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RIEMANN-STIELTJES INTEGRAL BOUNDARY VALUE PROBLEMS INVOLVING MIXED RIEMANN-LIOUVILLE AND CAPUTO FRACTIONAL DERIVATIVES

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Abstract. In this paper, we present the existence and uniqueness of solutions for a fractional integro-differential equation involving both Riemann-Liouville and Caputo derivatives equipped with non-conjugate Riemann-Stieltjes integro-multipoint boundary conditions. Our results are obtained by applying the modern methods of functional analysis. Examples are constructed for the illustration of our results.

Keywords. Fractional differential equation; Boundary value problem; Integro-differential equation; Caputo derivatives.

1. INTRODUCTION

Fractional calculus received considerable attention in view of its extensive applications in the study of many real world phenomena such as fractional dynamics [1], economic models [2], neural networks [3, 4], disease models [5, 6], etc. The nonlocal nature of fractional-order differential and integral operators contributed to the popularity of the subject. For details and examples, we refer the reader to [7, 8, 9].

Fractional-order boundary value problems have recently been studied by many researchers. One can find a variety of results for such problems involving different kinds of fractional differential equations and boundary conditions in [10]-[22] and the references cited therein.

In this paper, we introduce and investigate a new class of boundary value problems of fractional integro-differential equations involving both Riemann-Liouville and Caputo derivatives with non-conjugate Riemann-Stieltjes integro-multipoint boundary conditions. In precise terms,

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we discuss the existence and uniqueness of solutions for the following equation:

$$ {}^{RL}D^{q}\Big[({}^{c}D^{p}+\kappa)x(t)+\lambda I^{\gamma}h(t,x(t))\Big]=f(t,x(t)),\ 1<p,q\leq 2,\ t\in[a,T], \tag{1.1}$$

equipped with non-conjugate Riemann-Stieltjes integro-multipoint boundary conditions of the form:

$$x(a)=\sum_{i=1}^{n-2}\alpha_i x(\xi_i)+\int_a^T x(s)dA(s),\ x'(a)=0,\ x(T)=0,\ x'(T)=0, \tag{1.2}$$

where ${}^{c}D^{p}$ denotes the Caputo fractional differential operator of order $p\in(1,2]$, ${}^{RL}D^{q}$ denotes the Riemann-Liouville fractional differential operator of order $q\in(1,2]$, with $p+q>3$, I^{γ} is Riemann-Liouville fractional integral of order $\gamma>1$, κ, $\lambda\in\mathbb{R}$, $h,f:[a,T]\times\mathbb{R}\longrightarrow\mathbb{R}$ are given continuous functions, A is a function of bounded variation, $a<\xi_1<\xi_2<\cdots<\xi_{n-2}<T$, and $\alpha_i\in\mathbb{R},\ i=1,2,\cdots,n-2$. Existence results for problem (1.1)-(1.2) are proved by applying Krasnosel'skiĭ's fixed point theorem, nonlinear alternative of Leray-Schauder type and Leray-Schauder degree theory, while the uniqueness of solutions for problem (1.1)-(1.2) is proved by means of Banach's contraction mapping principle.

The rest of the paper is organized as follows. In Section 2, we recall some preliminary concepts of fractional calculus and present an auxiliary result concerning the linear variant of the problem (1.1)-(1.2). The existence results are presented in Section 3, while the uniqueness result is proved in Section 4, the last section. Illustrative examples for the main results are also included.

2. Preliminaries and basic results

Let us begin with some basic definitions of fractional calculus [7].

Definition 2.1. Let S be a locally integrable real-valued function on $-\infty\leq a<t<b\leq+\infty$. The Riemann-Liouville fractional integral I_a^{ω} of order $\omega\in\mathbb{R}$ $(\omega>0)$ for the function S is defined as

$$I_a^{\omega}S(t)=(S*K_{\omega})(t)=\frac{1}{\Gamma(\omega)}\int_a^t(t-u)^{\omega-1}S(u)du,$$

where $K_{\omega}=\frac{t^{\omega-1}}{\Gamma(\omega)}$, and Γ denotes the Euler gamma function.

Definition 2.2. The Riemann-Liouville fractional derivative of order $\omega>0,\ m=[\omega]+1$ (ω denotes the integer part of the real number ω) for a function $S:(a,\infty)\longrightarrow\mathbb{R}$ is defined as

$${}^{RL}D^{\omega}S(t)=\frac{1}{\Gamma(m-\omega)}\left(\frac{d}{dt}\right)^m\int_a^t(t-u)^{m-\omega-1}S(u)du=\left(\frac{d}{dt}\right)^m I^{m-\omega}S(t),$$

provided that it exists.

Definition 2.3. For (m-1)-times absolutely continuous differentiable function $S:[a,\infty)\longrightarrow\mathbb{R}$, the Caputo derivative of fractional order ω for the function S is defined as

$${}^{c}D^{\omega}S(t)=\frac{1}{\Gamma(m-\omega)}\int_a^t(t-u)^{m-\omega-1}S^{(m)}(u)du,\ m-1<\omega\leq m,\ m=[\omega]+1.$$

where $[\omega]$ denotes the integer part of the real number ω.

Lemma 2.4. [7] *For $m-1<\omega\leq m$, the general solution of the fractional differential equation ${}^{c}D^{\omega}S(t)=0,\ t\in[a,b]$, is*

$$S(t)=c_0+c_1(t-a)+c_2(t-a)^2+\ldots+c_{m-1}(t-a)^{m-1},$$

where $c_i\in\mathbb{R},\ i=0,1,\ldots,m-1$. Furthermore,

$$I^{\omega}\,{}^{c}D^{\omega}S(t)=S(t)+\sum_{i=0}^{m-1}c_i(t-a)^i.$$

Lemma 2.5. [7] *For $\omega>0$ and $S\in C(a,b)\cap L(a,b)$, the general solution of the equation ${}^{RL}D^{\omega}S(t)=0$ is*

$$S(t)=c_0(t-a)^{\omega-m}+c_1(t-a)^{\omega-m-1}+\cdots+c_{m-2}(t-a)^{\omega-2}+c_{m-1}(t-a)^{\omega-1},$$

where $c_i,\ i=0,1,\cdots,m-1$, are arbitrary real constants and

$$I^{\omega}\,{}^{RL}D^{\omega}S(t)=S(t)+c_0(t-a)^{\omega-m}+c_1(t-a)^{\omega-m-1}+\cdots+c_{m-2}(t-a)^{\omega-2}+c_{m-1}(t-a)^{\omega-1}.$$

On the other hand, ${}^{RL}D^{\omega}I^{\omega}S(t)=S(t)$.

In relation to problem (1.1)-(1.2), we define

$$\sigma=A_1A_4A_7-A_2A_3A_7+A_3A_6-A_4A_5\neq 0, \tag{2.1}$$

where

$$\begin{cases} A_1 = \dfrac{(T-a)^{p+q-1}\Gamma(q)}{\Gamma(p+q)},\ A_2=\dfrac{(T-a)^{p+q-2}\Gamma(q-1)}{\Gamma(p+q-1)},\\ A_3 = \dfrac{(T-a)^{p+q-2}\Gamma(q)}{\Gamma(p+q-1)},\ A_4=\dfrac{(T-a)^{p+q-3}\Gamma(q-1)}{\Gamma(p+q-2)},\\ A_5 = -\displaystyle\sum_{i=1}^{n-2}\alpha_i\frac{(\xi_i-a)^{p+q-1}\Gamma(q)}{\Gamma(p+q)}-\int_a^T\frac{(s-a)^{p+q-1}\Gamma(q)}{\Gamma(p+q)}dA(s),\\ A_6 = -\displaystyle\sum_{i=1}^{n-2}\alpha_i\frac{(\xi_i-a)^{p+q-2}\Gamma(q-1)}{\Gamma(p+q-1)}-\int_a^T\frac{(s-a)^{p+q-2}\Gamma(q-1)}{\Gamma(p+q-1)}dA(s),\\ A_7 = 1-\displaystyle\sum_{i=1}^{n-2}\alpha_i-\int_a^T dA(s). \end{cases} \tag{2.2}$$

In the following lemma, we solve a linear variant of problem (1.1)-(1.2).

Lemma 2.6. *Let $H,F\in C([a,T],\mathbb{R})$ and $\sigma\neq 0$, where σ is given by (2.1). Then the solution of the linear fractional integro-differential equation:*

$${}^{RL}D^{q}\left[({}^{c}D^{p}+\kappa)x(t)+\lambda I^{\gamma}H(t)\right]=F(t),\ 1<p,q\leq 2,\ t\in(a,T), \tag{2.3}$$

subject to the boundary conditions (1.2) is given by

$$\begin{aligned} x(t) &= -\kappa\int_a^t \frac{(t-s)^{p-1}}{\Gamma(p)}x(s)ds - \lambda\int_a^t \frac{(t-s)^{\gamma+p-1}}{\Gamma(\gamma+p)}H(s)ds \\ &\quad + \int_a^t \frac{(t-s)^{p+q-1}}{\Gamma(p+q)}F(s)ds + \varphi_1(t)\Big[\kappa\int_a^T \frac{(T-s)^{p-1}}{\Gamma(p)}x(s)ds \\ &\quad + \lambda\int_a^T \frac{(T-s)^{\gamma+p-1}}{\Gamma(\gamma+p)}H(s)ds - \int_a^T \frac{(T-s)^{p+q-1}}{\Gamma(p+q)}F(s)ds\Big] \\ &\quad + \varphi_2(t)\Big[\kappa\int_a^T \frac{(T-s)^{p-2}}{\Gamma(p-1)}x(s)ds + \lambda\int_a^T \frac{(T-s)^{\gamma+p-2}}{\Gamma(\gamma+p-1)}H(s)ds \\ &\quad - \int_a^T \frac{(T-s)^{p+q-2}}{\Gamma(p+q-1)}F(s)ds\Big] + \varphi_3(t)\Big[-\kappa\sum_{i=1}^{n-2}\alpha_i\int_a^{\xi_i}\frac{(\xi_i-s)^{p-1}}{\Gamma(p)}x(s)ds \\ &\quad - \lambda\sum_{i=1}^{n-2}\alpha_i\int_a^{\xi_i}\frac{(\xi_i-s)^{\gamma+p-1}}{\Gamma(\gamma+p)}H(s)ds + \sum_{i=1}^{n-2}\alpha_i\int_a^{\xi_i}\frac{(\xi_i-s)^{p+q-1}}{\Gamma(p+q)}F(s)ds \\ &\quad + \int_a^T\Big(-\kappa\int_a^s \frac{(s-u)^{p-1}}{\Gamma(p)}x(u)du - \lambda\int_a^s\frac{(s-u)^{\gamma+p-1}}{\Gamma(\gamma+p)}H(u)du \\ &\quad + \int_a^s \frac{(s-u)^{p+q-1}}{\Gamma(p+q)}F(u)du\Big)dA(s)\Big], \end{aligned} \tag{2.4}$$

where

$$\begin{cases} \varphi_1(t) = B_1\dfrac{(t-a)^{p+q-1}\Gamma(q)}{\Gamma(p+q)} + B_4\dfrac{(t-a)^{p+q-2}\Gamma(q-1)}{\Gamma(p+q-1)} + B_7, \\ \varphi_2(t) = B_2\dfrac{(t-a)^{p+q-1}\Gamma(q)}{\Gamma(p+q)} + B_5\dfrac{(t-a)^{p+q-2}\Gamma(q-1)}{\Gamma(p+q-1)} + B_8, \\ \varphi_3(t) = B_3\dfrac{(t-a)^{p+q-1}\Gamma(q)}{\Gamma(p+q)} + B_6\dfrac{(t-a)^{p+q-2}\Gamma(q-1)}{\Gamma(p+q-1)} + B_9, \end{cases} \tag{2.5}$$

$$\begin{cases} B_1 = \dfrac{A_4A_7}{\sigma},\ B_2 = \dfrac{A_6-A_2A_7}{\sigma},\ B_3 = \dfrac{-A_4}{\sigma},\ B_4 = \dfrac{-A_3A_7}{\sigma}, \\ B_5 = \dfrac{A_1A_7-A_5}{\sigma},\ B_6 = \dfrac{A_3}{\sigma},\ B_7 = \dfrac{A_3A_6-A_4A_5}{\sigma}, \\ B_8 = \dfrac{A_2A_5-A_1A_6}{\sigma},\ B_9 = \dfrac{A_1A_4-A_2A_3}{\sigma}, \end{cases} \tag{2.6}$$

where A_i $(i=1,\ldots,7)$ are given by (2.2).

Proof. Applying the integral operator I^q to both sides of (2.3) and using Lemma 2.5, we get

$$({}^cD^p+\kappa)x(t) + \lambda I^\gamma H(t) = I^qF(t) + c_1(t-a)^{q-1} + c_2(t-a)^{q-2}, \tag{2.7}$$

where c_1 and c_2 are unknown arbitrary constants. Now operating the integral operator I^p to (2.7) and using Lemma 2.4, we obtain

$$\begin{aligned} x(t) &= -\kappa\int_a^t \frac{(t-s)^{p-1}}{\Gamma(p)}x(s)ds - \lambda\int_a^t\frac{(t-s)^{\gamma+p-1}}{\Gamma(\gamma+p)}H(s)ds + \int_a^t\frac{(t-s)^{p+q-1}}{\Gamma(p+q)}F(s)ds \\ &\quad + c_1\frac{(t-a)^{p+q-1}\Gamma(q)}{\Gamma(p+q)} + c_2\frac{(t-a)^{p+q-2}\Gamma(q-1)}{\Gamma(p+q-1)} + c_3 + c_4(t-a), \end{aligned} \tag{2.8}$$

where c_3 and c_4 are unknown arbitrary constants. From (2.8), we have

$$\begin{aligned} x'(t) &= -\kappa\int_a^t \frac{(t-s)^{p-2}}{\Gamma(p-1)}x(s)ds - \lambda\int_a^t \frac{(t-s)^{\gamma+p-2}}{\Gamma(\gamma+p-1)}H(s)ds + \int_a^t \frac{(t-s)^{p+q-2}}{\Gamma(p+q-1)}F(s)ds \\ &\quad + c_1\frac{(t-a)^{p+q-2}\Gamma(q)}{\Gamma(p+q-1)} + c_2\frac{(t-a)^{p+q-3}\Gamma(q-1)}{\Gamma(p+q-2)} + c_4. \end{aligned} \tag{2.9}$$

Using the conditions (1.2) in (2.8) and (2.9), we get $c_4 = 0$ and a system of equtions in c_1, c_2 and c_3 given by

$$A_1c_1 + A_2c_2 + c_3 = I_1, \tag{2.10}$$
$$A_3c_1 + A_4c_2 = I_2, \tag{2.11}$$
$$A_5c_1 + A_6c_2 + A_7c_3 = I_3, \tag{2.12}$$

where A_i $(i = 1, \ldots, 7)$ are given by (2.2) and

$$\begin{aligned} I_1 &= \kappa\int_a^T \frac{(T-s)^{p-1}}{\Gamma(p)}x(s)ds + \lambda\int_a^T \frac{(T-s)^{\gamma+p-1}}{\Gamma(\gamma+p)}H(s)ds - \int_a^T \frac{(T-s)^{p+q-1}}{\Gamma(p+q)}F(s)ds, \\ I_2 &= \kappa\int_a^T \frac{(T-s)^{p-2}}{\Gamma(p-1)}x(s)ds + \lambda\int_a^T \frac{(T-s)^{\gamma+p-2}}{\Gamma(\gamma+p-1)}H(s)ds - \int_a^T \frac{(T-s)^{p+q-2}}{\Gamma(p+q-1)}F(s)ds, \\ I_3 &= -\kappa\sum_{i=1}^{n-2}\alpha_i\int_a^{\xi_i} \frac{(\xi_i-s)^{p-1}}{\Gamma(p)}x(s)ds - \lambda\sum_{i=1}^{n-2}\alpha_i\int_a^{\xi_i} \frac{(\xi_i-s)^{\gamma+p-1}}{\Gamma(\gamma+p)}H(s)ds \\ &\quad + \sum_{i=1}^{n-2}\alpha_i\int_a^{\xi_i} \frac{(\xi_i-s)^{p+q-1}}{\Gamma(p+q)}F(s)ds + \int_a^T \Big(-\kappa\int_a^s \frac{(s-u)^{p-1}}{\Gamma(p)}x(u)du \\ &\quad -\lambda\int_a^s \frac{(s-u)^{\gamma+p-1}}{\Gamma(\gamma+p)}H(u)du + \int_a^s \frac{(s-u)^{p+q-1}}{\Gamma(p+q)}F(u)du\Big)dA(s). \end{aligned} \tag{2.13}$$

Solving system (2.10)-(2.12), the values of c_1, c_2, and c_3 are formed to be

$$\begin{aligned} c_1 &= B_1I_1 + B_2I_2 + B_3I_3, \\ c_2 &= B_4I_1 + B_5I_2 + B_6I_3, \\ c_3 &= B_7I_1 + B_8I_2 + B_9I_3, \end{aligned}$$

where B_i $(i = 1\ldots9)$ are defined by (2.6). Inserting the values of c_1, c_2, c_3 and c_4 in (2.8) together with notations (2.5), we obtain the solution (2.4). The converse of the lemma follows by direct computation. The proof is completed. □

3. Existence results

Let $\mathscr{V} = C([a,T],\mathbb{R})$ denote the Banach space of all continuous functions from $[a,T] \longrightarrow \mathbb{R}$ endowed with the supremum norm defined by $\|x\| = \sup\{|x(t)|, t \in [a,T]\}$. In view of Lemma 2.6, we transform problem (1.1)-(1.2) into an equivalent fixed point problem as

$$x = \mathscr{Q}x, \tag{3.1}$$

where $\mathscr{Q}:\mathscr{V}\longrightarrow\mathscr{V}$ is defined by

$$\begin{aligned}(\mathscr{Q}x)(t) &= -\kappa\int_a^t \frac{(t-s)^{p-1}}{\Gamma(p)}x(s)ds-\lambda\int_a^t\frac{(t-s)^{\gamma+p-1}}{\Gamma(\gamma+p)}h(s,x(s))ds\\ &+\int_a^t\frac{(t-s)^{p+q-1}}{\Gamma(p+q)}f(s,x(s))ds+\varphi_1(t)\Big[\kappa\int_a^T\frac{(T-s)^{p-1}}{\Gamma(p)}x(s)ds\\ &+\lambda\int_a^T\frac{(T-s)^{\gamma+p-1}}{\Gamma(\gamma+p)}h(s,x(s))ds-\int_a^T\frac{(T-s)^{p+q-1}}{\Gamma(p+q)}f(s,x(s))ds\Big]\\ &+\varphi_2(t)\Big[\kappa\int_a^T\frac{(T-s)^{p-2}}{\Gamma(p-1)}x(s)ds+\lambda\int_a^T\frac{(T-s)^{\gamma+p-2}}{\Gamma(\gamma+p-1)}h(s,x(s))ds\\ &-\int_a^T\frac{(T-s)^{p+q-2}}{\Gamma(p+q-1)}f(s,x(s))ds\Big]+\varphi_3(t)\Big[-\kappa\sum_{i=1}^{n-2}\alpha_i\int_a^{\xi_i}\frac{(\xi_i-s)^{p-1}}{\Gamma(p)}x(s)ds\\ &-\lambda\sum_{i=1}^{n-2}\alpha_i\int_a^{\xi_i}\frac{(\xi_i-s)^{\gamma+p-1}}{\Gamma(\gamma+p)}h(s,x(s))ds+\sum_{i=1}^{n-2}\alpha_i\int_a^{\xi_i}\frac{(\xi_i-s)^{p+q-1}}{\Gamma(p+q)}f(s,x(s))ds\\ &+\int_a^T\Big(-\kappa\int_a^s\frac{(s-u)^{p-1}}{\Gamma(p)}x(u)du-\lambda\int_a^s\frac{(s-u)^{\gamma+p-1}}{\Gamma(\gamma+p)}h(u,x(u))du\\ &+\int_a^s\frac{(s-u)^{p+q-1}}{\Gamma(p+q)}f(u,x(u))du\Big)dA(s)\Big]. \qquad (3.2)\end{aligned}$$

Notice that problem (1.1)-(1.2) has solutions if $\mathscr{Q}$ has fixed points. For computational convenience, we set the notations:

$$\begin{aligned}\mathscr{E}_0 &= \frac{(T-a)^p}{\Gamma(p+1)}+\widetilde{\varphi}_1\frac{(T-a)^p}{\Gamma(p+1)}+\widetilde{\varphi}_2\frac{(T-a)^{p-1}}{\Gamma(p)}+\widetilde{\varphi}_3\Big(\sum_{i=1}^{n-2}|\alpha_i|\frac{(\xi_i-a)^p}{\Gamma(p+1)}\\ &+\int_a^T\frac{(s-a)^p}{\Gamma(p+1)}dA(s)\Big),\\ \mathscr{E}_1 &= \frac{(T-a)^{\gamma+p}}{\Gamma(\gamma+p+1)}+\widetilde{\varphi}_1\frac{(T-a)^{\gamma+p}}{\Gamma(\gamma+p+1)}+\widetilde{\varphi}_2\frac{(T-a)^{\gamma+p-1}}{\Gamma(\gamma+p)}+\widetilde{\varphi}_3\Big(\sum_{i=1}^{n-2}|\alpha_i|\frac{(\xi_i-a)^{\gamma+p}}{\Gamma(\gamma+p+1)}\\ &+\int_a^T\frac{(s-a)^{\gamma+p}}{\Gamma(\gamma+p+1)}dA(s)\Big),\\ \mathscr{E}_2 &= \frac{(T-a)^{q+p}}{\Gamma(q+p+1)}+\widetilde{\varphi}_1\frac{(T-a)^{q+p}}{\Gamma(q+p+1)}+\widetilde{\varphi}_2\frac{(T-a)^{q+p-1}}{\Gamma(q+p)}+\widetilde{\varphi}_3\Big(\sum_{i=1}^{n-2}|\alpha_i|\frac{(\xi_i-a)^{q+p}}{\Gamma(q+p+1)}\\ &+\int_a^T\frac{(s-a)^{q+p}}{\Gamma(q+p+1)}dA(s)\Big). \qquad (3.3)\end{aligned}$$

where $\widetilde{\varphi}_i=\sup_{t\in[a,T]}|\varphi_i(t)|,\ i=1,2,3.$

Now, we are in a position to prove the existence of solutions for problem (1.1)-(1.2). Our frist result relies on Krasnosel'skiĭ's fixed point theorem [23].

Theorem 3.1. *(Krasnosel'skiĭ's fixed point theorem* [23]*) Let $\mathscr{M}$ be a closed, convex, bounded and nonempty subset of a Banach space X and let $\mathscr{F}_1,\mathscr{F}_2$ be the operators defined from $\mathscr{M}$ to X such that: (i) $\mathscr{F}_1x+\mathscr{F}_2y\in\mathscr{M}$ wherever $x,y\in\mathscr{M}$; (ii) $\mathscr{F}_1$ is compact and continuous; (iii) $\mathscr{F}_2$ is a contraction. Then there exists $z\in\mathscr{M}$ such that $z=\mathscr{F}_1z+\mathscr{F}_2z$.*

Theorem 3.2. *Assume that $h, f : [a,T]\times\mathbb{R} \longrightarrow \mathbb{R}$ are continuous functions satisfying the condition:*
$\forall\,(t,x)\in[a,T]\times\mathbb{R}$, $\phi_1,\phi_2\in C([a,T],\mathbb{R}^+)$, *such that* $\sup_{t\in[a,T]}|\phi_i(t)| = \|\phi_i\|$, $i=1,2$ *and* $|h(t,x)|\le \phi_1(t)$, $|f(t,x)|\le\phi_2(t)$. *Then problem (1.1)-(1.2) has at least one solution on $[a,T]$ if*

$$|\kappa|\mathscr{E}_0 < 1, \tag{3.4}$$

Proof. Consider a closed ball $\mathbb{B}_\tau = \{x\in\mathscr{V} : \|x\|\le\tau\}$, with

$$\tau \ge \frac{|\lambda|\|\phi_1\|\mathscr{E}_1 + \|\phi_2\|\mathscr{E}_2}{1-|\kappa|\mathscr{E}_0}.$$

Next, we introduse the operators $\mathscr{Q}_1$ and $\mathscr{Q}_2$ from $\mathbb{B}_\tau$ to $\mathscr{V}$ as

$$\begin{aligned}
&(\mathscr{Q}_1x)(t)\\
&= -\lambda\int_a^t \frac{(t-s)^{\gamma+p-1}}{\Gamma(\gamma+p)}h(s,y(s))ds + \int_a^t\frac{(t-s)^{p+q-1}}{\Gamma(p+q)}f(s,y(s))ds\\
&+\varphi_1(t)\Big[\lambda\int_a^T\frac{(T-s)^{\gamma+p-1}}{\Gamma(\gamma+p)}h(s,x(s))ds - \int_a^T\frac{(T-s)^{p+q-1}}{\Gamma(p+q)}f(s,x(s))ds\Big]\\
&+\varphi_2(t)\Big[\lambda\int_a^T\frac{(T-s)^{\gamma+p-2}}{\Gamma(\gamma+p-1)}h(s,x(s))ds - \int_a^T\frac{(T-s)^{p+q-2}}{\Gamma(p+q-1)}f(s,x(s))ds\Big]\\
&+\varphi_3(t)\Big[-\lambda\sum_{i=1}^{n-2}\alpha_i\int_{a_1}^{\xi_i}\frac{(\xi_i-s)^{\gamma+p-1}}{\Gamma(\gamma+p)}h(s,x(s))ds\\
&+\sum_{i=1}^{n-2}\alpha_i\int_a^{\xi_i}\frac{(\xi_i-s)^{p+q-1}}{\Gamma(p+q)}f(s,x(s))ds + \int_a^T\Big(-\lambda\int_a^s\frac{(s-u)^{\gamma+p-1}}{\Gamma(\gamma+p)}h(u,x(u))du\\
&+\int_a^s\frac{(s-u)^{p+q-1}}{\Gamma(p+q)}f(u,x(u))du\Big)dA(s)\Big],\ t\in[a,T]
\end{aligned}$$

and

$$\begin{aligned}
(\mathscr{Q}_2x)(t) &= \kappa\Big[-\int_a^t\frac{(t-s)^{p-1}}{\Gamma(p)}x(s)ds + \varphi_1(t)\int_a^T\frac{(T-s)^{p-1}}{\Gamma(p)}x(s)ds\\
&+\varphi_2(t)\int_a^T\frac{(T-s)^{p-2}}{\Gamma(p-1)}x(s)ds - \varphi_3(t)\Big(\sum_{i=1}^{n-2}\alpha_i\int_a^{\xi_i}\frac{(\xi_i-s)^{p-1}}{\Gamma(p)}x(s)ds\\
&+\int_a^T\Big(\int_a^s\frac{(s-u)^{p-1}}{\Gamma(p)}x(u)dudA(s)\Big)\Big],\ t\in[a,T].
\end{aligned}$$

Here one can note that $\mathscr{Q} = \mathscr{Q}_1 + \mathscr{Q}_2$ on $\mathbb{B}_\tau$. Now, we verify the assumptions of Krasnosel'skiĭ fixed point theorem in three steps.

(i) For $x_1, x_2 \in \mathbb{B}_\tau$, $\mathscr{Q}_1 x_1 + \mathscr{Q}_2 x_2 \in \mathbb{B}_\tau$. Indeed, using (3.3), we get

$$\begin{aligned}
&\|\mathscr{Q}_1 x_1 + \mathscr{Q}_2 x_2\| \\
&\le \sup_{t\in[a,T]} \Big\{ |\lambda| \int_a^t \frac{(t-s)^{\gamma+p-1}}{\Gamma(\gamma+p)} |h(s,x_1(s))| ds \\
&+ \int_a^t \frac{(t-s)^{p+q-1}}{\Gamma(p+q)} |f(s,x_1(s))| ds + |\varphi_1(t)| \Big[|\lambda| \int_a^T \frac{(T-s)^{\gamma+p-1}}{\Gamma(\gamma+p)} |h(s,x_1(s))| ds \\
&+ \int_a^T \frac{(T-s)^{p+q-1}}{\Gamma(p+q)} |f(s,x_1(s))| ds \Big] + |\varphi_2(t)| \Big[|\lambda| \int_a^T \frac{(T-s)^{\gamma+p-2}}{\Gamma(\gamma+p-1)} |h(s,x_1(s))| ds \\
&+ \int_a^T \frac{(T-s)^{p+q-2}}{\Gamma(p+q-1)} |f(s,x_1(s))| ds \Big] + |\varphi_3(t)| \Big[|\lambda| \sum_{i=1}^{n-2} |\alpha_i| \int_a^{\xi_i} \frac{(\xi_i-s)^{\gamma+p-1}}{\Gamma(\gamma+p)} |h(s,x_1(s))| ds \\
&+ \sum_{i=1}^{n-2} |\alpha_i| \int_a^{\xi_i} \frac{(\xi_i-s)^{p+q-1}}{\Gamma(p+q)} |f(s,x_1(s))| ds + \int_a^T \Big(|\lambda| \int_a^s \frac{(s-u)^{\gamma+p-1}}{\Gamma(\gamma+p)} |h(u,x_1(u))| du \\
&+ \int_a^s \frac{(s-u)^{p+q-1}}{\Gamma(p+q)} |f(u,x_1(u))| du \Big) dA(s) \Big] + |\kappa| \Big[\int_a^t \frac{(t-s)^{p-1}}{\Gamma(p)} |x_2(s)| ds \\
&+ |\varphi_1(t)| \int_a^T \frac{(T-s)^{p-1}}{\Gamma(p)} |x_2(s)| ds + |\varphi_2(t)| \int_a^T \frac{(T-s)^{p-2}}{\Gamma(p-1)} |x_2(s)| ds \\
&+ |\varphi_3(t)| \Big(\sum_{i=1}^{n-2} |\alpha_i| \int_a^{\xi_i} \frac{(\xi_i-s)^{p-1}}{\Gamma(p)} |x_2(s)| ds + \int_a^T \Big(\int_a^s \frac{(s-u)^{p-1}}{\Gamma(p)} |x_2(u)| du \Big) \Big) dA(s) \Big] \Big\} \\
&\le |\lambda| \|\phi_1\| \Big\{ \frac{(T-a)^{\gamma+p}}{\Gamma(\gamma+p+1)} + \widetilde{\varphi}_1 \frac{(T-a)^{\gamma+p}}{\Gamma(\gamma+p+1)} + \widetilde{\varphi}_2 \frac{(T-a)^{\gamma+p-1}}{\Gamma(\gamma+p)} + \widetilde{\varphi}_3 \Big(\sum_{i=1}^{n-2} |\alpha_i| \frac{(\xi_i-a)^{\gamma+p}}{\Gamma(\gamma+p+1)} \\
&+ \int_a^T \frac{(s-a)^{\gamma+p}}{\Gamma(\gamma+p+1)} dA(s) \Big) \Big\} + \|\phi_2\| \Big\{ \frac{(T-a)^{q+p}}{\Gamma(q+p+1)} + \widetilde{\varphi}_1 \frac{(T-a)^{q+p}}{\Gamma(q+p+1)} \\
&+ \widetilde{\varphi}_2 \frac{(T-a)^{q+p-1}}{\Gamma(q+p)} + \widetilde{\varphi}_3 \Big(\sum_{i=1}^{n-2} |\alpha_i| \frac{(\xi_i-a)^{q+p}}{\Gamma(q+p+1)} + \int_a^T \frac{(s-a)^{q+p}}{\Gamma(q+p+1)} dA(s) \Big) \Big\} \\
&+ |\kappa| \|x\| \Big\{ \frac{(T-a)^{p}}{\Gamma(p+1)} + \widetilde{\varphi}_1 \frac{(T-a)^{p}}{\Gamma(p+1)} + \widetilde{\varphi}_2 \frac{(T-a)^{p-1}}{\Gamma(p)} + \widetilde{\varphi}_3 \Big(\sum_{i=1}^{n-2} |\alpha_i| \frac{(\xi_i-a)^{p}}{\Gamma(p+1)} \\
&+ \int_a^T \frac{(s-a)^{p}}{\Gamma(p+1)} dA(s) \Big) \Big\} \le |\lambda| \|\phi_1\| \mathscr{E}_1 + \|\phi_2\| \mathscr{E}_2 + |\kappa| \mathscr{E}_0 \tau \le \tau.
\end{aligned}$$

Hence $\mathscr{Q}_1 x_1 + \mathscr{Q}_2 x_2 \in \mathbb{B}_\tau$. (ii) $\mathscr{Q}_1$ is compact and continuous. Continuity of $\mathscr{Q}_1$ follows from continuity of h and f. Also, $\mathscr{Q}_1$ is uniformly bounded on $\mathbb{B}_\tau$ as

$$\begin{aligned}
\|\mathscr{Q}_1 x_1\| \le |\lambda| \|\phi_1\| \Big[&\frac{(T-a)^{\gamma+p}}{\Gamma(\gamma+p+1)} + \widetilde{\varphi}_1 \frac{(T-a)^{\gamma+p}}{\Gamma(\gamma+p+1)} + \widetilde{\varphi}_2 \frac{(T-a)^{\gamma+p-1}}{\Gamma(\gamma+p)} \\
&+ \widetilde{\varphi}_3 \Big(\sum_{i=1}^{n-2} |\alpha_i| \frac{(\xi_i-a)^{\gamma+p}}{\Gamma(\gamma+p+1)} + \int_a^T \frac{(s-a)^{\gamma+p}}{\Gamma(\gamma+p+1)} dA(s) \Big) \Big] + \|\phi_2\| \Big[\frac{(T-a)^{q+p}}{\Gamma(q+p+1)} \\
&+ \widetilde{\varphi}_1 \frac{(T-a)^{q+p}}{\Gamma(q+p+1)} + \widetilde{\varphi}_2 \frac{(T-a)^{q+p-1}}{\Gamma(q+p)} + \widetilde{\varphi}_3 \Big(\sum_{i=1}^{n-2} |\alpha_i| \frac{(\xi_i-a)^{q+p}}{\Gamma(q+p+1)}
\end{aligned}$$

$$+\int_a^T \frac{(s-a)^{\gamma+p}}{\Gamma(q+p+1)}dA(s)\Big)\Big] \leq |\lambda|\|\phi_1\|\mathscr{E}_1 + \|\phi_2\|\mathscr{E}_2 = \mathbf{Q}.$$

Furthermore, we show the compactness of the operator $\mathscr{Q}_1$. For $a < t_1 < t_2 < T$, we have

$$\begin{aligned}
& |(\mathscr{Q}_1 x)(t_2) - (\mathscr{Q}_1 x)(t_1)| \\
\leq\ & |\lambda|\Big[\int_a^{t_1} \frac{(t_2-s)^{\gamma+p-1}-(t_1-s)^{\gamma+p-1}}{\Gamma(\gamma+p)}|h(s,x(s))|ds \\
& + \int_{t_1}^{t_2} \frac{|(t_2-s)^{\gamma+p-1}|}{\Gamma(\gamma+p)}|h(s,x(s))|ds\Big] + \int_a^{t_1} \frac{(t_2-s)^{p+q-1}-(t_1-s)^{p+q-1}}{\Gamma(p+q)}|f(s,x(s))|ds \\
& + \int_{t_1}^{t_2} \frac{|(t_2-s)^{p+q-1}|}{\Gamma(p+q)}|f(s,x(s))|ds + |\varphi_1(t_2)-\varphi_1(t_1)|\Big[|\lambda|\int_a^T \frac{(T-a)^{\gamma+p-1}}{\Gamma+p)}|h(s,x(s))|ds \\
& + \int_a^T \frac{(T-s)^{p+q-1}}{\Gamma(p+q)}|f(s,x(s))|ds\Big] + |\varphi_2(t_2)-\varphi_2(t_1)|\Big[|\lambda|\int_a^T \frac{(T-s)^{\gamma+p-2}}{\Gamma(\gamma+p-1)}|h(s,x(s))|ds \\
& + \int_a^T \frac{(T-s)^{p+q-2}}{\Gamma(p+q-1)}|f(s,x(s))|ds\Big] + |\varphi_3(t_2)-\varphi_3(t_1)|\Big[|\lambda|\sum_{i=1}^{n-2}|\alpha_i|\int_a^{\xi_i} \frac{(\xi_i-s)^{\gamma+p-1}}{\Gamma(\gamma+p)} \\
& \times|h(s,x(s))|ds + \sum_{i=1}^{n-2}|\alpha_i|\int_a^{\xi_i} \frac{(\xi_i-s)^{p+q-1}}{\Gamma(p+q)}|f(s,x(s))|ds \\
& + \int_a^T \Big(|\lambda|\int_a^s \frac{(s-u)^{\gamma+p-1}}{\Gamma(\gamma+p)}|h(u,x(u))|)du + \int_a^s \frac{(s-u)^{p+q-1}}{\Gamma(p+q)}|f(u,x(u))|du\Big)dA(s)\Big] \\
\leq\ & \frac{|\lambda|\|\phi_1\|}{\Gamma(\gamma+p+1)}\Big((t_2-a)^{\gamma+p}-(t_1-a)^{\gamma+p}+2(t_2-t_1)^{\gamma+p}\Big) \\
& + \frac{\|\phi_2\|}{\Gamma(p+q+1)}\Big((t_2-a)^{p+q}-(t_1-a)^{p+q}+2(t_2-t_1)^{p+q}\Big) \\
& + |\varphi_1(t_2)-\varphi_1(t_1)|\Big[|\lambda|\frac{(T-a)^{\gamma+p}}{\Gamma(\gamma+p+1)}\|\phi_1\| + \frac{(T-a)^{p+q}}{\Gamma(p+q+1)}\|\phi_2\|\Big] \\
& + |\varphi_2(t_2)-\varphi_2(t_1)|\Big[|\lambda|\frac{(T-a)^{\gamma+p-1}}{\Gamma(\gamma+p)}\|\phi_1\| + \frac{(T-a)^{p+q-1}}{\Gamma(p+q)}\|\phi_2\|\Big] \\
& + |\varphi_3(t_2)-\varphi_3(t_1)|\Big[|\lambda|\sum_{i=1}^{n-2}|\alpha_i|\frac{(\xi_i-a)^{\gamma+p}}{\Gamma(\gamma+p+1)}\|\phi_1\| + \sum_{i=1}^{n-2}|\alpha_i|\frac{(\xi_i-a)^{p+q}}{\Gamma(p+q+1)}\|\phi_2\| \\
& + \int_a^T \Big(|\lambda|\frac{(s-a)^{\gamma+p|}}{\Gamma(\gamma+p+1)}\|\phi_1\|du + \int_a^s \frac{(s-u)^{p+q-1}}{\Gamma(p+q)}\|\phi_2\|du\Big)dA(s)\Big],
\end{aligned}$$

which tends to zero independently of x as $t_2 \longrightarrow t_1$. Thus, $\mathscr{Q}_1$ is equicontinuous on $\mathbb{B}_\tau$. Hence, by Arzelá-Ascoli theorem, the operator $\mathscr{Q}_1$ is compact on $\mathbb{B}_\tau$. (iii) $\mathscr{Q}_2$ is a contraction. In order to show that $\mathscr{Q}_2$ is a contraction, let us take $x_1, x_2 \in \mathbb{B}_\tau$, $t \in [a,T]$. Then

$$\begin{aligned}
\|\mathscr{Q}_2 x_1 - \mathscr{Q}_2 x_2\| \leq\ & \sup_{t\in[a,T]}\Big\{|\kappa|\Big[\int_a^t \frac{(t-s)^{p-1}}{\Gamma(p)}|x_1(s)-x_2(s)|ds \\
& + |\varphi_1(t)|\int_a^T \frac{(T-s)^{p-1}}{\Gamma(p)}|x_1(s)-x_2(s)|ds \\
& + |\varphi_2(t)|\int_a^T \frac{(T-s)^{p-2}}{\Gamma(p-1)}|x_1(s)-x_2(s)|ds
\end{aligned}$$

$$
\begin{aligned}
&+|\varphi_3(t)|\Big(\sum_{i=1}^{n-2}|\alpha_i|\int_a^{\xi_i}\frac{(\xi_i-s)^{p-1}}{\Gamma(p)}|x_1(s)-x_2(s)|ds\\
&+\int_a^T\Big(\int_a^s\frac{(s-u)^{p-1}}{\Gamma(p)}|x_1(u)-x_2(u)|du\Big)dA(s)\Big)\Big]\Big\}\\
\le\ & |\kappa|\Big[\frac{(T-a)^p}{\Gamma(p+1)}+\widetilde{\varphi}_1\frac{(T-a)^p}{\Gamma(p+1)}+\widetilde{\varphi}_2\frac{(T-a)^{p-1}}{\Gamma(p)}+\widetilde{\varphi}_3\Big(\sum_{i=1}^{n-2}|\alpha_i|\frac{(\xi_i-a)^p}{\Gamma(p+1)}\\
&+\int_a^T\frac{(s-a)^p}{\Gamma(p+1)}dA(s)\Big)\Big]\|x_1-x_2\|\\
\le\ & |\kappa|\mathscr{E}_0\|x_1-x_2\|,
\end{aligned}
$$

which, in view of (3.4), implies that $\mathscr{Q}_2$ is a contraction. Thus, all the conditions of Krasnosel'skiĭ's theorem are satisfied and consequently we deduce by its conclusion that there exists a solution for the problem (1.1)-(1.2) on $[a,T]$. The proof is finished. □

Example 3.3. Consider the following fractional integro-differential equations involving Rimann-Liouvill and Capouto derivetvies supplemented with integro-multipoint boundary conditions.

$$
\begin{cases}
{}^{RL}D^{\frac{7}{4}}\Big[\Big({}^{c}D^{\frac{9}{5}}+\frac{8}{27}\Big)x(t)+\frac{10}{33}I^{\frac{5}{3}}h(t,x(t))\Big]=f(t,x(t))\ ,\ t\in(0,1),\\
x(0)=\sum_{i=1}^{4}\alpha_i x(\xi_i)+\int_a^1 x(s)dA(s),\ x'(0)=0,\ x(T)=0,\ x'(T)=0,
\end{cases}
\tag{3.5}
$$

where $a=0,\ T=1,\ q=7/4,\ p=9/5,\ \gamma=5/3,\ \kappa=8/27,\ \lambda=10/33,\ \alpha_1=-1,\ \alpha_2=-1/2,\ \alpha_3=1/2,\ \alpha_4=3/4,\ \xi_1=1/3,\ \xi_2=1/2,\ \xi_3=2/3,\ \xi_4=3/4,$

$$
h(t,x(t))=\frac{(t+4)}{15}\Big(\frac{(x+3)^2}{8+(x+3)^2}+\cos x(t)\Big),
$$

and

$$
f(t,x(t))=\frac{1}{\sqrt{t+25}}+\frac{\arctan x(t)}{\sqrt{t^3+9}}.
$$

Let us take $A(s)=\frac{s^2}{2}+\frac{9}{29}$. Using the given data, we have that $A_1\simeq 0.261600$, $A_2\simeq 0.889440$, $A_3\simeq 0.667080$, $A_4\simeq 1.37863$, $A_5\simeq -0.160000$, $A_6\simeq -0.600959$, $A_7\simeq 0.750000$, $\sigma\simeq -0.354816$, $B_1\simeq -2.91410$, $B_2\simeq 3.57380$, $B_3\simeq 3.88548$, $B_4\simeq 1.41005$, $B_5\simeq -1.00390$, $B_6\simeq -1.88007$, $B_7\simeq 0.508170$, $B_8\simeq -0.041996$, $B_9\simeq 0.655771$, $\widetilde{\varphi}_1\simeq 1.000000$, $\widetilde{\varphi}_2\simeq 0.102500$, $\widetilde{\varphi}_3\simeq 0.655771$, $\mathscr{E}_0\simeq 1.78532$, $\mathscr{E}_1\simeq 0.250411$, and $\mathscr{E}_2\simeq 0.222752$.

Evidently, the hypothesis of Theorem 3.2 is satisfied with $\phi_1(t)=\frac{2t+8}{15}$, and

$$
\phi_2(t)=\frac{1}{\sqrt{t+25}}+\frac{\pi}{2\sqrt{t^2+9}}.
$$

Also, $|\kappa|\mathscr{E}_0\simeq 0.528984<1$. Therefore, from the conclusion of Theorem 3.2, there exists at least one solution for problem (3.5) on $[0,1]$.

Now, we show the existence results for problem (1.1)-(1.2) by applying Leray-Schauder nonlinear alternative criterion.

Lemma 3.4. *(Leray-Schauder nonlinear alternative [24]): Let $\mathscr{Y}$ be a closed, convex subset of a Banach space $\mathscr{U}$ and $\mathfrak{Z}$ be an open subset of $\mathscr{Y}$ with $0 \in \mathfrak{Z}$. Assume that $\mathfrak{J} : \bar{\mathfrak{Z}} \longrightarrow \mathscr{Y}$ is continuous and compact. Then either*

(i) *there is a $y \in \partial\,\mathfrak{Z}$ (the boundary of $\mathfrak{Z}$ in $\mathscr{Y}$) and $0 < \alpha < 1$ with $y = \alpha\mathfrak{J}(y)$, or*
(ii) *$\mathfrak{J}$ has a fixed point in $\bar{\mathfrak{Z}}$.*

Theorem 3.5. *Assume that (3.4) holds. In addition we suppose that $h, f : [a,T] \times \mathbb{R} \longrightarrow \mathbb{R}$ are continuous functions satisfying the following conditions:*

($\mathscr{O}_1$) *there exist functions $\rho_1, \rho_2 \in C([a,T] \times \mathbb{R}^+)$, and nondecreasing functions $\Omega_1, \Omega_2 : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ such that*

$$\begin{aligned} |h(t,x)| &\leq \rho_1(t)\Omega_1(\|x\|), \\ |f(t,x)| &\leq \rho_2(t)\Omega_2(\|x\|), \ \forall\, (t,x) \in [a,T] \times \mathbb{R}; \end{aligned}$$

($\mathscr{O}_2$) *there exists a constant $\mathscr{M} > 0$ such that*

$$\frac{(1 - |\kappa|\mathscr{E}_0)\mathscr{M}}{|\lambda|\|\rho_1\|\Omega_1(\mathscr{M})\mathscr{E}_1 + \|\rho_2\|\Omega_2(\mathscr{M})\mathscr{E}_2} > 1, \tag{3.6}$$

where $\mathscr{E}_i$ $(i = 0,1,2)$ are defined by (3.3).

Then problem (1.1)-(1.2) has at least one solution on $[a,T]$.

Proof. In the first step, it will be shown that the oprator $\mathscr{Q} : \mathscr{V} \longrightarrow \mathscr{V}$ defined by (3.2) maps bounded sets into bounded sets in $\mathscr{V}$. For $b > 0$, let $E_b = \{x \in \mathscr{V} : \|x\| \leq b\}$ be a bounded set in $\mathscr{V}$. Then, for $x \in E_b$, we have

$$\begin{aligned} \|\mathscr{Q}x\| &= \sup_{t\in[a,T]} |(\mathscr{Q}x)(t)| \\ &\leq \sup_{t\in[a,T]} \Big\{ |\kappa| \int_a^t \frac{(t-s)^{p-1}}{\Gamma(p)} |x(s)|ds + |\lambda| \int_a^t \frac{(t-s)^{\gamma+p-1}}{\Gamma(\gamma+p)} |h(s,x(s))|ds \\ &\quad + \int_a^t \frac{(t-s)^{p+q-1}}{\Gamma(p+q)} |f(s,x(s))|ds + |\varphi_1(t)| \Big[|\kappa| \int_a^T \frac{(T-s)^{p-1}}{\Gamma(p)} |x(s)|ds \\ &\quad + |\lambda| \int_a^T \frac{(T-s)^{\gamma+p-1}}{\Gamma(\gamma+p)} |h(s,x(s))|ds + \int_a^T \frac{(T-s)^{p+q-1}}{\Gamma(p+q)} |f(s,x(s))|ds \Big] \\ &\quad + |\varphi_2(t)| \Big[|\kappa| \int_a^T \frac{(T-s)^{p-2}}{\Gamma(p-1)} |x(s)|ds + |\lambda| \int_a^T \frac{(T-s)^{\gamma+p-2}}{\Gamma(\gamma+p-1)} |h(s,x(s))|ds \\ &\quad + \int_a^T \frac{(T-s)^{p+q-2}}{\Gamma(p+q-1)} |f(s,x(s))|ds \Big] + |\varphi_3(t)| \Big[|\kappa| \sum_{i=1}^{n-2} |\alpha_i| \int_a^{\xi_i} \frac{(\xi_i - s)^{p-1}}{\Gamma(p)} |x(s)|ds \\ &\quad + |\lambda| \sum_{i=1}^{n-2} |\alpha_i| \int_a^{\xi_i} \frac{(\xi_i - s)^{\gamma+p-1}}{\Gamma(\gamma+p)} |h(s,x(s))|ds \\ &\quad + \sum_{i=1}^{n-2} |\alpha_i| \int_a^{\xi_i} \frac{(\xi_i - s)^{p+q-1}}{\Gamma(p+q)} |f(s,x(s))|ds + \int_a^T \Big(|\kappa| \int_a^s \frac{(s-u)^{p-1}}{\Gamma(p)} |x(u)|du \\ &\quad + |\lambda| \int_a^s \frac{(s-u)^{\gamma+p-1}}{\Gamma(\gamma+p)} |h(u,x(u))|du + \int_a^s \frac{(s-u)^{p+q-1}}{\Gamma(p+q)} |f(u,x(u))|du \Big) dA(s) \Big] \Big\} \end{aligned}$$

$$
\begin{aligned}
\leq\ & |\kappa|\|x\|\Big\{\frac{(T-a)^p}{\Gamma(p+1)}+\widetilde{\varphi}_1\frac{(T-a)^p}{\Gamma(p+1)}+\widetilde{\varphi}_2\frac{(T-a)^{p-1}}{\Gamma(p)}+\widetilde{\varphi}_3\Big(\sum_{i=1}^{n-2}|\alpha_i|\frac{(\xi_i-a)^p}{\Gamma(p+1)}\\
&+\int_a^T\frac{(s-a)^p}{\Gamma(p+1)}dA(s)\Big)\Big\}+|\lambda|\|\rho_1\|\Omega_1(\|x\|)\Big\{\frac{(T-a)^{\gamma+p}}{\Gamma(\gamma+p+1)}\\
&+\widetilde{\varphi}_1\frac{(T-a)^{\gamma+p}}{\Gamma(\gamma+p+1)}+\widetilde{\varphi}_2\frac{(T-a)^{\gamma+p-1}}{\Gamma(\gamma+p)}+\widetilde{\varphi}_3\Big(\sum_{i=1}^{n-2}|\alpha_i|\frac{(\xi_i-a)^{\gamma+p}}{\Gamma(\gamma+p+1)}\\
&+\int_a^T\frac{(s-a)^{\gamma+p}}{\Gamma(\gamma+p+1)}dA(s)\Big)\Big\}+\|\rho_2\|\Omega_2(\|x\|)\Big\{\frac{(T-a)^{q+p}}{\Gamma(q+p+1)}+\widetilde{\varphi}_1\frac{(T-a)^{q+p}}{\Gamma(q+p+1)}\\
&+\widetilde{\varphi}_2\frac{(T-a)^{q+p-1}}{\Gamma(q+p)}+\widetilde{\varphi}_3\Big(\sum_{i=1}^{n-2}|\alpha_i|\frac{(\xi_i-a)^{q+p}}{\Gamma(q+p+1)}+\int_a^T\frac{(s-a)^{q+p}}{\Gamma(q+p+1)}dA(s)\Big)\Big\}.
\end{aligned}
$$

Using (3.3), the above estimate becomes

$$
\begin{aligned}
\|\mathscr{Q}x\| &\leq |\kappa|\|x\|\mathscr{E}_0+|\lambda|\|\rho_1\|\Omega_1(\|x\|)\mathscr{E}_1+\|\rho_2\|\Omega_2(\|x\|)\mathscr{E}_2\\
&\leq |\kappa|b\mathscr{E}_0+|\lambda|\|\rho_1\|\Omega_1(b)\mathscr{E}_1+\|\rho_2\|\Omega_2(b)\mathscr{E}_2,
\end{aligned}
$$

which implies that the operator $\mathscr{Q}$ is bounded in $\mathscr{V}$.

Now, we show that $\mathscr{Q}$ is equicontinuous on $[a,T]$. Let $t_1,t_2\in[a,T]$ with $a<t_1<t_2<T$, and $x\in E_b$. Then

$$
\begin{aligned}
&|(\mathscr{Q}x)(t_2)-(\mathscr{Q}x)(t_1)|\\
\leq\ & |\kappa|\Big[\int_a^{t_1}\frac{(t_2-s)^{p-1}-(t_1-s)^{p-1}}{\Gamma(p)}|x(s)|ds+\int_{t_1}^{t_2}\frac{(t_2-s)^{p-1}}{\Gamma(p)}|x(s)|ds\Big]\\
&+|\lambda|\Big[\int_a^{t_1}\frac{(t_2-s)^{\gamma+p-1}-(t_1-s)^{\gamma+p-1}}{\Gamma(\gamma+p)}|h(s,x(s))|ds\\
&+\int_{t_1}^{t_2}\frac{(t_2-s)^{\gamma+p-1}}{\Gamma(\gamma+p)}|h(s,x(s))|ds\Big]+\int_a^{t_1}\frac{(t_2-s)^{p+q-1}-(t_1-s)^{p+q-1}}{\Gamma(p+q)}|f(s,x(s))|ds\\
&+\int_{t_1}^{t_2}\frac{(t_2-s)^{p+q-1}}{\Gamma(p+q)}|f(s,x(s))|ds\\
&+|\varphi_1(t_2)-\varphi_1(t_1)|\Big[|\kappa|\int_a^T\frac{(T-s)^{p-1}}{\Gamma(p)}|x(s)|ds+|\lambda|\int_a^T\frac{(T-a)^{\gamma+p-1}}{\Gamma+p)}|h(s,x(s))|ds\\
&+\int_a^T\frac{(T-s)^{p+q-1}}{\Gamma(p+q)}|f(s,x(s))|ds\Big]+|\varphi_2(t_2)-\varphi_2(t_1)|\Big[|\kappa|\int_a^T\frac{(T-s)^{p-2}}{\Gamma(p-1)}|x(s)|ds\\
&+|\lambda|\int_a^T\frac{(T-s)^{\gamma+p-2}}{\Gamma(\gamma+p-1)}|h(s,x(s))|ds+\int_a^T\frac{(T-s)^{p+q-2}}{\Gamma(p+q-1)}|f(s,x(s))|ds\Big]\\
&+|\varphi_3(t_2)-\varphi_3(t_1)|\Big[|\kappa|\sum_{i=1}^{n-2}|\alpha_i|\int_a^{\xi_i}\frac{(\xi_i-s)^{p-1}}{\Gamma(p)}|x(s)|ds+|\lambda|\sum_{i=1}^{n-2}|\alpha_i|\int_a^{\xi_i}\frac{(\xi_i-s)^{\gamma+p-1}}{\Gamma(\gamma+p)}\\
&|h(s,x(s))|ds+\sum_{i=1}^{n-2}|\alpha_i|\int_a^{\xi_i}\frac{(\xi_i-s)^{p+q-1}}{\Gamma(p+q)}|f(s,x(s))|ds
\end{aligned}
$$

$$
\begin{aligned}
&+\int_a^T \Big(|\kappa|\int_a^s \frac{(s-u)^{p-1}}{\Gamma(p)}|x(u)|du+|\lambda|\int_a^s \frac{(s-u)^{\gamma+p-1}}{\Gamma(\gamma+p)}|h(u,x(u))|)du \\
&+\int_a^s \frac{(s-u)^{p+q-1}}{\Gamma(p+q)}|f(u,x(u))|du\Big)dA(s)\Big] \\
\leq\ & \frac{|\kappa|\|x\|}{\Gamma(p+1)}\Big((t_2-a)^p-(t_1-a)^p+2(t_2-t_1)^p\Big) \\
&+\frac{|\lambda|\|\rho_1\|\Omega_1(\|x\|)}{\Gamma(\gamma+p+1)}\Big((t_2-a)^{\gamma+p}-(t_1-a)^{\gamma+p}+2(t_2-t_1)^{\gamma+p}\Big) \\
&+\frac{\|\rho_2\|\Omega_2(\|x\|)}{\Gamma(p+q+1)}\Big((t_2-a)^{p+q}-(t_1-a)^{p+q}+2(t_2-t_1)^{p+q}\Big) \\
&+|\varphi_1(t_2)-\varphi_1(t_1)|\Big[|\kappa|\frac{(T-a)^p}{\Gamma(p+1)}b+|\lambda|\frac{(T-a)^{\gamma+p}}{\Gamma+p+1)}\|\phi_1\|+\frac{(T-a)^{p+q}}{\Gamma(p+q+1)}\|\phi_2\|\Big] \\
&+|\varphi_2(t_2)-\varphi_2(t_1)|\Big[|\kappa|\frac{(T-a)^{p-1}}{\Gamma(p)}b+|\lambda|\frac{(T-a)^{\gamma+p-1}}{\Gamma(\gamma+p)}\|\phi_1\|+\frac{(T-a)^{p+q-1}}{\Gamma(p+q)}\|\phi_2\|\Big] \\
&+|\varphi_3(t_2)-\varphi_3(t_1)|\Big[|\kappa|\sum_{i=1}^{n-2}|\alpha_i|\frac{(\xi_i-a)^p}{\Gamma(p+1)}b+|\lambda|\sum_{i=1}^{n-2}|\alpha_i|\frac{(\xi_i-a)^{\gamma+p}}{\Gamma(\gamma+p+1)}\|\phi_1\| \\
&+\sum_{i=1}^{n-2}|\alpha_i|\frac{(\xi_i-a)^{p+q}}{\Gamma(p+q+1)}\|\phi_2\|+\int_a^T\Big(|\kappa|\frac{(s-u)^{p-1}|}{\Gamma(p)}b+|\lambda|\int_a^s\frac{|(s-a)^{\gamma+p}}{\Gamma(\gamma+p+1)}\|\phi_1\| \\
&+\int_a^s \frac{(s-u)^{p+q-1}}{\Gamma(p+q)}\|\phi_2\|du\Big)dA(s)\Big].
\end{aligned}
$$

Notice that the right hand side of the above inequality tends to zero as $(t_2-t_1)\longrightarrow 0$ independent of $x\in E_b$, which shows that $\mathscr{Q}$ is equicontuous. Thus, by the Arzelá-Ascoli theorem, the operator $\mathscr{Q}:\mathscr{V}\longrightarrow\mathscr{V}$ is completely continuous.

Next, we show the boundedness of the set of all solutions to equation $x=\beta\mathscr{Q}x$ for $0<\beta<1$. For $t\in[a,T]$, by using the computations of the first step above, we have

$$
\begin{aligned}
|x(t)| &= |\beta(\mathscr{Q}x)(t)| \\
&\leq |\kappa|\|x\|\Big\{\frac{(T-a)^p}{\Gamma(p+1)}+\widetilde{\varphi}_1\frac{(T-a)^p}{\Gamma(p+1)}+\widetilde{\varphi}_2\frac{(T-a)^{p-1}}{\Gamma(p)}+\widetilde{\varphi}_3\Big(\sum_{i=1}^{n-2}|\alpha_i|\frac{(\xi_i-a)^p}{\Gamma(p+1)} \\
&+\int_a^T\frac{(s-a)^p}{\Gamma(p+1)}dA(s)\Big)\Big\}+|\lambda|\|\rho_1\|\Omega_1(\|x\|)\Big\{\frac{(T-a)^{\gamma+p}}{\Gamma(\gamma+p+1)} \\
&+\widetilde{\varphi}_1\frac{(T-a)^{\gamma+p}}{\Gamma(\gamma+p+1)}+\widetilde{\varphi}_2\frac{(T-a)^{\gamma+p-1}}{\Gamma(\gamma+p)}+\widetilde{\varphi}_3\Big(\sum_{i=1}^{n-2}|\alpha_i|\frac{(\xi_i-a)^{\gamma+p}}{\Gamma(\gamma+p+1)} \\
&+\int_a^T\frac{(s-a)^{\gamma+p}}{\Gamma(\gamma+p+1)}dA(s)\Big)\Big\}+\|\rho_2\|\Omega_2(\|x\|)\Big\{\frac{(T-a)^{q+p}}{\Gamma(q+p+1)}+\widetilde{\varphi}_1\frac{(T-a)^{q+p}}{\Gamma(q+p+1)} \\
&+\widetilde{\varphi}_2\frac{(T-a)^{q+p-1}}{\Gamma(q+p)}+\widetilde{\varphi}_3\Big(\sum_{i=1}^{n-2}|\alpha_i|\frac{(\xi_i-a)^{q+p}}{\Gamma(q+p+1)}+\int_a^T\frac{(s-a)^{q+p}}{\Gamma(q+p+1)}dA(s)\Big)\Big\},
\end{aligned}
$$

which yields

$$\|x\|\leq|\kappa|\|x\|\mathscr{E}_0+|\lambda|\|\rho_1\|\Omega_1(\|x\|)\mathscr{E}_1+\|\rho_2\|\Omega_2(\|x\|)\mathscr{E}_2,$$

or

$$\frac{(1-|\kappa|\mathscr{E}_0)\|x\|}{|\lambda|\|\rho_1\|\Omega_1(\|x\|)\mathscr{E}_1+\|\rho_2\|\Omega_2(\|x\|)\mathscr{E}_2} \leq 1.$$

By the condition $(\mathscr{O}_2)$, there exists $\mathscr{M}$ such that $\|x\| \neq \mathscr{M}$. Consider the set $\mathscr{U} = \{x \in \mathscr{V} : \|x\| < \mathscr{M}\}$. Observe that the operator $\mathscr{Q} : \bar{\mathscr{U}} \longrightarrow \mathscr{V}$ is continuous and completely continuous. Thus, from the choice of $\mathscr{U}$, there is no $x \in \partial\mathscr{U}$ such that $x = \beta\mathscr{Q}x$ for some $0 < \beta < 1$. Therefore, by Lemma 3.4 (Leray-Schauder nonlinear alternative), the operator $\mathscr{Q}$ has a fixed point $x \in \bar{\mathscr{U}}$, which implies that the problem (1.1)-(1.2) has at least one solution on $[a,T]$. This completes the proof. □

Example 3.6. Let us consider the problem (3.5) supplemented with the same boundary conditions and

$$h(t,x(t)) = \frac{7}{\sqrt{t^4+\frac{25}{9}}}\big(\sin x(t)+\ln 5\big),$$

and

$$f(t,x(t)) = \frac{18\cos x(t)}{e^{6t}+35},\ t \in (0,1)$$

Obviously, $|h(t,x)| \leq \frac{7}{\sqrt{t^4+\frac{25}{9}}}(\|x\|+\ln 5)$ and $|f(t,x)| \leq \frac{18}{e^{6t}+35}$. Let us choose $\rho_1 = \frac{7}{\sqrt{t^4+\frac{25}{9}}}$, $\Omega_1(\|x\|) = \|x\|+\ln 5$ and $\rho_2 = \frac{18}{e^{6t}+35}$, $\Omega_2(\|x\|) = 1$, $\|\rho_1\| = \frac{21}{5}$, $\|\rho_2\| = \frac{1}{2}$. Using the foregoing values and the data of Example 3.3 in the assumption $(\mathscr{O}_2)$, it is found that $\mathscr{M} > \mathscr{M}_1 \simeq 4.0989275$. Hence, by Theorem 3.5 there exists at least one solution for the problem (3.5) on $[0,1]$.

In the following result, we make use of Leray-Schauder degree theory [25] to establish the existence of solutions for the problem (1.1)-(1.2)

Theorem 3.7. *Suppose that there exist* $\delta_i > 0$, $\varpi_i > 0$ $(i = 1,2)$ *such that* $|h(t,x)| \leq \delta_1|x| + \varpi_1$, $|f(t,x)| \leq \delta_2|x| + \varpi_2$, $\forall\ (t,x) \in [a,T]\times\mathbb{R}$ *and*

$$0 < \Big(|\kappa|\mathscr{E}_0+|\lambda|\delta_1\mathscr{E}_1+\delta_2\mathscr{E}_2\Big) < 1.$$

Then there exist at least one solution for the problem (1.1)-(1.2) on $[a,T]$.

Proof. we need to show there exists $x \in \mathbb{R}$ satisfying (3.1). Consider a set $\mathbb{S}_m = \{x \in \mathscr{V} : \max_{t\in[a,T]} |x(t)| < m\}$, where $m > 0$ to be fixed later. Then, it is sufficient to show that $\mathscr{Q} : \bar{\mathbb{S}}_m \longrightarrow C([a,T],\mathbb{R})$ satisfies

$$x \neq \Lambda\mathscr{Q}x, \forall\, x \in \partial\mathbb{S}_m,\ \forall\, 0 \leq \Lambda \leq 1. \tag{3.7}$$

Let us set

$$\Psi(\Lambda,x) = \Lambda\mathscr{Q}x,\ x \in C(\mathbb{R}),\ 0 \leq \Lambda \leq 1.$$

Then, by Arzelá-Ascoil theorem, $\psi_\Lambda(x) = x - \Psi(\Lambda,x) = x - \Lambda\mathscr{Q}x$ is completely continuous. If (3.7) is true, then the following Leray-Schauder degrees are well defined and by the homotopy invariance of topological degree, it follows that

$$\begin{aligned}\deg(\psi_\Lambda,\mathbb{S}_m,0) &= \deg(I-\Lambda\mathscr{Q},\mathbb{S}_m,0)=\deg(\psi_1,\mathbb{S}_m,0)\\ &= \deg(\psi_0,\mathbb{S}_m,0)=\deg(I,\mathbb{S}_m,0)=1\neq 0,\ 0\in\mathbb{S}_m,\end{aligned}$$

where I denotes the unit operator. By the nonzero property of Leray-Schauder degrees, $\psi_1(t)=x-\Lambda\mathscr{Q}x=0$ for at least one $x\in\mathbb{S}_m$. In order to prove (3.7), we suppose that $x=\Lambda\mathscr{Q}x$ for some $0\le\Lambda\le 1$ and for all $t\in[a,T]$. Then

$$\begin{aligned}|x(t)| &\le |\kappa||x|\Big\{\frac{(T-a)^p}{\Gamma(p+1)}+\widetilde{\varphi}_1\frac{(T-a)^p}{\Gamma(p+1)}+\widetilde{\varphi}_2\frac{(T-a)^{p-1}}{\Gamma(p)}+\widetilde{\varphi}_3\Big(\sum_{i=1}^{n-2}|\alpha_i|\frac{(\xi_i-a)^p}{\Gamma(p+1)}\\ &\quad+\int_a^T\frac{(s-a)^p}{\Gamma(p+1)}dA(s)\Big)\Big\}+|\lambda|(\delta_1|x|+\varpi_1)\Big\{\frac{(T-a)^{\gamma+p}}{\Gamma(\gamma+p+1)}\\ &\quad+\widetilde{\varphi}_1\frac{(T-a)^{\gamma+p}}{\Gamma(\gamma+p+1)}+\widetilde{\varphi}_2\frac{(T-a)^{\gamma+p-1}}{\Gamma(\gamma+p)}+\widetilde{\varphi}_3\Big(\sum_{i=1}^{n-2}|\alpha_i|\frac{(\xi_i-a)^{\gamma+p}}{\Gamma(\gamma+p+1)}\\ &\quad+\int_a^T\frac{(s-a)^{\gamma+p}}{\Gamma(\gamma+p+1)}dA(s)\Big)\Big\}+(\delta_2|x|+\varpi_2)\Big\{\frac{(T-a)^{q+p}}{\Gamma(q+p+1)}\\ &\quad+\widetilde{\varphi}_1\frac{(T-a)^{q+p}}{\Gamma(q+p+1)}+\widetilde{\varphi}_2\frac{(T-a)^{q+p-1}}{\Gamma(q+p)}+\widetilde{\varphi}_3\Big(\sum_{i=1}^{n-2}|\alpha_i|\frac{(\xi_i-a)^{\gamma+p}}{\Gamma(q+p+1)}\\ &\quad+\int_a^T\frac{(s-a)^{\gamma+p}}{\Gamma(q+p+1)}dA(s)\Big)\Big\}\\ &\le \Big(|\kappa|\mathscr{E}_0+|\lambda|\delta_1\mathscr{E}_1+\delta_2\mathscr{E}_2\Big)|x|+\Big(|\lambda|\varpi_1\mathscr{E}_1+\varpi_2\mathscr{E}_2\Big),\end{aligned}$$

which, on taking the norm $t\in[a,T]$ and solving for $\|x\|$, we get $\|x\|\le\frac{\mathscr{F}_2}{1-\mathscr{F}_1}$, where $\mathscr{F}_1=|\kappa|\mathscr{E}_0+|\lambda|\delta_1\mathscr{E}_1+\delta_2\mathscr{E}_2$, $\mathscr{F}_2=|\lambda|\varpi_1\mathscr{E}_1+\varpi_2\mathscr{E}_2$. Letting $m=\dfrac{\mathscr{F}_2}{1-\mathscr{F}_1}+1$, (3.7) holds. This completes the proof. □

Example 3.8. Consider the problem (3.5) in Example 3.3 with

$$h(t,x(t))=\Big(\frac{2t^2}{\sqrt[4]{t^3+1296}}\Big)x(t)+\Big(\frac{t+1}{70}\Big)\tan^{-1}x(t),$$

and

$$f(t,x(t))=\frac{23x(t)}{(t+8)^2}+\frac{5}{31}.$$

Clearly, $|h(t,x)|\le\frac{1}{3}\|x\|+\frac{\pi}{70}$ and $|f(t,x)|\le\frac{23}{64}\|x\|+\frac{5}{31}$. So, $\delta_1=\frac{1}{3}$, $\delta_2=\frac{23}{64}$, $\varpi_1=\frac{\pi}{70}$, $\varpi_2=\frac{5}{31}$ and $\mathscr{F}_1\simeq 0.634330<1$. Therefore it follows by the conclusion of Theorem 3.7, there exists at least one solution for the problem (3.5) on $[0,1]$.

4. Uniqueness of solutions

In this section, we study the uniqueness result of solutions for the problem (1.1)-(1.2) by applying Banach contraction mapping principle.

Theorem 4.1. *Let $h, f : [a,T]\times\mathbb{R}\longrightarrow\mathbb{R}$ be continuous functions satisfying condition: $\forall t\in[a,T]$, $x_1,x_2\in\mathbb{R}$, there exist $\mathscr{L}_1,\mathscr{L}_2>0$ such that $|h(t,x_1)-h(t,x_2)|\leq\mathscr{L}_1|x_1-x_2|$, $|f(t,x_1)-f(t,x_2)|\leq\mathscr{L}_2|x_1-x_2|$. If*

$$|\kappa|\mathscr{E}_0+|\lambda|\mathscr{L}_1\mathscr{E}_1+\mathscr{L}_2\mathscr{E}_2<1, \tag{4.1}$$

where $\mathscr{E}_i$ $(i=0,1,,2)$ are given by (3.3), then there exists a unique solution for the problem (1.1)-(1.2) on $[a,T]$

Proof. Take $n_1=\sup_{t\in[a,T]}|h(t,0)|$, $n_2=\sup_{t\in[a,T]}|f(t,0)|$, and choose

$$\theta\geq\frac{|\lambda|n_1\mathscr{E}_1+n_2\mathscr{E}_2}{1-|\kappa|\mathscr{E}_0-|\lambda|\mathscr{L}_1\mathscr{E}_1-\mathscr{L}_2\mathscr{E}_2}.$$

We define $\mathbb{S}_\theta=\{x\in\mathscr{V}:\|x\|\leq\theta\}$ and show that $\mathscr{Q}\mathbb{S}_\theta\subset\mathbb{S}_\theta$, where the operator $\mathscr{Q}$ is defined by (3.2). For any $x\in\mathbb{S}_\theta$, $t\in[a,T]$, we have

$$\begin{aligned}|h(t,x(t))| &= |h(t,x(t))-h(t,0)+h(t,0)|\leq|h(t,x(t))-h(t,0)|+|h(t,0)|\\ &\leq \mathscr{L}_1|x(t)|+n_1\leq\mathscr{L}_1\|x\|+n_1\leq\mathscr{L}_1\theta+n_1\end{aligned}$$

and

$$\begin{aligned}|f(t,x(t))| &= |f(t,x(t))-f(t,0)+f(t,0)|\leq|f(t,x(t))-f(t,0)|+|f(t,0)|\\ &\leq \mathscr{L}_2|x(t)|+n_2\leq\mathscr{L}_2\|x\|+n_2\leq\mathscr{L}_2\theta+n_2.\end{aligned}$$

Then

$$\begin{aligned}\|\mathscr{Q}x\| \leq\ & \sup_{t\in[a,T]}\Big\{|\kappa|\int_a^t\frac{(t-s)^{p-1}}{\Gamma(p)}|x(s)|ds+|\lambda|\int_a^t\frac{(t-s)^{\gamma+p-1}}{\Gamma(\gamma+p)}|h(s,x(s))|ds\\ &+\int_a^t\frac{(t-s)^{p+q-1}}{\Gamma(p+q)}|f(s,x(s))|ds+|\varphi_1(t)|\Big[|\kappa|\int_a^T\frac{(T-s)^{p-1}}{\Gamma(p)}|x(s)|ds\\ &+|\lambda|\int_a^T\frac{(T-s)^{\gamma+p-1}}{\Gamma(\gamma+p)}|h(s,x(s))|ds+\int_a^T\frac{(T-s)^{p+q-1}}{\Gamma(p+q)}|f(s,x(s))|ds\Big]\\ &+|\varphi_2(t)|\Big[|\kappa|\int_a^T\frac{(T-s)^{p-2}}{\Gamma(p-1)}|x(s)|ds+|\lambda|\int_a^T\frac{(T-s)^{\gamma+p-2}}{\Gamma(\gamma+p-1)}|h(s,x(s))|ds\\ &+\int_a^T\frac{(T-s)^{p+q-2}}{\Gamma(p+q-1)}|f(s,x(s))|ds\Big]+|\varphi_3(t)|\Big[|\kappa|\sum_{i=1}^{n-2}|\alpha_i|\int_a^{\xi_i}\frac{(\xi_i-s)^{p-1}}{\Gamma(p)}|x(s)|ds\\ &+|\lambda|\sum_{i=1}^{n-2}|\alpha_i|\int_a^{\xi_i}\frac{(\xi_i-s)^{\gamma+p-1}}{\Gamma(\gamma+p)}|h(s,x(s))|ds\\ &+\sum_{i=1}^{n-2}|\alpha_i|\int_a^{\xi_i}\frac{(\xi_i-s)^{p+q-1}}{\Gamma(p+q)}|f(s,x(s))|ds+\int_a^T\Big(|\kappa|\int_a^s\frac{(s-u)^{p-1}}{\Gamma(p)}|x(u)|du\\ &+|\lambda|\int_a^s\frac{(s-u)^{\gamma+p-1}}{\Gamma(\gamma+p)}|h(u,x(u))|du+\int_a^s\frac{(s-u)^{p+q-1}}{\Gamma(p+q)}|f(u,x(u))|du\Big)dA(s)\Big]\Big\}\\ \leq\ & |\kappa|\theta\mathscr{E}_0+|\lambda|(\mathscr{L}_1\theta+n_1)\mathscr{E}_1+(\mathscr{L}_2\theta+n_2)\mathscr{E}_2\leq\theta.\end{aligned}$$

This shows that $\mathscr{Q}x\in\mathbb{S}_\theta$ for any $x\in\mathbb{S}_\theta$. Therefore, $\mathscr{Q}\mathbb{S}_\theta\subset\mathbb{S}_\theta$.

Now, we show that $\mathscr{Q}$ is a contraction. For $x_1, x_2 \in \mathscr{V}$ and $t \in [a,T]$, using (4.1), we obtain

$$
\begin{aligned}
&\|(\mathscr{Q}x_1)-(\mathscr{Q}x_2)\| = \sup_{t\in[a,T]}\Big|(\mathscr{Q}x_1)(t)-(\mathscr{Q}x_2)(t)\Big| \\
\le\ & \sup_{t\in[a,T]}\Big\{|\kappa|\int_a^t \frac{(t-s)^{p-1}}{\Gamma(p)}|x_1(s)-x_2(s)|ds \\
& +|\lambda|\int_a^t \frac{(t-s)^{\gamma+p-1}}{\Gamma(\gamma+p)}\Big|h(s,x_1(s))-h(s,x_2(s))\Big|ds+\int_a^t \frac{(t-s)^{p+q-1}}{\Gamma(p+q)}\Big|f(s,x_1(s)) \\
& -f(s,x_2(s))\Big|ds+|\varphi_1(t)|\Big[|\kappa|\int_a^T \frac{(T-s)^{p-1}}{\Gamma(p)}|x_1(s)-x_2(s)|ds \\
& +|\lambda|\int_a^T \frac{(T-s)^{\gamma+p-1}}{\Gamma(\gamma+p)}\Big|h(s,x_1(s))-h(s,x_2(s))\Big|ds \\
& +\int_a^T \frac{(T-s)^{p+q-1}}{\Gamma(p+q)}\Big|f(s,x_1(s))-f(s,x_2(s))\Big|ds\Big] \\
& +|\varphi_2(t)|\Big[|\kappa|\int_a^T \frac{(T-s)^{p-2}}{\Gamma(p-1)}|x_1(s)-x_2(s)|ds \\
& +|\lambda|\int_a^T \frac{(T-s)^{\gamma+p-2}}{\Gamma(\gamma+p-1)}\Big|h(s,x_1(s))-h(s,x_2(s))\Big|ds \\
& +\int_a^T \frac{(T-s)^{p+q-2}}{\Gamma(p+q-1)}\Big|f(s,x_1(s))-f(s,x_2(s))\Big|ds\Big] \\
& +|\varphi_3(t)|\Big[|\kappa|\sum_{i=1}^{n-2}|\alpha_i|\int_a^{\xi_i} \frac{(\xi_i-s)^{p-1}}{\Gamma(p)}|x_1(s)-x_2(s)|ds \\
& +|\lambda|\sum_{i=1}^{n-2}|\alpha_i|\int_a^{\xi_i} \frac{(\xi_i-s)^{\gamma+p-1}}{\Gamma(\gamma+p)}\Big|h(s,x_1(s))-h(s,x_2(s))\Big|ds \\
& +\sum_{i=1}^{n-2}|\alpha_i|\int_a^{\xi_i} \frac{(\xi_i-s)^{p+q-1}}{\Gamma(p+q)}\Big|f(s,x_1(s))-f(s,x_2(s))\Big|ds \\
& +\int_a^T\Big(|\kappa|\int_a^s \frac{(s-u)^{p-1}}{\Gamma(p)}|x_1(u)-x_2(u)|du \\
& +|\lambda|\int_a^s \frac{(s-u)^{\gamma+p-1}}{\Gamma(\gamma+p)}\Big|h(u,x_1(u))-h(u,x_2(u))\Big|du \\
& +\int_a^s \frac{(s-u)^{p+q-1}}{\Gamma(p+q)}\Big|f(u,x_1(u))-f(u,x_2(u))\Big|du\Big)dA(s)\Big]\Big\} \\
\le\ & (|\kappa|\mathscr{E}_0+|\lambda|\mathscr{L}_1\mathscr{E}_1+\mathscr{L}_2\mathscr{E}_2)\|x_1-x_2\|,
\end{aligned}
$$

which, in view of (4.1), implies that the operator $\mathscr{Q}$ is a contraction. Hence, we deduce by the conclusion of contraction mapping principle that problem (1.1)-(1.2) has a unique solution on $[a,T]$. The proof is complete. □

Example 4.2. Consider the same problem (3.5) with

$$h(t,x(t)) = \ln(3t^3+9) + \left(\frac{\sqrt{t^2+35}}{44}\right)\arctan x(t),$$

and

$$f(t,x(t)) = \frac{28\,x(t)}{e^{7t}+62} + (\sqrt{t}+8),\ t\in(0,1).$$

Obviosly, $|h(t,x_1)-h(t,x_2)| \leq \mathscr{L}_1\|x_1-x_2\|$ with $\mathscr{L}_1 = \frac{3}{22}$ and $|f(t,x_1)-f(t,x_2)| \leq \mathscr{L}_2\|x_1-x_2\|$ with $\mathscr{L}_2 = \frac{4}{9}$. Using the given data in Example (3.3) and (4.1), we find that $|\kappa|\mathscr{E}_0 + |\lambda|\mathscr{L}_1\mathscr{E}_1 + \mathscr{L}_2\mathscr{E}_2 \simeq 0.638333 < 1$. Clearly, the hypotheses of Theorem 4.1 are satisfied. Hence, by the conclusion of Theorem 4.1, we have that there is a unique solution to problem (3.5) on $[0,1]$.

5. Conclusions

In this paper, we studied the existence of solutions for a fractional differential equation involving both Riemann-Liouville and Caputo derivatives together with integral type nonlinearity in the sense of Riemann-Liouville fractional integral and non-integral type nonlinearity, complemented with Riemann-Stieltjes integro-multipoint boundary conditions on an arbitrary domain. We applied the tools of functional analysis to derive the desired results. It is imperative to note that our results correspond to the ones for the equation ${}^{RL}D^q({}^cD^p+\kappa)x(t) = f(t,x(t))$ supplemented with the given boundary conditions if we insert $\lambda = 0$ in the obtained results, which are indeed new. Moreover, our results become the ones with Riemann-Stieltjes boundary conditions when $\alpha_i = 0, i = 1,\ldots,n-2$. In the nutshell, our results are not only new in the given configuration but also can reduce to some special cases (new results) for appropriate choice of the parameters involved in the problem.

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