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CERTAIN INTEGRAL INEQUALITIES RELATED TO (φ,ρ^{α})-LIPSCHITZIAN MAPPINGS AND GENERALIZED h-CONVEXITY ON FRACTAL SETS

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Abstract. In this paper, we deal with certain new Hadamard's type inequalities involving $(\varphi,\sigma^{\alpha})$-Lipschitzian mappings on fractal sets $\mathbb{R}^{\alpha}(0\le\alpha\le 1)$. Also, some properties of generalized mappings H and F related to generalized h-convexity are given.

Keywords. Local fractional integrals; Generalized h-convexity; $(\varphi,\sigma^{\alpha})$-Lipschitzian; Hadamard's type inequality.

1. INTRODUCTION

Let us recall that a function $h:[\mu,\nu]\subset\mathbb{R}\to\mathbb{R}$ is said to be L-Lipschitzian if $|h(x)-h(y)|\le L|x-y|$, $L>0$, $x,y\in[\mu,\nu]$. In 2000, Dragomir et al. [1] established the following Hadamard's type inequalities by utilizing L-Lipschitzian mappings.

Theorem 1.1. *Let $h:[\mu,\nu]\to\mathbb{R}$ be a L-Lipschitzian mapping along with $\mu<\nu$. Then*

$$\left|\frac{1}{\nu-\mu}\int_{\mu}^{\nu}h(x)\mathrm{d}x-h\Big(\frac{\mu+\nu}{2}\Big)\right|\le\frac{L}{4}(\nu-\mu) \tag{1.1}$$

and

$$\left|\frac{1}{\nu-\mu}\int_{\mu}^{\nu}h(x)\mathrm{d}x-\frac{h(\mu)+h(\nu)}{2}\right|\le\frac{L}{3}(\nu-\mu). \tag{1.2}$$

Because of the extensive applications of Hadamard's type inequalities, many authors investigated them via L-Lipschitzian mappings. For example, Delavar et al. [2] estimated the difference between the right and middle part in Fejér inequality with L-Lisplitzian mappings. Du et al. [3] presented several estimation results on k-fractional integral inequalities through L-Lipschitzian mappings. Hsu [4] gave some Fejér inequalities related to L-Lipschitzian functions. For more results associated with L-Lipschitzian functions, we refer to [5, 6] and the references therein.

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In [7], Ahmad, Jleli and Samet introduced a class of (φ,σ)-Lipschitzian mappings as follows.

Definition 1.2. A function $h:[\mu,\nu]\subset\mathbb{R}\to\mathbb{R}$ is said to be (φ,ρ)-Lipschitzian, where $\varphi:[0,\infty)\to[0,\infty)$ and $\rho>0$, if

$$|h(x)-h(y)|\leq\varphi(|x-y|)+\rho, \tag{1.3}$$

holds for all $x,y\in[\mu,\nu]$.

The L-Lipschitzian functions appear as a special case of (φ,ρ)-Lipschitzian functions with $\varphi(x)=Lx$, $x\geq 0$ and $\rho=0$.

Next, we recall some concepts and known results associated with local fractional calculus. These theories of local fractional derivative and local fractional integral are primarily due to Yang [8].

Let μ^α, ν^α and ρ^α belong to the set $\mathbb{R}^\alpha(0<\alpha\leq 1)$. Then
(1) $\mu^\alpha+\nu^\alpha$ and $\mu^\alpha\nu^\alpha$ belong to the set $\mathbb{R}^\alpha$;
(2) $\mu^\alpha+\nu^\alpha=\nu^\alpha+\mu^\alpha=(\mu+\nu)^\alpha=(\nu+\mu)^\alpha$;
(3) $\mu^\alpha+(\nu^\alpha+\rho^\alpha)=(\mu^\alpha+\nu^\alpha)+\rho^\alpha$;
(4) $\mu^\alpha\nu^\alpha=\nu^\alpha\mu^\alpha=(\mu\nu)^\alpha=(\nu\mu)^\alpha$;
(5) $\mu^\alpha(\nu^\alpha\rho^\alpha)=(\mu^\alpha\nu^\alpha)\rho^\alpha$;
(6) $\mu^\alpha(\nu^\alpha+\rho^\alpha)=\mu^\alpha\nu^\alpha+\mu^\alpha\rho^\alpha$;
(7) $\mu^\alpha+0^\alpha=0^\alpha+\mu^\alpha=\mu^\alpha$ and $\mu^\alpha 1^\alpha=1^\alpha\mu^\alpha=\mu^\alpha$.

Definition 1.3. A non-differentiable function $g:\mathbb{R}\to\mathbb{R}^\alpha$, $\theta\to g(\theta)$ is said to be local fractional continuous at θ_0 if, for any $\varepsilon>0$, there exists $\delta>0$ satisfying that

$$|g(\theta)-g(\theta_0)|<\varepsilon^\alpha$$

for $|\theta-\theta_0|<\delta$. If $g(\theta)$ is local continuous on (μ,ν), then we denote that $g(\theta)\in C_\alpha(\mu,\nu)$.

Definition 1.4. The local fractional derivative of $g(\theta)$ of order α at $\theta=\theta_0$ is defined by the expression

$$g^{(\alpha)}(\theta_0)={}_{\theta_0}D_\theta^\alpha g(\theta)=\left.\frac{\mathrm{d}^\alpha g(\theta)}{\mathrm{d}\theta^\alpha}\right|_{\theta=\theta_0}=\lim_{\theta\to\theta_0}\frac{\Delta^\alpha\big(g(\theta)-g(\theta_0)\big)}{(\theta-\theta_0)^\alpha},$$

where $\Delta^\alpha\big(g(\theta)-g(\theta_0)\big)=\Gamma(\alpha+1)\big(g(\theta)-g(\theta_0)\big)$.

Let $g^{(\alpha)}(\theta)=D_\theta^\alpha g(\theta)$. If there exists

$$g^{((k+1)\alpha)}(\theta)=\overbrace{D_\theta^\alpha\cdots D_\theta^\alpha}^{(k+1)\text{ times}}g(\theta)$$

for any $\theta\in\mathscr{I}\subseteq\mathbb{R}$, then it is denoted by $g\in D_{(k+1)\alpha}(\mathscr{I})$, where $k=0,1,2,\cdots$.

Definition 1.5. Let $g(\theta)\in C_\alpha[\mu,\nu]$, and let $\triangle=\{t_0,t_1,\cdots,t_N\}$, $(N\in\mathbb{N})$ be a partition of $[\mu,\nu]$ which satisfies $\mu=t_0<t_1<\cdots<t_N=\nu$. Then the local fractional integral of g on $[\mu,\nu]$ of order α is defined as

$${}_\mu\mathscr{I}_\nu^{(\alpha)}g(\mu)=\frac{1}{\Gamma(1+\alpha)}\int_\mu^\nu g(t)(\mathrm{d}t)^\alpha:=\frac{1}{\Gamma(\alpha+1)}\lim_{\triangle t\to 0}\sum_{j=0}^{N-1}g(t_j)(\Delta t_j)^\alpha,$$

where $\Delta t:=\max\{\Delta t_1,\Delta t_2,\cdots,\Delta t_{N-1}\}$ and $\Delta t_j:=t_{j+1}-t_j$, $j=0,\cdots,N-1$.

Here, it follows that ${}_{\mu}\mathscr{I}_{\nu}^{(\alpha)}g(\theta)=0$ if $\mu=\nu$ and ${}_{\mu}\mathscr{I}_{\nu}^{(\alpha)}g(\theta)=-{}_{\nu}\mathscr{I}_{\mu}^{(\alpha)}g(\theta)$ if $\mu<\nu$. For any $\theta\in[\mu,\nu]$, if there exists ${}_{\mu}\mathscr{I}_{\theta}^{(\alpha)}g(\theta)$, then it is denoted by $g(\theta)\in\mathscr{I}_{\theta}^{\alpha}[\mu,\nu]$.

Lemma 1.6. [8] *The following identities are true.*
(i) *(Local fractional integration is anti-differentiation). If $g(\theta)=r^{(\alpha)}(\theta)\in C_{\alpha}[\mu,\nu]$, then we have*

$$ {}_{\mu}\mathscr{I}_{\nu}^{(\alpha)}g(\mu)=r(\nu)-r(\mu). $$

(ii) *(Local fractional integration by parts). If $g(\theta),r(\theta)\in D_{\alpha}[\mu,\nu]$ and $g^{(\alpha)}(\theta),r^{(\alpha)}(\theta)\in C_{\alpha}[\mu,\nu]$, then we have*

$$ {}_{\mu}\mathscr{I}_{\nu}^{(\alpha)}g(\theta)r^{(\alpha)}(\theta)=g(\theta)r(\theta)\Big|_{\mu}^{\nu}-{}_{\mu}\mathscr{I}_{\nu}^{(\alpha)}g^{(\alpha)}(\theta)r(\theta). $$

(iii) *(Local fractional derivative of $\theta^{k\alpha}$).*

$$ \frac{d^{\alpha}\theta^{k\alpha}}{d\theta^{\alpha}}=\frac{\Gamma(1+k\alpha)}{\Gamma\big(1+(k-1)\alpha\big)}\theta^{(k-1)\alpha},\ k\in\mathbb{R}. $$

(iv) *(Local fractional definite integrals of $\theta^{k\alpha}$).*

$$ \frac{1}{\Gamma(1+\alpha)}\int_{\mu}^{\nu}\theta^{k\alpha}(\mathrm{d}\theta)^{\alpha}=\frac{\Gamma(1+k\alpha)}{\Gamma\big(1+(k+1)\alpha\big)}\Big(\nu^{(k+1)\alpha}-\mu^{(k+1)\alpha}\Big),\ k\in\mathbb{R}. $$

Definition 1.7. [8] We say that $h(x)$ satisfies a local Lipschitz condition for some $L^{\alpha}>0^{\alpha}$ if

$$ \big|h(x)-h(y)\big|\leq L^{\alpha}|x^{\alpha}-y^{\alpha}| $$

for all $x,y\in[\mu,\nu]$.

In [9], Vivas, Hernández and Merentes defined the following generalized h-convex functions on fractal set $\mathbb{R}^{\alpha}(0<\alpha\leq 1)$.

Definition 1.8. [9] Let $h:J\subseteq\mathbb{R}\to\mathbb{R}^{\alpha}$ be a non-negative function with $h\not\equiv 0^{\alpha}$. We say that the function $\rho:I\subseteq\mathbb{R}\to\mathbb{R}^{\alpha}$ is generalized h-convex if ρ is non-negative, i.e., $\rho(\mu)\geq 0^{\alpha}$ and

$$ \rho\big(t\mu+(1-t)\nu\big)\leq h(t)\rho(\mu)+h(1-t)\rho(\nu) \tag{1.4} $$

holds for all $t\in[0,1]$ and $\mu,\nu\in I$.

Note that, in [9], the function h should have the property that $h\not\equiv 0^{\alpha}$ instead of $h\not\equiv 0$ in their original definition. The local gamma mapping and the local beta mapping are defined as follows.

Definition 1.9. [10] For all $0<\alpha\leq 1$ and $x\in\mathbb{R}^{+}$, the local gamma mapping is defined as

$$ \Gamma_{\alpha}(x):=\frac{1}{\alpha!}\int_{0}^{\infty}E_{\alpha}(-t^{\alpha})t^{(x-1)\alpha}(dt)^{\alpha}, $$

where $E_{\alpha}(t^{\alpha}):=\sum_{k=0}^{\infty}\frac{t^{\alpha k}}{\Gamma(1+k\alpha)}$ is the Mittag-Leffler function.

Definition 1.10. [10] For all $0<\alpha\leq 1$ and $x,y\in\mathbb{R}^{+}$, the local beta mapping with two parameters x and y is defined by

$$ B_{\alpha}(x,y):=\int_{0}^{1}t^{(x-1)\alpha}(1-t)^{(y-1)\alpha}(dt)^{\alpha}. $$

The great influence of local fractional calculus in various fields of mathematics science and engineering problems is undeniable. Recently, the study of some well-known integral inequalities for the local fractional integral have been performed by some researchers. For example, Mo, Sui and Yu [11] established the generalized Jensen's inequality and generalized Hermite-Hadamard's inequality for generalized convex mappings. Kiliçman and Saleh [12, 13] studied generalized Hadamard's type inequalities for generalized s-convex mappings. Erden and Sarikaya [14] gave some generalized Pompeiu's type inequalities associating with local fractional integrals. Luo, Wang and Du [15] investigated the Fejér type inequalites for generalized h-convex functions. For more results related to the local fractional integral inequalities, we refer to [16, 17, 18, 19] and the references cited therein.

Motivated by the above results mentioned, our principal goal is to introduce a new class of (φ,ρ^α)-Lipschitzian mappings and to establish certain Hadamard's type inequalities for such mappings on fractal set $\mathbb{R}^\alpha (0<\alpha\leq 1)$. Moreover, we give some properties of generalized mappings H and F, which are naturally related to the generalized h-convex functions. The results present in this paper extend the results obtained by Ahmad, Jleli and Samet [7] and Kiliçman and Saleh [12].

2. Main results

We now define (φ,ρ^α)-Lipschitzian mappings as follows.

Definition 2.1. The mapping $f:[\mu,\nu]\to\mathbb{R}^\alpha$ is said to be (φ,ρ^α)-Lipschitzian if the following inequality holds

$$\big|f(x)-f(y)\big|\leq \varphi\big(|x-y|\big)+\rho^\alpha, \tag{2.1}$$

where $x,y\in[\mu,\nu]$, $\rho^\alpha\geq 0^\alpha$ and $\varphi:[0,\infty)\to\mathbb{R}^\alpha,\varphi\geq 0^\alpha$.

In the case of $\varphi(x)=L^\alpha x^\alpha$, $x^\alpha\geq 0^\alpha$ and $\rho=0^\alpha$, the (φ,ρ^α)-Lipschitzian mappings recapture local Lipschitzian mappings.

Next, we establish the following theorem containing two Hadamard's type inequalities via local fractional integrals.

Theorem 2.2. *Let $f:[\mu,\nu]\subset\mathbb{R}\to\mathbb{R}^\alpha$ be a (φ,ρ^α)-Lipschitzian mapping, where $\varphi:[0,\infty)\to\mathbb{R}^\alpha,\varphi\geq 0^\alpha$ is local fractional continuous, with $\mu\leq\nu$ and $\rho^\alpha\geq 0^\alpha$. If $f(x)\in I_x^\alpha[\mu,\nu]$, then we have*

$$\left|\frac{{}_\mu I_\nu^{(\alpha)}f(x)}{(\nu-\mu)^\alpha}-\frac{f\left(\frac{\mu+\nu}{2}\right)}{\Gamma(1+\alpha)}\right|\leq\frac{1}{\Gamma(1+\alpha)}\int_0^1\varphi\left(\frac{(\nu-\mu)|1-2t|}{2}\right)(\mathrm{d}t)^\alpha+\frac{\rho^\alpha}{\Gamma(1+\alpha)} \tag{2.2}$$

and

$$\begin{aligned}&\left|\frac{{}_\mu I_\nu^{(\alpha)}f(x)}{(\nu-\mu)^\alpha}-\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\Big(f(\mu)+f(\nu)\Big)\right|\\&\leq\frac{1}{\Gamma(1+\alpha)}\int_0^1(2t)^\alpha\varphi\big((1-t)(\nu-\mu)\big)(\mathrm{d}t)^\alpha+\frac{\rho^\alpha}{\Gamma(1+\alpha)}.\end{aligned}\tag{2.3}$$

Proof. (i) For all $x, y \in [\mu, \nu]$, considering following inequality on $t \in [0,1]$, we obtain

$$\begin{aligned} &\Big|t^\alpha f(x) + (1-t)^\alpha f(y) - f\big(tx + (1-t)y\big)\Big| \\ &= \Big|t^\alpha\Big(f(x) - f\big(tx+(1-t)y\big)\Big) + (1-t)^\alpha\Big(f(y) - f\big(tx+(1-t)y\big)\Big)\Big| \\ &\le t^\alpha\Big|f(x) - f\big(tx+(1-t)y\big)\Big| + (1-t)^\alpha\Big|f(y) - f\big(tx+(1-t)y\big)\Big| \\ &\le t^\alpha\Big(\varphi\big((1-t)|x-y|\big) + \rho^\alpha\Big) + (1-t)^\alpha\Big(\varphi\big(t|x-y|\big) + \rho^\alpha\Big) \\ &= t^\alpha\Big(\varphi\big((1-t)|x-y|\big)\Big) + (1-t)^\alpha\Big(\varphi\big(t|x-y|\big)\Big) + \rho^\alpha. \end{aligned} \tag{2.4}$$

Taking $t = \frac{1}{2}$ in (2.4), we deduce

$$\left|\frac{f(x)+f(y)}{2^\alpha} - f\Big(\frac{x+y}{2}\Big)\right| \le \varphi\Big(\frac{|x-y|}{2}\Big) + \rho^\alpha. \tag{2.5}$$

Putting $x = t\mu + (1-t)\nu$ and $y = t\nu + (1-t)\mu$ for all $t \in (0,1)$ in inequality (2.5), we get

$$\left|\frac{f\big(t\mu+(1-t)\nu\big) + f\big(t\nu+(1-t)\mu\big)}{2^\alpha} - f\Big(\frac{\mu+\nu}{2}\Big)\right| \le \varphi\Big(\frac{(\nu-\mu)|1-2t|}{2}\Big) + \rho^\alpha. \tag{2.6}$$

Integrating inequality (2.6) with $t \in [0,1]$, we can obtain desired inequality (2.2).

(ii) Continuing from inequality (2.4), if we take $x = \mu$ and $y = \nu$, then

$$\begin{aligned} &|t^\alpha f(\mu) + (1-t)^\alpha f(\nu) - f\big(t\mu + (1-\nu)y\big)| \\ &\le t^\alpha\Big(\varphi\big((1-t)(\nu-\mu)\big)\Big) + (1-t)^\alpha\Big(\varphi\big(t(\nu-\mu)\big)\Big) + \rho^\alpha. \end{aligned} \tag{2.7}$$

Integrating inequality (2.7) respected to t over $(0,1)$, we can obtain desired inequality (2.3). Note that

$$\frac{1}{\Gamma(1+\alpha)}\int_0^1 t^\alpha \varphi\big((1-t)(b-a)\big)(\mathrm{d}t)^\alpha = \frac{1}{\Gamma(1+\alpha)}\int_0^1 (1-t)^\alpha \varphi\big(t(b-a)\big)(\mathrm{d}t)^\alpha.$$

This completes the proof.

Example 2.3. If $f(x) = x^{2\alpha}$, then f is a $\big(x^{2\alpha}, (\frac{1}{2})^\alpha\big)$-Lipschitzian function on $[0^\alpha, 1^\alpha]$. It follows that

$$\left|\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)} - \frac{(\frac{1}{4})^\alpha}{\Gamma(1+\alpha)}\right| \le \frac{(\frac{3}{4})^\alpha}{\Gamma(1+\alpha)} - \frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)} + \frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}$$

and

$$\begin{aligned} &\left|\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)} - \frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\right| \\ &\le 2^\alpha\left(\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)} - 2^\alpha\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)} + \frac{\Gamma(1+3\alpha)}{\Gamma(1+4\alpha)}\right) + \frac{(\frac{1}{2})^\alpha}{\Gamma(1+\alpha)}. \end{aligned}$$

Corollary 2.4. *Taking $\alpha = 1$ in Theorem 2.2, then we obtain the Theorem 2.2 in [7].*

Corollary 2.5. *Putting $\varphi(x) = L^\alpha x^{k\alpha}, k \ge 0$ and $L^\alpha > 0^\alpha$ in Theorem 2.2, we have*

$$\left|\frac{{}_\mu I_\nu^{(\alpha)} f(x)}{(\nu-\mu)^\alpha} - \frac{f\big(\frac{\mu+\nu}{2}\big)}{\Gamma(1+\alpha)}\right| \le \frac{L^\alpha(\nu-\mu)^{k\alpha}\Gamma(1+k\alpha)}{2^{k\alpha}\Gamma\big(1+(k+1)\alpha\big)} + \frac{\rho^\alpha}{\Gamma(1+\alpha)}$$

and

$$\left|\frac{{}_{\mu}I_{\nu}^{(\alpha)}f(x)}{(\nu-\mu)^{\alpha}}-\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\Big(f(\mu)+f(\nu)\Big)\right|$$
$$\leq \frac{2^{\alpha}L^{\alpha}(\nu-\mu)^{k\alpha}}{\Gamma(1+\alpha)}B_{\alpha}(2,k+1)+\frac{\rho^{\alpha}}{\Gamma(1+\alpha)}.$$

In particular, if $k=1$ *and* $\rho=0^{\alpha}$*, then*

$$\left|\frac{{}_{\mu}I_{\nu}^{(\alpha)}f(x)}{(\nu-\mu)^{\alpha}}-\frac{f\big(\frac{\mu+\nu}{2}\big)}{\Gamma(1+\alpha)}\right|\leq\frac{L^{\alpha}(\nu-\mu)^{\alpha}\Gamma(1+\alpha)}{2^{\alpha}\Gamma(1+2\alpha)}$$

and

$$\left|\frac{{}_{\mu}I_{\nu}^{(\alpha)}f(x)}{(\nu-\mu)^{\alpha}}-\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\Big(f(\mu)+f(\nu)\Big)\right|\leq 2^{\alpha}L^{\alpha}(\nu-\nu)^{\alpha}\left(\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}-\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\right).$$

Corollary 2.6. *Under the assumptions of Theorem 2.2, let* $g([\mu,\nu])\subseteq[\mu,\nu]$ *be a* (ϕ,θ)*-Lipschitzian mapping, where* $\phi:[0,\infty)\to[0,\infty)$ *is continuous, and* $\theta\geq 0$*. If* $\varphi(x)$ *is nondecreasing, then*

$$\left|\frac{{}_{\mu}I_{\nu}^{(\alpha)}f\circ g(x)}{(\nu-\mu)^{\alpha}}-\frac{f\circ g\big(\frac{\mu+\nu}{2}\big)}{\Gamma(1+\alpha)}\right|\leq\frac{1}{\Gamma(1+\alpha)}\int_{0}^{1}\chi\left(\frac{(\nu-\mu)|1-2t|}{2}\right)(\mathrm{d}t)^{\alpha}+\frac{\rho^{\alpha}}{\Gamma(1+\alpha)} \tag{2.8}$$

and

$$\begin{aligned}&\left|\frac{{}_{\mu}I_{\nu}^{(\alpha)}f\circ g(x)}{(\nu-\mu)^{\alpha}}-\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\Big(f\circ g(\mu)+f\circ g(\nu)\Big)\right|\\&\leq\frac{1}{\Gamma(1+\alpha)}\int_{0}^{1}(2t)^{\alpha}\chi\big((1-t)(\nu-\mu)\big)(\mathrm{d}t)^{\alpha}+\frac{\rho^{\alpha}}{\Gamma(1+\alpha)},\end{aligned} \tag{2.9}$$

where $f\circ g(x)=f\big(g(x)\big)$ *and* $\chi:[0,\infty)\to\mathbb{R}^{\alpha},\chi\geq 0^{\alpha}$ *is define as follows*

$$\chi(x)=\varphi\big(\phi(x)+\theta\big),\quad x\geq 0.$$

Proof. For all $x,y\in[\mu,\nu]$, we have

$$\begin{aligned}\Big|f\circ g(x)-f\circ g(y)\Big|&=\Big|f\big(g(x)\big)-f\big(g(y)\big)\Big|\\&\leq\varphi\Big(\big|g(x)-g(y)\big|\Big)+\rho^{\alpha}\\&\leq\varphi\Big(\phi(|x-y|)+\theta\Big)+\rho^{\alpha}.\end{aligned}$$

It is easy to see that $f\circ g(x)$ is a (χ,ρ^{α})-Lipschitzian function. From Theorem 2.2, we deduce inequality (2.8) and (2.9).

To obtain more results related to (φ,ρ^{α})-Lipschitzian mappings, let us define the generalized mappings H and generalized mappings F on the interval $[0,1]$

$$H(t):=\frac{{}_{\mu}I_{\nu}^{(\alpha)}f\big(tx+(1-t)\frac{\mu+\nu}{2}\big)}{(\nu-\mu)^{\alpha}} \tag{2.10}$$

and

$$F(t):=\frac{1}{\big(\Gamma(1+\alpha)\big)^{2}}\int_{\mu}^{\nu}\int_{\mu}^{\nu}\left(\frac{1}{\nu-\mu}\right)^{2\alpha}f\big(tx+(1-t)y\big)(\mathrm{d}x)^{\alpha}(\mathrm{d}y)^{\alpha}. \tag{2.11}$$

In what follows, we give some properties of the generalized mapping H and generalized mappings F through (φ,ρ^{α})-Lipschitzian mappings.

Theorem 2.7. *Let $f:[\mu,\nu]\subset\mathbb{R}\to\mathbb{R}^{\alpha}$ be a (φ,ρ^{α})-Lipschitzian mapping, where $\varphi:[0,\infty)\to\mathbb{R}^{\alpha},\varphi\geq 0^{\alpha}$ is local fractional continuous, with $\mu<\nu$ and $\rho^{\alpha}\geq 0^{\alpha}$. If $f(x)\in I_x^{\alpha}[\mu,\nu]$, then*
(i) $H(t)$ is a $\left(\phi,\frac{\rho^{\alpha}}{\Gamma(1+\alpha)}\right)$-Lipschizian mapping, where $\phi:[0,\infty)\to\mathbb{R}^{\alpha}$, $\phi\geq 0^{\alpha}$ is defined by

$$\phi(m)=\frac{{}_{\mu}I_{\nu}^{(\alpha)}\varphi\left(m\left|x-\frac{\mu+\nu}{2}\right|\right)}{(\nu-\mu)^{\alpha}},m\geq 0. \tag{2.12}$$

(ii) For all $t\in[0,1]$, we have

$$\left|H(t)-\frac{f\left(\frac{\mu+\nu}{2}\right)}{\Gamma(1+\alpha)}\right|\leq\phi(t)+\frac{\rho^{\alpha}}{\Gamma(1+\alpha)}. \tag{2.13}$$

(iii) For all $t\in[0,1]$, we obtain

$$\left|H(t)-\frac{{}_{\mu}I_{\nu}^{(\alpha)}f(x)}{\Gamma(1+\alpha)}\right|\leq\phi(1-t)+\frac{\rho^{\alpha}}{\Gamma(1+\alpha)}. \tag{2.14}$$

(iv) For all $t\in[0,1]$, we get

$$\left|H(t)-t^{\alpha}\frac{{}_{\mu}I_{\nu}^{(\alpha)}f(x)}{\Gamma(1+\alpha)}-(1-t)^{\alpha}f\left(\frac{\mu+\nu}{2}\right)\right|\leq t^{\alpha}\phi(1-t)+(1-t)^{\alpha}\phi(t)+\frac{\rho^{\alpha}}{\Gamma(1+\alpha)}. \tag{2.15}$$

Proof. (i) Let $t_1,t_2\in[0,1]$. Then

$$\begin{aligned}\left|H(t_1)-H(t_2)\right|&=\left|\frac{{}_{\mu}I_{\nu}^{(\alpha)}f\left(t_1x+(1-t_1)\frac{\mu+\nu}{2}\right)}{(\nu-\mu)^{\alpha}}-\frac{{}_{\mu}I_{\nu}^{(\alpha)}f\left(t_2x+(1-t_2)\frac{\mu+\nu}{2}\right)}{(\nu-\mu)^{\alpha}}\right|\\&\leq\frac{{}_{\mu}I_{\nu}^{(\alpha)}\left|f\left(t_1x+(1-t_1)\frac{\mu+\nu}{2}\right)-f\left(t_2x+(1-t_2)\frac{\mu+\nu}{2}\right)\right|}{(\nu-\mu)^{\alpha}}\\&\leq\frac{{}_{\mu}I_{\nu}^{(\alpha)}\left[\varphi\left(\left|t_1x+(1-t_1)\frac{\mu+\nu}{2}-t_2x-(1-t_2)\frac{\mu+\nu}{2}\right|\right)+\rho^{\alpha}\right]}{(\nu-\mu)^{\alpha}}\\&=\frac{{}_{\mu}I_{\nu}^{(\alpha)}\varphi\left(\left|(t_1-t_2)\left(x-\frac{\mu+\nu}{2}\right)\right|\right)}{(\nu-\mu)^{\alpha}}+\frac{\rho^{\alpha}}{\Gamma(1+\alpha)}\\&\leq\frac{{}_{\mu}I_{\nu}^{(\alpha)}\varphi\left(|t_1-t_2|\left|x-\frac{\mu+\nu}{2}\right|\right)}{(\nu-\mu)^{\alpha}}+\frac{\rho^{\alpha}}{\Gamma(1+\alpha)}\\&=\phi\left(|t_1-t_2|\right)+\frac{\rho^{\alpha}}{\Gamma(1+\alpha)},\end{aligned}$$

where $\phi:[0,\infty)\to\mathbb{R}^{\alpha},\phi\geq 0^{\alpha}$ is defined by (2.12). So, $H(t)$ is a $\left(\phi,\frac{\rho^{\alpha}}{\Gamma(1+\alpha)}\right)$-Lipschizian mapping.

(ii) Let $t_1 = t$ and $t_2 = 0$. Then

$$\begin{aligned}
\Big|H(t)-H(0)\Big| &= \left|H(t)-\frac{f\left(\frac{\mu+\nu}{2}\right)}{\Gamma(1+\alpha)}\right| \\
&= \left|\frac{{}_{\mu}I_{\nu}^{(\alpha)} f\left(tx+(1-t)\frac{\mu+\nu}{2}\right)}{(\nu-\mu)^{\alpha}} - \frac{{}_{\mu}I_{\nu}^{(\alpha)} f\left(\frac{\mu+\nu}{2}\right)}{(\nu-\mu)^{\alpha}}\right| \\
&\le \frac{{}_{\mu}I_{\nu}^{(\alpha)}\left[\varphi\left(\left|tx+(1-t)\frac{\mu+\nu}{2}-\frac{\mu+\nu}{2}\right|\right)+\rho^{\alpha}\right]}{(\nu-\nu)^{\alpha}} \\
&= \frac{{}_{\mu}I_{\nu}^{(\alpha)}\varphi\left(t\left|x-\frac{\mu+\nu}{2}\right|\right)}{(\nu-\mu)^{\alpha}} + \frac{\rho^{\alpha}}{\Gamma(1+\alpha)} \\
&= \phi(t) + \frac{\rho^{\alpha}}{\Gamma(1+\alpha)}.
\end{aligned}$$

(iii) Let $t_1 = t$ and $t_2 = 1$. Then

$$\begin{aligned}
\Big|H(t)-H(1)\Big| &= \left|H(t)-\frac{{}_{\mu}I_{\nu}^{(\alpha)} f(x)}{\Gamma(1+\alpha)}\right| \\
&= \left|\frac{{}_{\mu}I_{\nu}^{(\alpha)} f\left(tx+(1-t)\frac{\mu+\nu}{2}\right)}{(\nu-\mu)^{\alpha}} - \frac{{}_{\mu}I_{\nu}^{(\alpha)} f(x)}{(\nu-\mu)^{\alpha}}\right| \\
&\le \frac{{}_{\mu}I_{\nu}^{(\alpha)}\left[\varphi\left(\left|tx+(1-t)\frac{\mu+\nu}{2}-x\right|\right)+\rho^{\alpha}\right]}{(b-a)^{\alpha}} \\
&\le \frac{{}_{\mu}I_{\nu}^{(\alpha)}\varphi\left(|1-t|\left|x-\frac{\mu+\nu}{2}\right|\right)}{(\nu-\mu)^{\alpha}} + \frac{\rho^{\alpha}}{\Gamma(1+\alpha)} \\
&= \phi(1-t) + \frac{\rho^{\alpha}}{\Gamma(1+\alpha)}.
\end{aligned}$$

(iv) Multiplying both sides of inequality (2.13) and inequality (2.14) by $(1-t)^{\alpha}$ and t^{α}, respectively, and then adding the resulting expressions together, we deduce

$$\begin{aligned}
&\left|H(t)-t^{\alpha}\frac{{}_{\mu}I_{\nu}^{(\alpha)} f(x)}{\Gamma(1+\alpha)}-(1-t)^{\alpha} f\left(\frac{\mu+\nu}{2}\right)\right| \\
&\le (1-t)^{\alpha}\left|H(t)-f\left(\frac{\mu+\nu}{2}\right)\right| + t^{\alpha}\left|H(t)-\frac{{}_{\mu}I_{\nu}^{(\alpha)} f(x)}{\Gamma(1+\alpha)}\right| \\
&\le (1-t)^{\alpha}\left[\phi(t)+\frac{\rho^{\alpha}}{\Gamma(1+\alpha)}\right] + t^{\alpha}\left[\phi(1-t)+\frac{\rho^{\alpha}}{\Gamma(1+\alpha)}\right] \\
&= t^{\alpha}\phi(1-t)+(1-t)^{\alpha}\phi(t)+\frac{\rho^{\alpha}}{\Gamma(1+\alpha)}.
\end{aligned}$$

This completes the proof.

Corollary 2.8. *In Theorem 2.7, if $\varphi(x) = L^{\alpha}x^{k\alpha}, k \ge 0$ and $L \ge 0^{\alpha}$, then we have the following assertions:*

(i) For all $t_1, t_2 \in [0,1]$, we have

$$\big|H(t_1) - H(t_2)\big| \le \frac{\Gamma(1+k\alpha)}{\Gamma\big(1+(k+1)\alpha\big)} L^{\alpha} \big|t_1 - t_2\big|^{k\alpha} \Big(\frac{\nu-\mu}{2}\Big)^{k\alpha} + \frac{\rho^{\alpha}}{\Gamma(1+\alpha)}.$$

(ii) For all $t \in [0,1]$, we obtain

$$\left|H(t) - \frac{f\big(\frac{\mu+\nu}{2}\big)}{\Gamma(1+\alpha)}\right| \le \frac{\Gamma(1+k\alpha)}{\Gamma\big(1+(k+1)\alpha\big)} L^{\alpha} t^{k\alpha} \Big(\frac{\nu-\mu}{2}\Big)^{k\alpha} + \frac{\rho^{\alpha}}{\Gamma(1+\alpha)}.$$

(iii) For all $t \in [0,1]$, we get

$$\left|H(t) - \frac{{}_{\mu}I_{\nu}^{(\alpha)} f(x)}{\Gamma(1+\alpha)}\right| \le \frac{\Gamma(1+k\alpha)}{\Gamma\big(1+(k+1)\alpha\big)} L^{\alpha} (1-t)^{k\alpha} \Big(\frac{\nu-\mu}{2}\Big)^{k\alpha} + \frac{\rho^{\alpha}}{\Gamma(1+\alpha)}.$$

(iv) For all $t \in [0,1]$, we deduce

$$\begin{aligned}&\left|H(t) - t^{\alpha} \frac{{}_{\mu}I_{\nu}^{(\alpha)} f(x)}{\Gamma(1+\alpha)} - (1-t)^{\alpha} f\Big(\frac{\mu+\nu}{2}\Big)\right| \\ &\le \frac{\Gamma(1+k\alpha)}{\Gamma\big(1+(k+1)\alpha\big)} L^{\alpha} \Big(t^{\alpha}(1-t)^{k\alpha} + t^{k\alpha}(1-t)^{\alpha}\Big) \Big(\frac{\nu-\mu}{2}\Big)^{k\alpha} + \frac{\rho^{\alpha}}{\Gamma(1+\alpha)}.\end{aligned}$$

In particular, if $k = 1$ and $\rho = 0^{\alpha}$, then

$$\big|H(t_1) - H(t_2)\big| \le \frac{\Gamma(1+\alpha)}{\Gamma\big(1+2\alpha\big)} L^{\alpha} \big|t_1 - t_2\big|^{\alpha} \Big(\frac{\nu-\mu}{2}\Big)^{\alpha},$$

$$\left|H(t) - \frac{f\big(\frac{\mu+\nu}{2}\big)}{\Gamma(1+\alpha)}\right| \le \frac{\Gamma(1+\alpha)}{\Gamma\big(1+2\alpha\big)} L^{\alpha} t^{\alpha} \Big(\frac{\nu-\mu}{2}\Big)^{\alpha},$$

$$\left|H(t) - \frac{{}_{\mu}I_{\nu}^{(\alpha)} f(x)}{\Gamma(1+\alpha)}\right| \le \frac{\Gamma(1+\alpha)}{\Gamma\big(1+2\alpha\big)} L^{\alpha} (1-t)^{\alpha} \Big(\frac{\nu-\mu}{2}\Big)^{\alpha}$$

and

$$\left|H(t) - t^{\alpha} \frac{{}_{\mu}I_{\nu}^{(\alpha)} f(x)}{\Gamma(1+\alpha)} - (1-t)^{\alpha} f\Big(\frac{\mu+\nu}{2}\Big)\right| \le \frac{\Gamma(1+\alpha)}{\Gamma\big(1+2\alpha\big)} L^{\alpha} t^{\alpha} (1-t)^{\alpha} (\nu-\mu)^{\alpha},$$

for all $t, t_1, t_2 \in [0,1]$.

Corollary 2.9. *If $\alpha = 1$ in Theorem 2.7, then we obtain Theorem 3.1 in* [7].

Theorem 2.10. *Let $f : [\mu, \nu] \subset \mathbb{R} \to \mathbb{R}^{\alpha}$ be a (φ, ρ^{α})-Lipschitzian mapping, where $\varphi : [0,\infty) \to \mathbb{R}^{\alpha}, \varphi \ge 0^{\alpha}$ is local fractional continuous, with $\mu < \nu$ and $\rho^{\alpha} \ge 0^{\alpha}$. If $f(x) \in I_x^{\alpha}[\mu, \nu]$, then*
(i) $F(t)$ is a $\big(\rho, \frac{\rho^{\alpha}}{(\Gamma(1+\alpha))^2}\big)$-Lipschizian mapping, where $\rho : [0,\infty) \to \mathbb{R}^{\alpha}, \rho \ge 0^{\alpha}$ is defined as

$$\rho(n) = \frac{1}{\big(\Gamma(1+\alpha)\big)^2} \int_{\mu}^{\nu} \int_{\mu}^{\nu} \Big(\frac{1}{\nu-\mu}\Big)^{2\alpha} \varphi\big(n|x-y|\big) (\mathrm{d}x)^{\alpha} (\mathrm{d}y)^{\alpha}, \ \ n \ge 0. \tag{2.16}$$

(ii) For all $t \in [0,1]$, we have

$$\left|F(t) - \frac{1}{\big(\Gamma(1+\alpha)\big)^2} \int_{\mu}^{\nu} \int_{\mu}^{\nu} \Big(\frac{1}{\nu-\mu}\Big)^{2\alpha} f\Big(\frac{x+y}{2}\Big) (\mathrm{d}x)^{\alpha} (\mathrm{d}y)^{\alpha}\right| \le \rho\Big(\Big|t - \frac{1}{2}\Big|\Big) + \frac{\rho^{\alpha}}{\big(\Gamma(1+\alpha)\big)^2}. \tag{2.17}$$

(iii) For all $t \in [0,1]$, we obtain

$$\left|F(t)-\frac{{}_{\mu}I_{\nu}^{(\alpha)}f(x)}{\Gamma(1+\alpha)(\nu-\mu)^{\alpha}}\right| \le \rho(1-t)+\frac{\rho^{\alpha}}{\big(\Gamma(1+\alpha)\big)^{2}}. \tag{2.18}$$

(iv) For all $t \in [0,1]$, we get

$$|F(t)-H(t)| \le \phi(1-t)+\frac{\rho^{\alpha}}{\Gamma(1+\alpha)}. \tag{2.19}$$

Proof. (i) Let $t_1, t_2 \in [0,1]$. Then

$$\begin{aligned}
&\left|F(t_1)-F(t_2)\right| \\
&=\left|\frac{1}{\big(\Gamma(1+\alpha)\big)^{2}}\int_{\mu}^{\nu}\int_{\mu}^{\nu}\Big(\frac{1}{\nu-\mu}\Big)^{2\alpha} f\big(t_1x+(1-t_1)y\big)(\mathrm{d}x)^{\alpha}(\mathrm{d}y)^{\alpha}\right. \\
&\quad\left.-\frac{1}{\big(\Gamma(1+\alpha)\big)^{2}}\int_{\mu}^{\nu}\int_{\mu}^{\nu}\Big(\frac{1}{\nu-\mu}\Big)^{2\alpha} f\big(t_2x+(1-t_2)y\big)(\mathrm{d}x)^{\alpha}(\mathrm{d}y)^{\alpha}\right| \\
&\le \frac{1}{\big(\Gamma(1+\alpha)\big)^{2}}\int_{\mu}^{\nu}\int_{\mu}^{\nu}\Big(\frac{1}{\nu-\mu}\Big)^{2\alpha}\Big|f\big(t_1x+(1-t_1)y\big)-f\big(t_2x+(1-t_2)y\big)\Big|(\mathrm{d}x)^{\alpha}(\mathrm{d}y)^{\alpha} \\
&\le \frac{1}{\big(\Gamma(1+\alpha)\big)^{2}}\int_{\mu}^{\nu}\int_{\mu}^{\nu}\Big(\frac{1}{\nu-\mu}\Big)^{2\alpha}\Big[\varphi\big(|t_1-t_2||x-y|\big)+\rho^{\alpha}\Big](\mathrm{d}x)^{\alpha}(\mathrm{d}y)^{\alpha} \\
&=\rho(|t_1-t_2|)+\frac{\rho^{\alpha}}{(\Gamma(1+\alpha))^{2}},
\end{aligned}$$

where $\rho:[0,\infty)\to\mathbb{R}^{\alpha}$,$\rho \ge 0^{\alpha}$ is defined by (2.16). So, $F(t)$ is a $\big(\rho,\frac{\rho^{\alpha}}{(\Gamma(1+\alpha))^{2}}\big)$-Lipschizian mapping.

(ii) If $t_1 = t$ and $t_2 = \frac{1}{2}$, then

$$\begin{aligned}
\left|F(t)-F\Big(\frac{1}{2}\Big)\right| &= \left|F(t)-\frac{1}{\big(\Gamma(1+\alpha)\big)^{2}}\int_{\mu}^{\nu}\int_{\mu}^{\nu}\Big(\frac{1}{\nu-\mu}\Big)^{2\alpha} f\Big(\frac{x+y}{2}\Big)(\mathrm{d}x)^{\alpha}(\mathrm{d}y)^{\alpha}\right| \\
&\le \frac{1}{\big(\Gamma(1+\alpha)\big)^{2}}\int_{\mu}^{\nu}\int_{\mu}^{\nu}\Big(\frac{1}{\nu-\mu}\Big)^{2\alpha}\Big|f\big(tx+(1-t)y\big)-f\Big(\frac{x+y}{2}\Big)\Big|(\mathrm{d}x)^{\alpha}(\mathrm{d}y)^{\alpha} \\
&\le \frac{1}{\big(\Gamma(1+\alpha)\big)^{2}}\int_{\mu}^{\nu}\int_{\mu}^{\nu}\Big(\frac{1}{\nu-\mu}\Big)^{2\alpha}\Big[\varphi\Big(\Big|t-\frac{1}{2}\Big||x-y|\Big)+\rho^{\alpha}\Big](\mathrm{d}x)^{\alpha}(\mathrm{d}y)^{\alpha} \\
&= \rho\Big(\Big|t-\frac{1}{2}\Big|\Big)+\frac{\rho^{\alpha}}{(\Gamma(1+\alpha))^{2}}.
\end{aligned}$$

(iii) If $t_1 = t$ and $t_2 = 1$, then

$$\begin{aligned}\Big|F(t)-F(0)\Big| &= \left|F(t)-\frac{1}{\big(\Gamma(1+\alpha)\big)^2}\int_\mu^\nu\int_\mu^\nu\Big(\frac{1}{\nu-\mu}\Big)^{2\alpha}f(x)(\mathrm{d}x)^\alpha(\mathrm{d}y)^\alpha\right|\\ &\le \frac{1}{\big(\Gamma(1+\alpha)\big)^2}\int_\mu^\nu\int_\mu^\nu\Big(\frac{1}{\nu-\mu}\Big)^{2\alpha}\Big|f\big(tx+(1-t)y\big)-f(x)\Big|(\mathrm{d}x)^\alpha(\mathrm{d}y)^\alpha\\ &\le \frac{1}{\big(\Gamma(1+\alpha)\big)^2}\int_\mu^\nu\int_\mu^\nu\Big(\frac{1}{\nu-\mu}\Big)^{2\alpha}\Big[\varphi\big(|1-t||x-y|\big)+\rho^\alpha\Big](\mathrm{d}x)^\alpha(\mathrm{d}y)^\alpha\\ &= \rho(1-t)+\frac{\rho^\alpha}{(\Gamma(1+\alpha))^2}.\end{aligned}$$

(iv) Let $t\in[0,1]$ and $x,y\in[\mu,\nu]$. Since f is a $\big(\varphi,\rho^\alpha\big)$-Lipschitzian mapping, then we have

$$\left|f\big(tx+(1-t)y\big)-f\Big(tx+(1-t)\frac{\mu+\nu}{2}\Big)\right| \le \varphi\Big((1-t)\Big|y-\frac{\mu+\nu}{2}\Big|\Big)+\rho^\alpha. \tag{2.20}$$

Multiplying inequality (2.20) with $(\frac{1}{\nu-\mu})^{2\alpha}$ and integrating over the interval $[\mu,\nu]$ with respect to x and y, we can get the desired inequality (2.19). This completes the proof.

Corollary 2.11. *In Theorem 2.7, if we choose $\varphi(x)=L^\alpha x^{k\alpha}, k\ge 0$ and $L^\alpha \ge 0^\alpha$ then we have the following facts*
(i) For all $t_1,t_2\in[0,1]$, we have

$$\big|F(t_1)-F(t_2)\big| \le \frac{2^\alpha L^\alpha\Gamma(1+k\alpha)}{\Gamma\big(1+(k+2)\alpha\big)}(\nu-\mu)^\alpha\big|t_1-t_2\big|^{k\alpha}+\frac{\rho^\alpha}{(\Gamma(1+\alpha))^2}.$$

(ii) For all $t\in[0,1]$, we obtain

$$\begin{aligned}&\left|F(t)-\frac{1}{\big(\Gamma(1+\alpha)\big)^2}\int_\mu^\nu\int_\mu^\nu\Big(\frac{1}{\nu-\mu}\Big)^{2\alpha}f\Big(\frac{x+y}{2}\Big)(\mathrm{d}x)^\alpha(\mathrm{d}y)^\alpha\right|\\ &\qquad\le \frac{2^\alpha L^\alpha\Gamma(1+k\alpha)}{\Gamma\big(1+(k+2)\alpha\big)}(\nu-\mu)^\alpha\Big|t-\frac{1}{2}\Big|^{k\alpha}+\frac{\rho^\alpha}{(\Gamma(1+\alpha))^2}.\end{aligned}$$

(iii) For all $t\in[0,1]$, we get

$$\left|F(t)-\frac{{}_\mu I_\nu^{(\alpha)}f(x)}{\Gamma(1+\alpha)(\nu-\mu)^\alpha}\right| \le \frac{2^\alpha L^\alpha\Gamma(1+k\alpha)}{\Gamma\big(1+(k+2)\alpha\big)}(\nu-\mu)^\alpha(1-t)^{k\alpha}+\frac{\rho^\alpha}{(\Gamma(1+\alpha))^2}.$$

(iv) For all $t\in[0,1]$, we deduce

$$|F(t)-H(t)| \le \frac{\Gamma(1+k\alpha)}{\Gamma\big(1+(k+1)\alpha\big)}L^\alpha(1-t)^{k\alpha}\Big(\frac{\nu-\mu}{2}\Big)^{k\alpha}+\frac{\rho^\alpha}{\Gamma(1+\alpha)}.$$

In particualr, if $k=1$ and $\rho=0^\alpha$, then

$$\big|F(t_1)-F(t_2)\big| \le \frac{2^\alpha L^\alpha\Gamma(1+\alpha)}{\Gamma\big(1+3\alpha\big)}(\nu-\mu)^\alpha\big|t_1-t_2\big|^\alpha,$$

$$\left|F(t)-\frac{1}{\big(\Gamma(1+\alpha)\big)^2}\int_\mu^\nu\int_\mu^\nu\Big(\frac{1}{\nu-\mu}\Big)^{2\alpha}f\Big(\frac{x+y}{2}\Big)(\mathrm{d}x)^\alpha(\mathrm{d}y)^\alpha\right|$$
$$\le\frac{2^\alpha L^\alpha\Gamma(1+\alpha)}{\Gamma\big(1+3\alpha\big)}(\nu-\mu)^\alpha\Big|t-\frac{1}{2}\Big|^\alpha,$$

$$\left|F(t)-\frac{{}_\mu I_\nu^{(\alpha)}f(x)}{\Gamma(1+\alpha)(\nu-\mu)^\alpha}\right|\le\frac{2^\alpha L^\alpha\Gamma(1+\alpha)}{\Gamma\big(1+3\alpha\big)}(\nu-\mu)^\alpha(1-t)^\alpha$$

and

$$|F(t)-H(t)|\le\frac{\Gamma(1+\alpha)}{\Gamma\big(1+2\alpha\big)}L^\alpha(1-t)^\alpha\Big(\frac{\nu-\mu}{2}\Big)^\alpha,$$

for all $t,t_1,t_2\in[0,1]$.

Corollary 2.12. *If we choose* $\alpha=1$ *in Theorem 2.10, then we get the Theorem 4.1 in* [7].

3. Inequalities for generalized h-convex mappings

In this section, instead of the (φ,ρ^α)-Lipschizian mappings, we consider generalized h-convexity for the generalized mappings H and generalized mappings F to establish some new inequalities. To demonstrate the following results, we need [15, Remark 3.4].
The following inequality exists:

$$\frac{f\big(\frac{\mu+\nu}{2}\big)}{2^\alpha\Gamma(1+\alpha)h\big(\frac{1}{2}\big)}\le\frac{{}_\mu I_\nu^{(\alpha)}f(t)}{(\nu-\mu)^\alpha}\le\big[f(\mu)+f(\nu)\big]{}_0I_1^{(\alpha)}h(t). \tag{3.1}$$

If $f:[\mu,\nu]\to\mathbb{R}^\alpha$ is generalized h-convex and $f(x)\in I_x^\alpha[\mu,\nu]$.

Theorem 3.1. *Let* $h:J\subset\mathbb{R}\to\mathbb{R}^\alpha$ *be a non-negative mapping. If* $f:[\mu,\nu]\to\mathbb{R}^\alpha$ *is generalized* h*-convex with* $\mu<\nu$, $\mu,\nu\in\mathbb{R}_+$ *and* $f(x)\in I_x^\alpha[\mu,\nu]$, *then*

$$\frac{f\big(\frac{\mu+\nu}{2}\big)}{\Gamma(1+\alpha)}\le 2^\alpha h\Big(\frac{1}{2}\Big)H(t). \tag{3.2}$$

(ii) $H(t)\le\min\{H_1(t),H_2(t)\}$, $t\in[0,1]$, *where*

$$H_1(t)=\Big[f\Big(t\mu+(1-t)\frac{\mu+\nu}{2}\Big)+f\Big(t\nu+(1-t)\frac{\mu+\nu}{2}\Big)\Big]{}_0I_1^{(\alpha)}h(m) \tag{3.3}$$

and

$$H_2(t)=\frac{h(t)\,{}_\mu I_\nu^{(\alpha)}f(x)}{(\nu-\mu)^\alpha}+\frac{h(1-t)f\big(\frac{\mu+\nu}{2}\big)}{\Gamma(1+\alpha)}. \tag{3.4}$$

Proof. (i) Letting $t\in(0,1]$ and using the change of variable $m=tx+(1-t)\frac{\mu+\nu}{2}$, we have

$$H(t):=\frac{{}_\mu I_\nu^{(\alpha)}f\big(tx+(1-t)\frac{\mu+\nu}{2}\big)}{(\nu-\mu)^\alpha}=\frac{{}_{t\mu+(1-t)\frac{\mu+\nu}{2}}I^{(\alpha)}_{t\nu+(1-t)\frac{\mu+\nu}{2}}f(m)}{t^\alpha(\nu-\mu)^\alpha}.$$

Using the left-sided inequality of (3.1), we deduce

$$\begin{aligned}\frac{f\left(\frac{\mu+\nu}{2}\right)}{2^{\alpha}\Gamma(1+\alpha)h\left(\frac{1}{2}\right)} &= \frac{f\left(\frac{t\mu+(1-t)\frac{\mu+\nu}{2}+t\nu+(1-t)\frac{\mu+\nu}{2}}{2}\right)}{2^{\alpha}\Gamma(1+\alpha)h\left(\frac{1}{2}\right)} \\ &\leq \frac{{}_{t\mu+(1-t)\frac{\mu+\nu}{2}}I^{(\alpha)}_{t\nu+(1-t)\frac{\mu+\nu}{2}}f(m)}{\left[t\nu+(1-t)\frac{\mu+\nu}{2}-t\mu-(1-t)\frac{\mu+\nu}{2}\right]^{\alpha}} \\ &= \frac{{}_{t\mu+(1-t)\frac{\mu+\nu}{2}}I^{(\alpha)}_{t\nu+(1-t)\frac{\mu+\nu}{2}}f(m)}{t^{\alpha}(\nu-\mu)^{\alpha}} = H(t).\end{aligned}$$

This obtians (3.2) immediately.
(ii) By utilizing the right-sided inequality of (3.1), we have

$$\begin{aligned}H(t) &= \frac{{}_{t\mu+(1-t)\frac{\mu+\nu}{2}}I^{(\alpha)}_{t\nu+(1-t)\frac{\mu+\nu}{2}}f(m)}{\left[t\nu+(1-t)\frac{\mu+\nu}{2}-t\mu-(1-t)\frac{\mu+\nu}{2}\right]^{\alpha}} \\ &\leq \left[f\left(t\mu+(1-t)\frac{\mu+\nu}{2}\right)+f\left(t\nu+(1-t)\frac{\mu+\nu}{2}\right)\right]{}_0I_1^{(\alpha)}h(m) = H_1(t).\end{aligned}$$

According to the generalized h-convexity of mapping f, we obtain

$$f\left(tx+(1-t)\frac{\mu+\nu}{2}\right) \leq h(t)f(x)+h(1-t)f\left(\frac{\mu+\nu}{2}\right).$$

So, for all $t\in[0,1]$ and $x\in[\mu,\nu]$, we deduce

$$\begin{aligned}H(t) &= \frac{{}_{\mu}I_{\nu}^{(\alpha)}f\left(tx+(1-t)\frac{\mu+\nu}{2}\right)}{(\nu-\mu)^{\alpha}} \\ &\leq \frac{{}_{\mu}I_{\nu}^{(\alpha)}\left[h(t)f(x)+h(1-t)f\left(\frac{\mu+\nu}{2}\right)\right]}{(\nu-\mu)^{\alpha}} \\ &= \frac{h(t)\,{}_{\mu}I_{\nu}^{(\alpha)}f(x)}{(\nu-\mu)^{\alpha}}+\frac{h(1-t)f\left(\frac{\mu+\nu}{2}\right)}{\Gamma(1+\alpha)} = H_2(t).\end{aligned}$$

The proof is completed.

Corollary 3.2. *Under all assumptions of Theorem 3.1, if $h(t)=t^{\alpha}$, then*

$$\begin{aligned}H(t)\leq \min\Bigg\{&\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\left[f\left(t\mu+(1-t)\frac{\mu+\nu}{2}\right)+f\left(t\nu+(1-t)\frac{\mu+\nu}{2}\right)\right], \\ &\frac{t^{\alpha}\,{}_{\mu}I_{\nu}^{\alpha}f(x)}{(\nu-\mu)^{\alpha}}+\frac{(1-t)^{\alpha}f\left(\frac{\mu+\nu}{2}\right)}{\Gamma(1+\alpha)}\Bigg\}.\end{aligned}$$

Corollary 3.3. *Putting $h(t)=t^{s\alpha}$ for some fixed $0<s<1$ in Theorem 3.1, we have*

$$H(t)\le \min\left\{\frac{\Gamma(1+s\alpha)}{\Gamma\big(1+(s+1)\alpha\big)}\left[f\Big(t\mu+(1-t)\frac{\mu+\nu}{2}\Big)+f\Big(t\nu+(1-t)\frac{\mu+\nu}{2}\Big)\right],\right.$$
$$\left.\frac{t^{s\alpha}{}_{\mu}I_{\nu}^{\alpha}f(x)}{(\nu-\mu)^{\alpha}}+\frac{(1-t)^{s\alpha}f\big(\frac{\mu+\nu}{2}\big)}{\Gamma(1+\alpha)}\right\},$$

which reduces to the inequality (7) of Theorem 3.2 in [12].

Corollary 3.4. *Taking $h(t)=1^{\alpha}$ in Theorem 3.1, we have*

$$H(t)\le \min\left\{\frac{f\big(t\mu+(1-t)\frac{\mu+\nu}{2}\big)+f\big(t\nu+(1-t)\frac{\mu+\nu}{2}\big)}{\Gamma(1+\alpha)},\frac{{}_{\mu}I_{\nu}^{\alpha}f(x)}{(\nu-\mu)^{\alpha}}+\frac{f\big(\frac{\mu+\nu}{2}\big)}{\Gamma(1+\alpha)}\right\}.$$

Corollary 3.5. *If $h(t)=t^{\alpha}(1-t)^{\alpha}$ in Theorem 3.1, then*

$$H(t)\le \min\left\{\left(\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}-\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\right)\left[f\Big(t\mu+(1-t)\frac{\mu+\nu}{2}\Big)+f\Big(t\nu+(1-t)\frac{\mu+\nu}{2}\Big)\right],\right.$$
$$\left.t^{\alpha}(1-t)^{\alpha}\left[\frac{{}_{\mu}I_{\nu}^{\alpha}f(x)}{(\nu-\mu)^{\alpha}}+\frac{f\big(\frac{\mu+\nu}{2}\big)}{\Gamma(1+\alpha)}\right]\right\}.$$

We now present the following theorem for the generalized mapping F.

Theorem 3.6. *Let $h:J\subset\mathbb{R}\to\mathbb{R}^{\alpha}$ be a non-negative mapping. If $f:[\mu,\nu]\to\mathbb{R}^{\alpha}$ is generalized h-convex along with $\mu<\nu$, $\mu,\nu\in\mathbb{R}_{+}$ and $f(x)\in I_{x}^{\alpha}[\mu,\nu]$, then*
(i) $F\big(\frac{1}{2}\big)\le 2^{\alpha}h\big(\frac{1}{2}\big)F(t)$;
(ii) $F(t)\ge \frac{H(t)}{2^{\alpha}h(\frac{1}{2})\Gamma(1+\alpha)}$ for all $t\in[0,1]$;
(iii) $F(t)\le \min\big\{F_1(t),F_2(t)\big\}$ for all $t\in[0,1]$, where

$$F_1(t)=\frac{\big[h(t)+h(1-t)\big]}{\Gamma(1+\alpha)(\nu-\mu)^{\alpha}}{}_{\mu}I_{\nu}^{(\alpha)}f(x) \tag{3.5}$$

and

$$F_2(t)=\Big({}_{0}I_{1}^{(\alpha)}h(m)\Big)^{2}\Big[f(\mu)+f\big(t\mu+(1-t)\nu\big)+f\big(t\nu+(1-t)\mu\big)+f(\nu)\Big]. \tag{3.6}$$

Proof (i) Since f is generalized h-convex, we have

$$f\Big(\frac{x+y}{2}\Big)\le h\Big(\frac{1}{2}\Big)\Big[f\big(tx+(1-t)y\big)+f\big((1-t)y+tx\big)\Big]. \tag{3.7}$$

Integrating the variables x and y in inequality (3.7) over $[\mu,\nu]$, we obtain

$$\frac{1}{\big(\Gamma(1+\alpha)\big)^{2}}\int_{\mu}^{\nu}\int_{\mu}^{\nu}f\Big(\frac{x+y}{2}\Big)(\mathrm{d}x)^{\alpha}(\mathrm{d}y)^{\alpha}$$
$$\le\frac{1}{\big(\Gamma(1+\alpha)\big)^{2}}\int_{\mu}^{\nu}\int_{\mu}^{\nu}h\Big(\frac{1}{2}\Big)\Big[f\big(tx+(1-t)y\big)+f\big((1-t)y+tx\big)\Big](\mathrm{d}x)^{\alpha}(\mathrm{d}y)^{\alpha}.$$

Using the fact that

$$\frac{1}{\big(\Gamma(1+\alpha)\big)^2}\int_\mu^\nu\int_\mu^\nu f\big(tx+(1-t)y\big)(\mathrm{d}x)^\alpha(\mathrm{d}y)^\alpha = \frac{1}{\big(\Gamma(1+\alpha)\big)^2}\int_\mu^\nu\int_\mu^\nu f\big((1-t)x+ty\big)(\mathrm{d}x)^\alpha(\mathrm{d}y)^\alpha.$$

We deduce

$$\begin{aligned} F\Big(\frac{1}{2}\Big) &= \frac{1}{\big(\Gamma(1+\alpha)\big)^2}\int_\mu^\nu\int_\mu^\nu\Big(\frac{1}{\nu-\mu}\Big)^{2\alpha} f\Big(\frac{x+y}{2}\Big)(\mathrm{d}x)^\alpha(\mathrm{d}y)^\alpha \\ &\leq \frac{1}{\big(\Gamma(1+\alpha)\big)^2}\int_\mu^\nu\int_\mu^\nu\Big(\frac{1}{\nu-\mu}\Big)^{2\alpha} h\Big(\frac{1}{2}\Big)\Big[f\big(tx+(1-t)y\big)+f\big((1-t)x+ty\big)\Big](\mathrm{d}x)^\alpha(\mathrm{d}y)^\alpha \\ &= 2^\alpha h\Big(\frac{1}{2}\Big)F(t). \end{aligned}$$

(ii) We define the function $F_y(t):[0,1]\to\mathbb{R}^\alpha$ as follows:

$$F_y(t) = \frac{1}{\Gamma(1+\alpha)}\int_\mu^\nu\Big(\frac{1}{\nu-\mu}\Big)^\alpha f\big(tx+(1-t)y\big)(\mathrm{d}x)^\alpha.$$

Letting $t\in(0,1]$ and using the change of variable $m=tx+(1-t)y$, we have

$$F_y(t) = \frac{{}_{t\mu+(1-t)y}I^{(\alpha)}_{t\nu+(1-t)y}f(m)}{t^\alpha(\nu-\mu)^\alpha}$$

For all $x\in[\mu,\nu]$ and $t\in[0,1]$, utilizing the left-sided inequality of (3.1), we obtain

$$\begin{aligned} \frac{f\Big(\frac{t\mu+(1-t)y+t\nu+(1-t)y}{2}\Big)}{2^\alpha\Gamma(1+\alpha)h\big(\frac{1}{2}\big)} &= \frac{f\big(\frac{\mu+\nu}{2}t+(1-t)y\big)}{2^\alpha\Gamma(1+\alpha)h\big(\frac{1}{2}\big)} \\ &\leq \frac{{}_{t\mu+(1-t)y}I^{(\alpha)}_{t\nu+(1-t)y}f(m)}{t^\alpha(\nu-\mu)^\alpha} = F_y(t). \end{aligned} \tag{3.8}$$

Multiplying inequality (3.8) with $\big(\frac{1}{\nu-\mu}\big)^\alpha$ and integrating with respect to y over interval $[\mu,\nu]$, we deduce

$$\begin{aligned} F(t) &= \frac{1}{\big(\Gamma(1+\alpha)\big)^2}\int_\mu^\nu\int_\mu^\nu\Big(\frac{1}{\nu-\mu}\Big)^{2\alpha} f\big(tx+(1-t)y\big)(\mathrm{d}x)^\alpha(\mathrm{d}y)^\alpha \\ &= \frac{1}{\Gamma(1+\alpha)}\int_\mu^\nu\Big(\frac{1}{\nu-\mu}\Big)^\alpha F_y(t)(\mathrm{d}y)^\alpha \\ &\geq \frac{1}{2^\alpha h\big(\frac{1}{2}\big)(\Gamma(1+\alpha))^2}\int_\mu^\nu\Big(\frac{1}{\nu-\mu}\Big)^\alpha f\Big(\frac{\mu+\nu}{2}t+(1-t)y\Big)(\mathrm{d}y)^\alpha \\ &= \frac{H(1-t)}{2^\alpha h\big(\frac{1}{2}\big)\Gamma(1+\alpha)}. \end{aligned}$$

Since $F(t)$ is symmetric about $t=\frac{1}{2}$, i.e., $F(t)=F(1-t)$, we have (ii).
(iii) Utilizing the generalized h-convexity of f, we also have

$$f\big(tx+(1-t)y\big)\leq h(t)f(x)+h(1-t)f(y). \tag{3.9}$$

Integrating the variables x and y in inequality (3.9) over the interval $[\mu,\nu]$, and multiplying $\left(\frac{1}{\nu-\mu}\right)^{2\alpha}$, we get

$$\begin{aligned}
&\frac{1}{\big(\Gamma(1+\alpha)\big)^2}\int_\mu^\nu\int_\mu^\nu\left(\frac{1}{\nu-\mu}\right)^{2\alpha}f\big(tx+(1-t)y\big)(\mathrm{d}x)^\alpha(\mathrm{d}y)^\alpha\\
&\leq\frac{h(t)}{\big(\Gamma(1+\alpha)\big)^2(\nu-\mu)^\alpha}\int_\mu^\nu f(x)(\mathrm{d}x)^\alpha+\frac{h(1-t)}{\big(\Gamma(1+\alpha)\big)^2(\nu-\mu)^\alpha}\int_\mu^\nu f(y)(\mathrm{d}y)^\alpha\\
&=\frac{\big[h(t)+h(1-t)\big]}{\Gamma(1+\alpha)(\nu-\mu)^\alpha}{}_\mu I_\nu^{(\alpha)}f(x).
\end{aligned}$$

By using the right-sided inequality of (3.1), we deduce

$$\begin{aligned}
F_y(t)&=\frac{1}{\Gamma(1+\alpha)}\int_\mu^\nu\left(\frac{1}{\nu-\mu}\right)^\alpha f\big(tx+(1-t)y\big)(\mathrm{d}x)^\alpha\\
&=\frac{{}_{t\mu+(1-t)y}I_{t\nu+(1-t)y}^{(\alpha)}f(m)}{t^\alpha(\nu-\mu)^\alpha}\\
&\leq\Big[f\big(t\mu+(1-t)y\big)+f\big(\nu+(1-t)y\big)\Big]{}_0I_1^{(\alpha)}h(m).
\end{aligned} \tag{3.10}$$

Multiplying inequality (3.10) with $\left(\frac{1}{\nu-\mu}\right)^\alpha$ and integrating with respect to y over interval $[\mu,\nu]$, we conclude

$$\begin{aligned}
F(t)&\leq\frac{1}{\Gamma(1+\alpha)}\int_\mu^\nu\left(\frac{1}{\nu-\mu}\right)^\alpha\Big[f\big(t\mu+(1-t)y\big)+f\big(t\nu+(1-t)y\big)\Big](\mathrm{d}y)^\alpha{}_0I_1^{(\alpha)}h(m)\\
&=\frac{{}_\mu I_\nu^{(\alpha)}f\big(t\mu+(1-t)y\big)+{}_\mu I_\nu^{(\alpha)}f\big(t\nu+(1-t)y\big)}{(\nu-\mu)^\alpha}{}_0I_1^{(\alpha)}h(m).
\end{aligned}$$

Applying the right-sided inequality of (3.1) again, one has

$$\frac{{}_\mu I_\nu^{(\alpha)}f\big(t\mu+(1-t)y\big)}{(\nu-\mu)^\alpha}\leq{}_0I_1^{(\alpha)}h(m)\big[f(\mu)+f\big(t\mu+(1-t)\nu\big)\big]$$

and

$$\frac{{}_\mu I_\nu^{(\alpha)}f\big(t\nu+(1-t)y\big)}{(\nu-\mu)^\alpha}\leq{}_0I_1^{(\alpha)}h(m)\big[f(\nu)+f\big(t\nu+(1-t)\mu\big)\big].$$

They imply that

$$F(t)\leq\Big({}_0I_1^{(\alpha)}h(m)\Big)^2\Big[f(\mu)+f\big(t\mu+(1-t)\nu\big)+f\big(t\nu+(1-t)\mu\big)+f(\nu)\Big].$$

This completes the proof.

Corollary 3.7. *Under all assumptions of Theorem 3.6, if $h(t)=t^{\alpha}$, then*

$$F(t)\leq \min\left\{\frac{{}_{\mu}I_{\nu}^{(\alpha)}f(x)}{\Gamma(1+\alpha)(\nu-\mu)^{\alpha}},\left[\frac{\Gamma(1+\alpha)}{(\Gamma(1+2\alpha)}\right]^{2}\right.$$
$$\left.\times\Big[f(\mu)+f\big(t\mu+(1-t)\nu\big)+f\big(t\nu+(1-t)\mu\big)+f(\nu)\Big]\right\}.$$

Corollary 3.8. *Putting $h(t)=t^{s\alpha}$ for some fixed $0<s<1$ in Theorem 3.1, we have*

$$F(t)\leq \min\left\{\frac{t^{s\alpha}+(1-t)^{s\alpha}}{\Gamma(1+\alpha)(\nu-\mu)^{\alpha}}{}_{\mu}I_{\nu}^{(\alpha)}f(x),\left[\frac{\Gamma(1+s\alpha)}{\big(\Gamma(1+(s+1)\alpha\big)}\right]^{2}\right.$$
$$\left.\times\Big[f(\mu)+f\big(t\mu+(1-t)\nu\big)+f\big(t\nu+(1-t)\mu\big)+f(\nu)\Big]\right\},$$

which reduces to the inequality (9) of Theorem 3.2 given in [12].

Corollary 3.9. *Taking $h(t)=1^{\alpha}$ in Theorem 3.1, we have*

$$F(t)\leq \min\left\{\frac{2^{\alpha}{}_{\mu}I_{\nu}^{(\alpha)}f(x)}{\Gamma(1+\alpha)(\nu-\mu)^{\alpha}},\frac{1}{\big(\Gamma(1+\alpha)\big)^{2}}\right.$$
$$\left.\times\Big[f(\mu)+f\big(t\mu+(1-t)\nu\big)+f\big(t\nu+(1-t)\mu\big)+f(\nu)\Big]\right\}.$$

Corollary 3.10. *If $h(t)=t^{\alpha}(1-t)^{\alpha}$ in Theorem 3.1, then*

$$F(t)\leq \min\left\{\frac{2^{\alpha}t^{\alpha}(1-t)^{\alpha}{}_{\mu}I_{\nu}^{(\alpha)}f(x)}{\Gamma(1+\alpha)(\nu-\mu)^{\alpha}},\left(\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}-\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\right)^{2}\right.$$
$$\left.\times\Big[f(\mu)+f\big(t\mu+(1-t)\nu\big)+f\big(t\nu+(1-t)\mu\big)+f(\nu)\Big]\right\}.$$

4. Conclusion

In this paper, we introduced a class of (φ,ρ^{α})-Lipschitzian mappings and established some new generalized Hadamard's type inequalities for these mappings on fractal sets $R^{\alpha}(0<\alpha\leq 1)$. Furthermore, we discovered certain properties of generalized mappings H and generalized mappings F associated with generalized h-convexity mappings. The results proved in this paper generalize the corresponding results presented by Ahmad, Jleli and Samet [7] and Kiliçman and Saleh [12].

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