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INERTIAL ALGORITHM FOR SOLVING EQUILIBRIUM, VARIATIONAL INCLUSION AND FIXED POINT PROBLEMS FOR AN INFINITE FAMILY OF STRICT PSEUDOCONTRACTIVE MAPPINGS

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Abstract. In this paper, we study the problem of finding common solutions of equilibrium problems, variational inclusion problems and fixed point problems for an infinite family of strict pseudocontractive mappings. We propose an iterative algorithm, which combines inertial methods with viscosity methods, for approximating common solutions of the above problems. Under mild conditions, we prove a strong theorem in Hilbert spaces and apply our result to optimization problems. Finally, we present a numerical example to demonstrate the efficiency of our algorithm in comparison with other existing methods in the literature.

Keywords. Inertial technique; Viscosity method; Equilibrium problems; Inclusion problems; Strict pseudocontraction.

1. INTRODUCTION

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\|\cdot\|$. Let $A : H \to H$ be a single-valued operator, and let $B : H \to 2^H$ be a multi-valued operator. The *Variational Inclusion Problem* (VIP) is formulated as finding a point $\hat{x} \in H$ such that

$$0 \in (A+B)\hat{x}. \tag{1.1}$$

The solution set of VIP (1.1) is denoted by $(A+B)^{-1}(0)$ and referred to as the set of zero points of $A+B$. The VIP (1.1) includes, as special cases, convex programming, split feasibility problems, variational inequalities and minimization problems. More precisely, some concrete problems in machine learning, image processing and linear inverse problems can be modeled mathematically as VIP (1.1) [1, 2, 3, 4]. There are several methods for solving VIP (1.1).

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Among them, the forward-backward splitting method introduced in [5, 6] is efficient. Specifically, the forward-backward splitting method is presented as follows:

$$x_{n+1} = (I+\lambda_n B)^{-1}(I-\lambda_n A)(x_n),$$

where λ_n is a positive parameter, $(I-\lambda_n A)$ is the so-called forward operator and $(I+\lambda_n B)^{-1}$ is the resolvent operator, which was introduced in [7] and is often referred to as the backward operator. Recently, several authors studied and extended the forward-backward splitting method; see, e.g., [8, 9, 10] and the references therein.

Let C be a nonempty closed convex subset of a real Hilbert space H and let $F : C \times C \to \mathbb{R}$ be a bifunction. The *Equilibrium Problem* (shortly, EP) in the sense of Blum and Oettli [11] is to find $\hat{x} \in C$ such that

$$F(\hat{x}, y) \geq 0, \quad \forall\, y \in C. \tag{1.2}$$

The solution set of EP (1.2) is denoted by $EP(F)$. The EP attracts considerable research efforts and serves as a unifying framework for studying many well-known problems, such as, the Nonlinear Complementarity Problems (NCPs), Optimization Problems (OPs), Variational Inequality Problems (VIPs), Saddle Point Problems (SPPs), the Fixed Point Problem (FPP), the Nash equilibria and many others, and has many applications in physics and economics, (see, for example, [12, 13, 14] and the references therein). Several authors have studied and proposed various iterative algorithms for solving EPs and related Optimitization Problems; see, e.g., [15, 16, 17, 18] and the references therein.

Let $S : H \to H$ be a nonlinear mapping. A point $\hat{x} \in H$ is called a fixed point of S if $S\hat{x} = \hat{x}$. We denote by $F(S)$, the set of all fixed points of S, i.e., $F(S) = \{x^* \in H : S\hat{x} = \hat{x}\}$. Many problems in sciences and engineering can be formulated as finding solutions of the FPP of a nonlinear mapping. Recently, several authors have studied OPs dealing with finding a common solution of the set of fixed points of a nonlinear mapping and the set of solutions of variational inclusion problem/equilibrium problem. The motivation for studying such a common solution problem lies in its potential application to mathematical models whose constraints can be expressed as fixed point problem, variational inclusion problem and/equilibrium problem. This arises in practical problems such as signal processing, network resource allocation, image recovery. A scenario is in network bandwidth allocation problem for two services in a heterogeneous wireless access networks in which the bandwidth of the services are mathematically related (see, for instance, [19]).

In [20], Liu introduced the following algorithm for finding a common element of the set of solutions of the EP and set of fixed points of a k-strictly pseudocontractive mapping in the setting of real Hilbert spaces.

Algorithm 1.1.

$$F(u_n, y) + \frac{1}{r_n}(y-u_n, u_n-x_n) \geq 0, \quad \forall y \in C,$$
$$y_n = \beta_n u_n + (I-\beta_n)Su_n,$$
$$x_{n+1} = \alpha_n \gamma f(x_n) + (I-\alpha_n D)y_n, \quad \forall\, n \in \mathbb{N},$$

where $S : C \to H$ is a k-strictly pseudocontractive mapping, $f : H \to H$ is a contraction with constant $\rho \in (0,1)$ and D is a strongly positive bounded linear operator on H with coefficient $\bar{\gamma}$ and $0 < \gamma < \frac{\bar{\gamma}}{\rho}$. Under some conditions on the control parameters, the author proved that the sequence generated by the algorithm converges strongly to an element in the solution set, which also solves certain variational inequality. In 2011, Wang proposed the following general composite iterative method for approximating a common solution of an infinite family strict pseudo-contractions in Hilbert spaces:

Algorithm 1.2.

$$\begin{cases} x_1 \in C \\ y_n = \beta_n x_n + (1-\beta_n) W_n x_n, \\ x_{n+1} = \alpha_n \gamma f(x_n) + (I - \mu \alpha_n D) y_n, \ \forall\, n \geq 1, \end{cases}$$

where W_n is a mapping defined by (2.1), f is a contraction with constant $\rho \in (0,1), D$ is a k-Lipschitzian and η - strongly monotone operator with $0 < \mu < 2\eta/k^2$. Under appropriate conditions on the control parameters, they proved that the sequence generated by Algorithm 1.2 converges strongly to a common element of the fixed points of an infinite family of strict pseudo-contractions, which is a also a unique solution of certain variational inequality problem.

Based on the heavy ball methods of a two-order time dynamical system, Polyak [21] first proposed an inertial extrapolation as an acceleration process to solve the smooth convex minimization problem. The inertial algorithm is a two-step iteration where the next iterate is defined by making use of the previous two iterates. Recently, several researchers have constructed some fast iterative algorithms by using inertial extrapolation (see, e.g., [22, 23, 24]). In 2018, Cholamjiak *et al.* [25] introduced the following inertial forward-backward splitting algorithm, which combines Halpern and Mann iteration methods for solving inclusion problems in Hilbert spaces:

Algorithm 1.3.

$$\begin{aligned} y_n &= x_n + \alpha_n (x_n - x_{n-1}), \\ x_{n+1} &= \beta_n u + \xi_n y_n + \mu_n J^B_{\lambda_n}(y_n - \lambda_n A y_n), \quad n \geq 1, \end{aligned}$$

where $A : H \to H$ is a k-inverse strongly monotone operator and $B : H \to 2^H$ is a maximal monotone operator, $J^B_{\lambda_n} = (I + \lambda_n B)^{-1}, 0 < \lambda_n \leq 2k$, $\{\alpha_n\} \subset [0, \alpha]$ with $\alpha \in [0,1)$ and $\{\beta_n\}, \{\xi_n\}$ and $\{\mu_n\}$ are sequences in $[0,1]$ with $\beta_n + \xi_n + \mu_n = 1$. Under the following conditions on the control parameters:

(1) $\sum_{n=1}^{\infty} \alpha_n ||x_n - x_{n-1}|| < \infty$;
(2) $\lim_{n\to\infty} \beta_n = 0, \sum_{n=1}^{\infty} \beta_n = \infty$;
(3) $0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < 2k$;
(4) $\liminf_{n\to\infty} \mu_n > 0$,

they proved that the sequence generated by Algorithm 1.3 converges strongly to an element in the solution set. However, authors pointed out that the summability condition (1) of Algorithm 1.3 is a drawback in its implementation (see [26]). More recently, Thong and Vinh [27] studied the problem of finding a common element of the set of solutions of variational inclusion problems and the fixed points set of a nonexpansive mapping. They introduced the following modified inertial forward-backward splitting algorithm combined with viscosity techniques for finding a common solution of the problems in Hilbert spaces.

Algorithm 1.4.
Initialization: Select $x_0, x_1 \in H$ and set $n := 1$.

Step 1. Compute

$$w_n = x_n + \alpha_n(x_n - x_{n-1}),$$
$$z_n = (I + \lambda B)^{-1}(I - \lambda A)w_n.$$

If $z_n = w_n$ then stop (z_n is a solution to (1.1)). Otherwise, go to **Step 2**.

Step 2. Compute

$$x_{n+1} = \beta_n f(x_n) + (1 - \beta_n)Tz_n.$$

Let $n := n+1$ and return to **Step 1**.

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where $T : H \to H$ is a nonexpansive mapping, $f : H \to H$ is a contraction with constant $\rho \in [0,1), A : H \to H$ is a k-inverse strongly monotone operator, $B : H \to 2^H$ is a maximal monotone operator and $\lambda \in (0,2k)$ is the step size of the algorithm. Under the following conditions on the control sequences:

(1) $\{\beta_n\} \subset (0,1), \lim_{n\to\infty} \beta_n = 0, \sum_{n=1}^{\infty} \beta_n = \infty, \lim_{n\to\infty} \frac{\beta_{n-1}}{\beta_n} = 1;$
(2) $\{\alpha_n\} \subset [0,\alpha), \alpha > 0, \lim_{n\to\infty} \frac{\alpha_n}{\beta_n} ||x_n - x_{n-1}|| = 0,$

the authors proved that the sequence generated by Algorithm (1.4) converges strongly to an element in the solution set. We observe that the summability condition in Algorithm 1.3 has been dispensed in Algorithm 1.4. However, we point out that the step size of Algorithm 1.4 is a constant and hence admits the same value for each iteration. Moreover, additional restriction was imposed on the control parameter β_n, that is, $\lim_{n\to\infty} \frac{\beta_{n-1}}{\beta_n} = 1$. Therefore, from the above review it is natural to ask the following research question:
Question. Can we construct an inertial iterative scheme for finding solution of the VIP in which the conditions on the control parameters in Algorithm 1.3 and Algorithm 1.4 are relaxed and the step size is a sequence, and which gives a strong convergence result. We provide an affirmative answer to this question.

Motivated and inspired by the above results and the ongoing research in this direction, in this paper, we propose an inertial algorithm for approximating common solutions of the EP, VIP and FPP for an infinite family of strict pseudocontractive mappings in Hilbert spaces. We obtain strong convergence results for the proposed algorithm and apply our results to split feasibility problems and variationl inequality problems. A numerical example is provided to demonstrate the efficiency of our algorithm in comparison with other existing methods in the literature. The remaining sections of this paper are organized as follows. In Section 2, we recall some definitions and preliminary results employed in the convergence analysis of the proposed algorithm. We prove and analyze the convergence of the algorithm in Section 3. In Section 4, we apply our results to optimization problems. In Section 5, a numerical example is provided to illustrate the efficiency of our algorithm in comparison with other existing results in the literature. Finally, we give a concluding remark in Section 6, the last section.

2. PRELIMINARIES

In this section, we recall some useful definitions and lemmas required for establishing our main results. Let H be a real Hibert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let C be a nonempty closed convex subset of H. The weak and strong convergence of the sequence $\{x_n\}_{n=1}^{\infty}$ to x as

$n \to \infty$ is denoted by $x_n \rightharpoonup x$ and $x_n \to x$, respectively. For any $x \in H$, there exists a unique closest point in C denoted by $P_C x$ such that

$$||x - P_C x|| \leq ||x - y||, \quad \forall\, y \in C,$$

and the mapping P_C is called the metric projection of H onto C. It is known that P_C is nonexpansive and has the following properties:

(i) for $x \in H$ and $z \in C$, $z = P_C x \iff \langle x - z, z - y\rangle \leq 0$, $\forall y \in C$,
(ii) for $x \in H$, and $y \in C$, $||y - P_C x||^2 + ||x - P_C x||^2 \leq ||x - y||^2$,
(iii) $\langle x - y, P_C x - P_C y\rangle \geq ||P_C x - P_C y||^2, \quad \forall\, x, y \in H$.

Lemma 2.1. *Let H be a real Hilbert space, $\lambda \in (0,1)$, then $\forall x, y \in H$, we have*

(i) $\|x + y\|^2 = \|x\|^2 + 2\langle x, y\rangle + \|y\|^2$;
(ii) $\|x - y\|^2 = \|x\|^2 - 2\langle x, y\rangle + \|y\|^2$;
(iii) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle$.

Lemma 2.2. [28] *For each $x_1, \dots, x_m \in H$ and $\alpha_1, \dots, \alpha_m \in [0,1]$ with $\sum_{i=1}^m \alpha_i = 1$, the following holds:*

$$||\alpha_1 x_1 + \ldots + \alpha_m x_m||^2 = \sum_{i=1}^{m} \alpha_i ||x_i||^2 - \sum_{1 \leq i < j \leq m} \alpha_i \alpha_j ||x_i - x_j||^2.$$

Lemma 2.3. [29] *Let $\{a_n\}, \{c_n\} \subset \mathbb{R}_+, \{\sigma_n\} \subset (0,1)$ and $\{b_n\} \subset \mathbb{R}$ be sequences such that $a_{n+1} \leq (1 - \sigma_n)a_n + b_n + c_n$, for all $n \geq 0$. Assume $\sum_{n=0}^{\infty} |c_n| < \infty$. Then the following results hold:*

(1) If $b_n \leq \beta\sigma_n$ for some $\beta \geq 0$, then $\{a_n\}$ is a bounded sequence.
(2) If we have $\sum_{n=0}^{\infty} \sigma_n = \infty$ and $\limsup_{n\to\infty} \frac{b_n}{\sigma_n} \leq 0$, then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.4. [30] *Let $\{a_n\}$ be a sequence of non-negative real numbers, $\{\alpha_n\}$ be a sequence in $(0,1)$ with $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\{b_n\}$ be a sequence of real numbers. Assume that*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n b_n, \quad \forall n \geq 1,$$

if $\limsup_{k\to\infty} b_{n_k} \leq 0$ for every subsequence $\{a_{n_k}\}$ of $\{a_n\}$ satisfying $\liminf_{k\to\infty}(a_{n_{k+1}} - a_{n_k}) \geq 0$, then $\lim_{n\to\infty} a_n = 0$.

Definition 2.5. Let H be a real Hilbert space H. A mapping $T : H \to H$ is said to be:

(1) *L-Lipschitz continuous*, where $L > 0$, if

$$||Tx - Ty|| \leq L||x - y||, \quad \forall\, x, y \in H;$$

if $L \in [0,1)$, then T is called a *contraction mapping;*
(2) *nonexpansive* if T is $1-$Lipschitz continuous;
(3) *k-strictly pseudo-contractive* if there exists a constant $k \in [0,1)$ such that

$$||Tx - Ty||^2 \leq ||x - y||^2 + k||(I - T)x - (I - T)y||^2, \quad \forall\, x, y \in H;$$

(4) *monotone* if

$$\langle Tx - Ty, x - y\rangle \geq 0, \quad \forall\, x, y \in H;$$

(5) *k-inverse-strongly monotone* (k-ism) if there exists a constant $k > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq k\|Ax - Ay\|^2,\ \forall\, x, y \in H;$$

(6) *firmly nonexpansive* if

$$\|Tx - Ty\|^2 \leq \langle Tx - Ty, x - y\rangle, \ \forall\, x, y \in H,$$

or equivalently,

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 - \|(I - T)x - (I - T)y\|^2, \ \forall\, x, y \in H.$$

Observe that the class of k-strict pseudo-contractive mappings properly contains the class of nonexpansive mappings. That is, T is nonexpansive if and only if T is 0-strict pseudo-contractive. It is known that if T is a k-strict pseudo-contractiion and $F(T) \neq \emptyset$, then $F(T)$ is a closed convex subset of H (see [31]). Strict pseudo-contractions have many applications, due to their ties with inverse strongly monotone operators. It is known that, if B is a strongly monotone operator, then $T = I - B$ is a strict pseudo-contraction, and so we can recast a problem of zeros for B as a fixed point problem for T.

Lemma 2.6. [31] *Let C be a nonempty closed convex subset of a real Hilbert space H and $S : C \to C$ be a k-strict pseudo-contractive mapping. Define a mapping $T : C \to C$ by $Tx = \alpha x + (1 - \alpha)Sx$ for all $x \in C$ and $\alpha \in [k, 1)$. Then T is a nonexpansive mapping with $F(T) = F(S)$.*

Definition 2.7. [32] Let $\{S_n\}$ be a sequence of k_n-strict pseudo-contractions. Define $S'_n = t_n I + (1 - t_n)S_n, t_n \in [k_n, 1)$. Then, by Lemma 2.6, S'_n is nonexpansive. In this paper, we consider the mapping W_n defined by

$$\begin{cases} U_{n,n+1} = I, \\ U_{n,n} = \zeta_n S'_n U_{n,n+1} + (1 - \zeta_n)I, \\ U_{n,n-1} = \zeta_{n-1} S'_{n-1} U_{n,n} + (1 - \zeta_{n-1})I, \\ \cdots, \\ U_{n,k} = \zeta_k S'_k U_{n,k+1} + (1 - \zeta_k)I, \\ U_{n,k-1} = \zeta_{k-1} S'_{k-1} U_{n,k} + (1 - \zeta_{k-1})I, \\ \cdots, \\ U_{n,2} = \zeta_2 S'_2 U_{n,3} + (1 - \zeta_2)I, \\ W_n = U_{n,1} = \zeta_1 S'_1 U_{n,2} + (1 - \zeta_1)I. \end{cases} \tag{2.1}$$

where $\{\zeta_i\}$ is a sequence of real numbers such that $0 \leq \zeta_i \leq 1$ for all $i \geq 1$. For each $n \geq 1$, such a mapping W_n is nonexpansive.

We have the following lemmas relating to the mapping W_n, which are needed in proving our main results.

Lemma 2.8. [33] *Let $\{S'_i\}$ be an infinite family of nonexpansive mappings on a Hilbert space H such that $\bigcap_{i=1}^{\infty} F(S'_i) \neq \emptyset$ and $\{\zeta_i\}$ be a real sequence such that $0 < \zeta_i \leq b < 1$ for all $i \geq 1$. Then we have the following:*

(1) *W_n is nonexpansive and $F(W_n) = \bigcap_{i=1}^{n} F(S'_i)$ for each $n \geq 1$;*
(2) *for each $x \in H$ and for each positive integer k, the $\lim_{n\to\infty} U_{n,k}x$ exists;*
(3) *the mapping W defined by*

$$Wx := \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1}x \quad \text{for all } x \in H \tag{2.2}$$

is a nonexpansive mapping satisfying $F(W) = \bigcap_{i=1}^{\infty} F(S_i')$, which is called the modified W-mapping generated by $S_1, S_2, \cdots, \zeta_1, \zeta_2, \cdots$ and $t_1, t_2, \cdots$.

By combining Lemma 2.6 and Lemma 2.8, it follows that $F(W) = \bigcap_{i=1}^{\infty} F(S_i') = \bigcap_{i=1}^{\infty} F(S_i)$.

Lemma 2.9. [16] *Let $\{S_i'\}$ be an infinite family of nonexpansive mappings on a Hilbert space H such that $\bigcap_{i=1}^{\infty} F(S_i') \neq \emptyset$ and $\{\zeta_i\}$ be a real sequence such that $0 < \zeta_i \leq b < 1$ for all $i \geq 1$, where b is a positive real number. If K is any bounded subset of H, then $\lim_{n\to\infty} \sup_{x\in K} ||Wx - W_n x|| = 0$.*

Lemma 2.10. [34] *Each Hilbert space H satisfies the Opial condition, that is, for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality $\liminf_{n\to\infty} ||x_n - x|| < \liminf_{n\to\infty} ||x_n - y||$ holds for every $y \in H$ with $y \neq x$.*

Lemma 2.11. [31] *If S is a k-strict pseudo-contraction on closed convex subset C of a real Hilbert space H, then $I - S$ is demiclosed at any point $y \in H$.*

Lemma 2.12. [35] *Let $A : H \to H$ be a k-inverse-strongly monotone mapping. Then*

(1) A is $\frac{1}{k}$-Lipschitz continuous and monotone mapping;
(2) if λ is any constant in $(0, 2]$, then the mapping $I - \lambda A$ is nonexpansive , where I is the identity mapping on H.

Definition 2.13. A multi-valued mapping $B : H \to 2^H$ is called monotone if, for all $x, y \in H$, $u \in Bx$ and $v \in By$ implies that $\langle u - v, x - y\rangle \geq 0$. A multi-valued mapping $B : H \to 2^H$ is called maximal monotone if it is monotone and for any $(x, u) \in H \times H$, $\langle u - v, x - y\rangle \geq 0$ for every $(y, v) \in$ Graph (B) (the graph of mapping B) implies that $u \in Bx$.

Definition 2.14. Let $B : H \to 2^H$ be a multi-valued maximal monotone mapping. Then the resolvent mapping $J_\lambda^B : H \to H$ associated with B is defined by $J_\lambda^B(x) = (I + \lambda B)^{-1}(x)$, $\forall x \in H$, for some $\lambda > 0$, where I is the identity operator on H.

It is well known that if $B : H \to 2^H$ is a multi-valued maximal monotone mapping and $\lambda > 0$, then $\mathrm{Dom}(J_\lambda^B) = H$, and J_λ^B is single-valued and firmly nonexpansive mapping.

Assumption 2.15. *For solving the EP, we assume that the bifunction $F : C \times C \to \mathbb{R}$ satisfies the following conditions:*

(A1) $F(x, x) = 0$ for all $x \in C$;
(A2) F is monotone, that is, $F(x, y) + F(y, x) \leq 0$ for all $x, y \in C$;
(A3) F is upper hemicontinuous, that is, for all $x, y, z \in C$, $\lim_{t\downarrow 0} F\big(tz + (1-t)x, y\big) \leq F(x, y)$;
(A4) for each $x \in C, y \mapsto F(x, y)$ is convex and lower semicontinuous.

Lemma 2.16. [36] *Let C be a nonempty closed convex subset of a Hilbert space H and $F : C \times C \to \mathbb{R}$ be a bifunction satisfying Assumption 2.15. For $r > 0$ and $x \in H$, define a mapping $T_r^F : H \to C$ as follows:*

$$T_r^F(x) = \{z \in C : F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0,\ \forall\ y \in C\}. \tag{2.3}$$

Then T_r^F is well defined and the following hold:

(1) for each $x \in H, T_r^F(x) \neq \emptyset$;

(2) T_r^F *is single-valued;*
(3) T_r^F *is firmly nonexpansive, that is, for any* $x, y \in H$,

$$\|T_r^F x - T_r^F y\|^2 \le \langle T_r^F x - T_r^F y, x - y\rangle;$$

(4) $F(T_r^F) = EP(F)$;
(5) $EP(F)$ *is closed and convex.*

Lemma 2.17. [37] *Let* E *be a real Banach space. Let* $B : E \to 2^E$ *maximal monotone operator and* $A : E \to E$ *be a k-inverse strongly monotone mapping on* E*. Define* $T_\lambda = (I + \lambda B)^{-1}(I - \lambda A)$, $\lambda > 0$*. Then*

(i) $F(T_\lambda) = (A+B)^{-1}(0)$;
(ii) for $0 < s \le \lambda$ *and* $x \in E$, $\|x - T_s x\| \le 2\|x - T_\lambda x\|$.

Lemma 2.18. [38] *Let* $B : H \to 2^H$ *be a set-valued maximal monotone mapping and* $\lambda > 0$*. Then* J_λ^B *is a single-valued and firmly nonexapansive mapping.*

3. Main Results

In this section, we present the proposed algorithm and investigate its convergence. Let C be a nonempty closed convex subset of a real Hilbert space H. Let $A : H \to H$ be a $k-$ism and $B : H \to 2^H$ be a maximal monotone mapping. Let $f : H \to H$ be a contraction mapping with coefficient $\rho \in (0,1)$. Let $\{W_n\}$ be a sequence defined by (2.1) and let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying Assumption 2.15. Suppose that the solution set denoted by $\Gamma = (A+B)^{-1}(0) \cap EP(F) \cap \bigcap_{i=1}^{\infty} F(S_i)$ is nonempty, where $S_i : H \to H$ is an infinite family of k_i-strict pseudo-contractions. We establish the convergence of the algorithm under the following conditions on the control parameters:

(C1) Let $\{\delta_n\}, \{\xi_n\}, \{\mu_n\} \subset (0,1), \{\beta_n\} \subset (0,1)$ such that $\lim_{n\to\infty} \beta_n = 0$ and $\sum_{n=0}^{\infty} \beta_n = \infty$;
(C2) Let $\alpha > 0, \{\theta_n\}$ be a positive sequence such that $\lim_{n\to\infty} \frac{\theta_n}{\beta_n} = 0$;
(C3) $0 < \liminf_{n\to\infty} \lambda_n < \limsup_{n\to\infty} \lambda_n < 2k, \{r_n\} \subset (0,\infty)$ such that $\liminf_{n\to\infty} r_n > 0$.

Now, the proposed algorithm is presented as follows:

Algorithm 3.1.

Step 0 : Select initial data $x_0, x_1 \in H$ and set $n = 1$.
Step 1. Given the $(n-1)th$ and nth iterates, choose α_n such that $0 \le \alpha_n \le \hat{\alpha}_n$ with $\hat{\alpha}_n$ defined by

$$\hat{\alpha}_n = \begin{cases} \min\left\{\alpha, \frac{\theta_n}{\|x_n - x_{n-1}\|}\right\}, & \text{if } x_n \ne x_{n-1}, \\ \alpha, & \text{otherwise.} \end{cases} \tag{3.1}$$

Step 2: Compute

$$w_n = x_n + \alpha_n(x_n - x_{n-1}).$$

Step 3: Compute

$$F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - w_n\rangle \ge 0, \ \forall\, y \in H.$$

Step 4: Compute

$$v_n = \delta_n w_n + (1 - \delta_n)u_n.$$

Step 5: Compute

$$z_n = (I + \lambda_n B)^{-1}(I - \lambda_n A)v_n.$$

Step 6: Compute
$$x_{n+1} = \beta_n f(x_n) + \xi_n x_n + \mu_n W_n z_n.$$
Set n := n + 1 and return to **Step 1.**

Remark 3.2. By conditions (C1) and (C2), one can easily verify from (3.1) that
$$\lim_{n\to\infty} \alpha_n ||x_n - x_{n-1}|| = 0 \text{ and } \lim_{n\to\infty} \frac{\alpha_n}{\beta_n} ||x_n - x_{n-1}|| = 0. \tag{3.2}$$

Now, we state the strong convergence theorem as follows:

Theorem 3.3. *Suppose that $\{x_n\}$ is a sequence generated by Algorithm 3.1 such that conditions (C1)-(C3) are satisfied and $\Gamma \neq \emptyset$. Then the sequence $\{x_n\}$ converges strongly to an element $\hat{x} \in \Gamma$, where $\hat{x} = P_\Gamma \circ f(\hat{x})$.*

First, we prove some lemmas which will be employed in establishing Theorem 3.3.

Lemma 3.4. *Let $\{x_n\}$ be a sequence generated by Algorithm 3.1. Then $\{x_n\}$ is bounded.*

Proof. Let $q \in \Gamma$. Then, by Lemma 2.18 and the conditions on the control parameters, we have
$$\begin{aligned}
\|z_n - q\|^2 =& \|(I + \lambda_n B)^{-1}(I - \lambda_n A)v_n - (I - \lambda_n B)^{-1}(I - \lambda_n A)q\|^2 \\
&\leq \|v_n - q - \lambda_n(Av_n - Aq)\|^2 \\
&= \|v_n - q\|^2 - 2\lambda_n \langle Av_n - Aq, v_n - q\rangle + \lambda_n^2 \|Av_n - Aq\|^2 \\
&\leq \|v_n - q\|^2 - 2\lambda_n k\|Av_n - Aq\|^2 + \lambda_n^2 \|Av_n - Aq\| \\
&= \|v_n - q\|^2 - (2k - \lambda_n)\lambda_n \|Av_n - Aq\|^2 \qquad (3.3) \\
&\leq \|v_n - q\|^2. \qquad (3.4)
\end{aligned}$$
Thus, from (3.4), we have
$$\|z_n - q\| \leq \|v_n - q\|. \tag{3.5}$$
From the definition of w_n, we have
$$\begin{aligned}
\|w_n - q\| &\leq \|x_n - q\| + \alpha_n \|x_n - x_{n-1}\| \\
&= \|x_n - q\| + \beta_n \frac{\alpha_n}{\beta_n} \|x_n - x_{n-1}\|. \qquad (3.6)
\end{aligned}$$
From Remark 3.2, we have that $\lim_{n\to\infty} \frac{\alpha_n}{\beta_n} \|x_n - x_{n-1}\| = 0$. Then there exists a constant $L_1 > 0$ such that $\frac{\alpha_n}{\beta_n} \|x_n - x_{n-1}\| \leq L_1$, for all $n \geq 1$. Thus, from (3.6), we obtain
$$\|w_n - q\| \leq \|x_n - q\| + \beta_n L_1. \tag{3.7}$$
Let $T_{r_n}^F w_n = \{u_n \in C : F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - w_n\rangle \geq 0\}$. This implies that $u_n = T_{r_n}^F w_n$. Since $q \in \Gamma$, then $T_{r_n}^F q = q$. By the nonexpansiveness of $T_{r_n}^F$, we have
$$||u_n - q|| = ||T_{r_n}^F w_n - q|| \leq ||w_n - q||. \tag{3.8}$$
From the definition of v_n and (3.8), we have
$$\begin{aligned}
\|v_n - q\| &\leq \delta_n \|w_n - q\| + (1 - \delta_n)\|u_n - q\| \\
&\leq \delta_n \|w_n - q\| + (1 - \delta_n)\|w_n - q\| \\
&= \|w_n - q\|. \qquad (3.9)
\end{aligned}$$

By combining (3.5), (3.7) and (3.9), we have

$$\|z_n - q\| \le \|x_n - q\| + \beta_n L_1.$$

In view of (3.5), (3.6) and (3.9), we obtain

$$\begin{aligned}\|x_{n+1} - q\| &= ||\beta_n(f(x_n) - f(q)) + \beta_n(f(q) - q) + \xi_n(x_n - q) + \mu_n(W_n z_n - q)|| \\ &\le \beta_n\rho||x_n - q|| + \beta_n||fq - q|| + \xi_n||x_n - q|| + \mu_n||z_n - q|| \\ &\le \beta_n\rho||x_n - q|| + \beta_n||fq - q|| + \xi_n||x_n - q|| + \mu_n(||x_n - q|| + \beta_n L_1) \\ &= (1 - \beta_n(1-\rho))||x_n - q|| + \beta_n(1-\rho)\frac{||fq - q|| + \mu_n L_1}{1-\rho} \\ &\le (1 - \beta_n(1-\rho))||x_n - q|| + \beta_n(1-\rho)M^*,\end{aligned}$$

where $M^* := \sup_{n\in N}\{\frac{||fq-q||+\mu_n L_1}{1-\rho}\}$. Setting $a_n = ||x_n - q||$, $b_n = \beta_n(1-\rho)M^*$, $c_n = 0$, and $\sigma_n = \beta_n(1-\rho)$. By Lemma 2.3(1) and the assumptions on the control parameters, it follows that $\{||x_n - q||\}$ is bounded and thus $\{x_n\}$ is bounded. Consequently, $\{w_n\}, \{u_n\}, \{v_n\}, \{z_n\}, \{f(x_n)\}$ are all bounded. □

Lemma 3.5. *The following inequality holds for all $q \in \Gamma$ and $n \in \mathbb{N}$:*

$$\begin{aligned}||x_{n+1} - q||^2 \le \Big(1 - \frac{2\beta_n(1-\rho)}{(1-\beta_n\rho)}\Big)||x_n - q||^2 + \frac{2\beta_n(1-\rho)}{(1-\beta_n\rho)}\Big\{\frac{\beta_n}{2(1-\rho)}L_3 \\ + \frac{3L_2\mu_n(1-\beta_n)}{2(1-\rho)}\frac{\alpha_n}{\beta_n}||x_n - x_{n-1}|| + \frac{1}{(1-\rho)}\langle f(q) - q, x_{n+1} - q\rangle\Big\} \\ - \frac{\mu_n(1-\beta_n)}{(1-\beta_n\rho)}\Big\{(2k - \lambda_n)\lambda_n||Av_n - Aq||^2 + \delta_n(1-\delta_n)||w_n - u_n||^2\Big\}.\end{aligned}$$

Proof. Let $q \in \Gamma$. By applying the Cauchy-Schwartz inequality and Lemma 2.1(i), we have

$$\begin{aligned}||w_n - q||^2 &= ||x_n - q||^2 + \alpha_n^2||x_n - x_{n-1}||^2 + 2\alpha_n\langle x_n - q, x_n - x_{n-1}\rangle \\ &\le ||x_n - q||^2 + \alpha_n^2||x_n - x_{n-1}||^2 + 2\alpha_n||x_n - x_{n-1}||||x_n - q|| \\ &= ||x_n - q||^2 + \alpha_n||x_n - x_{n-1}||(\alpha_n||x_n - x_{n-1}|| + 2||x_n - q||) \\ &\le ||x_n - q||^2 + 3L_2\beta_n\frac{\alpha_n}{\beta_n}||x_n - x_{n-1}||,\end{aligned} \tag{3.10}$$

where $L_2 := \sup_{n\in\mathbb{N}}\{||x_n - q||, \alpha_n||x_n - x_{n-1}||\} > 0$. Also, by using Lemma 2.2 and (3.8), we obtain

$$\begin{aligned}||v_n - q||^2 &= \delta_n||w_n - q||^2 + (1-\delta_n)||u_n - q||^2 - \delta_n(1-\delta_n)||w_n - u_n||^2 \\ &\le \delta_n||w_n - q||^2 + (1-\delta_n)||w_n - q||^2 - \delta_n(1-\delta_n)||w_n - u_n||^2 \\ &= ||w_n - q||^2 - \delta_n(1-\delta_n)||w_n - u_n||^2.\end{aligned} \tag{3.11}$$

By invoking Lemma 2.1, and using (3.3), (3.10) and (3.11), we have

$$\begin{aligned}
&||x_{n+1}-q||^2\\
&\le ||\xi_n(x_n-q)+\mu_n(W_nz_n-q)||^2+2\beta_n\langle f(x_n)-q,x_{n+1}-q\rangle\\
&\le \xi_n^2||x_n-q||^2+\mu_n^2||W_nz_n-q||^2+2\xi_n\mu_n||x_n-q||||W_nz_n-q||+2\beta_n\langle f(x_n)-q,x_{n+1}-q\rangle\\
&\le \xi_n^2||x_n-q||^2+\mu_n^2||z_n-q||^2+\xi_n\mu_n(||x_n-q||^2+||z_n-q||^2)+2\beta_n\langle f(x_n)-q,x_{n+1}-q\rangle\\
&\le \xi_n(1-\beta_n)||x_n-q||^2+\mu_n(1-\beta_n)\Big\{||x_n-q||^2+3L_2\beta_n\frac{\alpha_n}{\beta_n}||x_n-x_{n-1}||\\
&\quad -(2k-\lambda_n)\lambda_n||Av_n-Aq||^2-\delta_n(1-\delta_n)||w_n-u_n||^2\Big\}+2\beta_n\langle f(x_n)-f(q),x_{n+1}-q\rangle\\
&\quad +2\beta_n\langle f(q)-q,x_{n+1}-q\rangle\\
&\le \xi_n(1-\beta_n)||x_n-q||^2+\mu_n(1-\beta_n)\Big\{||x_n-q||^2+3L_2\beta_n\frac{\alpha_n}{\beta_n}||x_n-x_{n-1}||\\
&\quad -(2k-\lambda_n)\lambda_n||Av_n-Aq||^2-\delta_n(1-\delta_n)||w_n-u_n||^2\Big\}+2\beta_n\rho||x_n-q||||x_{n+1}-q||\\
&\quad +2\beta_n\langle f(q)-q,x_{n+1}-q\rangle\\
&\le \big((1-\beta_n)^2+\beta_n\rho\big)||x_n-q||^2+\beta_n\rho||x_{n+1}-q||^2+3L_2\mu_n(1-\beta_n)\beta_n\frac{\alpha_n}{\beta_n}||x_n-x_{n-1}||\\
&\quad +2\beta_n\langle f(q)-q,x_{n+1}-q\rangle-\mu_n(1-\beta_n)\Big\{(2k-\lambda_n)\lambda_n||Av_n-Aq||^2+\delta_n(1-\delta_n)||w_n-u_n||^2\Big\}.
\end{aligned}$$

This implies that

$$\begin{aligned}
&||x_{n+1}-q||^2\\
&\le \frac{(1-2\beta_n+\beta_n^2+\beta_n\rho)}{(1-\beta_n\rho)}||x_n-q||^2+\frac{\beta_n}{(1-\beta_n\rho)}\Big\{3L_2\mu_n(1-\beta_n)\frac{\alpha_n}{\beta_n}||x_n-x_{n-1}||\\
&+2\langle f(q)-q,x_{n+1}-q\rangle\Big\}-\frac{\mu_n(1-\beta_n)}{(1-\beta_n\rho)}\Big\{(2k-\lambda_n)\lambda_n||Av_n-Aq||^2+\delta_n(1-\delta_n)||w_n-u_n||^2\Big\}\\
&\le \Big(1-\frac{2\beta_n(1-\rho)}{(1-\beta_n\rho)}\Big)||x_n-q||^2+\frac{2\beta_n(1-\rho)}{(1-\beta_n\rho)}\Big\{\frac{\beta_n}{2(1-\rho)}L_3+\frac{3L_2\mu_n(1-\beta_n)}{2(1-\rho)}\frac{\alpha_n}{\beta_n}||x_n-x_{n-1}||\\
&+\frac{1}{(1-\rho)}\langle f(q)-q,x_{n+1}-q\rangle\Big\}\\
&-\frac{\mu_n(1-\beta_n)}{(1-\beta_n\rho)}\Big\{(2k-\lambda_n)\lambda_n||Av_n-Aq||^2+\delta_n(1-\delta_n)||w_n-u_n||^2\Big\},
\end{aligned}$$

where $L_3:=\sup\{||x_n-q||^2:n\in\mathbb{N}\}$. This completes the proof. □

Lemma 3.6. *The following inequality holds for all $q\in\Gamma$ and $n\in\mathbb{N}$:*

$$\begin{aligned}
||x_{n+1}-q||^2\le(1-\beta_n)||x_n-q||^2+\beta_n\Big\{||f(x_n)-q||^2+3L_2\mu_n\frac{\alpha_n}{\beta_n}||x_n-x_{n-1}||\Big\}\\
+2L_4||Av_n-Aq||-\mu_n||v_n-z_n||^2-\xi_n\mu_n||W_nz_n-x_n||^2.
\end{aligned}$$

Proof. Applying the fact that $(I+\lambda_n B)^{-1}$ is firmly nonexpansive and $I-\lambda_n A$ is nonexpansive, we have

$$\begin{aligned}
&\|z_n-q\|^2\\
&\le \langle z_n-q,(I-\lambda_nA)v_n-(I-\lambda_nA)q\rangle\\
&\le \frac{1}{2}\|v_n-q\|^2+\frac{1}{2}\|z_n-q\|^2-\frac{1}{2}\|v_n-z_n-\lambda_n(Av_n-Aq)\|^2\\
&\le \frac{1}{2}\|v_n-q\|^2+\frac{1}{2}\|z_n-q\|^2-\frac{1}{2}\|v_n-z_n\|^2-\frac{1}{2}\lambda_n^2\|Av_n-Aq\|^2+\lambda_n\|v_n-z_n\|\|Av_n-Aq\|.
\end{aligned}$$

This implies that

$$\|z_n-q\|^2\le\|v_n-q\|^2-\|v_n-z_n\|^2+2\lambda_n\|v_n-z_n\|\|Av_n-Aq\|. \tag{3.12}$$

By applying Lemma 2.2 and using (3.9), (3.10) and (3.12), we have

$$\begin{aligned}
&||x_{n+1}-q||^2\\
&=\beta_n||f(x_n)-q||^2+\xi_n||x_n-q||^2+\mu_n||W_nz_n-q||^2-\xi_n\mu_n||W_nz_n-x_n||^2\\
&\le\beta_n||f(x_n)-q||^2+\xi_n||x_n-q||^2+\mu_n\Big\{||x_n-q||^2+3L_2\beta_n\frac{\alpha_n}{\beta_n}||x_n-x_{n-1}||-\|v_n-z_n\|^2\\
&+2\lambda_n\|v_n-z_n\|\|Av_n-Aq\|\Big\}-\xi_n\mu_n||W_nz_n-x_n||^2\\
&\le(1-\beta_n)||x_n-q||^2+\beta_n\Big\{||f(x_n)-q||^2+3L_2\mu_n\frac{\alpha_n}{\beta_n}||x_n-x_{n-1}||\Big\}+2L_4||Av_n-Aq||\\
&-\mu_n||v_n-z_n||^2-\xi_n\mu_n||W_nz_n-x_n||^2,
\end{aligned}$$

where $L_4:=\sup_{n\in\mathbb{N}}\{\mu_n\lambda_n||v_n-z_n||\}$. Hence, the desired result. □

Lemma 3.7. *Let $q\in\Gamma$. Suppose that $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$ such that $\liminf_{k\to\infty}(||x_{n_k+1}-q||-||x_{n_k}-q||)\ge 0$. Then $x_{n_k}\rightharpoonup x^*\in\Gamma$, i.e. $w_\omega(x_n)\subset\Gamma$.*

Proof. Suppose $q\in\Gamma$. Then, from Lemma 3.5, we obtain

$$\begin{aligned}
&\frac{\mu_{n_k}(1-\beta_{n_k})}{(1-\beta_{n_k}\rho)}\delta_{n_k}(1-\delta_{n_k})||w_{n_k}-u_{n_k}||^2\\
&\le\Big(1-\frac{2\beta_{n_k}(1-\rho)}{(1-\beta_{n_k}\rho)}\Big)||x_{n_k}-q||^2-||x_{n_k+1}-q||^2+\frac{2\beta_{n_k}(1-\rho)}{(1-\beta_{n_k}\rho)}\Big\{\frac{\beta_{n_k}}{2(1-\rho)}L_3\\
&+\frac{3L_2\mu_{n_k}(1-\beta_{n_k})}{2(1-\rho)}\frac{\alpha_{n_k}}{\beta_{n_k}}||x_{n_k}-x_{n_k-1}||+\frac{1}{(1-\rho)}\langle f(q)-q,x_{n_k+1}-q\rangle\Big\}.
\end{aligned}$$

By the hypothesis of Lemma 3.7 together with the fact that $\lim_{k\to\infty}\beta_{n_k}=0$, we have

$$\frac{\mu_{n_k}(1-\beta_{n_k})}{(1-\beta_{n_k}\rho)}\delta_{n_k}(1-\delta_{n_k})||w_{n_k}-u_{n_k}||^2\to 0,\quad k\to\infty.$$

Therefore, we have

$$||w_{n_k}-u_{n_k}||\to 0,\quad k\to\infty. \tag{3.13}$$

By following similar argument, we get from Lemma 3.5 that $(2k-\lambda_{n_k})\lambda_{n_k}||Av_{n_k}-Aq||^2\to 0$, $k\to\infty$. By the conditions on k and λ_n, it follows that

$$||Av_{n_k}-Aq||\to 0,\quad k\to\infty. \tag{3.14}$$

Also, it follows from Lemma 3.6 that

$$\mu_{n_k}||v_{n_k}-z_{n_k}||^2 \le (1-\beta_{n_k})||x_{n_k}-q||^2 - ||x_{n_k+1}-q||^2 + \beta_{n_k}\left\{||f(x_{n_k})-q||^2 + 3L_2\mu_{n_k}\frac{\alpha_{n_k}}{\beta_{n_k}}||x_{n_k}-x_{n_k-1}||\right\} + 2L_4||Av_{n_k}-Aq||.$$

Using (3.14) together with the condition of β_n, we conclude from the hypothesis of Lemma 3.7 that $\mu_{n_k}||v_{n_k}-z_{n_k}||^2 \to 0$, $k\to\infty$. By the condition on μ_n, it follows that

$$||v_{n_k}-z_{n_k}|| \to 0, \quad k\to\infty. \tag{3.15}$$

Following similar argument, we obtain from Lemma 3.6 that

$$||W_{n_k}z_{n_k}-x_{n_k}|| \to 0, \quad k\to\infty. \tag{3.16}$$

By Remark 3.2, we have

$$||w_{n_k}-x_{n_k}|| = \alpha_{n_k}||x_{n_k}-x_{n_k-1}|| \to 0, \quad k\to\infty. \tag{3.17}$$

By the definition of v_n and (3.13), we obtain

$$\begin{aligned}||v_{n_k}-w_{n_k}|| &= ||\delta_{n_k}w_{n_k}+(1-\delta_{n_k})u_{n_k}-w_{n_k}|| \\ &\le \delta_{n_k}||w_{n_k}-w_{n_k}||+(1-\delta_{n_k})||u_{n_k}-w_{n_k}|| \to 0, \quad k\to\infty.\end{aligned} \tag{3.18}$$

By applying (3.15), (3.16), (3.17) and (3.18), we obtain

$$||W_{n_k}z_{n_k}-z_{n_k}|| \to 0, \quad k\to\infty. \tag{3.19}$$

Combining (3.16) and (3.19), we have

$$||x_{n_k}-z_{n_k}|| \to 0, \quad k\to\infty. \tag{3.20}$$

Also, using (3.13), (3.15), (3.17) and (3.20), we get

$$||x_{n_k}-u_{n_k}|| \to 0, \qquad ||x_{n_k}-v_{n_k}|| \to 0, \quad k\to\infty. \tag{3.21}$$

By applying (3.16) and the condition on β_n, we get

$$\begin{aligned}||x_{n_k+1}-x_{n_k}|| &= ||\beta_{n_k}f(x_{n_k})+\xi_{n_k}x_{n_k}+\mu_{n_k}W_{n_k}z_{n_k}-x_{n_k}|| \\ &\le \beta_{n_k}||f(x_{n_k})-x_{n_k}||+\xi_{n_k}||x_{n_k}-x_{n_k}||+\mu_{n_k}||W_{n_k}z_{n_k}-x_{n_k}|| \to 0, \quad k\to\infty.\end{aligned} \tag{3.22}$$

We next show that $w_\omega(x_n) \subset \cap_{i=1}^{\infty}F(S_i) = F(W)$. Let $x^* \in w_\omega(x_n)$ and suppose that $x^* \notin F(W)$, that is, $Wx^* \neq x^*$. From (3.20), we have that $w_\omega(x_n) = w_\omega(z_n)$. By Lemma 2.10, we have

$$\begin{aligned}\liminf_{k\to\infty}||z_{n_k}-x^*|| &< \liminf_{k\to\infty}|z_{n_k}-Wx^*||| \\ &\le \liminf_{k\to\infty}\{||z_{n_k}-Wz_{n_k}||+||Wz_{n_k}-Wx^*||\} \\ &\le \liminf_{k\to\infty}\{||z_{n_k}-Wz_{n_k}||+||z_{n_k}-x^*||\}.\end{aligned} \tag{3.23}$$

Since $x_{n_k} \in K$ for all $k \ge 1$ and $\lim_{k\to\infty}||x_{n_k}-z_{n_k}|| = 0$, we obtain

$$\begin{aligned}||Wz_{n_k}-z_{n_k}|| &\le ||Wz_{n_k}-W_{n_k}z_{n_k}||+||W_{n_k}z_{n_k}-z_{n_k}|| \\ &\le \sup_{x\in K}||Wx-W_{n_k}x||+||W_{n_k}z_{n_k}-z_{n_k}||.\end{aligned}$$

By applying Lemma 2.9 and (3.19), we have $\lim_{k\to\infty}||Wz_{n_k}-z_{n_k}||=0$. Combining this with (3.23) yields $\liminf_{k\to\infty}||z_{n_k}-x^*||<\liminf_{k\to\infty}||z_{n_k}-x^*||$, which is a contradiction. Hence, we have

$$x^*\in F(W)=\cap_{i=1}^{\infty}F(S_i),\quad \text{i.e.,}\quad w_\omega(x_n)\subset F(W)=\bigcap_{i=1}^{\infty}F(S_i). \tag{3.24}$$

Next, we show that $x^*\in EP(F)$. From the definition of $T^F_{r_{n_k}}w_{n_k}$, we have that $r_{n_k}F(u_{n_k},y)+\langle y-u_{n_k},u_{n_k}-w_{n_k}\rangle\geq 0$, $\forall y\in C$. By the monotonicity of F, we have

$$\frac{1}{r_{n_k}}\langle y-u_{n_k},u_{n_k}-w_{n_k}\rangle\geq F(y,u_{n_k}),\quad \forall\, y\in C.$$

Since $x_{n_k}\rightharpoonup x^*$, then by (3.21) it follows that $u_{n_k}\rightharpoonup x^*$. By combining (3.17) and (3.21), and applying condition (A4) together with the fact that $\liminf_{k\to\infty}r_{n_k}>0$, we obtain

$$F(y,x^*)\leq 0,\quad \forall\, y\in C. \tag{3.25}$$

Let $y_t=ty+(1-t)x^*$, $\forall\, t\in(0,1]$ and $y\in C$. This implies that $y_t\in C$, and it follows from (3.25) that $F(y_t,x^*)\leq 0$. So, by applying conditions (A1)-(A4), we have

$$\begin{aligned}0&=F(y_t,y_t)\\&\leq tF(y_t,y)+(1-t)F(y_t,x^*)\\&\leq tF(y_t,y).\end{aligned}$$

Hence, we have $F(y_t,y)\geq 0$, $\forall\, y\in C$. Letting $t\to 0$, by condition (A3), we get $F(x^*,y)\geq 0$, $\forall\, y\in C$. This implies that

$$x^*\in EP(F). \tag{3.26}$$

Finally, we show that $x^*\in(A+B)^{-1}(0)$. Let $T_{n_k}=(I+\lambda_{n_k}B)^{-1}(I-\lambda_{n_k}A)$. From the definition of z_n and (3.15), we have

$$\lim_{k\to\infty}||T_{n_k}v_{n_k}-v_{n_k}||=\lim_{k\to\infty}||z_{n_k}-v_{n_k}||=0.$$

Since $\liminf_{k\to\infty}\lambda_{n_k}>0$, we have that there exists $\delta>0$ such that $\lambda_{n_k}\geq\delta$ for all $k\geq 1$. By Lemma 2.17(ii), we have

$$\lim_{k\to\infty}||T_\delta v_{n_k}-v_{n_k}||\leq 2\lim_{k\to\infty}||T_{n_k}v_{n_k}-v_{n_k}||=0.$$

By Lemma 2(ii) and Lemma 2.18, we have that T_δ is nonexpansive and $v_{n_k}\rightharpoonup x^*$. By the demi-closedness of $I-T_\delta$, we have that $x^*\in F(T_\delta)$. By Lemma 2.17(i), we obtain $x^*\in(A+B)^{-1}(0)$. Hence, by combining (3.24) and (3.26), we have that $w_\omega(x_n)\subset\Gamma$ as required. □

Now, we prove the strong convergence result Theorem 3.3.

Proof of Theorem 3.3.

Let $\hat{x}=P_\Gamma\circ f(\hat{x})$. Then it follows from Lemma 3.5 that

$$\begin{aligned}||x_{n+1}-\hat{x}||^2\leq\Big(1-\frac{2\beta_n(1-\rho)}{(1-\beta_n\rho)}\Big)||x_n-\hat{x}||^2+\frac{2\beta_n(1-\rho)}{(1-\beta_n\rho)}\Big\{\frac{\beta_n}{2(1-\rho)}L_3\\+\frac{3L_2\mu_n(1-\beta_n)}{2(1-\rho)}\frac{\alpha_n}{\beta_n}||x_n-x_{n-1}||+\frac{1}{(1-\rho)}\langle f(\hat{x})-\hat{x},x_{n+1}-\hat{x}\rangle\Big\}.\end{aligned} \tag{3.27}$$

Now, we claim that the sequence $\{||x_n-\hat{x}||\}$ converges to zero. In order to establish this, by Lemma 2.4, it suffices to show that $\limsup_{k\to\infty}\langle f(\hat{x})-\hat{x},x_{n_k+1}-\hat{x}\rangle\leq 0$ for every subsequence

$\{||x_{n_k}-\hat{x}||\}$ of $\{||x_n-\hat{x}||\}$ satisfying $\liminf_{k\to\infty}(||x_{n_k+1}-\hat{x}||-||x_{n_k}-\hat{x}||)\geq 0$. Suppose that $\{||x_{n_k}-\hat{x}||\}$ is a subsequence of $\{||x_n-\hat{x}||\}$ such that $\liminf_{k\to\infty}(||x_{n_k+1}-\hat{x}||-||x_{n_k}-\hat{x}||)\geq 0$. Then, by Lemma 3.7, we have that $w_\omega\{x_n\}\subset\Gamma$. It also follows from (3.20) that $w_\omega\{z_n\}=w_\omega\{x_n\}$. By the boundedness of $\{x_{n_k}\}$, there exists a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ such that $x_{n_{k_j}}\rightharpoonup x^\dagger$ and

$$\lim_{j\to\infty}\langle f(\hat{x})-\hat{x},x_{n_{k_j}}-\hat{x}\rangle=\limsup_{k\to\infty}\langle f(\hat{x})-\hat{x},x_{n_k}-\hat{x}\rangle=\limsup_{k\to\infty}\langle f(\hat{x})-\hat{x},z_{n_k}-\hat{x}\rangle. \tag{3.28}$$

Since $\hat{x}=P_\Gamma\circ f(\hat{x})$, it follows from (3.28) that

$$\limsup_{k\to\infty}\langle f(\hat{x})-\hat{x},x_{n_k}-\hat{x}\rangle=\lim_{j\to\infty}\langle f(\hat{x})-\hat{x},x_{n_{k_j}}-\hat{x}\rangle=\langle f(\hat{x})-\hat{x},x^\dagger-\hat{x}\rangle\leq 0. \tag{3.29}$$

Hence, by (3.22) and (3.29), we have

$$\begin{aligned}\limsup_{k\to\infty}\langle f(\hat{x})-\hat{x},x_{n_k+1}-\hat{x}\rangle&\leq\limsup_{k\to\infty}\langle f(\hat{x})-\hat{x},x_{n_k+1}-x_{n_k}\rangle+\limsup_{k\to\infty}\langle f(\hat{x})-\hat{x},x_{n_k}-\hat{x}\rangle\\&=\langle f(\hat{x})-\hat{x},x^\dagger-\hat{x}\rangle\leq 0.\end{aligned} \tag{3.30}$$

Applying Lemma 2.4 to (3.27), and using (3.30) together with Remark 3.2 and the condition on β_n, we deduce that $\lim_{n\to\infty}||x_n-\hat{x}||=0$ as required. Hence, that completes the proof.

Taking $S_n=S$ for all $n\geq 1$ in Theorem 3.3, we obtain the following consequent result.

Corollary 3.8. *Let C be a nonempty closed convex subset of a real Hilbert space H and let $f:H\to H$ be a contraction mapping with coefficient $\rho\in(0,1)$. Let $\{W_n\}$ be a sequence defined by (2.1) and let $F:C\times C\to\mathbb{R}$ be a bifunction satisfying Assumption 2.15. Suppose that the solution set denoted by $\Gamma\neq\emptyset$ and let $\{x_n\}$ be a sequence generated as follows:*

Algorithm 3.9.

Step 0 : Select initial data $x_0,x_1\in H$ and set $n=1$.

Step 1. Given the $(n-1)th$ and nth iterates, choose α_n such that $0\leq\alpha_n\leq\hat{\alpha}_n$ with $\hat{\alpha}_n$ defined by

$$\hat{\alpha}_n=\begin{cases}\min\left\{\alpha,\ \frac{\theta_n}{||x_n-x_{n-1}||}\right\}, & \text{if } x_n\neq x_{n-1},\\ \alpha, & \text{otherwise.}\end{cases} \tag{3.31}$$

Step 2: Compute

$$w_n=x_n+\alpha_n(x_n-x_{n-1}).$$

Step 3: Compute

$$F(u_n,y)+\frac{1}{r_n}\langle y-u_n,u_n-w_n\rangle\geq 0,\ \forall\, y\in H.$$

Step 4: Compute

$$v_n=\delta_n w_n+(1-\delta_n)u_n.$$

Step 5: Compute

$$z_n=(I+\lambda_n B)^{-1}(I-\lambda_n A)v_n.$$

Step 6: Compute

$$x_{n+1}=\beta_n f(x_n)+\xi_n x_n+\mu_n S z_n.$$

Set n := n + 1 and return to **Step 1.**

Suppose that conditions (C1)-(C3) *are satisfied. Then the sequence $\{x_n\}$ generated by Algorithm 3.9 converges strongly to a point $\hat{x}\in\Gamma$, where $\hat{x}=P_\Gamma\circ f(\hat{x})$.*

4. Applications

In this section, we present some applications of our results to related optimization problems.

4.1. **Variational inequality problems.** Here, we apply our result to common solutions of variational inclusion, variational inequality and fixed point problems.

Let C be a nonempty closed convex subset of a real Hilbert space H, and let $P : H \to H$ be a single-valued mapping. The *Variational Inequality Problem* (VIP) is defined as follows:

$$\text{Find } x^* \in C \text{ such that } \langle y - x^*, Px^* \rangle \geq 0, \quad \forall\, y \in C. \tag{4.1}$$

The solution set of the VIP is denoted by $VI(C,P)$. The variational inequality was first introduced independently by Stampacchia [39]. The VIP is a useful mathematical model that unifies many important concepts in applied mathematics, such as necessary optimality conditions, complementarity problems, network equilibrium problems, and systems of nonlinear equations. Several methods have been proposed and analyzed by authors for solving VIP and related optimization problems in the literature. If we take $F(x,y) := \langle y - x, Px \rangle$, then the VIP (4.1) becomes the EP (1.2). Moreover, all the conditions of Theorem 3.3 are satisfied. Hence, Theorem 3.3 provides a strong convergence theorem for approximating common solutions of variational inclusion, variational inequality and fixed point problems for an infinite family of strict pseudocontractions.

4.2. **Split feasibility and fixed point problems.** In this subsection, we derive a scheme for approximating common solutions of split feasibility problems, equilibrium problems and fixed point problems from Algorithm 3.1.

Let H_1 and H_2 be two real Hilbert spaces and let C and Q be nonempty closed convex subsets of H_1 and H_2, respectively. The Split Feasibility Problem (SFP) is defined as follows:

$$\text{Find } x^* \in C \text{ such that } Ax^* \in Q, \tag{4.2}$$

where $A : H_1 \to H_2$ is a bounded linear operator. Let the solution set of SFP (4.2) be denoted by Ω. In 1994, the Split Feasibility Problem (SFP) was introduced by Censor and Elfving [40] in finite dimensional Hilbert spaces for modelling inverse problems, which arise from phase retrievals and in medical image reconstruction [41]. Furthermore, problem (4.2) is also useful in various disciplines such as computer tomography, image restoration, and radiation therapy treatment planning [42, 43]. Let f be a proper, lower semi-continuous convex function of H into $(-\infty, \infty)$. Then the *subdifferential* ∂f of f is defined as follows:

$$\partial f(x) = \{z \in H : f(x) - f(y) \leq \langle z, x - y \rangle, \forall y \in H, \}, \quad \forall x \in H.$$

Let C be a nonempty closed convex subset of a real Hilbert space H and let i_c be the indicator function on C, that is,

$$i_c(x) = \begin{cases} 0, & \text{if } x \in C; \\ \infty, & \text{if } x \notin C. \end{cases}$$

Moreover, we define the *normal cone* $N_C u$ of C at $u \in C$ as follows:

$$N_C u = \{z \in H : \langle z, v - u \rangle \leq 0, \forall v \in C\}.$$

It is known that i_C is a proper, lower semi-continuous and convex function on H. Hence, the subdifferential ∂i_C of i_C is a maximal monotone operator. Therefore, we define the resolvent $J_r^{\partial i_C}$ of ∂i_C, $\forall r > 0$ as follows:

$$J_r^{\partial i_C} x = (I + r\partial i_C)^{-1} x, \forall x \in H.$$

Moreover, for each $x \in C$, we have

$$\begin{aligned} \partial i_C x &= \{z \in H : i_C x + \langle z, u - x\rangle \leq i_C u, \forall u \in H\} \\ &= \{z \in H : \langle z, u - x\rangle \leq 0, \forall u \in C\} \\ &= N_C x. \end{aligned}$$

Hence, for all $\alpha > 0$, we derive

$$\begin{aligned} u = \partial i_C x &\iff x \in u + r\partial i_C u \\ &\iff x - u \in r\partial i_C u \\ &\iff u = P_C x. \end{aligned}$$

It is known that $A^*(I - P_Q)A$ is $1/||A||^2$-inverse strongly monotone [41]. Hence, by applying Theorem 3.3, we obtain the following strong convergence theorem for approximating common solutions of the SFP, EP and FPP for an infinite family of strict pseudocontractive mappings.

Theorem 4.1. *Let C and Q be nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively, and $f : H_1 \to H_1$ be a contraction mapping with coefficient $\rho \in (0,1)$. Let $\{W_n\}$ be a sequence defined by (2.1) and $F : C \times C \to \mathbb{R}$ be a bifunction satisfying Assumption 2.15. Suppose that the solution set denoted by $\Gamma = \Omega \cap EP(F) \cap \bigcap_{i=1}^{\infty} F(S_i)$ is nonempty and let $\{x_n\}$ be a sequence generated as follows:*

Algorithm 4.2.

Step 0 : Select initial data $x_0, x_1 \in H$ and set $n = 1$.
Step 1. Given the $(n-1)th$ and nth iterates, choose α_n such that $0 \leq \alpha_n \leq \hat{\alpha}_n$ with $\hat{\alpha}_n$ defined by

$$\hat{\alpha}_n = \begin{cases} \min\left\{\alpha, \ \frac{\theta_n}{||x_n - x_{n-1}||}\right\}, & \text{if } x_n \neq x_{n-1}, \\ \alpha, & \text{otherwise}. \end{cases} \tag{4.3}$$

Step 2: Compute

$$w_n = x_n + \alpha_n(x_n - x_{n-1}).$$

Step 3: Compute

$$F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - w_n\rangle \geq 0, \ \forall\, y \in H.$$

Step 4: Compute

$$v_n = \delta_n w_n + (1 - \delta_n)u_n.$$

Step 5: Compute

$$z_n = P_C\big[v_n - \lambda_n A^*(I - P_Q)Av_n\big].$$

Step 6: Compute

$$x_{n+1} = \beta_n f(x_n) + \xi_n x_n + \mu_n W_n z_n.$$

Set n := n + 1 and return to **Step 1.**

Suppose that conditions (C1)-(C3) *are satisfied. Then the sequence* $\{x_n\}$ *generated by Algorithm 4.2 converges strongly to a point* $\hat{x} \in \Gamma$, *where* $\hat{x} = P_\Gamma \circ f(\hat{x})$.

5. Numerical example

In this section, we provide numerical example to illustrate the efficiency of our algorithm in comparison with Algorithm 1.3 and Algorithm 1.4 in the literature.

Example 5.1. Let $H = (l_2(\mathbb{R}), \|\cdot\|_2)$, where $l_2(\mathbb{R}) := \{x = (x_1, x_2, \ldots, x_n, \ldots), x_j \in \mathbb{R} : \sum_{j=1}^{\infty} |x_j|^2 < \infty\}, ||x||_2 = (\sum_{j=1}^{\infty} |x_j|^2)^{\frac{1}{2}}$ for all $x \in l_2(\mathbb{R})$. Let $A : H \to H$ be defined by $Ax = \frac{x}{2}$ for all $x \in H$, and let $B : H \to H$ be defined by $Bx = \frac{3}{2}x$. Define the bifunction F by $F(x,y) = x(y-x)$. It can be verified that

$$T_r^F x = \frac{x}{1+r} \quad \text{for all } x \in H,$$

and

$$J_{\lambda_n}^B x = \frac{x}{1+\frac{3}{2}\lambda_n} \quad \text{for all } x \in H.$$

Define an infinite family of mappings $S_n : H \to H$ by

$$S_n x := -\frac{2}{n}x \quad \text{for all } x \in H.$$

It can easily be verified that S_n is k_n-strict pseudo-contractive for each $n \in \mathbb{N}$. Define $S_n' = t_n I + (1-t_n)S_n, t_n \in [k_n, 1)$. Let $\{\zeta_n\}$ be a sequence of nonnegative real numbers defined by $\zeta_n = \{\frac{n}{3n-1}\}$ for all $n \in \mathbb{N}$ and W_n be generated by $\{S_n\}, \{\zeta_n\}$ and $\{t_n\}$. Let $f(x) = \frac{1}{3}x$. Then, $\rho = \frac{1}{3}$ is the Lipschitz constant for f. Choose $\alpha = 0.8, \beta_n = \frac{1}{n+2}, \xi_n = \mu_n = \frac{n+1}{2(n+2)}, \theta_n = \frac{1}{(n+2)^2}, \delta_n = \frac{n}{2n+1}, \lambda_n = \frac{n+1}{2n+3}, r_n = \frac{n}{2n+3}, t_n = \frac{1}{n+3}$ in Algorithm 3.1 and we take $\alpha_n = \frac{1}{10n+1}, u = (1, -\frac{1}{2}, \frac{1}{4}, \cdots)$ in Algorithm 1.3 and $\lambda = 0.01$ in Algorithm 1.4. It can easily be verified that all the conditions of Theorem 3.3 are satisfied.

We choose different initial values as follows:
Case I: $x_0 = (-2, 1, -\frac{1}{2}, \cdots), x_1 = (\frac{1}{5}, -\frac{1}{10}, \frac{1}{20}, \cdots)$,
Case II: $x_0 = (-4, 1, -\frac{1}{4}, \cdots), x_1 = (1, \frac{1}{5}, \frac{1}{25}, \cdots)$,
Case III: $x_0 = (-\frac{5}{2}, \frac{5}{4}, -\frac{5}{8}, \cdots), x_1 = (\frac{1}{3}, \frac{1}{6}, \frac{1}{12}, \cdots)$,
Case IV: $x_0 = (-5, 1, -\frac{1}{5}, \cdots), x_1 = (1, -0.1, 0.01, \cdots)$.
Using MATLAB 2019(b), we compare the performance of Algorithm 3.1 with Algorithm 1.3 and Algorithm 1.4. The stopping criterion used for our computation is $||x_{n+1} - x_n|| < 10^{-4}$. We plot the graphs of errors against the number of iterations in each case. The numerical result is reported in Figure 1 and Table 1.

6. Conclusion

In this paper, we studied the common solutions to equilibrium problems, variational inequality problems and fixed point problems with an infinite family of strict pseudocontractive mappings. We introduced an inertial viscosity method for common solutions of the above problems and established strong convergence theorems in Hilbert spaces. Some interesting examples are also considered to support our main results. Our results extend and improve corresponding results in [25] and [27], and several other results in the literature.

TABLE 1. **Numerical results for Example 5.1**

		Alg. 1.3	Alg. 1.4	Alg. 3.1
Case I	CPU time (sec)	0.0518	0.0216	0.0536
	No. of Iter.	200	105	14
Case II	CPU time (sec)	0.0320	0.0172	0.0266
	No. of Iter.	200	130	13
Case III	CPU time (sec)	0.0303	0.0151	0.0259
	No. of Iter.	200	119	14
Case IV	CPU time (sec)	0.0468	0.0198	0.0236
	No. of Iter.	200	155	13

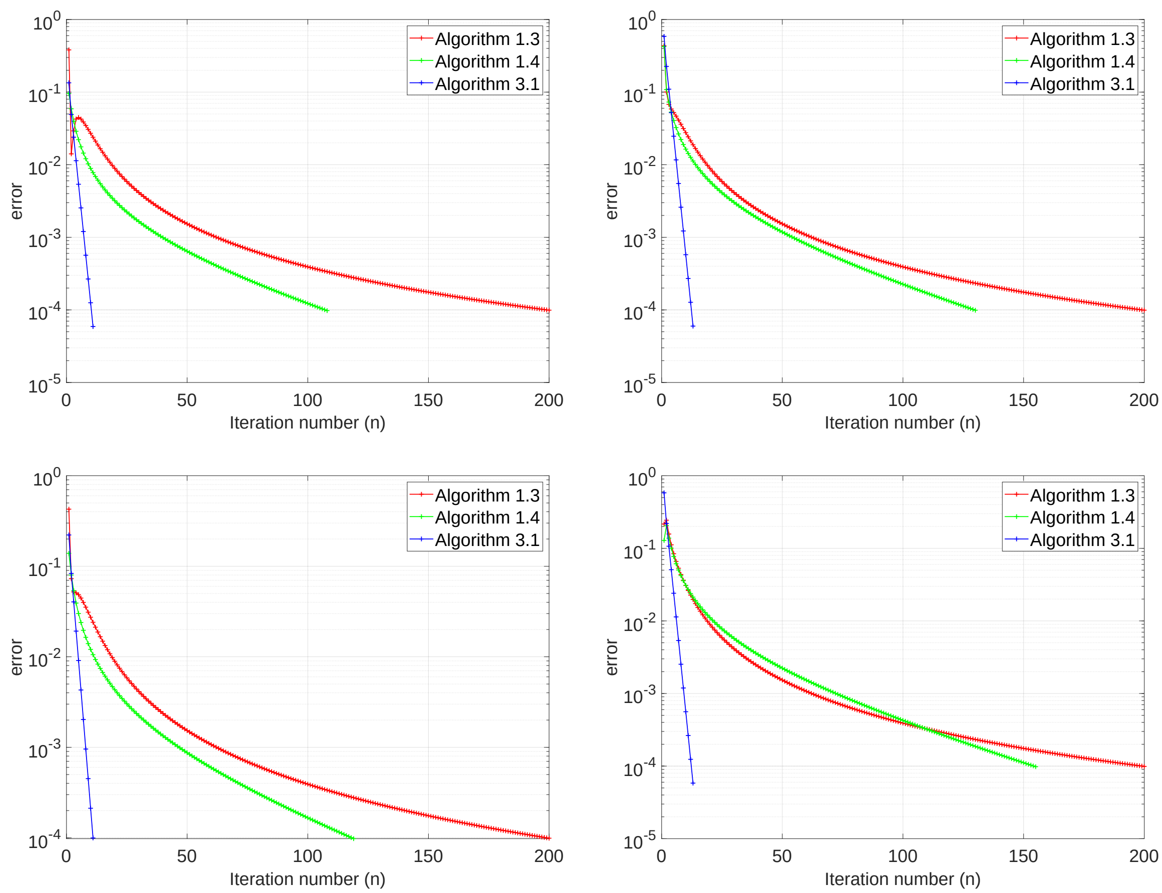

FIGURE 1. Top left: Case I; Top right: Case II; Bottom left: Case III; Bottom right: Case IV.

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REFERENCES

[1] P.L. Combettes, V.R. Wajs, Signal recovery by proximal forward-backward splitting, Multiscale Model. Simul. 4 (2005), 1168-1200.

[2] F. Cui, Y. Tang, Y. Yang, An inertial three-operator splitting algorithm with applications to image inpainting, Appl. Set-Valued Anal. Optim. 1 (2019), 113-134.

[3] X. Qin, N.T. An, Smoothing algorithms for computing the projection onto a Minkowski sum of convex sets, Comput. Optim. Appl. 74 (2019), 821-850.

[4] Y. Censor, S. Petra, C. Schnorr, Superiorization vs. accelerated convex optimization: The superiorized / regularized least-squares case, J. Appl. Numer. Optim. 2 (2020), 15-62.

[5] P.L. Lions, B. Mercier, Splitting algorithms for the sum of two nonlinear operators, SIAM J. Numer. Anal. 16 (1979), 964-979.

[6] G.B. Passty, Ergodic convergence to a zero of the sum of monotone operators in Hilbert space, J. Math. Anal. Appl. 72 (1979), 383-390.

[7] J.J. Moreau, Proximité et dualité dans un espace Hilbertien, Bull. Soc. Math. France 93 (1965), 273-299.

[8] T.O. Alakoya, L.O. Jolaoso, O.T. Mewomo, A self adaptive inertial algorithm for solving split variational inclusion and fixed point problems with applications, J. Ind. Manag. Optim. (2020), DOI:10.3934/jimo.2020152.

[9] Y. Kimura, K. Nakajo, Strong convergence for a modified forward-backward splitting method in Banach spaces, J. Nonlinear Var. Anal. 3 (2019), 5-18.

[10] X. Qin, J.C. Yao, Projection splitting algorithms for nonself operators, J. Nonlinear Convex Anal. 18 (2017), 925–935.

[11] E. Blum, W. Oettli, From optimization and variational inequalities to equilibrium problems, Math. Stud. 63 (1994), 123-145.

[12] L.O. Jolaoso, T.O. Alakoya, A. Taiwo, O.T. Mewomo, A parallel combination extragradient method with Armijo line searching for finding common solution of finite families of equilibrium and fixed point problems, Rend. Circ. Mat. Palermo II 69 (2020), 475-495.

[13] G. N. Ogwo, C. Izuchukwu, K.O. Aremu, O.T. Mewomo, On θ-generalized demimetric mappings and monotone operators in Hadamard spaces, Demonstr. Math. 53 (2020), 95–111.

[14] M.O. Osilike, E.E. Chima, Mixed equilibrium and multiple-set split feasibility problems for asymptotically k-strictly pseudononspreading mappings, Commun. Optim. Theory, 2020 (2020), Article ID 14.

[15] T.O. Alakoya, L.O. Jolaoso, O.T. Mewomo, A general iterative method for finding common fixed point of finite family of demicontractive mappings with accretive variational inequality problems in Banach spaces, Nonlinear Stud. 27 (2020), 1-24.

[16] S.S. Chang, H. W.J. Lee, C.K. Chan, A new method for solving equilibrium problem fixed point problem and variational inequality problem with application to optimization, Nonlinear Anal. 70 (2009), 3307-3319.

[17] X. Qin, Y.J. Cho, S.M. Kang, Convergence theorems of common elements for equilibrium problems and fixed point problems in Banach spaces, J. Comput. Appl. Math. 225 (2009), 20-30.

[18] A. Taiwo, L.O. Jolaoso, O.T. Mewomo, A. Gibali, On generalized mixed equilibrium problem with α,β,μ bifunction and μ,τ monotone mapping, J. Nonlinear Convex Anal. 21 (2020), 1381-1401.

[19] H. Iiduka, Fixed point optimization algorithm and its application to network bandwidth allocation, J. Comput. Appl. Math. 236 (2012), 1733–1742.

[20] Y. Liu, A general iterative method for equilibrium problems and strict pseudo-contractions in Hilbert spaces, Nonlinear Anal. 71 (2009), 4852–4861.

[21] B.T. Polyak, Some methods of speeding up the convergence of iteration methods, Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys. 4 (1964), 1-17.

[22] T.O. Alakoya, L.O. Jolaoso, O.T. Mewomo, Two modifications of the inertial Tseng extragradient method with self-adaptive step size for solving monotone variational inequality problems, Demonstr. Math. (2020), doi.org/10.1515/dema-2020-0013.

[23] Q. Dong, D. Jiang, P. Cholamjiak, A. Shehu, strong convergence result involving an inertial forward–backward algorithm for monotone inclusions, J. Fixed Point Theory Appl. 19 (2017), 3097-3118.

[24] L.O. Jolaoso, T.O. Alakoya, A. Taiwo, O.T. Mewomo, Inertial extragradient method via viscosity approximation approach for solving Equilibrium problem in Hilbert space, Optimization, (2020), DOI:10.1080/02331934.2020.1716752.

[25] W. Cholamjiak, P. Cholamjiak, S. Suantai, An inertial forward-backward splitting method for solving inclusion problems in Hilbert spaces, J Fixed Point Theory Appl. 20 (2018), 42.

[26] A. Moudafi, M. Oliny, Convergence of a splitting inertial proximal method for monotone operators, J. Comput. Appl. Math. 155 (2003), 447-454.

[27] D.V. Thong, N.T. Vinh, Inertial methods for fixed point problems and zero point problems of the sum of two monotone mappings, Optimization 68 (2019), 1037-1072.

[28] A. Taiwo, L.O. Jolaoso, O.T. Mewomo, Inertial-type algorithm for solving split common fixed-point problem in Banach spaces, J. Sci. Comput. (2020) DOI: 10.1007/s10915-020-01385-9.

[29] P.E. Maingé, Approximation methods for common fixed points of nonexpansive mappings in Hilbert spaces, J. Math. Anal. Appl. 325 (2007), 469-479.

[30] S.S. Chang, Some problems and results in the study of nonlinear analysis, Nonlinear Anal. 30 (1997), 4197-4208.

[31] H. Zhou, Fixed Points of Nonlinear Operators. Iterative Methods, De Gruyter, Berlin, 2020.

[32] S. Wang, A general iterative method for obtaining an infinite family of strictly pseudo-contractive mappings in Hilbert spaces, Appl. Math. Lett. 24 (2011), 901-907.

[33] K. Shimoji, W. Takahashi, Strong convergence to common fixed points of infinite nonexpansive mappings and applications, Taiwanese J. Math. 5 (2001), 387-404.

[34] Z. Opial, Weak convergence of the sequence of successive approximation for nonexpansive mappings, Bull. Amer. Math. Soc. 73 (1967), 591-597.

[35] W. Takahashi, M. Toyoda, Weak convergence theorems for nonexpansive mappings and monotone mappings, J. Optim. Theory Appl. 118 (2003), 417–428.

[36] Z. Ma, L. Wang, S.S. Chang, W. Duan, Convergence theorems for split equality mixed equilibrium problems with applications, Fixed Point Theory Appl. 2015 (2015), Art. ID 31.

[37] G. López, M.V. Márquez, F. Wang, H.K. Xu, Forward-backward splitting methods for accretive operators in Banach spaces, Abstr. Appl. Anal. 2012 (2012), Art. ID 109236.

[38] W. Takahashi, Nonlinear functional analysis-fixed point theory and its applications, Yokohama Publishers, Yokohama, 2000.

[39] G. Stampacchia, Formes bilineaires coercitives sur les ensembles convexes, C. R. Acad. Sci. Paris 258 (1964), 4413-4416.

[40] Y. Censor, T. Elfving, A multiprojection algorithm using Bregman projections in a product space, Numer. Algo. 8 (1994), 221-239.

[41] C. Byrne, A unified treatment of some iterative algorithms in signal processing and image reconstruction, Inverse Probl. 20 (2004), 103-120.

[42] Y. Censor, T. Bortfeld, B. Martin, A. Trofimov, A unified approach for inversion problems in intensity modulated radiation therapy, Phys. Med. Biol. 51 (2006), 2353-2365.

[43] E. Bonacker, A. Gibali, K.H. Kufer, Nesterov perturbations and projection methods applied to IMRT, J. Nonlinear Var. Anal. 4 (2020), 63-86.