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SYNCHRONIZATION ON THE NON-AUTONOMOUS CELLULAR NEURAL NETWORKS WITH TIME DELAYS

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Abstract. This paper is concerned with a general decay synchronization (GDS) between two delayed non-autonomous cellular neural networks. A non-autonomous case and infinite delays are taken into consideration. By using the Lyapunov stability theory and employing useful inequality techniques, some sufficient conditions on the GDS of the considered system are established based on a type of nonlinear control. In addition, an example with numerical simulations is provided to demonstrate the effectiveness and feasibility of the obtained results.
Keywords. Cellular neural networks; General decay synchronization; Time-varying coefficients; Time delays.

1. Introduction

In the past few decades, the dynamical properties of neural networks (NNs) including stability, synchronization, bifurcation, periodic attractors, and chaotic attractors have attracted considerable attention since they have been successfully applied in many areas, such as, image processing, pattern recognition, associative memory, and optimization problems; see, e.g., [1]-[8] and the references therein. In particular, the research on the synchronization of NNs has arrested much attention ([9]-[15]) as there are many benefits of having synchronization or chaos synchronization in some engineering applications, such as, secure communication, information science, harmonic oscillation generation, language emergence and development [3]. In addition, the synchronization, as a typical collective behavior, has been observed in biological systems, such as, synchronous fireflies, swarming of fishes, flocking of birds [4]. Therefore, it is important to investigate the synchronization behaviors in neural networks.

It is well known that one of the central issues of the synchronization studies in neural networks is how to achieve the synchronization between drive-response systems. However, when

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we investigate the synchronization behaviors of the NNs, we should introduce time delay into the model foundation due to the fact that the time delay is inevitable and will occur during the signal transmission among the neurons, which will affect the stability of the neural system and may lead to some complex dynamic behaviors such as periodic oscillation, bifurcation, or chaos [5]. In recent years, as an important neural network model, the delayed neural networks (DNNs) have been studied in many books and papers (see, e.g., [16]-[25] and the references therein). Also, the synchronization study of delayed neural networks (DNNs) has attracted considerable attention of many researchers from various fields, and many important results were established. In these studies, many methods were exploited, such as, the Lyapunov function approach [1, 2], linear matrix inequality [3], matrix measure strategy and Lyapunov approach [4], Lyapunov theory and fractional-order differential inequalities [6], and generalized Halanay inequalities and matrix measure approaches [7]. And, several control strategies, such as, sampled-data control[1, 8, 9, 10], linear feedback control[2], pinning control[4, 11], impulsive control[12] and fuzzy control[13] are widely applied to controlling and synchronizing the DNNs.

In general, there are two types of neural networks in recent studies. The first type is autonomous NNs and the second type is non-autonomous NNs. The main difference between the autonomous NNs and the non-autonomous NNs is that the intrinsic parameters of autonomous NNs are constants and has no input effect. But the intrinsic parameters of the non-autonomous NNs are variables and have input effect. Up to now, the majority of existing results are devoted to the autonomous NNs [12]-[26], and there are few papers considered the non-autonomous NNs [27, 28]. In [27, 28], the authors considered the following non-autonomous cellular neural networks(CNNs) with time variable delays and infinite delays

$$\begin{aligned} \dot{x}_i(t) = &- c_i(t)x_i(t) + \sum_{j=1}^{n} a_{ij}(t) f_j(x_j(t-\tau_{ij}(t))) \\ &+ \sum_{j=1}^{n} b_{ij}(t) \int_0^{\infty} K_{ij}(u) g_j(x_j(t-u))du + I_i(t), \end{aligned} \tag{1.1}$$

The authors obtained some sufficient conditions on the exponential convergence for system (1.1) by using the differential inequality strategies. But, in [27, 28], the authors did not consider the estimation of exponential convergent rate. On the other hand, in the process of investigating neural networks, the estimate of the convergent rate of the synchronization is one of the major concerns and useful for studying the synchronization of chaotic systems [22]. However, in some cases, the convergence rate of the synchronization can not be obtained or it is very difficult to estimate. For example, consider the following equation [23]

$$\dot{x}(t) = -\frac{1}{2}x^3, \quad t \geq 0.$$

Although, we can find that the above equation is asymptotically stable, we can not be able to estimate the convergent rate of its solution. Recently, a new concept of the synchronization, namely, the general decay synchronization (GDS) was introduced for a class of chaotic NNs by Wang, Shen and Zhang [24, 25] and this GDS can deal with the above mentioned problem. There has been some literatures related to the study of the general decay synchronization for neural networks with time delays [21]-[26]. However, these models in [21]-[26] are all autonomous and without infinite delays. Besides, there is no theoretical result published on the research on the general decay synchronization for delayed cellular neural networks with time-varying coefficients and infinite delays.

Motive and inspired by the above results, in this paper, we extend system (1.1) to the more general case and investigate the following non-autonomous CNNs system with time variable delays and infinite delays

$$\begin{aligned} \dot{x}_i(t) = & -c_i(t)x_i(t) + \sum_{j=1}^{n} d_{ij}(t) f_j(x_j(t)) + \sum_{j=1}^{n} a_{ij}(t) f_j(x_j(t-\tau_{ij}(t))) \\ & + \sum_{j=1}^{n} b_{ij}(t) \int_0^{\infty} K_{ij}(u) g_j(x_j(t-u)) du + I_i(t). \end{aligned} \tag{1.2}$$

The main purpose of this paper is to establish some sufficient conditions on the general decay synchronization for system (1.2). The method used in this paper is mainly motivated by the work done by Wang, Shen and Zhang [24, 25] and the Lyapunov-Krasovskii function method.

The main contributions of this paper are:

1) The general non-autonomous CNNs system with time variable delays and infinite delays is considered.

2) Compared with previous related results, we, for the first time, consider the GDS problem for non-autonomous CNNs system with time variable delays and infinite delays, and derive some sufficient conditions, which guarantee the considered system to achieve GDS. In addition, we are able to estimate the convergence rate of the synchronization by GDS.

3) Since the general decay synchronization may be specialized as exponential synchronization, polynomial synchronization and logarithmic synchronization, the results presented in this paper can be seen as the improvement and extension of the previously known results.

2. Preliminaries

In system (1.2), $i \in \Pi \triangleq \{1, 2, \cdots, n\}$, n corresponds to the number of units in a neural network; $x_i(t)$ denotes the state vector of the ith unit at the time t; $c_i(t) > 0$ denotes the rate with which the ith unit will reset its potential to the resting state in isolation when disconnected from the network and external inputs at the time t; $a_{ij}(t)$, $b_{ij}(t)$ and $d_{ij}(t)$ denote the connection weights between the ith neuron and the jth neuron at time t; $f_i(\cdot)$ and $g_i(\cdot)$ are the nonlinear activation functions of signal transmission; $I_i(t)$ denotes the external bias on the ith unit at the time t; $K_{ij}(u)$ corresponds to the transmission delay kernel satisfying $\int_0^{\infty} K_{ij}(s) du = 1$, and $\tau_{ij}(t)$ denotes the transmission time-varying delays of the ith unit along the axon of the jth unit at the time t satisfying $0 \le \tau_{ij}(t) \le \tau_{ij}$.

In this paper, we always use $\Pi \triangleq \{1, 2, \cdots, n\}$ and $R^+ = [0, +\infty)$, unless otherwise stated. The initial conditions associated with system (1.2) are given by

$$x_i(s) = \varphi_i(s),\ s \in (-\infty, 0],\ i = 1, 2, \cdots, n,$$

where $\varphi(s) = (\varphi_1(s), \varphi_2(s), \cdots, \varphi_n(s)) \in C((-\infty, 0], R^n)$ denotes the Banach space of all continuous functions mapping $(-\infty, 0]$ into R^n with norm defined by

$$\|\varphi\| = \sup_{s \in [-\tau, 0]} \|\varphi(s)\|$$

with $\|\varphi(s)\| = \max_{i \in \Pi} |\varphi_i(s)|$.

For a continuous and bounded function $f(t)$, we define

$$f^- = \inf_{t\in R}\{f(t)\}$$

and

$$f^+ = \sup_{t\in R}\{f(t)\}.$$

Throughout this paper, we assume that the following assumptions are satisfied.

$\mathbf{H}_1$ Activation functions $f_j(u)$ and $g_j(u)$ are continuous and there exist nonnegative constants $H_j, K_j, M_j, O_j \geq 0$ such that, for any $v_1, v_2 \in R$,

$$|f_j(v_1) - f_j(v_2)| \leq H_j|v_1 - v_2| + M_j,\ |g_j(v_1) - g_j(v_2)| \leq K_j|v_1 - v_2| + O_j.$$

$\mathbf{H}_2$ Time-varying delay $\tau_{ij}(t)$ is differentiable, and there exists a real number $0 \leq \zeta_{ij} \leq 1$ such that, for any $t \in R^+$,

$$0 \leq \dot{\tau}_{ij}(t) \leq \zeta_{ij}.$$

In this paper, we consider system (1.2) as the drive system, and the response system is given as follows

$$\begin{aligned}\dot{y}_i(t) = &- c_i(t)y_i(t) + \sum_{j=1}^{n} d_{ij}(t)f_j(y_j(t)) + \sum_{j=1}^{n} a_{ij}(t)f_j(y_j(t-\tau_{ij}(t)))\\ &+ \sum_{j=1}^{n} b_{ij}(t)\int_0^{\infty} K_{ij}(u)g_j(y_j(t-u))du + I_i(t) + u_i(t),\end{aligned} \tag{2.1}$$

where $u_i(t)$ is the controller to be designed.

Let $e_i(t) = y_i(t) - x_i(t)$. From (1.2) and (2.1), the error dynamical system is expressed as

$$\begin{aligned}\dot{e}_i(t) = &- c_i(t)e_i(t) + \sum_{j=1}^{n} d_{ij}(t)\widetilde{f}_j(e_j(t)) + \sum_{j=1}^{n} a_{ij}(t)\widetilde{f}_j\left(e_j(t-\tau_{ij}(t))\right)\\ &+ \sum_{j=1}^{n} b_{ij}(t)\int_0^{\infty} K_{ij}(u)\widetilde{g}_j(e_j(t-u))du + u_i(t),\end{aligned} \tag{2.2}$$

where $\tilde{f}_j(e_j(t)) = f_j(y_j(t)) - f_j(x_j(t))$.

Now, we present the definitions of the ψ-type function and the GDS.

Definition 2.1. [24, 25] A function $\psi : R^+ \to [1, +\infty)$ is said to be $\psi-$type function if it satisfies the following conditions

1) It is differentiable and nondecreasing;

2) $\psi(0) = 1$ and $\psi(+\infty) = +\infty$;

3) $\tilde{\psi}(t) = \dot{\psi}(t)/\psi(t)$ is nondecreasing and $\psi^* = \sup_{t\geq 0}\tilde{\psi}(t) < +\infty$, where $\dot{\psi}(t)$ is the time derivative of $\psi(t)$;

4) For any $t,\ s \geq 0,\ \psi(t+s) \leq \psi(t)\psi(s)$.

It is not difficult to check that $\psi(t) = e^{\alpha t}$ and $\psi(t) = (1+t)^{\alpha}$, for any $\alpha > 0$, satisfy the above four conditions. Thus, they can be seen as ψ-type functions.

Definition 2.2. [24, 25] The drive-response systems (1.2) and (2.1) are said to be general decay synchronized if there exist a constant $\varepsilon > 0$ and a ψ-type function ψ such that, for any solutions $x(t) = (x_1(t), x_2(t), \cdots, x_n(t))$ of system (1.2) and $y(t) = (y_1(t), y_2(t), \cdots, y_n(t))$ of system (2.1),

$$\limsup_{t\to+\infty} \frac{\log\|y(t)-x(t)\|}{\log\psi(t)} \le -\varepsilon,$$

where $\varepsilon > 0$ can be seen as the convergence rate as synchronization error approaches zero.

$\mathbf{H}_3$: For the function $\psi(t)$ given in Definition 2.1, there exist a function $\rho(t) \in C(R, R^+)$ and a constant δ such that, for any $t \ge 0$,

$$\tilde{\psi}(t) \le 1, \qquad \sup_{t\in[0,+\infty)} \int_0^t \psi^{\delta}(s)\rho(s)ds < +\infty.$$

Now, we present a useful lemma, which is essential to our later study.

Lemma 2.3. [24, 25] *Under assumption $\mathbf{H}_3$, assume that the synchronization error $e(t) = y(t) - x(t)$ of driver-response systems (1.2) and (2.1) satisfies the differential equation $\dot{e}(t) = g(t, e_t)$, where $e_t = e(t+s)$ for $s \in [-\tau, 0]$, and function $g(t, e_t)$ is locally bounded. If there exist a differentiable functional $V(t, e_t) : R^+ \times C \to R^+$, and positive constants λ_1, λ_2 such that, for any $(t, e_t) \in R^+ \times C$,*

$$(\lambda_1\|e(t)\|)^2 \le V(t, e_t), \qquad \frac{dV(t,e_t)}{dt}\bigg|_{(4)} \le -\delta V(t, e_t) + \lambda_2\rho(t), \tag{2.3}$$

where $x(t)$ and $y(t)$ are solutions of systems (1.2) and (2.1) respectively, $\delta > 0$ and $\rho(t)$ are defined in $\mathbf{H}_3$. Then the driver-response systems (1.2) and (2.1) are general decay synchronized in the sense of Definition 2.2, and the convergence rate is $\delta/2$.

3. Main results

First, under assumption $\mathbf{H}_3$, we design the controller $u_i(t)$ of response system (2.1) as follows:

$$u_i(t) = -\alpha_i(t)\text{sign}(e_i(t)) - \frac{\beta_i(t)\|e(t)\|e_i(t)}{\|e(t)\|+\rho(t)}, \quad i \in \Pi, \tag{3.1}$$

where $\beta_i(t)$ and $\alpha_i(t)$ for $i \in \Pi$ are control gains satisfying

$$\begin{aligned} &E_i \triangleq c_i^- + \beta_i^- - \sum_{j=1}^{n} \Big(\frac{A_{ji}}{(1-\zeta_{ji})} + B_{ji}\int_0^{\infty} K_{ji}(u)du + D_{ji} + \tau_{ji}A_{ji}\Big) > 0, \\ &\alpha_i^- - \sum_{j=1}^{n} \big(M_i d_{ji}^+ + M_i a_{ji}^+ + O_i b_{ji}^+\big) > 0, \end{aligned} \tag{3.2}$$

where $D_{ij} = d_{ij}^+ H_j$, $A_{ij} = a_{ij}^+ H_j$ and $B_{ij} = b_{ij}^+ K_j$.

Theorem 3.1. *Let $\mathbf{H_1} - \mathbf{H_3}$ hold. Then the response network (2.1) can be general decay synchronized with the drive network (1.2) under nonlinear controller (3.1) if the control gains $\beta_i(t)$ and $\alpha_i(t)$ satisfy inequality (3.2) .*

Proof. First, we construct the following Lyapunov-Krasovskii functional

$$\begin{aligned} V_1(t) = & \sum_{i=1}^{n} |e_i(t)| + \sum_{i=1}^{n}\sum_{j=1}^{n} \int_{t-\tau_{ij}(t)}^{t} \frac{A_{ij}}{(1-\zeta_{ij})} |e_j(s)| ds \\ & + \sum_{i=1}^{n}\sum_{j=1}^{n} B_{ij} \int_0^{\infty} \int_{t-u}^{t} |e_j(s)| ds du. \end{aligned} \tag{3.3}$$

Calculating the derivative of $V_1(t)$ along system (2.2), we get

$$\begin{aligned} \dot{V}_1(t) = & \sum_{i=1}^{n} \text{sign}(e_i(t)) \Bigg\{ -c_i(t)e_i(t) + \sum_{j=1}^{n} d_{ij}(t)\widetilde{f}_j(e_j(t)) + \sum_{j=1}^{n} a_{ij}(t)\widetilde{f}_j\left(e_j(t-\tau_{ij}(t))\right) \\ & + \sum_{j=1}^{n} b_{ij}(t) \int_0^{\infty} K_{ij}(u)\widetilde{g}_j\left(e_j(t-u)\right) du - \alpha_i(t)\text{sign}(e_i(t)) - \frac{\beta_i(t)\|e(t)\|e_i(t)}{\|e(t)\|+\rho(t)} \Bigg\} \\ & + \sum_{i=1}^{n}\sum_{j=1}^{n} A_{ij} \left(\frac{1}{(1-\zeta_{ij})}|e_j(t)| - \frac{(1-\dot{\tau}_{ij}(t))}{(1-\zeta_{ij})}|e_j(t-\tau_{ij}(t))| \right) \\ & + \sum_{i=1}^{n}\sum_{j=1}^{n} B_{ij} \left[\int_0^{\infty} K_{ij}(u)\left(|e_j(t)| - |e_j(t-u)|\right) duds \right] \\ \le & \sum_{i=1}^{n} \Bigg\{ -c_i^-|e_i(t)| + \sum_{j=1}^{n} d_{ij}^+|\widetilde{f}_j(e_j(t))| + \sum_{j=1}^{n} a_{ij}^+|\widetilde{f}_j\left(e_j(t-\tau_{ij}(t))\right)| \\ & + \sum_{j=1}^{n} b_{ij}^+ \int_0^{\infty} K_{ij}(u)|\widetilde{g}_j(e_j(t-u))| du - \alpha_i(t) - \frac{\beta_i(t)\|e(t)\||e_i(t)|}{\|e(t)\|+\rho(t)} \\ & + \sum_{j=1}^{n} \frac{A_{ij}}{(1-\zeta_{ij})}|e_j(t)| - \sum_{j=1}^{n} A_{ij}|e_j(t-\tau_{ij}(t))| + \sum_{j=1}^{n} B_{ij} \int_0^{\infty} K_{ij}(u)du|e_j(t)| \\ & - \sum_{j=1}^{n} B_{ij} \int_0^{\infty} K_{ij}(u)|e_j(t-u)| du \Bigg\}. \end{aligned} \tag{3.4}$$

Now, using $\mathbf{H_1}$, we have

$$\begin{aligned} \sum_{i=1}^{n}\sum_{j=1}^{n} d_{ij}^+|\widetilde{f}_j(e_j(t))| & \le \sum_{i=1}^{n}\sum_{j=1}^{n} d_{ij}^+(H_j|e_j(t)| + M_j) \\ & = \sum_{i=1}^{n}\sum_{j=1}^{n} D_{ij}|e_j(t)| + \sum_{i=1}^{n}\sum_{j=1}^{n} M_j d_{ij}^+. \end{aligned} \tag{3.5}$$

Similarly, we have

$$\begin{aligned} \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}^+|\widetilde{f}_j(e_j(t-\tau_{ij}(t)))| & \le \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}^+(H_j|e_j(t-\tau_{ij}(t))| + M_j) \\ & = \sum_{i=1}^{n}\sum_{j=1}^{n} A_{ij}|e_j(t-\tau_{ij}(t))| + \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}^+ M_j, \end{aligned} \tag{3.6}$$

and

$$\begin{aligned}&\sum_{i=1}^{n}\sum_{j=1}^{n} b_{ij}^{+}\int_{0}^{\infty}K_{ij}(u)|\widetilde{g}_j(e_j(t-u))|du\\&\leq\sum_{i=1}^{n}\sum_{j=1}^{n} B_{ij}\int_{0}^{\infty}K_{ij}(u)|e_j(t-u)|du+\sum_{i=1}^{n}\sum_{j=1}^{n} b_{ij}^{+}O_j.\end{aligned}\tag{3.7}$$

Using $\mathbf{H_2}$, (3.2), (3.4)-(3.7), we have

$$\begin{aligned}\dot{V}_1(t)\leq&\sum_{i=1}^{n}\Bigg\{-c_i^{-}|e_i(t)|+\sum_{j=1}^{n}D_{ij}|e_j(t)|+\sum_{j=1}^{n}M_jd_{ij}^{+}+\sum_{j=1}^{n}A_{ij}|e_j(t-\tau_{ij}(t))|\\&+\sum_{j=1}^{n}a_{ij}^{+}M_j+\sum_{j=1}^{n}B_{ij}\int_{0}^{\infty}K_{ij}(u)|e_j(t-u)|du+\sum_{j=1}^{n}b_{ij}^{+}O_j-\alpha_i(t)-\frac{\beta_i(t)\|e(t)\||e_i(t)|}{\|e(t)\|+\rho(t)}\\&+\sum_{j=1}^{n}\frac{A_{ij}}{(1-\zeta_{ij})}|e_j(t)|-\sum_{j=1}^{n}A_{ij}|e_j(t-\tau_{ij}(t))|+\sum_{j=1}^{n}B_{ij}\int_{0}^{\infty}K_{ij}(u)du|e_j(t)|\\&-\sum_{j=1}^{n}B_{ij}\int_{0}^{\infty}K_{ij}(u)|e_j(t-u)|du\Bigg\}\\\leq&\sum_{i=1}^{n}\Bigg\{-\Bigg[c_i^{-}+\beta_i^{-}-\sum_{j=1}^{n}\Big(\frac{A_{ji}}{(1-\zeta_{ji})}+B_{ji}\int_{0}^{\infty}K_{ji}(u)du+D_{ji}\Big)\Bigg]|e_j(t)|\\&-\Bigg[\alpha_i^{-}-\sum_{j=1}^{n}\big(M_id_{ji}^{+}+M_ia_{ji}^{+}+O_ib_{ji}^{+}\big)\Bigg]+\beta_i(t)|e_i(t)|-\frac{\beta_i(t)\|e(t)\||e_i(t)|}{\|e(t)\|+\rho(t)}\Bigg\}\\\leq&\sum_{i=1}^{n}-\Bigg[c_i^{-}+\beta_i^{-}-\sum_{j=1}^{n}\Big(\frac{A_{ji}}{(1-\zeta_{ji})}+B_{ji}\int_{0}^{\infty}K_{ji}(u)du+D_{ji}\Big)\Bigg]|e_j(t)|\\&+\sum_{i=1}^{n}\frac{\beta_i^{+}\rho(t)|e_i(t)|}{\|e(t)\|+\rho(t)}.\end{aligned}\tag{3.8}$$

Next, we construct the following Lyapunov-Krasovskii functional:

$$V_2(t)=\sum_{i=1}^{n}\sum_{j=1}^{n}A_{ij}\int_{-\tau_{ij}}^{0}\int_{t+s}^{t}|e_j(\theta)|d\theta ds.$$

Calculating the derivative of $V_2(t)$, we get

$$\dot{V}_2(t)=\sum_{i=1}^{n}\sum_{j=1}^{n}\Bigg[A_{ij}\big(\tau_{ij}|e_j(t)|-\int_{t-\tau_{ij}}^{t}|e_j(s)|ds\big)\Bigg]=\sum_{i=1}^{n}\sum_{j=1}^{n}\tau_{ij}A_{ij}|e_j(t)|-A,\tag{3.9}$$

where

$$A=\sum_{i=1}^{n}\sum_{j=1}^{n}A_{ij}\int_{t-\tau_{ij}}^{t}|e_j(s)|ds.$$

Finally, we construct the following Lyapunov-Krasovskii functional:

$$V(t)=V_1(t)+V_2(t).$$

Then, there exists a scalar $\chi > 1$ such that

$$\sum_{i=1}^{n}|e_i(t)| \leq V(t) \leq \chi \sum_{i=1}^{n}|e_i(t)| + \frac{\chi}{E}A, \tag{3.10}$$

where $E = \min_{i\in\Pi}\{E_i\}$. Calculating the derivative of $V(t)$, and using (3.8) and (3.9), we get

$$\begin{aligned}\dot{V}(t) \leq & \sum_{i=1}^{n} -\Big[c_i^- + \beta_i^- - \sum_{j=1}^{n}\big(\frac{A_{ji}}{(1-\zeta_{ji})} + B_{ji}\int_0^\infty K_{ji}(u)du + D_{ji} + \tau_{ji}A_{ji}\big)\Big]|e_j(t)| \\ & + \sum_{i=1}^{n}\frac{\beta_i(t)\rho(t)\|e_i(t)\|}{\|e(t)\|+\rho(t)} - A \\ \leq & \sum_{i=1}^{n} -E_i|e_i(t)| + \max_{i\in\Pi}\{\beta_i^+\}\frac{\|e(t)\|\rho(t)}{\|e(t)\|+\rho(t)} - A. \end{aligned} \tag{3.11}$$

Letting $\beta = \max_{i\in\Pi}\{\beta_i^+\} > 0$, and using the inequality $0 \leq ab/(a+b) \leq a$ for any $a > 0$, $b > 0$, we have

$$\dot{V}(t) \leq \sum_{i=1}^{n} -E_i|e_i(t)| + \beta\rho(t) - A. \tag{3.12}$$

Taking a small enough δ such that $\delta\chi < E$, and using (3.10) and (3.12), we get

$$\begin{aligned}\frac{d}{dt}V(t) + \delta V(t) \leq & \sum_{i=1}^{n} -E_i|e_i(t)| + \beta\rho(t) - A + \delta\Big(\chi\sum_{i=1}^{n}|e_i(t)| + \frac{\chi}{E}A\Big) \\ \leq & (\delta\chi - E)\sum_{i=1}^{n}|e_i(t)| + \Big(\frac{\delta\chi}{E} - 1\Big)A + \beta\rho(t) \\ \leq & \beta\rho(t),\end{aligned}$$

which means that

$$\dot{V}(t) + \delta V(t) \leq \beta\rho(t). \tag{3.13}$$

Then, from Lemma 2.3, the drive-response systems (1.2) and (2.1) achieve GDS under the adaptive nonlinear controller (3.1). The convergence rate of $e(t)$ approaching zero is $\delta/2$. The proof is completed. □

If, in $\mathbf{H}_1$, $f_j(u), g_j(u)$ are globally Lipschitz, i.e., $M_j = O_j = 0$, then the $\mathbf{H}_1$ becomes

$\mathbf{H}_1^*$: $f_j(u), g_j(u)$ are globally Lipschitz continuous, i.e., there exist constants $H_j, K_j > 0$ such that

$$|f_j(v_1) - f_j(v_2)| \leq H_j|v_1 - v_2|, |g_j(v_1) - g_j(v_2)| \leq K_j|v_1 - v_2|,$$

where $v_1, v_2 \in R$.

In addition, the controller (3.1) in system (1.2) becomes

$$u_i(t) = -\frac{\beta_i(t)\|e(t)\|e_i(t)}{e_i(t)+\rho(t)}, \quad i \in \Pi. \tag{3.14}$$

From the proof of Theorem 3.1 and $\mathbf{H}_1^*$, we have the following result.

Theorem 3.2. *Let* $\mathbf{H}_1^*, \mathbf{H}_2, \mathbf{H}_3$ *hold. Then the response network (2.1) can be general decay synchronized with the drive network (1.2) under the nonlinear controller (3.14) if the control gains* $\beta_i(t)$ *satisfy the following inequality*

$$c_i^- + \beta_i^- - \sum_{j=1}^{n} \Big(\frac{A_{ji}}{(1-\zeta_{ji})} + B_{ji} \int_0^\infty K_{ji}(u)du + D_{ji} + \tau_{ji} A_{ji} \Big) > 0.$$

Proof. First, we construct the following Lyapunov-Krasovskii functional:

$$\begin{aligned} V_1(t) =& \sum_{i=1}^{n} |e_i(t)| + \sum_{i=1}^{n}\sum_{j=1}^{n} \int_{t-\tau_{ij}(t)}^{t} \frac{A_{ij}}{(1-\zeta_{ij})} |e_j(s)| ds \\ &+ \sum_{i=1}^{n}\sum_{j=1}^{n} B_{ij} \int_0^\infty \int_{t-u}^{t} |e_j(s)| ds du. \end{aligned} \tag{3.15}$$

Calculating the derivative of $V_1(t)$ along system (2.2), we get

$$\begin{aligned} \dot{V}_1(t) \le& \sum_{i=1}^{n} \Bigg\{ -c_i^- |e_i(t)| + \sum_{j=1}^{n} d_{ij}^+ |\widetilde{f}_j(e_j(t))| + \sum_{j=1}^{n} a_{ij}^+ |\widetilde{f}_j\left(e_j(t-\tau_{ij}(t))\right)| \\ &+ \sum_{j=1}^{n} b_{ij}^+ \int_0^\infty K_{ij}(u) |\widetilde{g}_j(e_j(t-u))| du - \alpha_i(t) - \frac{\beta_i(t)\|e(t)\| |e_i(t)|}{\|e(t)\| + \rho(t)} \\ &+ \sum_{j=1}^{n} \frac{A_{ij}}{(1-\zeta_{ij})} |e_j(t)| - \sum_{j=1}^{n} A_{ij} |e_j(t-\tau_{ij}(t))| + \sum_{j=1}^{n} B_{ij} \int_0^\infty K_{ij}(u) du |e_j(t)| \\ &- \sum_{j=1}^{n} B_{ij} \int_0^\infty K_{ij}(u) |e_j(t-u)| du \Bigg\}. \end{aligned} \tag{3.16}$$

Now, using $\mathbf{H_1}$, we have

$$\sum_{i=1}^{n}\sum_{j=1}^{n} d_{ij}^+ |\widetilde{f}_j(e_j(t))| \le \sum_{i=1}^{n}\sum_{j=1}^{n} d_{ij}^+ H_j |e_j(t)| = \sum_{i=1}^{n}\sum_{j=1}^{n} D_{ij} |e_j(t)|. \tag{3.17}$$

Similarly, we have

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}^+ |\widetilde{f}_j(e_j(t-\tau_{ij}(t)))| \le \sum_{i=1}^{n}\sum_{j=1}^{n} A_{ij} |e_j(t-\tau_{ij}(t))|. \tag{3.18}$$

and

$$\sum_{i=1}^{n}\sum_{j=1}^{n} b_{ij}^+ \int_0^\infty K_{ij}(u) |\widetilde{g}_j(e_j(t-u))| du \le \sum_{i=1}^{n}\sum_{j=1}^{n} B_{ij} \int_0^\infty K_{ij}(u) |e_j(t-u)| du. \tag{3.19}$$

Using $\mathbf{H_2}$, and (3.14)-(3.19), we have

$$\begin{aligned} \dot{V}_1(t) \le& \sum_{i=1}^{n} - \Bigg[c_i^- + \beta_i^- - \sum_{j=1}^{n} \Big(\frac{A_{ji}}{(1-\zeta_{ji})} + B_{ji} \int_0^\infty K_{ji}(u) du + D_{ji} \Big) \Bigg] |e_j(t)| \\ &+ \sum_{i=1}^{n} \frac{\beta_i^+ \rho(t) |e_i(t)|}{\|e(t)\| + \rho(t)}. \end{aligned} \tag{3.20}$$

Next, we construct the following Lyapunov-Krasovskii functional:

$$V_2(t) = \sum_{i=1}^{n}\sum_{j=1}^{n} A_{ij} \int_{-\tau_{ij}}^{0}\int_{t+s}^{t} |e_j(\theta)| d\theta ds.$$

Calculating the derivative of $V_2(t)$, we get

$$\dot{V}_2(t) = \sum_{i=1}^{n}\sum_{j=1}^{n}\left[A_{ij}\big(\tau_{ij}|e_j(t)| - \int_{t-\tau_{ij}}^{t}|e_j(s)|ds\big)\right] = \sum_{i=1}^{n}\sum_{j=1}^{n}\tau_{ij}A_{ij}|e_j(t)| - A, \tag{3.21}$$

where

$$A = \sum_{i=1}^{n}\sum_{j=1}^{n} A_{ij}\int_{t-\tau_{ij}}^{t}|e_j(s)|ds.$$

Finally, we construct the following Lyapunov-Krasovskii functional $V(t) = V_1(t) + V_2(t)$. Then, there exists a scalar $\chi > 1$ such that

$$\sum_{i=1}^{n}|e_i(t)| \leq V(t) \leq \chi\sum_{i=1}^{n}|e_i(t)| + \frac{\chi}{E}A, \tag{3.22}$$

where $E = \min_{i\in\Pi}\{E_i\}$. Calculating the derivative of $V(t)$, and using (3.20)-(3.21), we get

$$\begin{aligned}
\dot{V}(t) \leq & \sum_{i=1}^{n} -\left[c_i^- + \beta_i^- - \sum_{j=1}^{n}\big(\frac{A_{ji}}{(1-\zeta_{ji})} + B_{ji}\int_0^\infty K_{ji}(u)du + D_{ji} + \tau_{ji}A_{ji}\big)\right]|e_j(t)| \\
& + \sum_{i=1}^{n}\frac{\beta_i(t)\rho(t)\|e_i(t)\|}{\|e(t)\| + \rho(t)} - A \\
\leq & \sum_{i=1}^{n} -E_i|e_i(t)| + \max_{i\in\Pi}\{\beta_i^+\}\frac{\|e(t)\|\rho(t)}{\|e(t)\| + \rho(t)} - A.
\end{aligned} \tag{3.23}$$

Letting $\beta = \max_{i\in\Pi}\{\beta_i^+\} > 0$ and using the inequality $0 \leq ab/(a+b) \leq a$ for any $a > 0$, $b > 0$, we have

$$\dot{V}(t) \leq \sum_{i=1}^{n} -E_i|e_i(t)| + \beta\rho(t) - A. \tag{3.24}$$

Now taking a small enough δ such that $\delta\chi < E$, we conclude from (3.22) and (3.24) that

$$\begin{aligned}
\frac{d}{dt}V(t) + \delta V(t) \leq & \sum_{i=1}^{n} -E_i|e_i(t)| + \beta\rho(t) - A + \delta\left(\chi\sum_{i=1}^{n}|e_i(t)| + \frac{\chi}{E}A\right) \\
\leq & (\delta\chi - E)\sum_{i=1}^{n}|e_i(t)| + \left(\frac{\delta\chi}{E} - 1\right)A + \beta\rho(t) \\
\leq & \beta\rho(t),
\end{aligned}$$

which means that $\dot{V}(t) + \delta V(t) \leq \beta\rho(t)$. Then, from Lemma 2.3, the drive-response systems (1.2) and (2.1) achieve GDS under the adaptive nonlinear controller (3.14). The convergence rate of $e(t)$ approaching zero is $\delta/2$. The proof is completed. □

Remark 3.3. [25] The function ψ was used as the decay function. so, the ψ-type stability is also said to be stability with general decay rate. If $\psi(t) = e^{\alpha t}$, $\psi(t) = (1+t)^\alpha$ and $\psi(t) = 1 + \alpha\log(1+t)$ for any $\alpha > 0$, then the ψ-type stability can be specialized as exponential synchronization, polynomial synchronization and logarithmic synchronization, respectively. In this paper, if $\rho(t) = 0$, which is given in controller (3.1), then the GDS can be specialized as exponential synchronization.

4. Numerical Simulations

In this section we give one example to illustrate the results obtained in this paper.

Example 4.1. For $n = 2$, we consider the following delayed cellular neural networks with time-varying coefficients

$$\begin{aligned} \dot{x}_i(t) = &- c_i(t)x_i(t) + \sum_{j=1}^{2} d_{ij}(t)f_j(x_j(t)) + \sum_{j=1}^{2} a_{ij}(t)f_j(x_j(t-\tau_{ij}(t))) \\ &+ \sum_{j=1}^{2} b_{ij}(t)\int_0^{\infty} K_{ij}(u)g_j(x_j(t-u))du + I_i(t), \end{aligned} \tag{4.1}$$

where $f_1(x) = f_2(x) = \tanh(x),\ g_1(t) = g_2(t) = \tanh(x) - \sin(x)$ and

$$\begin{bmatrix} c_1(t) \\ c_2(t) \end{bmatrix} = \begin{bmatrix} \dfrac{11+sin(t)}{10} \\ \dfrac{115+15sin(t)}{100} \end{bmatrix}, \quad \begin{bmatrix} d_{11}(t) & d_{12}(t) \\ d_{21}(t) & d_{22}(t) \end{bmatrix} = \begin{bmatrix} 2+0.01sin(t) & -0.11+0.001sin(t) \\ -2.5+0.01sin(t) & 3.5+0.01sin(t) \end{bmatrix},$$

$$\begin{bmatrix} I_1(t) \\ I_2(t) \end{bmatrix} = \begin{bmatrix} \dfrac{1+sin(t)}{1000} \\ \dfrac{2+1.5sin(t)}{1000} \end{bmatrix}, \quad \begin{bmatrix} a_{11}(t) & a_{12}(t) \\ a_{21}(t) & a_{22}(t) \end{bmatrix} = \begin{bmatrix} -1.6+0.001sin(t) & -0.1+0.001sin(t) \\ -0.18+0.001sin(t) & -2.4+0.01sin(t) \end{bmatrix},$$

$$\begin{bmatrix} K_{11}(u) & K_{12}(u) \\ K_{21}(u) & K_{22}(u) \end{bmatrix} = \begin{bmatrix} e^u & e^u \\ e^u & e^u \end{bmatrix}, \quad \begin{bmatrix} \tau_{11}(t) & \tau_{12}(t) \\ \tau_{21}(t) & \tau_{22}(t) \end{bmatrix} = \begin{bmatrix} \dfrac{e^t}{1.5+e^t} & \dfrac{e^t}{1.4+e^t} \\ \dfrac{e^t}{1.4+e^t} & \dfrac{e^t}{1.5+e^t} \end{bmatrix},$$

$$\begin{bmatrix} b_{11}(t) & b_{12}(t) \\ b_{21}(t) & b_{22}(t) \end{bmatrix} = \begin{bmatrix} 0.2+0.01sin(t) & 0.01+0.001sin(t) \\ -0.2+0.001sin(t) & 0.16+0.01sin(t) \end{bmatrix}.$$

The corresponding response system is described by

$$\begin{aligned} \dot{y}_i(t) = &- c_i(t)y_i(t) + \sum_{j=1}^{2} d_{ij}(t)f_j(y_j(t)) + \sum_{j=1}^{2} a_{ij}(t)f_j(y_j(t-\tau_{ij}(t))) \\ &+ \sum_{j=1}^{2} b_{ij}(t)\int_0^{\infty} K_{ij}(u)g_j(y_j(t-u))du + I_i(t) + u_i(t), \end{aligned} \tag{4.2}$$

where $c_i(t),\ a_{ij}(t),\ b_{ij}(t), d_{ij}(t), f_j(t), g_j(t),\ \tau_{ij}(t)$ and $I_i(t)$ are the same as in system (4.1).

The numerical simulations of the system (4.1) and the system (4.2) with initial values $x_1(s) = 0.2$, $x_2(s) = 0.5$ and $y_1(s) = -1.3$, $y_2(s) = 2.1$ for $s \in [-1, 0]$ are represented in

From Fig.4.1 and Fig.4.2, we can see that system (4.1) and system (4.2) have chaotic attractors.

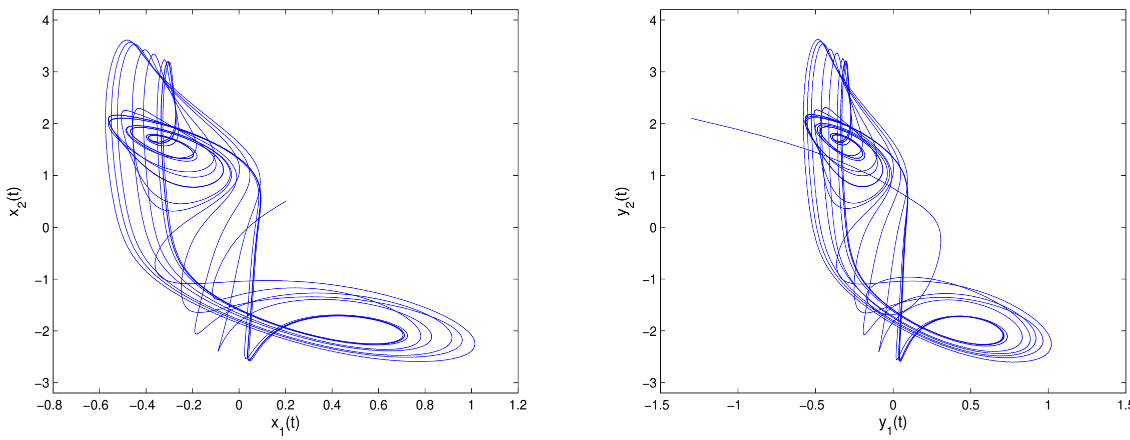

Fig. 4.1. The chaotic behavior of delayed cellular neural network system (4.1) and (4.2).

The nonlinear controller $u_i(t)$ is designed as follows

$$u_i(t) = -\alpha_i(t)\text{sign}(e_i(t)) - \frac{\beta_i(t)\|e(t)\|e_i(t)}{\|e(t)\| + \rho(t)}, \quad i \in \Pi, \tag{4.3}$$

where $e_i(t) = y_i(t) - x_i(t)$ for $i = 1, 2$.

It is easy to estimate that $H_j = K_j = 1$, $M_j = 0.045, O_j = 0.035$ and $\tau_{ij} = 1$. Thus, the assumptions $\mathbf{H_1}$ and $\mathbf{H_2}$ are satisfied. Let $\rho(t) = e^{-0.1t}$ and choose $\alpha_1^- = 0.5, \alpha_2^- = 0.6$, $\beta_1^- = 7.5, \beta_2^- = 6$. Then, the assumption $\mathbf{H_3}$ and the inequality (3.2) of Theorem 3.1 are satisfied. Therefore, according to Theorem 3.1, the drive-response systems (4.1) and (4.2) can be achieved by GDS under the controller (4.3). The time evolution of synchronization errors between systems (4.1) and (4.2) are demonstrated in Fig.4.2. The synchronization curves between systems (4.1) and (4.2) are shown in Fig.4.3.

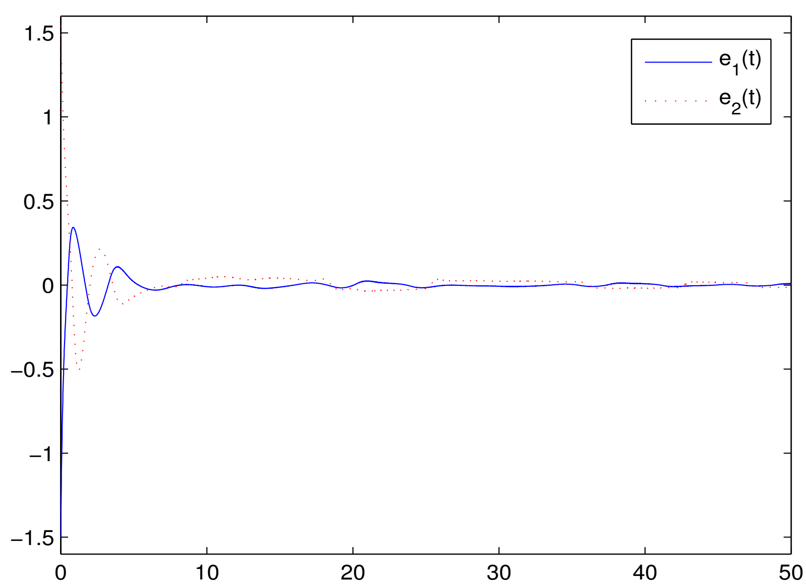

Fig.4.2. The evaluation of synchronization error $e_1(t)$ and $e_2(t)$.

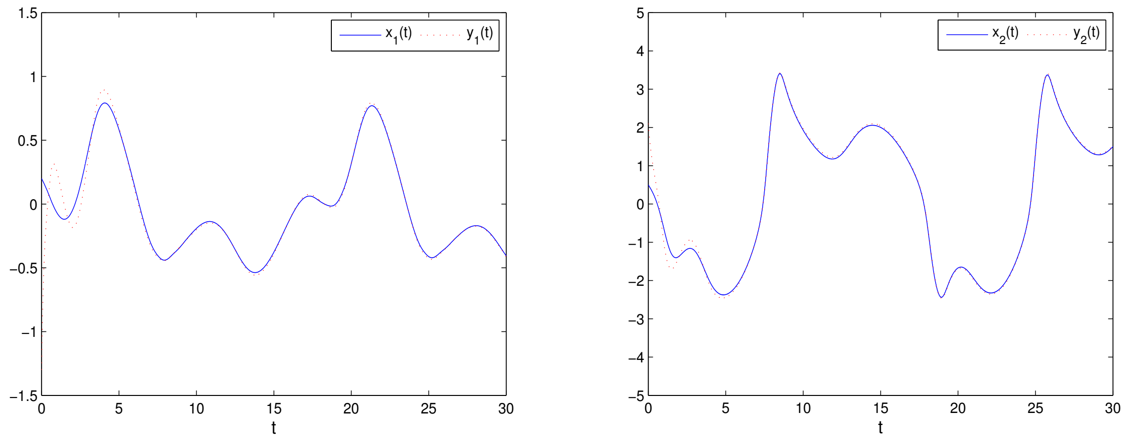

Fig.4.3. Synchronization curves of $x_1(t)$, $y_1(t)$ and $x_2(t)$, $y_2(t)$.

5. THE CONCLUSION

In the present analysis, to the best knowledge of the authors, this is the first result on the GDS problem for a class of general non-autonomous cellular neural networks with time variable delays and infinite delays. By constructing suitable Lyapunov-Krasovskii functionals and applying the method given in [24, 25], we obtained some new sufficient conditions on the general decay synchronization of the drive-response systems (1.2) and (2.1). In addition, an example and its numerical simulations are provided to validate the correctness of the theoretical results in this paper. In comparison to previous results presented in [21]-[28], results of this paper are less conservative and more general. Moreover, the theoretical results in this paper can be seen as a complement and an extension to the previous works [21]-[28].

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