INFINITELY MANY FAST HOMOCLINIC ORBITS FOR A CLASS OF SUPERQUADRATIC DAMPED VIBRATION SYSTEMS

MOHSEN TIMOUMI

Department of Mathematics, Faculty of Sciences, Monastir University, Tunisia

Abstract. In this paper, we consider the following damped vibration system $\ddot{u}(t) + q(t)\dot{u}(t) - L(t)u(t) + \nabla W(t,u(t)) = 0$, $\forall t \in \mathbb{R}$, where $q \in C(\mathbb{R},\mathbb{R})$, $L \in C(\mathbb{R},\mathbb{R}^{N^2})$ is a symmetric matrix valued function and $W(t,x) \in C^1(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$. We prove the existence of infinitely many fast homoclinic solutions for the system when $Q(t) = \int_0^t q(s)ds \to +\infty$ as $|t| \to \infty$, L is neither coercive nor uniformly positive definite and $W(t,x)$ is superquadratic at infinity in the second variable but does not satisfy the well-known superquadratic growth conditions like the Ambrosetti-Rabinowitz or the Fei's conditions.

Keywords. Damped vibration systems; Fast homoclinic solutions; Variational methods; Variant Fountain Theorem.

1. Introduction

In this paper, we consider the following damped vibration system

$$(\mathscr{DV}) \qquad\qquad \ddot{u}(t) + q(t)\dot{u}(t) - L(t)u(t) + \nabla W(t,u(t)) = 0, \ \forall t \in \mathbb{R}$$

where $q : \mathbb{R} \to \mathbb{R}$ is a continuous function, $L \in C(\mathbb{R},\mathbb{R}^{N^2})$ is a symmetric matrix-valued function and $W : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a continuous function, differentiable in the second variable with continuous derivative $\nabla W(t,x) = \frac{\partial W}{\partial x}(t,x)$. If $q(t) = 0$ for all $t \in \mathbb{R}$, then $(\mathscr{DV})$ is just the following second order Hamiltonian system

$$(\mathscr{HS}) \qquad\qquad \ddot{u}(t) - L(t)u(t) + \nabla W(t,u(t)) = 0, \ \forall t \in \mathbb{R}.$$

As a special case of dynamical systems, Hamiltonian systems play an important role in practical problems concerning relativistic mechanics, gaz dynamics, nuclear physics, fluid mechanics and others. A solution u of $(\mathscr{HS})$ is said to be homoclinic (to 0) if $u(t) \to 0$ and $\dot{u}(t) \to 0$ as $|t| \to \infty$. u is nontrivial if $u \neq 0$. With the aids of variational methods and critical point theory, the existence and the multiplicity of homoclinic orbits for $(\mathscr{HS})$ have been extensively investigated recently. Let us briefly comment some known results on homoclinic solutions for

---

$(\mathscr{HS})$. Under the assumptions that $L(t)$ and $W(t,x)$ are independent of t or $T-$periodic in t, the existence of homoclinic solutions for Hamiltonian system $(\mathscr{HS})$ was extensively studied, see, for example, [1, 2, 3, 4, 5, 6]. In this case, the existence of homoclinic solutions can be obtained by taking the limit of $2kT$-periodic solutions of approximating problems. The periodic assumptions are very important in the study of homoclinic orbits for $(\mathscr{HS})$ since the periodicity is used to control the lack of compactness due to the fact that $(\mathscr{HS})$ is set on all $\mathbb{R}$. If $L(t)$ and $W(t,x)$ are neither autonomous nor periodic in t, the problem of the existence of homoclinic solutions of $(\mathscr{HS})$ is quite different from the ones just described because of the lack of compactness of the Sobolev embedding. In 1991, Rabinowitz and Tanaka [7] introduced a type of coercive condition on the matix L:

$$l(t) = \inf_{|\xi|=1} L(t)\xi \cdot \xi \to +\infty \; as \; |t| \to \infty, \qquad (1.1)$$

where "$\cdot$" denotes the standard inner product in $\mathbb{R}^N$ and $|\cdot|$ is the induced norm. They established a compactness lemma under the nonperiodic case and obtained the existence of homoclinic orbits for system $(\mathscr{HS})$ under the well-known Ambrosetti-Rabinowitz superquadratic growth (in short $(\mathscr{AR})$) condition. Later, Ding [8] strengthened condition (1.1) by the following: there exists a constant $\alpha > 0$ such that

$$l(t)|t|^{-\alpha} \to +\infty \; as \; |t| \to \infty. \qquad (1.2)$$

Under condition (1.2) and some subquadratic conditions on $W(t,x)$, the author proved the existence and multiplicity of homoclinic orbits for the system $(\mathscr{HS})$. Subsequently, conditions (1.1) and (1.2) have been extensively used, see, e.g., [9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19] and the references cited therein. However, it is worth pointing out that all of the assumptions on L in the works mentioned above indicate that the smallest eigenvalue $l(t) = \inf_{|\xi|=1} L(t)\xi \cdot \xi$ of $L(t)$ is coercive, i.e., $l(t) \to +\infty$ as $|t| \to \infty$. The above condition does not seem to be natural. Recently, Tang [20] showed that it can be dropped. More precisely, he proved the existence of infinitely many homoclinic solutions for the Hamiltonian system $(\mathscr{HS})$ when the potential W satisfies some superquadratic conditions at infinity and the matrix-valued function L is neither coercive nor uniformly positive definite. More precisely, L is required to satisfy
(L) The smallest eigenvalue $l(t)$ of $L(t)$ is bounded from below;
(L_0) There exists a constant $r_0 > 0$ such that

$$\lim_{|s|\to\infty} meas(\{t \in]s-r_0, s+r_0[/L(t) < bI_N\}) = 0, \; \forall b > 0,$$

where *meas* denotes the Lebesgue's measure on $\mathbb{R}$ and I_N is the $N \times N$ identity matrix. Here, for two $N \times N$ symmetric matrices M_1 and M_2, we say that $M_1 < M_2$ if

$$\min_{x \in \mathbb{R}^N, |x|=1} (M_1 - M_2)x \cdot x < 0,$$

and $M_1 \geq M_2$ if $M_1 < M_2$ does not hold. When $q \neq 0$, some authors have studied homoclinic orbits of $(\mathscr{DV})$, see, e.g., [21, 22, 23, 24]. In all these results, the matrix-valued function $L(t)$ is assumed to be coercive. Recently, Timoumi [23] obtained the existence of infinitely many fast homoclinic orbits for $(\mathscr{DV})$ (see Definition 2.1), where $L(t)$ satisfies condition (1.2), and $W(t,x)$ satisfies the Fei's superquadratic growth (in short $(\mathscr{F})$) conditions.

In this paper, motivated by [20, 23], we will study the existence of infinitely many solutions for $(\mathscr{DV})$, which are fast homoclinics, where L is neither coercive nor uniformly positive definite and $W(t,x)$ satisfies some weak superquadratic conditions at infinity with respect to the second variable and does not check either $(\mathscr{AR})$-condition or $(\mathscr{F})-$condition. Before presenting our main result, we make our hypotheses:

(Q) $\|q\|_\infty < \infty$ and $Q(t) = \int_0^t q(s)ds \to +\infty$ as $|t| \to \infty$;

(L_Q) there exists a constant $r_0 > 0$ such that

$$\lim_{|s|\to\infty} meas_Q(\{t \in]s-r_0, s+r_0[/L(t) < bI_N\}) = 0, \ \forall b > 0,$$

where $meas_Q$ denotes the Lebesgue's measure on $\mathbb{R}$ with density $e^{Q(t)}$, i.e. $meas_Q(A) = \int_A e^{Q(t)}dt$;

(W_1) $W(t,0) = 0$ and there exist constants $c > 0$ and $v > 2$ such that

$$|\nabla W(t,x)| \leq c(|x| + |x|^{v-1}), \ \forall(t,x) \in \mathbb{R} \times \mathbb{R}^N;$$

(W_2) $\qquad W(t,x) \geq 0, \ \forall(t,x) \in \mathbb{R} \times \mathbb{R}^N$ and $\lim_{|x|\to\infty} \dfrac{W(t,x)}{|x|^2} = +\infty, \ uniformly \ for \ t \in \mathbb{R}$;

(W_3) there exists a constant $\sigma \geq 1$ such that

$$\sigma\widehat{W}(t,x) \geq \widehat{W}(t,sx), \ \forall(s,t,x) \in [0,1] \times \mathbb{R} \times \mathbb{R}^N,$$

where $\widehat{W}(t,x) = \nabla W(t,x) \cdot x - 2W(t,x)$;

(W_4) $\qquad\qquad\qquad W(t,-x) = W(t,x), \ \forall(t,x) \in \mathbb{R} \times \mathbb{R}^N.$

Our main result reads as follows.

Theorem 1.1. *Assume that (Q), (L), (L_Q) and $(W_1) - (W_4)$ are satisfied. Then system $(\mathscr{DV})$ possesses a sequence of nontrival fast homoclinic orbits (u_k) satisfying*

$$\frac{1}{2}\int_\mathbb{R} e^{Q(t)}[|\dot{u}_k|^2 + L(t)u_k \cdot u_k]dt - \int_\mathbb{R} e^{Q(t)}W(t,u_k)dt \to +\infty \ as \ k \to \infty.$$

Remark 1.2. a) In our result, $L(t)$ is unnecassarily required to be either uniformly positive definite or coercive. For example, let $L(t) = (t^2cos^2t - 1)I_N$. Then L satisfies (L) and (L_Q) but it does'nt satisfy neither (1.1) nor (1.2).

b) Let

$$W(t,x) = a(t)[|x|^2 ln(e + |x|) - \frac{1}{2}|x|^2 + e|x| - e^2(ln(e + |x|) - 1)],$$

where a is a continuous bounded function with a positive lower bound. Then an easy computation shows that W satisfies $(W_1) - (W_4)$. However, W does not satisfy the $(\mathscr{AR})$-condition or the $(\mathscr{F})$-conditions. The remainder of this paper is organized as follows. In Section 2, some preliminary results are presented, in particular, a compact lemma. The third Section is devoted to the proof of our main result.

2. PRELIMINARIES

In order to introduce the concept of fast homoclinic solutions for $(\mathscr{DV})$ conveniently, we firstly describe some properties of the weighted Sobolev space E on which the certain variational functional associated with $(\mathscr{DV})$ is defined and the fast homoclinic solutions of $(\mathscr{DV})$ are the critical points of such functional. We shall use $L_Q^2(\mathbb{R})$ to denote the Hilbert space of measurable functions from $\mathbb{R}$ into $\mathbb{R}^N$ under the inner product

$$< u,v >_{L_Q^2} = \int_{\mathbb{R}} e^{Q(t)} u(t) \cdot v(t) dt$$

and the induced norm

$$\|u\|_{L_Q^2} = (\int_{\mathbb{R}} e^{Q(t)} |u(t)|^2 dt)^{\frac{1}{2}}.$$

Similarly, $L_Q^p(\mathbb{R})$ $(1 \leq p < \infty)$ denotes the Banach space of functions on $\mathbb{R}$ with values in $\mathbb{R}^N$ under the norm

$$\|u\|_{L_Q^p} = (\int_{\mathbb{R}} e^{Q(t)} |u(t)|^p dt)^{\frac{1}{p}}$$

and $L_Q^\infty(\mathbb{R})$ denotes the Banach space of functions on $\mathbb{R}$ with values in $\mathbb{R}^N$ under the norm

$$\|u\|_{L_Q^\infty} = esssup \left\{ e^{\frac{Q(t)}{2}} |u(t)| / t \in \mathbb{R} \right\}.$$

In this section, we assume that L satisfies the following condition instead of condition (L)

$$(L') \qquad\qquad l(t) = \inf_{|\xi|=1} L(t)\xi \cdot \xi \geq 1, \ \forall t \in \mathbb{R}$$

and we introduce the Hilbert space

$$E = \left\{ u \in H_Q^1(\mathbb{R}) / \int_{\mathbb{R}} e^{Q(t)} L(t) u(t) \cdot u(t) dt < \infty \right\}$$

equipped with the following inner product

$$< u,v > = \int_{\mathbb{R}} e^{Q(t)} \left[\dot{u}(t) \cdot \dot{v}(t) + L(t) u(t) \cdot v(t) \right] dt$$

and the induced norm $\|u\| = < u,u >^{\frac{1}{2}}$. Here, $H_Q^1(\mathbb{R})$ denotes the Sobolev space

$$H_Q^1(\mathbb{R}) = \left\{ u \in L_Q^2(\mathbb{R}) / \dot{u} \in L_Q^2(\mathbb{R}) \right\}.$$

Evidently, E is continuously embedded into $L_Q^p(\mathbb{R})$ for $2 \leq p \leq \infty$, i.e., for all $2 \leq p \leq \infty$, there exists a constant $\eta_p > 0$ such that

$$\|u\|_{L_Q^p} \leq \eta_p \|u\|, \ \forall u \in E. \tag{2.1}$$

Definition 2.1. A solution u of $(\mathscr{DV})$ is called a fast homoclinic orbit if $u \in E$.

Lemma 2.2. *Assume that* (Q), (L') *and* (L_Q) *are satisfied. Then E is compactly embedded in* $L_Q^p(\mathbb{R})$ *for all* $2 \leq p < \infty$.

Proof. Let $(u_n) \subset E$ be a bounded sequence such that $u_n \rightharpoonup u$ in E. We shall show that $u_n \to u$ in $L_Q^p(\mathbb{R})$ for all $2 \leq p < \infty$.

Step 1. Consider the case $p = 2$. Suppose, without loss of generality, that $u_n \rightharpoonup 0$ in E. For any $s \in \mathbb{R}$, we denote $I_{r_0}(s)$ the open interval in $\mathbb{R}$ centered at s with radius r_0, i.e., $I_{r_0}(s) =]s - r_0, s + r_0[$, where r_0 is the constant given in (L_Q). Let $(s_i)_{i \in \mathbb{N}} \subset \mathbb{R}$ be a sequence of points such that $\mathbb{R} = \cup_{i=1}^\infty I_{r_0}(s_i)$ and each $t \in \mathbb{R}$ is contained in at most two such intervals. For any $r > 0$ and $b > 0$, let

$$C(r,b) = \{t \in \mathbb{R} \setminus]-r, r[/ L(t) \geq bI_N\},$$

$$D(r,b) = \{t \in \mathbb{R} \setminus]-r, r[/ L(t) < bI_N\}.$$

Choosing $b_\varepsilon > \frac{4}{\varepsilon} \sup_{n \in \mathbb{N}} \|u_n\|^2$, we have

$$\int_{C(r,b_\varepsilon)} e^{Q(t)} |u_n|^2 \, dt \leq \frac{1}{b_\varepsilon} \int_{C(r,b_\varepsilon)} e^{Q(t)} L(t) u_n(t) \cdot u_n(t) dt$$
$$\leq \frac{1}{b_\varepsilon} \int_{\mathbb{R}} e^{Q(t)} L(t) u_n(t) \cdot u_n(t) dt \leq \frac{1}{b_\varepsilon} \|u_n\|^2 < \frac{\varepsilon}{4}. \tag{2.2}$$

Now, we have

$$\int_{D(r,b_\varepsilon)} e^{Q(t)} |u_n|^2 \, dt \leq \sum_{i=1}^\infty \int_{D(r,b_\varepsilon) \cap I_{r_0}(s_i)} e^{Q(t)} |u_n|^2 \, dt$$
$$\leq \sum_{i=1}^\infty \left(\int_{D(r,b_\varepsilon) \cap I_{r_0}(s_i)} e^{Q(t)} |u_n|^4 \, dt \right)^{\frac{1}{2}} [meas_Q(D(r,b_\varepsilon) \cap I_{r_0}(s_i))]^{\frac{1}{2}}$$
$$\leq a_r \sum_{i=1}^\infty \left(\int_{I_{r_0}(s_i)} e^{Q(t)} |u_n|^4 \, dt \right)^{\frac{1}{2}},$$

where $a_r = \sup_{i \in \mathbb{N}} [meas_Q(D(r,b_\varepsilon) \cap I_{r_0}(s_i))]^{\frac{1}{2}}$. By inequality (2.1), we have

$$\left(\int_{I_{r_0}(s_i)} e^{Q(t)} |u_n|^4 \, dt \right)^{\frac{1}{4}} = \left(\int_{\mathbb{R}} e^{Q(t)} \left| \chi_{|I_{r_0}(s_i)} u_n \right|^4 dt \right)^{\frac{1}{4}}$$
$$\leq \eta_4 \left\| \chi_{|I_{r_0}(s_i)} u_n \right\| \leq \eta_4 \|u_n\|.$$

Hence

$$\int_{D(r,b_\varepsilon)} e^{Q(t)} |u_n|^2 \, dt \leq \eta_4^2 a_r \sum_{i=1}^\infty \left\| \chi_{|I_{r_0}(s_i)} u_n \right\|^2$$
$$= \eta_4^2 a_r \sum_{i=1}^\infty \int_{I_{r_0}(s_i)} e^{Q(t)} [|\dot{u}_n|^2 + L(t) u_n(t) \cdot u_n(t)] dt$$
$$\leq 2\eta_4^2 a_r \sup_{n \in \mathbb{N}} \|u_n\|^2.$$

By an easy computation, we show that $a_r \to 0$ as $r \to \infty$. Therefore there exists $r_\varepsilon > 0$ such that

$$\int_{D(r_\varepsilon,b_\varepsilon)} e^{Q(t)} |u_n|^2 \, dt < \frac{\varepsilon}{4}, \tag{2.3}$$

which together with (2.2) implies

$$\int_{\mathbb{R}\setminus]-r_\varepsilon,r_\varepsilon[} e^{Q(t)} |u_n|^2 \, dt = \int_{C(r_\varepsilon,b_\varepsilon)} e^{Q(t)} |u_n|^2 \, dt + \int_{D(r_\varepsilon,b_\varepsilon)} e^{Q(t)} |u_n|^2 \, dt$$
$$< \frac{\varepsilon}{2}. \tag{2.4}$$

By Sobolev's embedding theorem, we have that $u_k \to 0$ uniformly on $[-r_\varepsilon, r_\varepsilon]$. Then there exists $k_0 > 0$ such that

$$\int_{[-r_\varepsilon,r_\varepsilon]} e^{Q(t)} |u_k|^2 \, dt < \frac{\varepsilon}{2}, \ \forall k \geq k_0. \tag{2.5}$$

Combining (2.4) with (2.5), by the arbitrary of ε, we can obtain that $u_k \to 0$ in $L_Q^2(\mathbb{R})$.

Step 2. Case: $2 < p < \infty$. By [13, Lemma 2.3], for any $v \in H^1(\mathbb{R})$, we have

$$\|v\|_{L^\infty} \leq \frac{\sqrt{2}}{2}\left[\int_{\mathbb{R}} (|\dot{v}|^2 + |v|^2) dt\right]^{\frac{1}{2}}. \tag{2.6}$$

Let $u \in E$ and set $v(t) = e^{\frac{Q(t)}{2}} u(t)$. By (L'), one has

$$\int_{\mathbb{R}} (|\dot{v}|^2 + |v|^2) dt = \int_{\mathbb{R}} e^{Q(t)} \left(\left|\dot{u} + \frac{q}{2}u\right|^2 + |u|^2\right) dt$$
$$\leq \int_{\mathbb{R}} e^{Q(t)} [2|\dot{u}|^2 + (1 + \frac{\|q\|_\infty^2}{2})|u|^2] dt$$
$$\leq \int_{\mathbb{R}} e^{Q(t)} [2|\dot{u}|^2 + (1 + \frac{\|q\|_\infty^2}{2})L(t)u(t) \cdot u(t)] dt \tag{2.7}$$
$$\leq c^2(q) \|u\|^2,$$

where $c^2(q) = \max\left\{2, (1 + \frac{\|q\|_\infty^2}{2})\right\}$. Combining (2.6) and (2.7) yields

$$\|u\|_{L_Q^\infty} \leq \frac{\sqrt{2}}{2} c(q) \|u\|. \tag{2.8}$$

Hence

$$\left(\int_{\mathbb{R}} e^{Q(t)} |u|^p \, dt\right)^{\frac{1}{p}} = \left(\int_{\mathbb{R}} e^{Q(t)} |u|^2 |u|^{p-2} \, dt\right)^{\frac{1}{p}}$$
$$\leq \|u\|_{L^\infty}^{\frac{p-2}{2}} \|u\|_{L_Q^2}^{\frac{2}{p}}$$
$$\leq \left(\frac{1}{m_0}\right)^{\frac{p-2}{2p}} \|u\|_{L_Q^\infty}^{\frac{p-2}{p}} \|u\|_{L_Q^2}^{\frac{2}{p}} \tag{2.9}$$
$$\leq \left(\frac{\sqrt{2}}{2m_0} c^2(q)\right)^{\frac{p-2}{2p}} \|u\|^{\frac{p-2}{p}} \|u\|_{L_Q^2}^{\frac{2}{p}},$$

where $m_0 = \min_{t \in \mathbb{R}} e^{Q(t)}$. By virtue of Step 1 and (2.9), we obtain $u_n \to 0$ in $L_Q^p(\mathbb{R})$. The proof is completed. $\square$

To study the critical points of the variational functional associated with $(\mathscr{DV})$, we need the following Variant Fountain Theorem established by Zou [25].

Let E be a Banach space with the norm $\|.\|$ and $E = \overline{\oplus_{j \in \mathbb{N}} X_j}$ with $dim X_j < \infty$ for any $j \in \mathbb{N}$. Set $Y_k = \oplus_{j=1}^k X_j$ and $Z_k = \overline{\oplus_{j=k}^\infty X_j}$. Consider a family of functionals $f_\lambda \in C^1(E, \mathbb{R})$ of the type

$$f_\lambda(u) = A(u) - \lambda B(u), \ u \in E, \ \lambda \in [1,2].$$

Lemma 2.3. *(Variant Fountain Theorem)* [25]. *Assume that the functionals f_λ satisfy*
(a) f_λ maps bounded sets into bounded sets for all $\lambda \in [1,2]$ and

$$f_\lambda(-u) = f_\lambda(u) \ for \ all \ (\lambda, u) \in [1,2] \times E;$$

(b) $B(u) \geq 0$ for all $u \in E$ and $A(u) \to +\infty$ or $B(u) \to +\infty$ as $\|u\| \to \infty$;
(c) There exist $\rho_k > r_k > 0$ such that for all $\lambda \in [1,2]$

$$\alpha_k(\lambda) = \inf_{u \in Z_k, \|u\| = r_k} f_\lambda(u) > \beta_k(\lambda) = \max_{u \in Y_k, \|u\| = \rho_k} f_\lambda(u).$$

Then

$$\alpha_k(\lambda) \leq \xi_k(\lambda) = \inf_{\gamma \in \Gamma_k} \max_{u \in B_k} f_\lambda(\gamma(u)), \ \forall \lambda \in [1,2],$$

where

$$B_k = \{u \in Y_k / \|u\| \leq \rho_k\} \ and \ \Gamma_k = \left\{ \gamma \in C(B_k, E) / \gamma \ is \ odd, \ \gamma_{|\partial B_k} = id \right\}.$$

Moreover, for almost every $\lambda \in [1,2]$, there exists a sequence $(u_m^k(\lambda))_{m \in \mathbb{N}}$ such that

$$\sup_{m \in \mathbb{N}} \left\| u_m^k(\lambda) \right\| < \infty, \ f_\lambda'(u_m^k(\lambda)) \to 0, \ f_\lambda(u_m^k(\lambda)) \to \xi_k(\lambda) \ as \ m \to \infty.$$

3. PROOF OF THEOREM 1.1

First, note that (L) implies that there exists a constant $a_0 > 0$ such that $L(t) + 2a_0 I_N \geq I_N$ for all $t \in \mathbb{R}$. Let $\widetilde{L}(t) = L(t) + 2a_0 I_N$ and $\widetilde{W}(t,x) = W(t,x) + a_0 |x|^2$. Consider the following system

$$\ddot{u}(t) + q(t)\dot{u}(t) - \widetilde{L}(t)u(t) + \nabla \widetilde{W}(t,u(t)) = 0, \forall t \in \mathbb{R}. \tag{3.1}$$

Then (3.1) is equivalent to $(\mathscr{D}\mathscr{V})$. Moreover, it is easy to check that the hypotheses $(W_1) - (W_4)$ still hold for $\widetilde{W}(t,x)$ provided that those hold for $W(t,x)$ and the function $\widetilde{L}$ satisfies (L') and (L_Q). Hence, in what follows, we always assume, without loss of generality, that L satisfies (L') and (L_Q).

Consider the variational function f associated to the system (DV):

$$f(u) = \frac{1}{2} \int_{\mathbb{R}} e^{Q(t)} (|\dot{u}(t)|^2 + L(t)u(t) \cdot u(t)) dt - \int_{\mathbb{R}} e^{Q(t)} W(t,u(t)) dt$$

defined on the Hilbert space E introduced in Section 2. Set

$$g(u) = \int_{\mathbb{R}} e^{Q(t)} W(t,u) dt, \ u \in E.$$

Lemma 3.1. *Assume that (Q), (L'), (L_Q) and (W_1) are satisfied. Then $g \in C^1(E, \mathbb{R})$ and $g':$ $E \to E^*$ is compact, and $f \in C^1(E, \mathbb{R})$. Moreover, for all $u, v \in E$,*

$$g'(u)v = \int_{\mathbb{R}} e^{Q(t)} \nabla W(t,u) \cdot v dt \tag{3.2}$$

and

$$f'(u)v = \int_{\mathbb{R}} e^{Q(t)}[\dot{u} \cdot \dot{v} + L(t)u \cdot v]dt - \int_{\mathbb{R}} e^{Q(t)} \nabla W(t,u) \cdot v dt$$

$$= \, <u,v> - \int_{\mathbb{R}} e^{Q(t)} \nabla W(t,u) \cdot v dt. \tag{3.3}$$

Proof. By (W_1), for any $s \in [0,1]$ and $u,v \in E$, we have

$$|\nabla W(t,u+sv)v| \leq c(|u+sv| + |u+sv|^{\nu-1})|v|$$

$$\leq c[|u| + |v| + 2^{\nu-2}(|u|^{\nu-1} + |v|^{\nu-1})]|v| \tag{3.4}$$

$$\leq c2^{\nu-2}[|u||v| + |v|^2 + |u|^{\nu-1}|v| + |v|^{\nu}].$$

The Hölder's inequality implies

$$\int_{\mathbb{R}} e^{Q(t)}(|u||v| + |u|^{\nu-1}|v|)dt \leq \|u\|_{L_Q^2}\|v\|_{L_Q^2} + \|u\|_{L_Q^\nu}^{\nu-1}\|v\|_{L_Q^\nu}. \tag{3.5}$$

Hence, by (3.4), (3.5), the Mean Value Theorem and Lebesgue's Dominated Convergence Theorem, we get, for all $u,v \in E$,

$$\lim_{s \to 0} \frac{g(u+sv) - g(u)}{s} = \lim_{s \to 0} \int_{\mathbb{R}} \int_0^1 e^{Q(t)} \nabla W(t,u+\theta sv) \cdot v d\theta dt$$

$$= \int_{\mathbb{R}} e^{Q(t)} \nabla W(t,u) \cdot v dt \tag{3.6}$$

$$= J(u,v).$$

Moreover, it follows from (2.1), (W_1) and (3.5) that

$$|J(u,v)| \leq \int_{\mathbb{R}} e^{Q(t)} |\nabla W(t,u)||v| dt$$

$$\leq c \int_{\mathbb{R}} e^{Q(t)}(|u||v| + |u|^{\nu-1}|v|)dt$$

$$\leq c(\|u\|_{L_Q^2}\|v\|_{L_Q^2} + \|u\|_{L_Q^\nu}^{\nu-1}\|v\|_{L_Q^2})$$

$$\leq c(\eta_2^2\|u\| + \eta_\nu^\nu\|u\|^{\nu-1})\|v\|.$$

Therefore, $J(u,.)$ is linear and bounded, and $J(u,.)$ is the Gâteaux derivative of g at u.

Next, we prove that $J(u,.)$ is weakly continuous in u. To this end, we first claim that if $u_n \rightharpoonup u$ in E, then $\nabla W(t,u_n) \to \nabla W(t,u)$ in $L_Q^2(\mathbb{R})$. Arguing indirectly, by Lemma 2.2, we may assume that there exists a subsequence (u_{n_k}) such that

$$u_{n_k} \to u \text{ in both } L_Q^2(\mathbb{R}) \text{ and } L_Q^{2(\nu-1)}(\mathbb{R}) \text{ and } u_{n_k} \to u \text{ a.e. in } \mathbb{R} \text{ as } k \to \infty \tag{3.7}$$

and

$$\int_{\mathbb{R}} e^{Q(t)} |\nabla W(t,u_{n_k}) - \nabla W(t,u)|^2 dt \geq \varepsilon_0, \; \forall k \in \mathbb{N} \tag{3.8}$$

for some positive constant ε_0. By (3.7) and up to a subsequence if necessary, we can assume that $\sum_{k=1}^\infty \|u_{n_k} - u\|_{L_Q^2} < \infty$ and $\sum_{k=1}^\infty \|u_{n_k} - u\|_{L_Q^{2(\nu-1)}} < \infty$. Let $w(t) = \sum_{k=1}^\infty |u_{n_k}(t) - u(t)|$ for all

$t \in \mathbb{R}$. Then $w \in L_Q^2(\mathbb{R}) \cap L_Q^{2(\nu-1)}(\mathbb{R})$. By (W_1), there holds for all $k \in \mathbb{N}$ and $t \in \mathbb{R}$

$$|\nabla W(t, u_{n_k}) - \nabla W(t, u)|^2$$
$$\leq (|\nabla W(t, u_{n_k})| + |\nabla W(t, u)|)^2$$
$$\leq 2(|\nabla W(t, u_{n_k})|^2 + |\nabla W(t, u)|^2)$$
$$\leq 2c^2[(|u_{n_k}| + |u_{n_k}|^{\nu-1})^2 + (|u| + |u|^{\nu-1})^2]$$
$$\leq 2^2 c^2[|u_{n_k}|^2 + |u_{n_k}|^{2(\nu-1)} + |u|^2 + |u|^{2(\nu-1)}]$$
$$\leq 2^2 c^2[(|u_{n_k} - u| + |u|)^2 + (|u_{n_k} - u| + |u|)^{2(\nu-1)} + |u|^2 + |u|^{2(\nu-1)}]$$
$$\leq 2^2 c^2[2(|u_{n_k} - u|^2 + |u|^2) + 2^{2\nu-3}(|u_{n_k} - u|^{2(\nu-1)} + |u|^{2(\nu-1)}) + |u|^2 + |u|^{2(\nu-1)}]$$
$$\leq c_1(|w|^2 + |u|^2 + |w|^{2(\nu-1)} + |u|^{2(\nu-1)}),$$

where c_1 is a positive constant. Combining this with (3.7), Lebesgue's Dominated Convergence Theorem implies

$$\lim_{k \to \infty} \int_{\mathbb{R}} e^{Q(t)} |\nabla W(t, u_{n_k}) - \nabla W(t, u)|^2 dt = 0$$

which contradicts to (3.8). Hence the claim above is true.

Now, we suppose $u_n \rightharpoonup u$ in E. Then $\nabla W(t, u_n) \to \nabla W(t, u)$ in $L_Q^2(\mathbb{R})$. By Hölder's inequality and (2.1), we have

$$\|J(u_n, .) - J(u, .)\|_{E^*} = \sup_{\|v\|=1} \int_{\mathbb{R}} e^{Q(t)}(\nabla W(t, u_n) - \nabla W(t, u)) \cdot v dt$$
$$\leq \eta_2 \left(\int_{\mathbb{R}} e^{Q(t)} |\nabla W(t, u_n) - \nabla W(t, u)|^2 dt \right)^{\frac{1}{2}} \to 0 \ as \ n \to \infty.$$

This means that $u \longmapsto J(u, .)$ is weakly continuous and then it is continuous in E. Therefore $g \in C^1(E, \mathbb{R})$ and (3.2) is verified. Furthermore, g' is compact by the weak continuity of g' since E is reflexive. Due to the form of f, (3.3) is also verified and $f \in C^1(E, \mathbb{R})$.

Finally, let $u \in E$ be a critical point of f. A standard argument shows that $u \in C^2(\mathbb{R})$ and satisfies equation $(\mathscr{D}\mathscr{V})$. The proof is completed. $\qquad \square$

In order to apply the Variant Fountain Theorem to prove our main result, we choose an orthonormal basis $(e_n)_{n \in \mathbb{N}}$ of E and let $X_j = span\{e_j\}$ for all $j \in \mathbb{N}$. Define Y_k and Z_k by

$$Y_k = \oplus_{j=1}^k X_j, \ Z_k = \overline{\oplus_{j=k}^\infty X_j}$$

and the functionals A, B and f_λ on our working space E by

$$A(u) = \frac{1}{2}\|u\|^2, \ B(u) = g(u) = \int_{\mathbb{R}} e^{Q(t)} W(t, u) dt, \ f_\lambda(u) = A(u) - \lambda B(u),$$

for all $(\lambda, u) \in [1, 2] \times E$.

Assumption (W_1) and property (2.1) imply that f_λ maps bounded sets into bounded sets uniformly for $\lambda \in [1, 2]$. Note that $W(t, -x) = W(t, x)$. So $f_\lambda(-u) = f_\lambda(u)$ for all $(\lambda, u) \in [1, 2] \times E$. Thus condition a) of Lemma 2.3 holds. Since $W(t, x) \geq 0$ for all $(t, x) \in \mathbb{R} \times \mathbb{R}^N$, it is clear that condition b) is also satisfied. To verify condition c), we need to establish the three following lemmas.

Lemma 3.2. *Suppose that* (Q), (L') *and* (L_Q) *hold. Then, for any* $p \in [2, \infty]$,

$$l_p(k) = \sup_{u \in Z_k, \|u\|=1} \|u\|_{L_Q^p} \to 0 \ as \ k \to \infty. \tag{3.9}$$

Proof. It is clear that $0 < l_p(k+1) \le l_p(k)$ so that $l_p(k) \to \bar{l}_p$ as $k \to \infty$. For every $k \ge 1$, there exists $u_k \in Z_k$ such that $\|u_k\| = 1$ and $\|u_k\|_{L_Q^p} > \frac{1}{2} l_p(k)$. For any $v \in E$, let $v = \sum_{i=1}^{\infty} v_i e_i$. By the Cauchy-Schartz inequality, one has

$$\begin{aligned}
|<u_k, v>| &= \left| <u_k, \sum_{i=1}^{\infty} v_i e_i> \right| \\
&= \left| <u_k, \sum_{i=k+1}^{\infty} v_i e_i> \right| \\
&\le \|u_k\| \left\| \sum_{i=k+1}^{\infty} v_i e_i \right\| \\
&\le \sum_{i=k+1}^{\infty} |v_i| \|e_i\| \to 0 \ as \ k \to \infty,
\end{aligned}$$

which implies that $u_k \rightharpoonup 0$. Without loss of generality, Lemma 2.2 implies that $u_k \to 0$ in $L_Q^p(\mathbb{R})$. Thus we have proved that $\bar{l}_p(k) = 0$. The proof is completed. $\qquad \square$

Lemma 3.3. *Assume that* (Q), (L'), (L_Q), (W_1) *and* (W_2) *are satisfied. Then there exist a positive integer* k_0 *and a sequence* $r_k \to +\infty$ *as* $k \to \infty$ *such that*

$$\alpha_k(\lambda) = \inf_{u \in Z_k, \|u\|=r_k} f_\lambda(u) > 0, \ \forall k \ge k_0. \tag{3.10}$$

Proof. From (W_1) and the fact that $W(t,x) \ge 0$, we have

$$\begin{aligned}
f_\lambda(u) &\ge \frac{1}{2} \|u\|^2 - 2 \int_{\mathbb{R}} e^{Q(t)} W(t, u) dt \\
&\ge \frac{1}{2} \|u\|^2 - 2c(\|u\|_{L_Q^2}^2 + \|u\|_{L_Q^\nu}^\nu), \forall (\lambda, u) \in [1, 2] \times E.
\end{aligned} \tag{3.11}$$

Combining (3.9) and (3.11) yields

$$f_\lambda(u) \ge \frac{1}{2} \|u\|^2 - 2c l_2^2(k) \|u\|^2 - 2c l_\nu^\nu(k) \|u\|^\nu, \ \forall (\lambda, u) \in [1, 2] \times E. \tag{3.12}$$

In view of (3.9), there exists an integer k_0 such that

$$2c l_2^2(k) \le \frac{1}{4}, \forall k \ge k_0. \tag{3.13}$$

For any $k \ge k_0$, let us define

$$r_k = (16 c l_\nu^\nu(k))^{\frac{1}{2-\nu}}. \tag{3.14}$$

Since $\nu > 2$, we have $r_k \to +\infty$ as $k \to \infty$. From (3.12), (3.13) and (3.14), we deduce that, for all $k \ge k_0$,

$$\inf_{u \in Z_k, \|u\|=r_k} f_\lambda(u) \ge \frac{1}{2} r_k^2 - \frac{1}{4} r_k^2 - \frac{1}{8} r_k^{2-\nu} r_k^\nu = \frac{1}{8} r_k^2 > 0, \tag{3.15}$$

which completes the proof. $\qquad \square$

Lemma 3.4. *Assume that* (Q), (L'), (L_Q), (W_1) *and* (W_2) *are satisfied. Then, for any* $k \geq k_0$, *there exists* $\rho_k > r_k$ *such that*

$$\beta_k(\lambda) = \max_{u \in Y_k, \|u\| = \rho_k} f_\lambda(u) < 0,$$

where k_0 *is the positive integer obtained in Lemma 3.3.*

Proof. Firstly, we claim that for any finite-dimensional subspace $F \subset E$, there exists a constant $\varepsilon_0 > 0$ such that

$$meas_Q(\{t \in \mathbb{R} / |u(t)| \geq \varepsilon_0 \|u\|\}) \geq \varepsilon_0, \ \forall u \in F \setminus \{0\}. \tag{3.16}$$

In not, for any $n \in \mathbb{N}$, there exists $u_n \in F \setminus \{0\}$ such that

$$meas_Q\left(\left\{t \in \mathbb{R} / |u_n(t)| \geq \frac{1}{n} \|u_n\|\right\}\right) < \frac{1}{n}.$$

Let $v_n = \frac{u_n}{\|u_n\|} \in F$. Then $\|v_n\| = 1$ and

$$meas_Q\left(\left\{t \in \mathbb{R} / |v_n(t)| \geq \frac{1}{n}\right\}\right) < \frac{1}{n}, \forall n \in \mathbb{N}. \tag{3.17}$$

Since F is finite-dimensional, then up to a subsequence if necessary, we may assume that $v_n \to v_0$ in E for some $v_0 \in F$. Evidently, $\|v_0\| = 1$. Note that, since any two norms on F are equivalent, we have

$$\int_{\mathbb{R}} e^{Q(t)} |v_n - v_0| dt \to 0 \ as \ n \to \infty. \tag{3.18}$$

The fact that $\|v_0\| = 1$ implies $\|v_0\|_{L_Q^\infty} > 0$. By the definition of $\|.\|_{L_Q^\infty}$, there exists a constant $\delta_0 > 0$ such that

$$meas_Q(\{t \in \mathbb{R} / |v_0(t)| \geq \delta_0\}) \geq \delta_0. \tag{3.19}$$

Otherwise, for each fixed $n \in \mathbb{N}$ and $m > n$, we have

$$meas_Q\left(\left\{t \in \mathbb{R} / |v_0(t)| \geq \frac{1}{n}\right\}\right) \leq meas_Q\left(\left\{t \in \mathbb{R} / |v_0(t)| \geq \frac{1}{m}\right\}\right)$$

$$\leq \frac{1}{m}.$$

Letting $m \to \infty$, we obtain $meas_Q(\{t \in \mathbb{R} / |v_0(t)| \geq \frac{1}{n}\}) = 0$. Consequently

$$0 \leq meas_Q(\{t \in \mathbb{R} / |v_0(t)| \neq 0\})$$

$$= meas_Q\left(\cup_{n=1}^\infty \left\{t \in \mathbb{R} / |v_0(t)| \geq \frac{1}{n}\right\}\right)$$

$$\leq \sum_{n=1}^\infty meas_Q\left(\left\{t \in \mathbb{R} / |v_0(t)| \geq \frac{1}{n}\right\}\right) = 0,$$

which yields $v_0 = 0$ and contradicts $\|v_0\| = 1$. Then (3.19) holds. For any $n \in \mathbb{N}$, let

$$\Lambda_n = \left\{t \in \mathbb{R} / |v_n(t)| < \frac{1}{n}\right\}, \Lambda_0 = \{t \in \mathbb{R} / |v_0(t)| \geq \delta_0\}.$$

Then, for *n* large enough, we have from (3.17) and (3.19) that

$$meas_Q(\Lambda_n \cap \Lambda_0) \geq meas_Q(\Lambda_0) - meas_Q(\Lambda_n^c)$$

$$\geq \delta_0 - \frac{1}{n}$$

$$\geq \frac{\delta_0}{2}.$$

Consequently, for *n* large enough, there holds

$$\int_{\mathbb{R}} e^{Q(t)} |v_n - v_0| \, dt \geq \int_{\Lambda_n \cap \Lambda_0} e^{Q(t)} |v_n - v_0| \, dt$$

$$\geq \int_{\Lambda_n \cap \Lambda_0} e^{Q(t)} (|v_0| - |v_n|) dt$$

$$\geq (\delta_0 - \frac{1}{n}) meas_Q(\Lambda_n \cap \Lambda_0)$$

$$\geq \frac{\delta_0^2}{4} > 0.$$

This contradicts to (3.18). Therefore (3.16) holds.

Now, note that, for any $k \in \mathbb{N}$, Y_k is finite-dimensional. So there exists a constant $\varepsilon_k > 0$ such that

$$meas_Q(\Lambda_u^k) \geq \varepsilon_k, \ \forall u \in Y_k \setminus \{0\}, \tag{3.20}$$

where

$$\Lambda_u^k = \{t \in \mathbb{R} / |u(t)| \geq \varepsilon_k \|u\|\}$$

for all $k \in \mathbb{N}$ and $u \in Y_k \setminus \{0\}$. By (W_2), for any $k \in \mathbb{N}$, there exists a constant $R_k > 0$ such that

$$W(t,x) \geq \frac{|x|^2}{\varepsilon_k^3}, \ \forall t \in \mathbb{R} \ and \ |x| \geq R_k. \tag{3.21}$$

Combining (3.20) with (3.21), for any $k \in \mathbb{N}$ and $\lambda \in [1,2]$, we have

$$f_\lambda(u) \leq \frac{1}{2} \|u\|^2 - \int_{\Lambda_u^k} e^{Q(t)} W(t,u) dt$$

$$\leq \frac{1}{2} \|u\|^2 - \int_{\Lambda_u^k} e^{Q(t)} \frac{|u|^2}{\varepsilon_k^3} dt$$

$$\leq \frac{1}{2} \|u\|^2 - \varepsilon_k^2 \|u\|^2 meas_Q(\Lambda_u^k) \frac{1}{\varepsilon_k^3} \tag{3.22}$$

$$\leq \frac{1}{2} \|u\|^2 - \|u\|^2$$

$$= -\frac{1}{2} \|u\|^2$$

for all $u \in Y_k$ with $\|u\| \geq \frac{R_k}{\varepsilon_k}$. For any $k \geq k_0$, we choose $\rho_k > \max\left\{r_k, \frac{R_k}{\varepsilon_k}\right\}$. It follows from (3.22) that

$$\rho_k(\lambda) = \max_{u \in Y_k, \|u\| = \rho_k} f_\lambda(u) \leq -\frac{1}{2}\rho_k^2.$$

The proof is now completed.   □

Consequently, Lemmas 3.3, 3.4 show that condition c) of Lemma 2.3 is satisfied for all $k \geq k_0$. From the above, all the conditions of Lemma 2.3 hold for all $k \geq k_0$. Therefore, for any $k \geq k_0$ and $\lambda \in [1,2]$, there exists a sequence $(u_n^k(\lambda)) \subset E$ such that

$$\sup_{n \in \mathbb{N}} \left\| u_n^k(\lambda) \right\| < \infty, \ f_\lambda'(u_n^k(\lambda)) \to 0 \ and \ f_\lambda(u_n^k(\lambda)) \to \xi_k(\lambda) \tag{3.23}$$

as $n \to \infty$, where

$$\xi_k(\lambda) = \inf_{\gamma \in \Gamma_k} \max_{u \in B_k} f_\lambda(u), \ \forall \lambda \in [1,2]$$

with

$$B_k = \{u \in Y_k / \|u\| \leq \rho_k\} \ and \ \Gamma_k = \left\{ \gamma \in C(B_k, E) / \gamma \ is \ odd, \ \gamma_{|\partial B_k} = id \right\}.$$

From (3.15) and Lemma 2.3, we infer that

$$\xi_k(\lambda) \in [\overline{\alpha}_k, \overline{\xi}_k], \ \forall k \geq k_0 \ and \ \lambda \in [1,2], \tag{3.24}$$

where $\overline{\xi}_k = \max_{u \in B_k} f_1(u)$ and $\overline{\alpha}_k = \frac{r_k^2}{8} \to \infty$ as $k \to \infty$.

$$\sup_{m \in \mathbb{N}} \left\| u_m^k(\lambda_n) \right\| < \infty, \ f_{\lambda_n}'(u_m^k(\lambda_n)) \to 0 \ and \ f_{\lambda_n}(u_m^k(\lambda_n)) \to \xi_k(\lambda_n), \ as \ m \to \infty. \tag{3.25}$$

Lemma 3.5. *For any $n \in \mathbb{N}$ and $k \geq k_0$, there exists $u_n^k \in E$ such that*

$$\lim_{m \to \infty} u_m^k(\lambda_n) = u_n^k \ in \ E. \tag{3.26}$$

Proof. Throughout this proof and for the sake of simplicity, we shall let $u_m = u_m^k(\lambda_n)$ for $m \in \mathbb{N}$. Without loss of generality, we may assume by (3.25) that

$$u_m \rightharpoonup u \ as \ m \to \infty \tag{3.27}$$

for some $u \in E$. Using (3.3), we get

$$\|u_m - u\|^2 = f_{\lambda_n}'(u_m)(u_m - u) - f_{\lambda_n}'(u)(u_m - u)$$
$$+ \lambda_n \int_\mathbb{R} e^{Q(t)} (\nabla W(t, u_m) - \nabla W(t, u)) \cdot (u_m - u) dt. \tag{3.28}$$

By (3.25), one has

$$f_{\lambda_n}'(u_m)(u_m - u) \to 0 \ as \ m \to \infty. \tag{3.29}$$

Moreover (3.27) implies

$$f_{\lambda_n}'(u)(u_m - u) \to 0 \ as \ m \to \infty. \tag{3.30}$$

Now, by (2.1) and Hölder's inequality, we have

$$\left| \int_\mathbb{R} e^{Q(t)} (\nabla W(t, u_m) - \nabla W(t, u)) \cdot (u_m - u) dt \right|$$
$$\leq \left(\int_\mathbb{R} e^{Q(t)} |\nabla W(t, u_m) - \nabla W(t, u)|^2 dt \right)^{\frac{1}{2}} \left(\int_\mathbb{R} e^{Q(t)} |u_m - u|^2 dt \right)^{\frac{1}{2}} \tag{3.31}$$
$$\leq \eta_2 \left(\int_\mathbb{R} e^{Q(t)} |\nabla W(t, u_m) - \nabla W(t, u)|^2 dt \right)^{\frac{1}{2}} \|u_m - u\|.$$

As in the proof of Lemma 3.1, by passing to a subsequence if necessary, we may assume that $\int_\mathbb{R} e^{Q(t)} |\nabla W(t, u_m) - \nabla W(t, u)|^2 dt \to 0$ as $m \to \infty$. Hence (3.31) implies

$$\int_\mathbb{R} e^{Q(t)} (\nabla W(t, u_m) - \nabla W(t, u)) \cdot (u_m - u) dt \to 0 \ as \ m \to \infty. \tag{3.32}$$

Combining (3.28), (3.29), (3.30) and (3.32) yields $u_m \to u$ as $m \to \infty$ in E. The proof is now completed. $\square$

Note that (3.24), (3.25) and (3.26) imply

$$f'_{\lambda_n}(u_n^k) = 0, \ f_{\lambda_n}(u_n^k) \in [\overline{\alpha}_k, \overline{\xi}_k], \ \forall n \in \mathbb{N} \text{ and } k \geq k_0. \tag{3.33}$$

Lemma 3.6. *For any $k \geq k_0$, the sequence $(u_n^k)_{n \in \mathbb{N}}$ obtained above is bounded.*

Proof. For notational simplicity, we set $u_n = u_n^k$ for all $n \in \mathbb{N}$. Assuming indirectly that (u_n) is unbounded. By going to a subsequence if necessary, we may assume

$$\|u_n\| \to \infty \text{ and } v_n = \frac{u_n}{\|u_n\|} \rightharpoonup v \text{ as } n \to \infty. \tag{3.34}$$

Without loss of generality, Lemma 2.2 and (3.24) imply

$$v_n \to v \text{ both in } L_Q^2(\mathbb{R}) \text{ and } L_Q^{\nu}(\mathbb{R}), \text{ and } v_n(t) \to v(t) \text{ a.e. } t \in \mathbb{R} \text{ as } n \to \infty. \tag{3.35}$$

First case: $v = 0$. Let (s_n) be a sequence such that

$$f_{\lambda_n}(s_n u_n) = \max_{s \in [0,1]} f_{\lambda_n}(s u_n), \ \forall n \in \mathbb{N}. \tag{3.36}$$

For $R > 0$, let $w_n = 2\sqrt{R} v_n$. By (3.35), we have

$$w_n \to 2\sqrt{R} v = 0 \text{ both in } L_Q^2(\mathbb{R}) \text{ and } L_Q^{\nu}(\mathbb{R}), \tag{3.37}$$

which together with (W_1) implies

$$\left| \int_{\mathbb{R}} e^{Q(t)} W(t, w_n) dt \right| \leq c \int_{\mathbb{R}} e^{Q(t)} (|w_n|^2 + |w_n|^{\nu}) dt \to 0 \text{ as } n \to \infty. \tag{3.38}$$

Note that (3.34) implies that $0 < \frac{2\sqrt{R}}{\|u_n\|} < 1$ for n large enough. This together with (3.36) and (3.38) implies

$$\begin{aligned}
f_{\lambda_n}(s_n u_n) &\geq f_{\lambda_n}(w_n) \\
&= \frac{1}{2} \|u_n\|^2 - \lambda_n \int_{\mathbb{R}} e^{Q(t)} W(t, w_n) dt \\
&\geq 2R - 2 \int_{\mathbb{R}} e^{Q(t)} W(t, w_n) dt \geq R
\end{aligned}$$

for n large enough. Since R is arbitrarily, it follows that

$$\lim_{n \to \infty} f_{\lambda_n}(s_n u_n) = +\infty. \tag{3.39}$$

Note that, since $f_{\lambda_n}(0) = 0$ and $f_{\lambda_n}(u_n) \in [\overline{\alpha}_k, \overline{\xi}_k]$, then $s_n \in]0, 1[$ in (3.36) for n large enough. Therefore

$$0 = s_n \frac{d}{ds}(f_{\lambda_n})(s u_n)_{|s = s_n} = f'_{\lambda_n}(s_n u_n) s_n u_n. \tag{3.40}$$

Combining (3.33), (3.39) and (W_3) yields

$$\begin{aligned}
f_{\lambda_n}(u_n) - \frac{1}{2}f'_{\lambda_n}(u_n)u_n &= \frac{\lambda_n}{2}\int_{\mathbb{R}} e^{Q(t)}\widehat{W}(t,w_n)dt \\
&\geq \frac{\lambda_n}{2\sigma}\int_{\mathbb{R}} e^{Q(t)}\widehat{W}(t,s_n w_n)dt \\
&= \frac{1}{\sigma}[f_{\lambda_n}(s_n u_n) - \frac{1}{2}f'_{\lambda_n}(s_n u_n)s_n u_n] \\
&= \frac{1}{\sigma}f_{\lambda_n}(s_n u_n) \to +\infty \; as \; n \to \infty,
\end{aligned}$$

a contradiction with (3.33).

Second case: $v \neq 0$. The set $\Lambda = \{t \in \mathbb{R}/v(t) \neq 0\}$ has a positive measure with respect to $meas_Q$. By (3.34), it holds that

$$|u_n(t)| \to \infty \; as \; n \to \infty, \; \forall t \in \Lambda. \tag{3.41}$$

Combining (3.35), (3.41) and (W_2), Fatou's lemma implies

$$\begin{aligned}
\frac{1}{2} - \frac{f_{\lambda_n}(u_n)}{\|u_n\|^2} &= \lambda_n \int_{\mathbb{R}} e^{Q(t)}\frac{W(t,u_n)}{\|u_n\|^2}dt \\
&\geq \int_{\Lambda} e^{Q(t)}|v_n|^2 \frac{W(t,u_n)}{|u_n|^2}dt \to \infty \; as \; n \to \infty,
\end{aligned}$$

which provides a contradiction with (3.33) and (3.34). The proof is now completed. $\qquad\square$

Finally, using the similar arguments in the proof of Lemma 3.5 and in view of Lemma 3.6 and (3.33), we can show that, for any $k \geq k_0$, the sequence $(u_n^k)_{n \in \mathbb{N}}$ possesses a strong convergent subsequence with the limit u^k being a critical point of $f = f_1$. Since $\overline{\alpha}_k \to +\infty$ as $k \to \infty$ and $f(u^k) \in [\overline{\alpha}_k, \overline{\xi}_k]$ for all $k \geq k_0$, we have that f has infinitely many critical points. Consequently, $(\mathscr{DV})$ has infinitely many nontrivial fast homoclinic solutions.

REFERENCES

[1] M. Izydorek, J. Janczewska, Homoclinic solutions for a class of the second order Hamiltonian systems, J. Differential Equations, 219 (2005), 375-389.

[2] J. Jiang, S. Lu, X. Lv, Homoclinic solutions for a class of second order Hamiltonian systems, Nonlinear Anal. 13 (2012), 176-185.

[3] S. Lu, X. Lv, P. Yan, Existence of homoclinics for a class of Hamiltonian systems, Nonlinear Anal. 72 (2010), 390-398.

[4] P.H. Rabinowitz, Homoclinic orbits for a class of Hamiltonian systems, Proc. Roy. Soc. Edinburg 114 A (1990), 33-38.

[5] X.H. Tang, L. Xiao, Homoclinic solutionsd for a class of second order Hamiltonian systems, Nonlinear Anal. 71 (2009), 1140-1152.

[6] R. Yuan, Z. Zhang, Homoclinic solutions for some second order nonautonomous hamiltonian systems without the globally superquadratic condition, Nonlinear Anal. 72 (2010), 1809-1819.

[7] P.H. Rabinowitz, K. Tanaka, Some results on connecting orbits for a class of Hamiltonian systems, Math. Z. 206 (1991), 473-499.

[8] Y. Ding, Existence and multiplicity results for homoclinic solutions to a class of Hamiltonian systems, Nonlinear Anal. 25 (1995), 1095 -1113.

[9] G. Chen, Homoclinic orbits for second order Hamiltonian systems with asymptotically linear terms at infinity, Adv. Difference Equations 2014 (2014), Article ID 114.

[10] G. Chen, Z. He, Infinitely many homoclinic solutions for a class of second order Hamiltonian systems, Adv. Difference Equations 2014 (2014), 161.

[11] L. Chu, Q. Zhang, Homoclinic solutions for a class of second order Hamiltonian systems with locally defined potentials, Nonlinear Anal. 75 (2012), 3188-3197.

[12] X. Lin, X.H. Tang, Homoclinic solutions for a class of second order Hamiltonian systems, J. Math. Anal. Appl. 354 (2009), 539-549.

[13] X. Lin, X.H. Tang, Infinitely may homoclinic orbits for Hamiltonian systems with indefinite sign subquadratic potentials, Nonlinear Anal. 74 (2011), 6314-6325.

[14] C. Liu, Q. Zhang, Infinitely many homoclinic solutions for second order Hamiltonian systems, Nonlinear Anal. 72 (2010), 894-903.

[15] J. Sun, T-F. Wu, Homoclinic solutions for a second order Hamiltonian system with a positive semi-definite matrix, Chaos, Solitons and Fractals 76 (2015), 24-31.

[16] J. Sun, T-F. Wu, Multiplicity and concentration of homoclinic solutions for some second order Hamiltonian systems, Nonlinear Anal. 114 (2015), 105-115.

[17] C.L. Tang, L.-L. Wan, Existence and homoclinic orbits for second order Hamiltonian systems without (AR) condition, Nonlinear Anal. 74 (2011), 5303-5313.

[18] J. Wei, J. Wang, Infinitely many homoclinic orbits for the second order Hamiltonian systems with general potentials, J. Math. Anal. Appl. 366 (2010), 694-699.

[19] Z. Zhang; Existence of homoclinic solutions for second order Hamiltonian systems with general potentials, J. Appl. Math. Comput. 44 (2014), 263-272.

[20] X.H. Tang, Infinitely many homoclinic solutions for a second-order Hamiltonian system, Math. Nach. 289 (2016), 116-127.

[21] R.P. Agarwal, P. Chen, X. Tang, Fast homoclinic solutions for a class of damped vibration problems, Appl. Math. Comput. 219 (2013), 6053-6065.

[22] P. Chen, X.H. Tang, Fast homoclinic solutions for a class of damped vibration problems with sub-quadratic potentials, Math. Nachr. 286 (2013), 4-16.

[23] M. Timoumi, Infinitely many homoclinic solutions for a class of superquadratic fourth-order differential equations, J. Nonlinear Funct. Anal. 2018 (2018), Article ID 20.

[24] R. Yuan, Z. Zhang, Fast homoclinic solutions for some second order non-autonomous systems, J. Math. Anal. Appl. 376 (2011), 51-63.

[25] W. Zoum Variant fountain theorems and their applications, Manuscripta Math. 104 (2001) 343-358.