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THE OBSTACLE PROBLEM FOR DEGENERATE ANISOTROPIC ELLIPTIC EQUATIONS WITH VARIABLE EXPONENTS AND L^1-DATA

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Abstract. In this paper, we prove the existence of entropy solutions for the obstacle problem associated with nonlinear degenerate anisotropic elliptic equations with L^1-data. The functional framework involves anisotropic Sobolev spaces with variable exponents as well as weak Lebesgue (Marcinkiewicz) spaces with variable exponents. Our results are a natural generalization of some existing ones in the context of constant isotropic exponents.

Keywords. Obstacle problem; Entropy solutions; Degenerate anisotropic elliptic equations; Variable exponents; L^1-data.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$ $(N \geq 2)$ with smooth boundary $\partial\Omega$ and $f \in L^1(\Omega)$. We consider the following nonlinear anisotropic problem

$$\begin{cases} -\sum\limits_{i=1}^{N} D_i\left(\frac{a_i(x,\nabla u)}{(1+|u|)^{\gamma_i(x)}}\right) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $p_i : \overline{\Omega} \to (1,+\infty)$, and $\gamma_i : \overline{\Omega} \to [0,+\infty)$ for $i = 1,\dots,N$ are continuous functions such that

$$1 < \overline{p}(x) < N \quad \text{for all } x \in \overline{\Omega}, \tag{1.2}$$

where $\frac{1}{\overline{p}(x)} = \frac{1}{N}\sum\limits_{i=1}^{N}\frac{1}{p_i(x)}$. We assume, for $i = 1,\dots,N$, that $a_i : \Omega \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function satisfying, for almost every $x \in \Omega$ and for every $\xi = (\xi_1,\dots,\xi_N),\ \xi' = (\xi'_1,\dots,\xi'_N) \in \mathbb{R}^N$ with $\xi_i \neq \xi'_i$, the following assumptions

$$|a_i(x,\xi)| \leq \beta|\xi_i|^{p_i(x)-1}, \tag{1.3}$$

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$$a_i(x,\xi)\xi_i \geq \alpha|\xi_i|^{p_i(x)}, \tag{1.4}$$

and

$$\left[a_i(x,\xi)-a_i(x,\xi')\right]\left[\xi_i-\xi_i'\right] > 0, \tag{1.5}$$

where $\alpha > 0$, and $\beta > 0$.

We define, for $u \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$, the nonlinear elliptic operator

$$Au = -\sum_{i=1}^{N} D_i\left(\frac{a_i(x,\nabla u)}{(1+|u|)^{\gamma_i(x)}}\right),$$

which, thanks to (1.3) and (1.4), maps $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ into its dual space $W^{-1,\overrightarrow{p}'(\cdot)}(\Omega)$, but its coercivity can degenerate when u is too big. Due to the lack of coercivity, the standard theorem for variational inequalities involving pseudomonotone operators cannot be applied even if the data f is sufficiently regular.

For a given function $\psi \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)\cap L^\infty(\Omega)$, we define the following convex set

$$K_\psi = \left\{v \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega) : v \geq \psi \text{ a.e. in } \Omega\right\}.$$

The obstacle problem relative to the operator A with obstacle ψ and datum f (denoted by (A,f,ψ)) can be formulated using the duality between $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$, and $W^{-1,\overrightarrow{p}'(\cdot)}(\Omega)$ in terms of the variational inequality

$$\begin{cases} u \in K_\psi, \\ \displaystyle\sum_{i=1}^{N}\int_\Omega \frac{a_i(x,\nabla u)D_i(u-v)}{(1+|u|)^{\gamma_i(x)}}dx \leq \langle f, u-v\rangle, \ \forall v \in K_\psi, \end{cases} \tag{1.6}$$

whenever $f \in W^{-1,\overrightarrow{p}'(\cdot)}(\Omega)$. In light of (1.2), if $f \in L^1(\Omega)$, then the formulation (1.6) does not remain valid since both sides of inequality (1.6) are maybe meaningless. This leads, following [1, 2], to introduce a more general formulation of obstacle problem (A,f,ψ) using the truncation function at level $k>0$, $T_k : \mathbb{R} \to \mathbb{R}$ defined by

$$T_k(s) = \begin{cases} s, & \text{if } |s| \leq k, \\ k\frac{s}{|s|}, & \text{if } |s| > k. \end{cases}$$

Definition 1.1. An entropy solution of obstacle problem (A,f,ψ) associated to (1.1) is a measurable function u such that

$$\begin{cases} u \geq \psi \text{ a.e. in}\,\Omega, \\ T_k(u) \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega), \ \forall k>0, \\ \displaystyle\sum_{i=1}^{N}\int_\Omega \frac{a_i(x,\nabla u)D_iT_k(u-v)}{(1+|u|)^{\gamma_i(x)}}dx \leq \int_\Omega f\,T_k(u-v)dx, \ \forall v \in K_\psi \cap L^\infty(\Omega). \end{cases} \tag{1.7}$$

We point out that, in the case of constant exponents, the existence and regularity of entropy solutions to obstacle problem (A,f,ψ) were obtained in [3, 4, 5].

Under uniform ellipticity condition, namely, when $\gamma_i \equiv 0$ for all $i=1,\dots,N$, the existence of entropy solutions to the obstacle problems associated with the operator A has been widely investigated in literature (see, for example, [6, 7] and references therein).

In this paper, using the techniques presented in [8], we establish the existence of an entropy solution u to obstacle problem (A, f, ψ) such that $u \in M^{q(\cdot)}(\Omega)$, and $|D_i u|^{\xi_i(x)} \in M^{q(\cdot)}(\Omega)$ for each $i = 1, \ldots, N$ with

$$q(\cdot) = p_+(\cdot)\left(1 - \frac{1+\gamma_+^+}{p_-^-}\right) \text{ and } \xi_i(\cdot) = \frac{p_i(\cdot)}{q(\cdot)+1+\gamma_i(\cdot)}.$$

The paper is organized as follow. In Section 2, we recall some results on some variable exponent function spaces. In Section 3, we state the main results in this paper. In Section 4, we consider the sequence of approximating obstacle problems, and establish the uniform estimates for approximate solutions. In Section 5, we prove the strong convergence of the truncations of these solutions. The proof of main results is given in Section 6. Finally, this paper is ended by an appendix in which we give a proof of the existence of solutions to the approximate obstacle problems.

2. Mathematical Preliminaries

In this section, we recall the definitions and basic properties of some variable exponent function spaces which we need in the sequel. For further details on this topic, we refer to [9, 10] and the references therein. Hereinafter, we denote

$$P_+(\Omega) = \{h : \Omega \to \mathbb{R} \text{ is measurable} : \ 0 < h^- \le h^+ < \infty\} \text{ and } C_+(\overline{\Omega}) = \{h \in C(\overline{\Omega}) : \ h^- > 1\},$$

where

$$h^- = \operatorname*{ess\,inf}_{x\in\Omega} h(x) \text{ and } h^+ = \operatorname*{ess\,sup}_{x\in\Omega} h(x).$$

For any $p \in P_+(\Omega)$, the Lebesgue space with variable exponent $L^{p(\cdot)}(\Omega)$ is the set of all measurable functions $u : \Omega \to \mathbb{R}$ for which the modular

$$\rho_{p(\cdot)}(u) = \int_\Omega |u|^{p(x)} dx,$$

is finite. The space $L^{p(\cdot)}(\Omega)$ equipped with the Luxemburg-Nakano quasi-norm

$$\|u\|_{L^{p(\cdot)}(\Omega)} = \inf\left\{\lambda > 0 : \ \rho_{p(\cdot)}\left(\frac{u}{\lambda}\right) \le 1\right\},$$

is a quasi-Banach space (see [9, 11]). In particular, if $p^- \ge 1$, then the above expression defines a norm in $L^{p(\cdot)}(\Omega)$. In this case, the space $L^{p(\cdot)}(\Omega)$ becomes a separable Banach space (see, e.g., [10]). Moreover, if $p^- > 1$, then $L^{p(\cdot)}(\Omega)$ is reflexive and its dual space can be identified with $L^{p'(\cdot)}(\Omega)$ with $\frac{1}{p(\cdot)} + \frac{1}{p'(\cdot)} = 1$, and for all $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$, the following Hölder's inequality holds (see [10])

$$\left|\int_\Omega uv dx\right| \le 2\|u\|_{L^{p(\cdot)}(\Omega)}\|v\|_{L^{p'(\cdot)}(\Omega)}. \tag{2.1}$$

The norm and the modular are related by the following inequalities.

Proposition 2.1 ([9]). *Let $p \in P_+(\Omega)$. Then, for every $u \in L^{p(\cdot)}(\Omega)$, one has* $\rho_{p(\cdot)}(u) < 1 (> 1; = 1)$ *if and only if* $\|u\|_{L^{p(\cdot)}(\Omega)} < 1 (> 1; = 1)$. *Further,*

$$\text{if } \|u\|_{L^{p(\cdot)}(\Omega)} < 1 \text{ then } \|u\|^{p^+}_{L^{p(\cdot)}(\Omega)} \le \rho_{p(\cdot)}(u) \le \|u\|^{p^-}_{L^{p(\cdot)}(\Omega)}, \tag{2.2}$$

$$\text{if } \|u\|_{L^{p(\cdot)}(\Omega)} > 1 \text{ then } \|u\|^{p^-}_{L^{p(\cdot)}(\Omega)} \le \rho_{p(\cdot)}(u) \le \|u\|^{p^+}_{L^{p(\cdot)}(\Omega)}. \tag{2.3}$$

The above proposition states that the norm convergence and modular convergence are equivalent, that is, if u_n and u belong to $L^{p(\cdot)}(\Omega)$, then

$$\|u_n - u\|_{L^{p(\cdot)}(\Omega)} \to 0 \text{ if and only if } \rho_{p(\cdot)}(u_n - u) \to 0.$$

Next, we define Marcinkiewicz (weak Lebesgue) spaces with variable exponents and we investigate their relation with variable exponent Lebesgue spaces.

Definition 2.2 ([11]). Let $p \in P_+(\Omega)$. We say that a measurable function $u : \Omega \to \mathbb{R}$ belongs to the Marcinkiewicz space $M^{p(\cdot)}(\Omega)$ if

$$\|u\|_{M^{p(\cdot)}(\Omega)} = \sup_{\lambda > 0} \lambda \|\chi_{\{|u|>\lambda\}}\|_{L^{p(\cdot)}(\Omega)} < \infty, \tag{2.4}$$

where χ_E denotes the characteristic function of a measurable set E.

Note that inequalities (2.2) and (2.3) imply that (2.4) is equivalent to the fact that there exists a positive constant C such that

$$\int_{\{|u|>\lambda\}} \lambda^{p(x)} dx \le C, \quad \text{for all } \lambda > 0. \tag{2.5}$$

If $p, q \in P_+(\Omega)$ with $q \le p$, then we have the following two inclusions (see [9]):

$$L^{p(\cdot)}(\Omega) \subseteq L^{q(\cdot)}(\Omega) \text{ and } L^{p(\cdot)}(\Omega) \subset M^{p(\cdot)}(\Omega) \subseteq M^{q(\cdot)}(\Omega).$$

The following result is from [8, Proposition 2.5].

Proposition 2.3. *Let $p, q \in P_+(\Omega)$ such that $(q - p) \in P_+(\Omega)$. Then,*

$$M^{p(\cdot)}(\Omega) \subset L^{q(\cdot)}(\Omega).$$

We will need the following lemma proved in [12].

Lemma 2.4. *Let $u \in M^{p(\cdot)}(\Omega)$ with $p \in P_+(\Omega)$. Then there exists a constant $c > 0$ such that*

$$\text{meas}\{|u| > \lambda\} \le \frac{c}{\lambda^{p^-}} \quad \textit{for all } \lambda > 0.$$

Let $\overrightarrow{p}(\cdot) : \overline{\Omega} \longrightarrow \mathbb{R}^N$ be a vector-valued function defined by

$$\overrightarrow{p}(\cdot) = (p_1(\cdot), ..., p_N(\cdot)),$$

for all $i \in \{1, ..., N\}$, $p_i \in C_+(\overline{\Omega})$. We define the anisotropic variable exponent Sobolev space $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$, as the completion of $C_0^\infty(\Omega)$ with respect to the norm $\sum_{i=1}^{N} \|D_i u\|_{L^{p_i(\cdot)}(\Omega)}$. Equipped with this norm, $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ is a separable and reflexive Banach space (see [13]).

We introduce the following notations:

$$\overline{p}^\star(x) = \frac{N\overline{p}(x)}{N - \overline{p}(x)}, \; p_+(x) = \max\{p_1(x), \dots, p_N(x)\},$$
$$p_+^- = \max\{p_1^-, \dots, p_N^-\}, \text{ and } p_-^- = \min\{p_1^-, \dots, p_N^-\}.$$

We have the following embedding results [13, 14].

Lemma 2.5. *Let Ω be a smooth bounded domain in $\mathbb{R}^N$ and $\overrightarrow{p}(\cdot) \in \left(C_+(\overline{\Omega})\right)^N$. Assume that (1.2) is fulfilled. Then for any $q \in C_+\left(\overline{\Omega}\right)$ satisfying*

$$q(x) < \max\{(\overline{p}^-)^*, p_+^-\} \quad \text{for all } x \in \overline{\Omega},$$

the embedding

$$W_0^{1,\overrightarrow{p}(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega), \tag{2.6}$$

is continuous and compact. Moreover, if

$$p_+(x) < \overline{p}^*(x) \quad \text{for all } x \in \overline{\Omega}, \tag{2.7}$$

then the following Poincaré-type inequality holds

$$\|u\|_{L^{p_+(\cdot)}(\Omega)} \leq C \sum_{i=1}^{N} \|D_i u\|_{L^{p_i(\cdot)}(\Omega)} \quad \text{for all } u \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega), \tag{2.8}$$

where C is a positive constant independent of u.

We denote by $T_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ the set of all measurable functions $u : \Omega \to \mathbb{R}$ such that $T_k(u) \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ for any $k > 0$. Note that $T_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ is not contained in the Sobolev space $W_0^{1,1}(\Omega)$. However, the following proposition gives a sense to the partial derivatives of $u \in T_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$.

Proposition 2.6. *Let $u \in T_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$. Then, for each $i = 1, \ldots, N$, there exists a unique measurable function $v_i : \Omega \to \mathbb{R}$ such that, for any $k > 0$,*

$$D_i T_k(u) = v_i \chi_{\{|u| \leq k\}} \text{ a.e. in } \Omega.$$

The functions v_i are called the weak partial derivatives of u and are still denoted by $D_i u$. Moreover, if u belongs to $W_0^{1,1}(\Omega)$, then v_i coincides with the standard distributional derivatives of u, that is $v_i = D_i u$.

3. Statement of Main Results

We are in a position to state our main results.

Theorem 3.1. *Suppose Assumptions (1.2)-(1.5) hold and that*

$$0 \leq \gamma_+^+ < \overline{p}^- - 1, \tag{3.1}$$

where $\gamma_+^+ = \max\{\gamma_1^+, \ldots, \gamma_N^+\}$. Then, the obstacle problem (A, f, ψ) has at least one entropy solution $u \in M^q(\Omega)$, and $D_i u \in M^{q_i}(\Omega)$ for each $i = 1, \ldots, N$, with

$$q = \frac{N(\overline{p}^- - 1 - \gamma_+^+)}{N - \overline{p}^-} \text{ and } q_i = \frac{N(\overline{p}^- - 1 - \gamma_+^+)}{\overline{p}^-(N - 1 - \gamma_+^+)} p_i^-. \tag{3.2}$$

Remark 3.2. Notice that, in the isotropic constant case, namely, when $p_i(\cdot) = p$ and $\gamma_i(\cdot) = \gamma$, the exponents in (3.2) reduce to

$$q = \frac{N(p-1-\gamma)}{N-p} \quad \text{and} \quad q_1 = \frac{N(p-1-\gamma)}{N-1-\gamma},$$

which are the same as the ones obtained in [15, 16] while studying Problem (1.1) in the context of constant isotropic exponents.

The above theorem, in which the exponents q and q_i, $i=1,\dots,N$, are constants not variables, is optimal in light of the existing results on embeddings of anisotropic variable exponent Sobolev spaces. However, if we take advantage of the inequality (2.8), we can prove the following result under some more restrictive assumptions on the exponents.

Theorem 3.3. *Suppose Assumptions* (1.2)-(1.5) *and* (2.7) *hold, and that*

$$0 \le \gamma_+^+ < p_-^- - 1. \tag{3.3}$$

Then, obstacle problem (A, f, ψ) *has at least one entropy solution* $u \in M^{q(\cdot)}(\Omega)$ *and* $|D_i u|^{\xi_i(x)} \in M^{q(\cdot)}(\Omega)$ *for each* $i=1,\dots,N$, *with*

$$q(x) = p_+(x)\left(1 - \frac{1+\gamma_+^+}{p_-^-}\right) \text{ and } \xi_i(x) = \frac{p_i(x)}{q(x)+1+\gamma_i(x)}.$$

Remark 3.4. Note that conditions (3.1) and (3.3) are nothing else than $0 \le \theta < 1$ if $\gamma_i(x) = \theta(p-1)$, which is a crucial condition for the existence of the solutions in [3, 4, 15].

4. The Approximate Problem and Uniform Estimates

In order to prove our main results, let us consider the sequence of approximate problems

$$\begin{cases} u_n \in K_\psi, \\ \displaystyle\sum_{i=1}^{N} \int_\Omega \frac{a_i(x,\nabla u_n) D_i(u_n - v)}{(1+|T_n(u_n)|)^{\gamma_i(x)}} dx \le \int_\Omega f_n (u_n - v) dx, \ \forall v \in K_\psi, \end{cases} \tag{4.1}$$

where $f_n = T_n(f) \in L^\infty(\Omega)$.

Lemma 4.1. *Under our standing assumptions* (1.2)-(1.5), *there exists at least one solution* $u_n \in K_\psi$ *to Problem* (4.1). *Moreover, there exists a positive constant c, not depending on n, such that*

$$\sum_{i=1}^{N} \int_{A_k^n} |\nabla u_n|^{p_i(x)} dx \le c(1+k)^{1+\gamma_+^+}, \ \forall k > 0, \tag{4.2}$$

where $A_k^n = \{x \in \Omega : \ |u_n(x)| < k\}$.

Proof. Thanks to Lemma 7.3 (see Appendix), Problem (4.1) has at least one solution $u_n \in K_\psi$. Let us now prove the estimate (4.2). We consider the function

$$v = u_n - T_k(u_n - \psi), \ k > 0.$$

It is easy to see that $v \in K_\psi$. Hence, taking v as a test function in (4.1), we obtain

$$\sum_{i=1}^{N} \int_\Omega \frac{a_i(x,\nabla u_n) D_i T_k(u_n - \psi)}{(1+|T_n(u_n)|)^{\gamma_i(x)}} dx \le \int_\Omega f_n T_k(u_n - \psi) dx,$$

By assumption (1.4), we get

$$\alpha \sum_{i=1}^{N} \int_{\{u_n - \psi < k\}} \frac{|\nabla u_n|^{p_i(x)}}{(1+|T_n(u_n)|)^{\gamma_i(x)}} dx \le k\|f\|_{L^1(\Omega)} + \sum_{i=1}^{N} \int_{\{u_n - \psi < k\}} \frac{a_i(x,\nabla u_\varepsilon) D_i \psi}{(1+|T_n(u_n)|)^{\gamma_i(x)}} dx. \tag{4.3}$$

Now we estimate the second term on the right-hand side of (4.3). By using (1.3) and Young's inequality with $\eta > 0$, we get, for each $i = 1, \dots, N$,

$$\begin{aligned}
&\int_{\{u_n-\psi<k\}} \frac{a_i(x,\nabla u_n)D_i\psi}{(1+|T_n(u_n)|)^{\gamma_i(x)}}dx \le \beta \int_{\{u_n-\psi<k\}} \frac{|D_iu_n|^{p_i(x)-1}|D_i\psi|}{(1+|T_n(u_n)|)^{\gamma_i(x)}}dx,\\
&\le \beta\eta \int_{\{u_n-\psi<k\}} \frac{|D_iu_n|^{p_i(x)}}{(1+|T_n(u_n)|)^{\gamma_i(x)}}dx + C(\eta)\int_{\{u_n-\psi<k\}} \frac{|D_i\psi|^{p_i(x)}}{(1+|T_n(u_n)|)^{\gamma_i(x)}}dx \\
&\le \beta\eta \int_{\{u_n-\psi<k\}} \frac{|D_iu_n|^{p_i(x)}}{(1+|T_n(u_n)|)^{\gamma_i(x)}}dx + C.
\end{aligned} \tag{4.4}$$

Combining (4.3) and (4.4) gives

$$\begin{aligned}
&\alpha\sum_{i=1}^N \int_{\{u_n-\psi<k\}} \frac{|D_iu_n|^{p_i(x)}}{(1+|T_n(u_n)|)^{\gamma_i(x)}}dx\\
&\le k\|f\|_{L^1(\Omega)} + \beta\eta\sum_{i=1}^N \int_{\{u_n-\psi<k\}} \frac{|D_iu_n|^{p_i(x)}}{(1+|T_n(u_n)|)^{\gamma_i(x)}}dx + C.
\end{aligned}$$

Choosing η such that $\alpha = 2\beta\eta$, we find

$$\frac{\alpha}{2}\sum_{i=1}^N \int_{\{u_n-\psi<k\}} \frac{|D_iu_n|^{p_i(x)}}{(1+|T_n(u_n)|)^{\gamma_i(x)}}dx \le k\|f\|_{L^1(\Omega)} + C. \tag{4.5}$$

Replacing k by $k+\|\psi\|_{L^\infty(\Omega)}$ in (4.5) and noting that $\{|u_n|<k\} \subset \{|u_n-\psi| < k+\|\psi\|_{L^\infty(\Omega)}\}$, we get

$$\frac{\alpha}{2}\sum_{i=1}^N \int_{\{|u_n|<k\}} \frac{|D_iu_n|^{p_i(x)}}{(1+|T_n(u_n)|)^{\gamma_i(x)}}dx \le C(1+k),$$

which yields (4.2). This completes the proof of this Lemma. □

The following lemma was proved in [17].

Lemma 4.2. *Let the hypothesis of Theorem 3.1 be satisfied. Then there exist two positive constants c_1 and c_2 such that*

$$\|u_n\|_{M^q(\Omega)} \le c_1 \tag{4.6}$$

$$\|D_iu_n\|_{M^{q_i}(\Omega)} \le c_2, \quad \text{for } i = 1,\dots,N, \tag{4.7}$$

with $q = (\overline{p}^-)^*\left(1 - \frac{1+\gamma_+^+}{\overline{p}^-}\right)$, *and* $q_i = \frac{N(\overline{p}^- - 1 - \gamma_+^+)}{\overline{p}^-(N-1-\gamma_+^+)}p_i^-$.

The following lemma can be proved as [17, Lemma 4.3], so we omit the proof.

Lemma 4.3. *Let the hypothesis of Theorem 3.3 be satisfied. Then there exists a positive constant C independent of n, such that*

$$u_n \in M^{q(\cdot)}(\Omega) \text{ and } \int_{\{|u_n|>k\}} k^{q(x)}dx \le C, \ \forall k > 0,$$

with $q(x) = p_+(x)\left(1 - \frac{1+\gamma_+^+}{p_-^-}\right)$.

The following lemma can be proved exactly as in [17].

Lemma 4.4. *If there is a positive constant C_1, independent of n, such that*

$$\int_{\{|u_n|>k\}} k^{q(x)}dx \leq C_1,\ \forall k>0,$$

for some $q^->0$. Then, under the estimate (4.2), there holds $|D_iu_n|^{\xi_i(x)} \in M^{q(\cdot)}(\Omega)$ for all $i = 1,\cdots,N$, where $\xi_i(x) = \frac{p_i(x)}{q(x)+1+\gamma_i(x)}$. Moreover, there exists a positive constant C_2, independent of n, such that

$$\sum_{i=1}^{N} \int_{\{|D_iu_n|^{\xi_i(x)}>k\}} k^{q(x)}dx \leq C_2,\ \forall k>0. \tag{4.8}$$

5. The Strong Convergence of the Truncations

By using the uniform estimates obtained in the previous section, we are able to get the strong compactness of the truncations.

Proposition 5.1. *Let the assumptions of Theorem 3.3 be satisfied. If u_n is a sequence of solutions of Problem (4.1). Then, there exists a subsequence of u_n (still denoted by u_n) and a function $u \in T_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ such that*

$$T_k(u_n) \longrightarrow T_k(u) \text{ strongly in } W_0^{1,\overrightarrow{p}(\cdot)}(\Omega) \text{ and a.e. in } \Omega. \tag{5.1}$$

as $n \to +\infty$, for every $k>0$.

Proof. We proceed in two steps.

Step 1. The almost everywhere convergence of u_n in Ω.

We will show that $(u_n)_n$ is a Cauchy sequence in measure. Letting $\delta>0$, we have

$$\{|u_n-u_m|>\delta\} \subset \{|u_n|>k\} \cup \{|u_m|>k\} \cup \{|T_k(u_n)-T_k(u_m)|>\delta\},$$

which implies that

$$\begin{aligned} \operatorname{meas}\{|u_n-u_m|>\delta\} \quad &\leq \quad \operatorname{meas}\{|u_n|>k\}+\operatorname{meas}\{|u_m|>k\} \\ &\quad\quad +\operatorname{meas}\{|T_k(u_n)-T_k(u_m)|>\delta\}. \end{aligned}$$

Let $\varepsilon>0$. By invoking Lemma 2.4, we may choose $k=k(\varepsilon)$ large enough such that

$$\operatorname{meas}\{|u_n|>k\} \leq \frac{\varepsilon}{3} \text{ and } \operatorname{meas}\{|u_m|>k\} \leq \frac{\varepsilon}{3}. \tag{5.2}$$

From estimate (4.2), it follows that $\{T_k(u_n)\}_n$ is bounded in $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$. Then up to a subsequence (not relabeled)

$$T_k(u_n) \rightharpoonup \eta_k \text{ weakly in } W_0^{1,\overrightarrow{p}(\cdot)}(\Omega) \text{ as } n\to+\infty.$$

Thanks to the compact embedding (2.6), we get

$$T_k(u_n) \to \eta_k \text{ strongly in } L^1(\Omega) \text{ and a.e in } \Omega.$$

Consequently, we can assume that $\{T_k(u_n)\}_n$ is a Cauchy sequence in measure. Thus,

$$\operatorname{meas}\{|T_k(u_n)-T_k(u_m)|>\delta\} \leq \frac{\varepsilon}{3} \text{ for all } m,n \geq n_0(\varepsilon,\delta). \tag{5.3}$$

Combining this with (5.2) yields

$$\forall \delta,\varepsilon>0, \exists n_0(\varepsilon,\delta)\in\mathbb{N}, \forall n,m \geq n_0(\varepsilon,\delta);\ \operatorname{meas}\{|u_n-u_m|>\delta\} \leq \varepsilon,$$

which proves that (u_n) is a Cauchy sequence in measure. Then it converges almost everywhere to some measurable function u, and

$$T_k(u_n) \rightharpoonup T_k(u) \text{ weakly in } W_0^{1,\vec{p}(\cdot)}(\Omega) \ \text{ as } n \to +\infty. \tag{5.4}$$

Step 2. ***The strong convergence of the truncation*** $T_k(u_n)$. For a fixed integer $m > 0$, we introduce the function $\varphi_m : \mathbb{R} \to \mathbb{R}$ given by:

$$\varphi_m(s) = T_{m+1}(s) - T_m(s) = \begin{cases} 1, & \text{if } s \geq m+1, \\ s-k, & \text{if } m \leq s < m+1, \\ 0, & \text{if } 0 \leq s < m, \\ -\varphi_m(-s), & \text{if } s < 0. \end{cases} \tag{5.5}$$

Letting $m > k \geq \|\psi\|_\infty$, we set $h_m(s) = 1 - |\varphi_m(s)|$ and $v = u_n - h_m(u_n)\,(T_k(u_n) - T_k(u))$. Since $v \in W_0^{1,\vec{p}(\cdot)}(\Omega)$ and $v \geq \psi$, then v is an admissible test function for (4.1), we have

$$\overbrace{\sum_{i=1}^N \int_\Omega \frac{a_i(x,\nabla u_n) D_i u_n h_m'(u_n)(T_k(u_n) - T_k(u))}{(1+|T_n(u_n)|)^{\gamma_i(x)}} dx}^{I_1}$$
$$+ \overbrace{\sum_{i=1}^N \int_\Omega \frac{a_i(x,\nabla u_n) D_i(T_k(u_n) - T_k(u)) h_m(u_n)}{(1+|T_n(u_n)|)^{\gamma_i(x)}} dx}^{I_2}$$
$$\leq \overbrace{\int_\Omega f_n h_m(u_n)(T_k(u_n) - T_k(u))dx}^{I_3}.$$

Hereafter, we denote $\omega(n,m)$ for all quantities, possibly different, such that $\lim\limits_{m\to+\infty}\lim\limits_{n\to+\infty} \omega(n,m) = 0$. That is to say, in the limit process for $\omega(n,m)$, first let $n \to +\infty$ for fixed m, then let m tends to infinity. Similarly, the notation $\omega(n)$ represents all quantities, maybe different, such that $\lim\limits_{n\to+\infty} \omega(n) = 0$. Our aim is to prove that, for all $k > 0$ and for each $i = 1, \ldots, N$

$$\lim_{n\to+\infty} \int_\Omega [a_i(x,\nabla T_k(u_n)) - a_i(x,\nabla T_k(u))]\, D_i(T_k(u_n) - T_k(u))dx = 0. \tag{5.6}$$

First of all, using the fact that $h_m(u_n)\,(T_k(u_n) - T_k(u)) \rightharpoonup 0$ weakly* in $L^\infty(\Omega)$ and $f_n \to f$ strongly in $L^1(\Omega)$, we get

$$I_3 = \omega(n).$$

Next, we estimate I_1. If we take $v = u_n - \varphi_m(u_n)$ as a test function in (4.1), the almost everywhere convergence of u_n to u implies $\varphi_m(u_n) \to \varphi_m(u)$ as $n \to +\infty$, and $\varphi_m(u_n) \to 0$ as $m \to +\infty$. Thanks to (1.4), we get

$$\begin{aligned} &\alpha \sum_{i=1}^N \int_\Omega \frac{|D_i \varphi_m(u_n)|^{p_i(x)}}{(1+|T_n(u_n)|)^{\gamma_i(x)}} dx \leq \int_\Omega f_n \varphi_m(u_n) dx \\ &= \int_\Omega (f_n - f)\varphi_m(u_n)dx + \int_\Omega f(\varphi_m(u_n) - \varphi_m(u))dx + \int_\Omega f\varphi_m(u)dx \\ &= \omega(n,m). \end{aligned} \tag{5.7}$$

Using Young's inequality, We employ (1.3) and (5.7) to obtain

$$\begin{aligned} |I_1| &\leq \sum_{i=1}^{N}\int_{\Omega}\frac{|h'_m(u_n)|\,|D_iu_n|^{p_i(x)}\,|T_k(u_n)-T_k(u)|}{(1+|T_n(u_n)|)^{\gamma_i(x)}}dx \\ &\leq 2k\int_{\Omega}\frac{|D_i\varphi_m(u_n)|^{p_i(x)}}{(1+|T_n(u_n)|)^{\gamma_i(x)}}dx \\ &= \omega(n,m). \end{aligned}$$

We are left with the estimate of I_2 which can be split as follows

$$\begin{aligned} I_2 = &\overbrace{\sum_{i=1}^{N}\int_{\Omega}\frac{a_i(x,\nabla T_k(u_n))D_i(T_k(u_n)-T_k(u))h_m(u_n)}{(1+|T_n(u_n)|)^{\gamma_i(x)}}dx}^{J_1} \\ &-\overbrace{\sum_{i=1}^{N}\int_{\{|u_n|>k\}}\frac{a_i(x,\nabla u_n)D_iT_k(u)h_m(u_n)}{(1+|T_n(u_n)|)^{\gamma_i(x)}}dx}^{J_2}. \end{aligned}$$

Let $n>m+1$. Since h_m has a compact support, and the integral J_2 is taken on the subset $\{|u_n|\leq m+1\}$, then J_2 can be written as follows

$$J_2 = \sum_{i=1}^{N}\int_{\Omega}\frac{a_i(x,\nabla T_{m+1}(u_n))D_iT_k(u)h_m(u_n)\chi_{\{|u_n|>k\}}}{(1+|T_n(u_n)|)^{\gamma_i(x)}}dx.$$

From (5.4), we deduce that, for each $i=1,\dots,N$, the sequence

$$\left\{\frac{a_i(x,\nabla T_{m+1}(u_n))h_m(u_n)}{(1+|T_n(u_n)|)^{\gamma_i(x)}}\right\}_n,$$

is weakly converges in $L^{p'_i(\cdot)}(\Omega)$. On the other hand, $D_iT_k(u)\chi_{\{|u_n|>k\}}$ strongly converges to zero in $L^{p_i(\cdot)}(\Omega)$. It follows that

$$J_2 = \omega(n).$$

Noting that, for $n>m+1>m>k$, $h_m(u_n)=1$ on the set $\{|u_n|\leq k\}$, we have that J_1 simplified to

$$\begin{aligned} J_1 &= \sum_{i=1}^{N}\int_{\Omega}\frac{a_i(x,\nabla T_k(u_n))D_i(T_k(u_n)-T_k(u))}{(1+|T_n(u_n)|)^{\gamma_i(x)}}dx \\ &= \overbrace{\sum_{i=1}^{N}\int_{\{|u_n|\leq k\}}\frac{[a_i(x,\nabla T_k(u_n))-a_i(x,\nabla T_k(u))]\,D_i(T_k(u_n)-T_k(u))}{(1+|T_n(u_n)|)^{\gamma_i(x)}}dx}^{K_1} \\ &+\overbrace{\sum_{i=1}^{N}\int_{\Omega}\frac{a_i(x,\nabla T_k(u))D_i(T_k(u_n)-T_k(u))\chi_{\{|u_n|\leq k\}}}{(1+|T_n(u_n)|)^{\gamma_i(x)}}dx}^{K_2}. \end{aligned}$$

In view of (1.3) and using Lebesgue's dominated convergence theorem, we conclude that

$$\left\{\frac{a_i(x,\nabla T_k(u))}{(1+|T_n(u_n)|)^{\gamma_i(x)}}\right\}_n,$$

which converges strongly in $L^{p_i'(\cdot)}(\Omega)$. By (5.4), we also deduce that $K_2=\omega(n)$.

Based on the previous estimates, by (1.5) and the fact that the integral K_1 is taken on the subset $\{|u_n|\leq k\}$, we finally get

$$\omega(n,m)=K_1\geq\frac{1}{(1+k)^{\gamma_+^+}}\sum_{i=1}^N\int_\Omega[a_i(x,\nabla T_k(u_n))-a_i(x,\nabla T_k(u))]D_i(T_k(u_n)-T_k(u))dx\geq 0.$$

Therefore, (5.6) is proved. Under the assumptions (1.3)-(1.5), it is well known that (5.6) implies

$$T_k(u_n)\to T_k(u)\text{ strongly in }W_0^{1,\vec{p}(\cdot)}(\Omega)\text{ for all }k>0.$$

This affirms that, for each $i=1,\ldots,N$, $D_iu_n\to D_iu$ a.e. in Ω. □

6. The Proof of the Main Results

Let u_n be a solution of (4.1), and let $v\in W_0^{1,\vec{p}(\cdot)}(\Omega)\cap L^\infty(\Omega)$ with $v(x)\geq\psi(x)$ in Ω. For a fixed $k>0$, the function

$$u_n-T_k(u_n-v),$$

is an admissible test function in (4.1). With this choice of the test function, we get

$$\sum_{i=1}^N\int_\Omega\frac{a_i(x,\nabla u_n)D_iT_k(u_n-v)}{(1+|T_n(u_n)|)^{\gamma_i(x)}}dx\leq\int_\Omega f_nT_k(u_n-v)dx. \tag{6.1}$$

On the set $\{x\in\Omega;\ |u_n-v|<k\}$, one has $|u_n|\leq h=k+\|v\|_{L^\infty(\Omega)}$. Therefore (6.1) can be written as

$$\begin{aligned}\sum_{i=1}^N\int_\Omega\chi_n\frac{a_i(x,\nabla u_n)D_iu_n}{(1+|T_n(u_n)|)^{\gamma_i(x)}}dx&+\sum_{i=1}^N\int_\Omega A_i(x,u_n,\nabla u_n)D_ivdx\\&\leq\int_\Omega f_nT_k(u_n-v)dx,\end{aligned}\tag{6.2}$$

where $\chi_n(x)=\chi_{\{|u_n-v|<k\}}(x)$, and

$$A_i(x,u_n,\nabla u_n)=\frac{a_i(x,\nabla T_h(u_n))}{(1+|T_n(u_n)|)^{\gamma_i(x)}}\chi_n.$$

Now, let us pass to the limit in (6.2). In the right-hand, it is easy since f_n converges strongly to f in $L^1(\Omega)$ and $T_k(u_n-v)$ converges to $T_k(u-v)$ weakly$*$ in $L^\infty(\Omega)$. As for the first term on the left-hand side, we have, by using Fatou's lemma, that

$$\int_\Omega\chi\frac{a_i(x,\nabla u)D_iu}{(1+|u|)^{\gamma_i(x)}}dx\leq\liminf_{n\to+\infty}\int_\Omega\chi_n\frac{a_i(x,\nabla u_n)D_iu_n}{(1+|T_n(u_n)|)^{\gamma_i(x)}}dx,$$

where $\chi(x) = \chi_{\{|u-v|<k\}}(x)$. In the second term of the left-hand side, we have from (1.3) that the boundedness of sequence $T_h(u_n)$ in $W_0^{1,\overrightarrow{p}}(\Omega)$, and the almost everywhere convergence of ∇u_n in Ω to ∇u. It follows that, for each $i = 1, \ldots, N$,

$$\|A_i(x,u_n,\nabla u_n)\|_{L^{p_i'(\cdot)}(\Omega)} \le C, \tag{6.3}$$

and

$$A_i(x,u_n,\nabla u_n) \to A_i(x,u,\nabla u) \quad \text{a.e. in } \Omega, \tag{6.4}$$

where $A_i(x,u,\nabla u) = \frac{a_i(x,\nabla T_h(u))}{\left(1+|u|\right)^{\gamma_i(x)}}\chi(x)$. By (6.3), (6.4), and the Vitali's theorem, we can conclude that

$$A_i(x,u_n,\nabla u_n) \to A_i(x,u,\nabla u) \quad \text{weakly in } L^{p_i'(\cdot)}(\Omega).$$

Hence,

$$\int_\Omega A_i(x,u_n,\nabla u_n) D_i v dx \to \int_\Omega A_i(x,u,\nabla u) D_i v dx.$$

Letting n tend to infinity in (6.2) yields

$$\sum_{i=1}^N \int_\Omega \chi \frac{a_i(x,\nabla u) D_i u}{\left(1+|u|\right)^{\gamma_i(x)}} dx + \int_\Omega \chi \frac{a_i(x,\nabla T_h(u)) D_i v}{\left(1+|u|\right)^{\gamma_i(x)}} dx \\ \le \int_\Omega f T_k(u-v) dx.$$

Since $T_h(u) = u$ on the set $\{x \in \Omega;\ |u-v| < k\}$, the above inequality can be simplified to

$$\sum_{i=1}^N \int_\Omega \frac{a_i(x,\nabla u) D_i T_k(u-v)}{\left(1+|u|\right)^{\gamma_i(x)}} dx \le \int_\Omega f T_k(u-v) dx.$$

which is the desired result.

7. Appendix

In this section, we prove the existence of solutions to approximate obstacle problem (4.1) by using the celebrated theorem of abstract variational inequalities involving pseudomonotone operators. First, we recall the following definition.

Definition 7.1 ([18])**.** Let V be a separable and reflexive Banach space. An operator $A : V \to V'$ is pseudomonotone if

(1) A is bounded, that is, the image of a bounded subset of V is a bounded subset of V';
(2) if $u_m \rightharpoonup u$ weakly in V and if $\limsup_{m\to+\infty} \langle Au_m, u_m - v\rangle \le 0$, then

$$\liminf_{m\to+\infty} \langle Au_m, u_m - v\rangle \ge \langle Au, u - v\rangle,$$

for every v in V, where $\langle \cdot,\cdot\rangle$ refers to the duality product between V' and V.

We will use the following theorem, which is from [18].

Theorem 7.2. *Let V be a separable and reflexive Banach space, and let K be a nonempty, closed convex subset of V. Let $A : V \to V'$ be the operator, which is pseudomonotone and coercive in the following sense: there exists $v_0 \in K$ such that*

$$\lim_{\|v\| \to +\infty} \frac{\langle Av, v - v_0 \rangle}{\|v\|_V} = +\infty \text{ for } v \in K.$$

Then, for every f in V', there exists u in K such that

$$\forall v \in K : \ \langle Au, u - v \rangle \leq \langle f, u - v \rangle.$$

Lemma 7.3. *Suppose Assumptions (1.2)-(1.5) hold. Then approximate problem (4.1) admits at least one solution $u_n \in K_\psi$.*

Proof. We apply Theorem 7.2 to the operator

$$A_n : W_0^{1,\overrightarrow{p}(\cdot)}(\Omega) \longrightarrow \left(W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)\right)'$$
$$u \mapsto A_n u,$$

defined by

$$\langle A_n u, \varphi \rangle = \sum_{i=1}^{N} \int_\Omega \frac{a_i(x, \nabla u) D_i \varphi}{(1 + |T_n(u)|)^{\gamma_i(x)}} dx,$$

for any $\varphi \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$.

The proof is based on the assertion that the operator A_n is pseudomonotone and coercive.

(1) **The operator A_n is coercive**. Let $v_0 \in K_\psi$. Thanks to (1.3) and (1.4), we have, for any $v \in K_\psi$,

$$\begin{aligned}
\langle A_n v, v - v_0 \rangle &= \sum_{i=1}^{N} \int_\Omega \frac{a_i(x, \nabla v) D_i (v - v_0)}{(1 + |T_n(v)|)^{\gamma_i(x)}} dx \\
&= \sum_{i=1}^{N} \int_\Omega \frac{a_i(x, \nabla v) D_i v}{(1 + |T_n(v)|)^{\gamma_i(x)}} dx - \sum_{i=1}^{N} \int_\Omega \frac{a_i(x, \nabla v) D_i v_0}{(1 + |T_n(v)|)^{\gamma_i(x)}} dx \\
&\geq \frac{\alpha}{(1+n)^{\gamma_+^+}} \sum_{i=1}^{N} \int_\Omega |D_i v|^{p_i(x)} dx - \beta \sum_{i=1}^{N} \int_\Omega |D_i v|^{p_i(x)-1} |D_i v_0| dx.
\end{aligned}$$

By using Young's inequality, we get

$$\langle A_n v, v - v_0 \rangle \geq \left(\frac{\alpha}{(1+n)^{\gamma_+^+}} - \varepsilon\beta \right) \sum_{i=1}^{N} \int_\Omega |D_i v|^{p_i(x)} dx - \beta C(\varepsilon) \sum_{i=1}^{N} |D_i v_0|^{p_i(x)} dx.$$

Choosing ε such that $\frac{\alpha}{(1+n)^{\gamma_+^+}} - \beta\varepsilon = C_1 > 0$ and $\beta C(\varepsilon) = C_2 > 0$, we get

$$\langle A_n v, v - v_0 \rangle \geq \frac{C_1}{N^{p_-^- - 1}} \|v\|^{p_-^-}_{W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)} - N(C_1 + C_2) + C_2 \|v_0\|^{p_+^+}_{W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)},$$

which implies

$$\frac{\langle A_n v, v - v_0 \rangle}{\|v\|_{W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)}} \longrightarrow +\infty$$

as $\|v\|_{W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)} \longrightarrow +\infty$ since $p_-^- > 1$.

(2) **The operator A_n is pseudomonotone.**

(i) A_n **is bounded.** Indeed, let u be a bounded function in $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$. In view of (1.3), for all $v \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$, we have

$$|\langle A_n u, v\rangle| \quad \leq \quad \beta \sum_{i=1}^{N} \int_{\Omega} \left| |D_i u|^{p_i(x)-2} D_i u D_i v \right| dx.$$

From this and Hölder's inequality (2.1), it follows that

$$\begin{aligned}|\langle A_n u, v\rangle| &\leq C \sum_{i=1}^{N} \| \, |D_i u|^{p_i(\cdot)-1} \, \|_{L^{p_i'(\cdot)}(\Omega)} \|D_i v\|_{L^{p_i(\cdot)}(\Omega)} \\ &\leq C\|v\|_{W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)}.\end{aligned}$$

Therefore, we conclude that $\|A_n u\|_{\left(W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)\right)'}$ is bounded if $\|u\|_{W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)}$ is bounded.

(ii) If $u_m \rightharpoonup u$ weakly in $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$, as $m \to +\infty$, and for any $v \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$

$$\begin{aligned}0 &\geq \limsup_{m\to+\infty} \langle A_n u_m, u_m - v\rangle \\ &= \limsup_{m\to+\infty} \sum_{i=1}^{N} \int_{\Omega} \frac{a_i(x, \nabla u_m) D_i(u_m - v)}{(1+|T_n(u_m)|)^{\gamma_i(x)}} dx,\end{aligned}$$

then

$$\liminf_{m\to+\infty} \langle A_n u_m, u_m - v\rangle \geq \langle A_n u, u - v\rangle.$$

Indeed, the compact embedding (2.6) yields that $u_m \to u$ in $L^1(\Omega)$ for a subsequence still denoted as (u_m). Moreover, we assume that $u_m \to u$ a.e. in Ω.
Let us first prove that

$$\sum_{i=1}^{N} \int_{\Omega} \left[\frac{a_i(x, \nabla u_m)}{(1+|T_n(u_m)|)^{\gamma_i(x)}} - \frac{a_i(x, \nabla u)}{(1+|T_n(u)|)^{\gamma_i(x)}} \right] D_i(u_m - u) dx \xrightarrow{m\to+\infty} 0. \tag{7.1}$$

By invoking (1.3) and using the fact that $D_i u_n \rightharpoonup D_i u$ weakly in $L^{p_i(\cdot)}(\Omega)$, we get

$$\limsup_{m\to+\infty} \sum_{i=1}^{N} \int_{\Omega} \left[\frac{a_i(x, \nabla u_m)}{(1+|T_n(u_m)|)^{\gamma_i(x)}} - \frac{a_i(x, \nabla u)}{(1+|T_n(u)|)^{\gamma_i(x)}} \right] D_i(u_m - u) dx \leq 0.$$

We have, for all $i = 1, \ldots, N$,

$$
\begin{aligned}
&\int_\Omega \left[\frac{a_i(x,\nabla u_m)}{(1+|T_n(u_m)|)^{\gamma_i(x)}} - \frac{a_i(x,\nabla u)}{(1+|T_n(u)|)^{\gamma_i(x)}}\right] D_i(u_m-u)dx \\
&= \int_\Omega \frac{1}{(1+|T_n(u_m)|)^{\gamma_i(x)}} \left[a_i(x,\nabla u_m) - a_i(x,\nabla u)\right] D_i(u_m-u)dx \\
&\quad + \int_\Omega \left[\frac{1}{(1+|T_n(u_m)|)^{\gamma_i(x)}} - \frac{1}{(1+|T_n(u)|)^{\gamma_i(x)}}\right] a_i(x,\nabla u) D_i(u_m-u)dx \\
&\geq \int_\Omega \left[\frac{1}{(1+|T_n(u_m)|)^{\gamma_i(x)}} - \frac{1}{(1+|T_n(u)|)^{\gamma_i(x)}}\right] a_i(x,\nabla u) D_i(u_m-u)dx.
\end{aligned} \tag{7.2}
$$

Using (1.3) and Lebesgue's dominated convergence theorem, we obtain

$$
\frac{a_i(x,\nabla u)}{(1+|T_n(u_m)|)^{\gamma_i(x)}} \to \frac{a_i(x,\nabla u)}{(1+|T_n(u)|)^{\gamma_i(x)}},
$$

strongly in $L^{p_i'(\cdot)}(\Omega)$. We have thus proved (7.1). This and (7.2) imply that, for each $i = 1, \ldots, N$,

$$
\int_\Omega \frac{\left[a_i(x,\nabla u_m) - a_i(x,\nabla u)\right] D_i(u_m-u)}{(1+|T_n(u_m)|)^{\gamma_i(x)}} dx \xrightarrow{m\to+\infty} 0. \tag{7.3}
$$

Now, let us prove that

$$
\liminf_{m\to+\infty} \langle A_n u_m, u_m - v\rangle \geq \langle A_n u, u - v\rangle,\ \forall v \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega).
$$

From (7.3), we deduce up to a subsequence

$$
\nabla u_m \to \nabla u \text{ a.e in } \Omega.
$$

Therefore, for each $i = 1, \ldots, N$,

$$
\frac{a_i(x,\nabla u_m)}{(1+|T_n(u_m)|)^{\gamma_i(x)}} \rightharpoonup \frac{a_i(x,\nabla u)}{(1+|T_n(u)|)^{\gamma_i(x)}},
$$

weakly in $L^{p_i'(\cdot)}(\Omega)$ and a.e in Ω. Thus

$$
\lim_{m\to+\infty} \int_\Omega \frac{a_i(x,\nabla u_m) D_i v}{(1+|T_n(u_m)|)^{\gamma_i(x)}} dx = \int_\Omega \frac{a_i(x,\nabla u) D_i v}{(1+|T_n(u)|)^{\gamma_i(x)}} dx, \tag{7.4}
$$

for all $v \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$. By virtue of Fatou's lemma, we get

$$
\liminf_{m\to+\infty} \int_\Omega \frac{a_i(x,\nabla u_m) D_i u_m}{(1+|T_n(u_m)|)^{\gamma_i(x)}} dx \geq \int_\Omega \frac{a_i(x,\nabla u) D_i u}{(1+|T_n(u)|)^{\gamma_i(x)}} dx. \tag{7.5}
$$

Finally, combining (7.4) and (7.5), we obtain

$$\begin{aligned}\liminf_{m\to+\infty}\langle A_n u_m, u_m - v\rangle &= \liminf_{m\to+\infty}\sum_{i=1}^{N}\int_{\Omega}\frac{a_i(x,\nabla u_m)D_i(u_m-v)}{(1+|T_n(u_m)|)^{\gamma_i(x)}}dx\\ &\geq \sum_{i=1}^{N}\int_{\Omega}\frac{a_i(x,\nabla u)D_i(u-v)}{(1+|T_n(u)|)^{\gamma_i(x)}}dx\\ &= \langle A_n u, u-v\rangle.\end{aligned}$$

Therefore A_n is pseudomonotone. Then, according to Theorem 7.2, there exists at least one solution $u_n \in K_\psi$ to Problem (4.1).

□

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