,\ x_2 = 0$. But if $x_1(t) = E_1$, we know

$$\int_0^T g(t,E_1)dt = 0.$$

From assumption (H_2), we have $x_1(t) \leq d_2 \leq E_1$, which yields a contradiction. We also have $QNx \neq 0$, i.e., $\forall\ x \in \partial\Omega \cap Ker\ L,\ x \notin Im\ L$. So, conditions (1) and (2) of Lemma 2.1 are both satisfied. Finally, we show that the (3) of Lemma 2.1 is also satisfied. In fact, from condition (H_2), we have

$$g(t,E_5) < 0 \text{ and } g(t,E_1) > 0.$$

So the condition (3) of Lemma 2.1 is satisfied. Applying Lemma 2.1, equation (1.5) has a positive T-periodic solution. □

Similarly, we get the existence of a positive periodic solution for equation (1.5) in the case that $c_* > 1$.

Theorem 3.2. *Assume that conditions $c_* > 1$ and $(H_1)-(H_3)$ hold. Then equation (1.5) has at least one positive T-periodic solution if the following condition hold:*

$$0 < \frac{aT^{p+1}\left(\frac{T}{2\pi}\right)^{n+p-5}}{2^p(1-\sigma')\left(c_* - 1 - \left(\frac{T^2}{4\pi}c_2 + Tc_1 + Tc_1\tau_1 + \frac{Tc^*\tau_2}{2} + c^*\tau_1^2 + 2c^*\tau_1\right)\right)^{p-1}} < 1,$$

Proof. We follow the same strategy and notation as in the proof of Theorem 3.1. From $c_* > 1$ and equation (3), applying Lemma 2.1, we see that

$$\begin{aligned}
\|x_1''\| &= \max_{t\in[0,T]} \left|A^{-1}Ax_1''(t)\right| \\
&\le \frac{\max_{t\in[0,T]} |Ax_1''(t)|}{c_* - 1} \\
&\le \frac{\phi_q(\|x_2\|) + c_2\|x_1\| + (2c_1 + 2c_1\tau_1 + c^*\tau_2)\|x_1'\| + (c^*\tau_1^2 + 2c^*\tau_1)\|x_1''\|}{c_* - 1} \\
&\le \frac{\phi_q(\|x_2\|) + \left(\frac{T^2}{4\pi}c_2 + Tc_1 + Tc_1\tau_1 + \frac{T}{2}c^*\tau_2 + c^*\tau_1^2 + 2c^*\tau_1\right)\|x_1''\| + c_2d_2}{c_* - 1}.
\end{aligned}$$

Since $c_* - 1 - \left(\frac{T^2}{4\pi}c_2 + Tc_1 + Tc_1\tau_1 + \frac{T}{2}c^*\tau_2 + c^*\tau_1^2 + 2c^*\tau_1\right) > 0$, we have

$$\|x_1''\| \le \frac{\phi_q(\|x_2\|) + c_2d_2}{c_* - 1 - \left(\frac{T^2}{4\pi}c_2 + Tc_1 + Tc_1\tau_1 + \frac{Tc^*\tau_2}{2} + c^*\tau_1^2 + 2c^*\tau_1\right)}.$$

Similarly, it is easy to verify that $\|x_2\| \le M_3$. The rest of the proof is the same as Theorem 3.1. $\square$

We next illustrate our results with an example.

Example 3.3. Consider the following 6-th singular differential equation with time-dependent deviating argument.

$$\begin{aligned}
&\left(\phi_p\left(x(t) - \frac{1}{64}\sin(4t)x\left(t - \frac{1}{32}\cos 4t\right)\right)''\right)^{(4)} + (\sin^2 2t + 3)\sin x'(t) \\
&\quad + \frac{1}{4\pi^2}(\cos 4t + 1)x^3\left(t - \frac{1}{16}\sin 4t\right) - \frac{1}{x^\kappa\left(t - \frac{1}{16}\sin 4t\right)} = 4\sin 4t,
\end{aligned} \tag{3.26}$$

where $\kappa \ge 1$ and $p = 4$ is a constant. It is clear that $T = \frac{\pi}{2}$, $n = 6$, $c(t) = \frac{1}{64}\sin 4t$, $\tau(t) = \frac{1}{32}\cos 4t$, $\sigma(t) = \frac{1}{16}\sin 4t$, $p(t) = \sin 4t$, $c_1 = \max_{t\in[0,T]}|\frac{1}{16}\cos 4t| = \frac{1}{16}$, $c_2 = \max_{t\in[0,T]}|-\frac{1}{4}\sin 4t| = \frac{1}{4}$, $\tau_1 = \max_{t\in[0,T]}|-\frac{1}{8}\sin 4t| = \frac{1}{8}$, $\tau_2 = \max_{t\in[0,T]}|-\frac{1}{2}\cos 4t| = \frac{1}{2}$, $\sigma' = \max_{t\in[0,T]}|\frac{1}{4}\cos 4t| = \frac{1}{4} < 1$, $g(t,x) = \frac{1}{4\pi^2}(\cos 4t + 1)x^3 - \frac{1}{x^\kappa}$, $a = \frac{1}{2\pi^2}$, $b = 1$, $f(t,v) = (\sin^2 2t + 3)\sin v$, where $N = 4$. It is obviously that $(H_1)-(H_3)$ hold. Now we consider the assumption condition

$$\begin{aligned}
&\frac{aT^{p+1}\left(\frac{T}{2\pi}\right)^{n+p-5}}{2^p(1-\sigma')\left(1-c^*-\left(\frac{T^2}{4\pi}c_2+Tc_1+Tc_1\tau_1+\frac{Tc^*\tau_2}{2}+c^*\tau_1^2+2c^*\tau_1\right)\right)^{p-1}}\\
&=\frac{\pi^5}{8\times 98304\times\left(1-\frac{1}{64}-\left(\frac{\pi}{64}+\frac{\pi}{32}+\frac{\pi}{256}+\frac{\pi}{512}+\frac{1}{64}\times\frac{1}{64}+\frac{1}{32}\times\frac{1}{8}\right)\right)^3}\\
&\approx 0.0868<1.
\end{aligned}$$

So, by Theorem 3.1, we obtain that equation (3.26) has at least one $\frac{\pi}{2}$ -periodic solution.

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