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METRIC SPACES WITH ASYMPTOTIC PROPERTY C AND FINITE DECOMPOSITION COMPLEXITY

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Abstract. We construct a class of metric spaces $X_{\omega+k}$ whose transfinite asymptotic dimension and complementary-finite asymptotic dimension are both $\omega+k$ for any $k\in\mathbb{N}$, where ω is the smallest infinite ordinal number and a metric space $Y_{2\omega}$ whose transfinite asymptotic dimension and complementary-finite asymptotic dimension are both 2ω. Finally, we introduce a geometric property called decomposition dimension (decodim). Using decomposition dimension, we prove that the metric spaces $X_{\omega+k}$ and $Y_{2\omega}$ have finite decomposition complexity.

Keywords. Asymptotic property C; Complementary-finite asymptotic dimension; Finite decomposition complexity; Transfinite asymptotic dimension;.

1. Introduction

The asymptotic dimension is a coarse invariant of metric spaces introduced by Gromov [1]. Recently, it has received a great deal of attention; see, e.g., [2, 3, 4, 5] and the references therein. Generalizing finite asymptotic dimension, Dranishnikov introduced the notion of the asymptotic property C in [6], which covers a large family of metric spaces with the infinite asymptotic dimension. In 2010, Radul generalized the asymptotic dimension of a metric space X to the transfinite asymptotic dimension, which is denoted by $\mathrm{trasdim}(X)$, and proved that, for a metric space X, X has the asymptotic property C is equivalent to $\mathrm{trasdim}(X)<\infty$ [7]. There are examples of the metric space with trasdim$=\infty$ and the metric space with trasdim$=\omega$, where ω is the smallest infinite ordinal number [7]. But, if there is a metric space X with $\omega<\mathrm{trasdim}(X)<\infty$ was unknown until we constructed a metric space whose transfinite asymptotic dimension is $\omega+1$ [8]. In [9], we introduced another approach to classify the metric spaces with the infinite asymptotic dimension, which is called the complementary-finite asymptotic dimension (coasdim).

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Inspired by the method in [8], we construct a class of metric spaces $X_{\omega+k}$ with

$$\operatorname{trasdim}(X_{\omega+k}) = \operatorname{coasdim}(X_{\omega+k}) = \omega + k \text{ for any } k \in \mathbb{N},$$

which generalizes the result in [8]. We use these metric spaces to construct a metric space $Y_{2\omega}$ with $\operatorname{trasdim}(Y_{2\omega}) = \operatorname{coasdim}(Y_{2\omega}) = 2\omega$.

In 2012, Guentner, Tessera and Yu [10] introduced the notion of finite decomposition complexity to study the topological rigidity of manifolds, and they proved that every metric space of bounded geometry with finite decomposition complexity has property A [11]. Dranishnikov proved that every metric space of bounded geometry with asymptotic property C has property A [6]. The relation between asymptotic property C and finite decomposition complexity was studied by Dranishnikov and Zarichnyi [5]. There is no example of group known which makes a difference between asymptotic property C and the finite decomposition complexity. Here we prove that $X_{\omega+k} \in \mathscr{D}_{\omega+k}$, and $Y_{2\omega} \in \mathscr{D}_{2\omega}$. Therefore, $X_{\omega+k}$ and $Y_{2\omega}$ have both asymptotic property C, and the finite decomposition complexity.

2. Preliminaries

Let (X,d) be a metric space, and $U,V \subseteq X$. Define

$$\operatorname{diam} U = \sup\{d(x,y) | x,y \in U\} \text{ and } d(U,V) = \inf\{d(x,y) | x \in U, y \in V\}.$$

Let R and r be positive numbers, and let $\mathscr{U}$ be a family of subsets of X. We say that $\mathscr{U}$ is *R-bounded* if

$$\operatorname{diam} \mathscr{U} \doteq \sup\{\operatorname{diam} U \mid U \in \mathscr{U}\} \leq R.$$

In this case, $\mathscr{U}$ is said to be *uniformly bounded*. We say that $\mathscr{U}$ is *r-disjoint* if

$$d(U,V) \geq r \text{ for every } U,V \in \mathscr{U} \text{ with } U \neq V.$$

Let A be a subset of X and $\varepsilon > 0$. Define

$$N_\varepsilon(A) = \{x \in X \mid d(x,A) < \varepsilon\} \text{ and } \overline{N_\varepsilon(A)} = \{x \in X \mid d(x,A) \leq \varepsilon\}.$$

Letting $\delta > 0$, we denote

$$N_\delta(\mathscr{U}) = \{N_\delta(U) \mid U \in \mathscr{U}\},$$

and

$$\bigcup \mathscr{U} = \bigcup\{U \mid U \in \mathscr{U}\} \text{ and } \mathscr{U}_1 \cup \mathscr{U}_2 = \{U \mid U \in \mathscr{U}_1 \text{ or } U \in \mathscr{U}_2\}.$$

Definition 2.1. [4] The asymptotic dimension of a metric space X does not exceed n and write $\operatorname{asdim}(X) \leq n$ if there exists $n \in \mathbb{N}$ such that, for every $r > 0$, there exists a sequence of uniformly bounded families $\{\mathscr{U}_i\}_{i=0}^n$ of subsets of X such that $\bigcup_{i=0}^n \mathscr{U}_i$ covers X, and each $\mathscr{U}_i$ is r-disjoint for $i = 0,1,\cdots,n$. In this case, we say that X has *finite asymptotic dimension*.

We say that $\operatorname{asdim}(X) = n$ if $\operatorname{asdim}(X) \leq n$ and $\operatorname{asdim}(X) \leq n-1$ is not true.

We say that $\operatorname{asdim}(X) < \infty$ if $\operatorname{asdim}(X) \leq m$ for some $m \in \mathbb{N}$, and $\operatorname{asdim}(X) = \infty$ if $\operatorname{asdim}(X) \leq n$ is not true for any $n \in \mathbb{N}$.

Definition 2.2. [7] Let $Fin\mathbb{N}$ be the collection of all finite, nonempty subsets of $\mathbb{N}$, and let $M \subseteq Fin\mathbb{N}$. For $\sigma \in \{\varnothing\} \bigcup Fin\mathbb{N}$, let

$$M^\sigma = \{\tau \in Fin\mathbb{N} \mid \tau \cup \sigma \in M \text{ and } \tau \cap \sigma = \varnothing\}.$$

Let M^a abbreviate $M^{\{a\}}$ for $a \in \mathbb{N}$. Define the ordinal number Ord$M$ inductively as follows:

$$\begin{aligned}
\mathrm{Ord}M = 0 &\quad\Leftrightarrow\quad M = \varnothing,\\
\mathrm{Ord}M \le \alpha &\quad\Leftrightarrow\quad \forall\, a \in \mathbb{N},\ \text{there exists } \beta < \alpha,\ \text{ such that } \mathrm{Ord}M^a \le \beta,\\
\mathrm{Ord}M = \alpha &\quad\Leftrightarrow\quad \mathrm{Ord}M \le \alpha \text{ and } \mathrm{Ord}M \le \beta \text{ is not true for any } \beta < \alpha,\\
\mathrm{Ord}M = \infty &\quad\Leftrightarrow\quad \mathrm{Ord}M \le \alpha \text{ is not true for any ordinal number } \alpha.
\end{aligned}$$

Definition 2.3. [7] Given a metric space X, let

$$\begin{aligned}
A(X) = \{\sigma \in Fin\mathbb{N} \mid\ &\text{there are no uniformly bounded families } \mathscr{U}_i \text{ for } i \in \sigma\\
&\text{such that each } \mathscr{U}_i \text{ is } i\text{-disjoint and } \bigcup_{i\in\sigma} \mathscr{U}_i \text{ covers } X\}.
\end{aligned}$$

The *transfinite asymptotic dimension* of X is defined as trasdim(X)=Ord$A(X)$.

Remark 2.4. Note that trasdim$(X) \le n$ if and only if asdim$(X) \le n$.

Lemma 2.5. *(see* [9]*, Proposition 2.1) Let X be a metric space, and let $l \in \mathbb{N}\cup\{0\}$. Then the following conditions are equivalent:*

1. *trasdim$(X) \le \omega + l$;*
2. *for every $k \in \mathbb{N}$, there exists $m = m(k) \in \mathbb{N}$ such that for every $n \in \mathbb{N}$, there are uniformly bounded families $\mathscr{U}_{-l}, \mathscr{U}_{-l+1}, \cdots, \mathscr{U}_m$ satisfying $\mathscr{U}_i$ is k-disjoint for $i \in \{-l, \cdots, 0\}$, $\mathscr{U}_j$ is n-disjoint for $j \in \{1, 2, \cdots, m\}$ and $\bigcup_{i=-l}^m \mathscr{U}_i$ covers X.*

Definition 2.6. [9] Every ordinal number γ can be represented as $\gamma = \lambda(\gamma) + n(\gamma)$, where $\lambda(\gamma)$ is the limit ordinal or 0 and $n(\gamma) \in \mathbb{N}\cup\{0\}$. Let X be a metric space, we define *complementary-finite asymptotic dimension* coasdim(X) inductively as follows:

- coasdim$(X) = -1$ if and only if $X = \emptyset$,
- coasdim$(X) \le \gamma$ if and only if for every $r > 0$ there exist r-disjoint and uniformly bounded families $\mathscr{U}_0, \ldots, \mathscr{U}_{n(\gamma)}$ of subsets of X such that coasdim$(X \setminus \bigcup(\bigcup_{i=0}^{n(\gamma)} \mathscr{U}_i)) \le \alpha$ for some $\alpha < \lambda(\gamma)$,
- coasdim$(X) = \gamma$ if and only if coasdim$(X) \le \gamma$ and for any $\beta < \gamma$, coasdim$(X) \le \beta$ is not true.

Remark 2.7. Note that coasdim$(X) \le n$ if and only if asdim$(X) \le n$.

Lemma 2.8. *(see* [9]*, Theorem 3.3) Let X be a metric space with $X_1, X_2 \subseteq X$. Then coasdim$(X_1 \cup X_2) \le max\{coasdim(X_1), coasdim(X_2)\}$.*

Definition 2.9. Let X and Y be metric spaces. $f : X \to Y$ is said to be *coarse embedding* if there are nondecreasing functions $p_1, p_2 : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\lim_{x\to+\infty} p_1(x) = +\infty$ and for every $x_1, x_2 \in X$,

$$p_1(d(x_1,x_2)) \le d(f(x_1), f(x_2)) \le p_2(d(x_1,x_2)).$$

If, additionally, there exists $R > 0$ such that $Y \subseteq N_R(f[X])$, then f is called *coarse equivalence* and metric spaces X and Y are said to be coarse equivalent. Coarse equivalence is an equivalence relation.

Transfinite asymptotic dimension and complementary-finite asymptotic dimension are coarsely invariant properties of metric spaces.

Lemma 2.10. *(see [9, 12]) Let X and Y be metric spaces, and let ξ be an countable ordinal number. There is a coarse embedding $\phi : X \to Y$ from X to Y. If coasdim$(Y) \leq \xi$ and trasdim$(Y) \leq \xi$, then*

$$\text{coasdim}(X) \leq \text{coasdim}(Y) \text{ and } \text{trasdim}(X) \leq \text{trasdim}(Y).$$

Consequently, if X and Y are coarse equivalent, coasdim$(Y) \leq \xi$, and trasdim$(Y) \leq \xi$, then

$$\text{trasdim}\, X = \text{trasdim}\, Y \text{ and } \text{coasdim}\, X = \text{coasdim}\, Y.$$

3. Main Results

3.1. **A metric space with transfinite asymptotic dimension $\omega + k$.** In this section, we will construct a metric space $X_{\omega+k}$, and prove that trasdim$(X_{\omega+k}) = \omega + k$ for every $k \in \mathbb{N}$, which are inspired by the method in [8].

Definition 3.1. ([13]) Let X be a metric space, and let A, B be a pair of disjoint subsets of X. We say that a subset $L \subset X$ is a *partition* of X between A and B if there exist open sets $U, W \subset X$ satisfying the following conditions

$$A \subset U, B \subset W \text{ and } X = U \sqcup L \sqcup W.$$

Definition 3.2. Let X be a metric space, and let A, B be a pair of disjoint subsets of X. For any $\varepsilon > 0$, we say that a subset $L \subset X$ is an ε-*partition* of X between A and B if there exist open sets $U, W \subset X$ satisfying the following conditions

$$A \subset U, B \subset W, X = U \sqcup L \sqcup W, d(L, A) > \varepsilon \text{ and } d(L, B) > \varepsilon.$$

Clearly, an ε-partition L of X between A and B is a partition of X between A and B.

Lemma 3.3. *([14]) Let $L_0 \doteq [0, B]^n$ for some $B > 0$. Let F_i^+ and F_i^- be the pairs of opposite faces of L_0, where $i = 1, 2, \cdots, n$, and let $0 < \varepsilon < \frac{1}{6}B$. For $k = 1, 2, \cdots, n$, let $\mathscr{U}_k$ be an ε-disjoint and $\frac{1}{3}B$-bounded family of subsets of $[0, B]^n$. Then there exists a decreasing sequence of closed sets $[0, B]^n = L_0 \supset L_1 \supset L_2 \cdots \supset L_n$ such that L_{k+1} is an ε-partition of L_k between $F_{k+1}^+ \cap L_k$ and $F_{k+1}^- \cap L_k$, and $L_{k+1} \subseteq L_k \cap (\bigcup \mathscr{U}_{k+1})^c$ for $k = 0, 1, 2, \cdots, n-1$.*

To prove the main results, we will use a version of the Lebesgue theorem.

Lemma 3.4. *(see [13], Lemma 1.8.19) Let F_i^+, F_i^-, where $i \in \{1, \ldots, n\}$, be the pairs of opposite faces of $I^n \doteq [0, 1]^n$. If $I^n = L'_0 \supset L'_1 \supset \ldots \supset L'_n$ is a decreasing sequence of closed sets such that L'_i is a partition of L'_{i-1} between $L'_{i-1} \cap F_i^+$ and $L'_{i-1} \cap F_i^-$ for $i \in \{1, 2, \ldots, n\}$, then $L'_n \neq \emptyset$.*

As it is similar with [8], we will use the asymptotic union to construct examples.

Definition 3.5. Let $\{Z_i\}_{i=1}^{\infty}$ be a sequence of subspaces of a metric space (Z, d_Z). Let

$$X = \bigsqcup_{i=1}^{\infty} (0, \cdots, 0, Z_i, 0, \cdots).$$

For every $x, y \in X$, there exist unique $l, k \in \mathbb{N}$, $x_l \in Z_l$ and $y_k \in Z_k$ such that $x = (0, \cdots, 0, x_l, 0, \cdots)$ and $y = (0, \cdots, 0, y_k, 0, \cdots)$. Assume that $l \leq k$, put $c = 0$ if $l = k$, and $c = l + (l+1) + \ldots + (k-1)$ if $l < k$. Define a metric on X by

$$d(x, y) = d_Z(x_l, y_k) + c.$$

We say that (X,d) is *asymptotic union* of $\{Z_i\}_{i=1}^{\infty}$, which is denoted by as$\bigsqcup_{i=1}^{\infty} Z_i$. And we denote $as\bigsqcup_{i=n}^{\infty} Z_i$ as a subspace of $as\bigsqcup_{i=1}^{\infty} Z_i$.

For every $k,i\in\mathbb{N}$, let

$$X_{\omega+k}^{(i)}=\{(x_1,\dots,x_i)\in\mathbb{R}^i \big| \, |\{j \mid x_j\notin 2^i\mathbb{Z}\}|\le k\}.$$

Note that $X_{\omega+k}^{(i)}\subset\mathbb{R}^i$ for each $i\in\mathbb{N}$. Let $X_{\omega+k}=\text{as}\bigsqcup_{i=1}^{\infty} X_{\omega+k}^{(i)}$, where $X_{\omega+k}^{(i)}$ is a subspace of the metric space $(\bigoplus\mathbb{R},d_{\max})$ for each $i\in\mathbb{N}$, and $d_{\max}$ is the maximum metric.

Proposition 3.6. *For any $k\in\mathbb{N}$, trasdim$(X_{\omega+k})\le\omega+k-1$ is not true.*

Proof. Suppose that trasdim$(X_{\omega+k})\le\omega+k-1$. By Lemma 2.5, for every $n\in\mathbb{N}$, there exists $m=m(n)\in\mathbb{N}$ such that there exist B-bounded families $\mathscr{U}_{-k+1},\mathscr{U}_{-k+2},\dots,\mathscr{U}_{m-1},\mathscr{U}_m$ satisfying $\mathscr{U}_i$ is n-disjoint for $i=-k+1,\dots,0$, $\mathscr{U}_j$ is 2^{m+k+2}-disjoint for $j=1,2,\dots,m$ and $\bigcup_{i=-k+1}^{m}\mathscr{U}_i$ covers $X_{\omega+k}$ and hence covers $[0,6B]^{m+k}\cap X_{\omega+k}^{(m+k)}$. Without lose of generality, we can assume $B=B(n)>\max\{n,2^{m+k+2}\}$.

We assume that $p=\frac{6B}{2^{m+k}}\in\mathbb{N}$. Taking a bijection $\psi:\{1,2,\cdots,p^{m+k}\}\to\{0,1,2,\cdots,p-1\}^{m+k}$, let

$$Q(t)=\prod_{j=1}^{m+k}[2^{m+k}\psi(t)_j,2^{m+k}(\psi(t)_j+1)],\text{in which } \psi(t)_j \text{ is the } j\text{th coordinate of } \psi(t).$$

Let $\mathscr{Q}=\{Q(t)\mid t\in\{1,2,\cdots,p^{m+k}\}\}$. Then, $[0,6B]^{m+k}=\bigcup_{Q\in\mathscr{Q}}Q$. Note that

$$[0,6B]^{m+k}\cap X_{\omega+k}^{(m+k)}=\bigcup_{Q\in\mathscr{Q}}\partial_k Q,$$

where $\partial_k Q$ is the k-dimensional skeleton of Q.

Let $L_0=[0,6B]^{m+k}$. By Lemma 3.3, since $N_{2^{m+k}}(\mathscr{U}_1)$ is 2^{m+k}-disjoint and $(2^{m+k+1}+B)$-bounded, there exists a 2^{m+k}-partition L_1 of $[0,6B]^{m+k}$ such that

$$L_1\subset(\bigcup N_{2^{m+k}}(\mathscr{U}_1))^c\cap[0,6B]^{m+k},$$

and $d(L_1,F_1^{+/-})>2^{m+k}$. Since L_1 is a 2^{m+k}-partition of $[0,6B]^{m+k}$ between F_1^+ and F_1^-, then $[0,6B]^{m+k}=L_1\sqcup A_1\sqcup B_1$ such that A_1, B_1 are open in $[0,6B]^{m+k}$, and A_1, B_1 contain two opposite facets F_1^-, F_1^+ respectively.

Let $\mathscr{M}_1=\{Q\in\mathscr{Q}|Q\cap L_1\neq\emptyset\}$, and $M_1=\bigcup\mathscr{M}_1$. Since L_1 is a 2^{m+k}-partition of $[0,6B]^{m+k}$ between F_1^+ and F_1^-, then M_1 is a partition of $[0,6B]^{m+k}$ between F_1^+ and F_1^-, i.e., $[0,6B]^{m+k}=M_1\sqcup A_1'\sqcup B_1'$ such that A_1' and B_1' are open in $[0,6B]^{m+k}$, and A_1' and B_1' contain two opposite facets F_1^-, F_1^+, respectively. Let

$$L_1'=\partial_{m+k-1}M_1=\bigcup\{\partial_{m+k-1}Q|Q\in\mathscr{M}_1\}.$$

Then, $[0,6B]^{m+k}\setminus(L_1'\sqcup A_1'\sqcup B_1')$ is the union of some disjoint open $(m+k)$-dimensional cubes with length of edge $=2^{m+k}$. So L_1' is a partition of $[0,6B]^{m+k}$ between F_1^+ and F_1^-, and $L_1'\subset(\bigcup\mathscr{U}_1)^c\cap[0,6B]^{m+k}$.

For $N_{2^{m+k}}(\mathscr{U}_2)$, there exists a 2^{m+k}-partition L_2 of L_1' such that

$$L_2\subset(\bigcup N_{2^{m+k}}(\mathscr{U}_2))^c\cap[0,6B]^{m+k},$$

and $d(L_2, F_2^{+/-}) > 2^{m+k}$. Since L_2 is a 2^{m+k}-partition of L_1' between $L_1' \cap F_2^+$ and $L_1' \cap F_2^-$, then $L_1' = L_2 \sqcup A_2 \sqcup B_2$ such that A_2 and B_2 are open in L_1', and A_2 and B_2 contain two opposite facets $L_1' \cap F_2^-$, $L_1' \cap F_2^+$, respectively, and $d(L_2, F_2^{+/-}) > 2^{m+k}$.

Let $\mathscr{M}_2 = \{Q \in \mathscr{M}_1 \mid Q \cap L_2 \neq \emptyset\}$, and $M_2 = \bigcup \mathscr{M}_2$. Since L_2 is a 2^{m+k}-partition of L_1' between $L_1' \cap F_2^+$ and $L_1' \cap F_2^-$, then $M_2 \cap L_1'$ is a partition of L_1' between $L_1' \cap F_2^+$ and $L_1' \cap F_2^-$, i.e., $L_1' = (M_2 \cap L_1') \sqcup A_2' \sqcup B_2'$ such that A_2', B_2' are open in L_1', and A_2' and B_2' contain two opposite facets $L_1' \cap F_2^-$ and $L_1' \cap F_2^+$, respectively. Let

$$L_2' = \partial_{m+k-2} M_2 \doteq \bigcup \{\partial_{m+k-2} Q | Q \in \mathscr{M}_2\}.$$

Then $L_1' \setminus (L_2' \sqcup A_2' \sqcup B_2')$ is the union of some disjoint open $(m+k-1)$-dimensional cubes with length of edge = 2^{m+k}. So L_2' is also a partition of L_1' between $L_1' \cap F_2^+$ and $L_1' \cap F_2^-$, and $L_2' \subset (\bigcup(\mathscr{U}_1 \cup \mathscr{U}_2))^c \cap [0, 6B]^{m+k}$. After m steps above, we have L_m' to be a partition of L_{m-1}' and $L_m' \subset (\bigcup(\mathscr{U}_1 \cup \ldots \cup \mathscr{U}_m))^c \cap [0, 6B]^{m+k}$. Note that $L_m' \subset X_{\omega+k}^{(m+k)}$ and hence

$$L_m' \subset (\bigcup(\mathscr{U}_{-k+1} \cup \ldots \cup \mathscr{U}_0)) \cap [0, 6B]^{m+k}.$$

For $j = 1, 2, \cdots, k$, there exists a partition L_{m+j}' of L_{m+j-1}' between $L_{m+j-1}' \cap F_{m+j}^+$ and $L_{m+j-1}' \cap F_{m+j}^-$ such that $L_{m+j}' \subseteq L_{m+j-1}' \cap (\bigcup(\mathscr{U}_{-j+1} \cup \ldots \cup \mathscr{U}_m))^c$ due to Lemma 3.3. It follows that

$$L_{m+k}' \subseteq L_{m+k-1}' \cap (\bigcup(\mathscr{U}_{-k+1} \cup \ldots \cup \mathscr{U}_0))^c = \emptyset,$$

which is a contradiction with Lemma 3.4. So $\mathrm{trasdim}(X_{\omega+k}) \leq \omega + k - 1$ is not true. □

Lemma 3.7. *(see [15], Proposition 3.1) Let X be a metric space if $\mathrm{coasdim}(X) \leq \gamma$ for some ordinal number γ, then $\mathrm{trasdim}(X) \leq \gamma$.*

For every $n, i, k \in \mathbb{N}$, let

$$X_{\omega+k}^{(i,n)} = \{(x_1, \ldots, x_i) \in \mathbb{R}^i \mid |\{j \mid x_j \notin 2^n \mathbb{Z}\}| \leq k\}.$$

Note that $X_{\omega+k}^{(i)} = X_{\omega+k}^{(i,i)}$.

Lemma 3.8. *For every $r \in \mathbb{N}$ with $r \geq 4$, there exist $n = r \in \mathbb{N}$ and r-disjoint uniformly bounded families $\mathscr{U}_0, \mathscr{U}_1$ such that $\mathscr{U}_0 \cup \mathscr{U}_1$ covers as $\bigsqcup_{i=n}^{\infty} X_{\omega+1}^{(i,n)}$.*

Proof. For every $r \in \mathbb{N}$ and $r \geq 4$, choose $n = r \in \mathbb{N}$. For every $i \geq n$, let

$$\mathscr{U}_0^{(i)} = \{\big(\prod_{t=1}^{i}(n_t 2^n - r, n_t 2^n + r)\big) \cap X_{\omega+1}^{(i,n)} \mid n_t \in \mathbb{Z}\},$$

and

$$\mathscr{U}_1^{(i)} = \{\big(\prod_{t=1}^{j-1}(n_t 2^n - r, n_t 2^n + r) \times [n_j 2^n + r, (n_j + 1)2^n - r] \times \prod_{t=j+1}^{i}(n_t 2^n - r, n_t 2^n + r)\big) \cap X_{\omega+1}^{(i,n)} \mid n_t \in \mathbb{Z}, 1 \leq j \leq i\}.$$

It is easy to see that $\mathscr{U}_0^{(i)}$ and $\mathscr{U}_1^{(i)}$ are r-disjoint and 2^n-bounded families. Now, for every $x = (x_1, \ldots, x_i) \in X_{\omega+1}^{(i,n)} \setminus (\bigcup \mathscr{U}_0^{(i)})$, there exists unique $j \in \{1, 2, \cdots, i\}$ such that $x_j \in [n_j 2^n + r, (n_j + 1)2^n - r]$. It follows that $x \in \mathscr{U}_1^{(i)}$. Therefore, $\mathscr{U}_0^{(i)} \cup \mathscr{U}_1^{(i)}$ covers $X_{\omega+1}^{(i,n)}$. Let $\mathscr{U}_0 = \bigcup_{i \geq n} \mathscr{U}_0^{(i)}$, and $\mathscr{U}_1=\bigcup_{i\geq n}\mathscr{U}_1^{(i)}$. Since $d(X_{\omega+1}^{(i,n)},X_{\omega+1}^{(j,n)})\geq n=r$ for every $i,j\geq n$ and $i\neq j$, then $\mathscr{U}_0,\mathscr{U}_1$ are r-disjoint and 2^n-bounded families such that $\mathscr{U}_0\cup\mathscr{U}_1$ covers $as\bigsqcup_{i=n}^{\infty}X_{\omega+1}^{(i,n)}$. □

Remark 3.9. By Lemma 3.8, for every $r\in\mathbb{N}$ and $r>1$, there exist $3r$-disjoint and uniformly bounded families $\mathscr{U}_0,\mathscr{U}_1$ such that $\mathscr{U}_0\cup\mathscr{U}_1$ covers $as\bigsqcup_{i=3r}^{\infty}X_{\omega+1}^{(i,3r)}$. Let $\mathscr{V}_0=\{N_r(U)|\ U\in\mathscr{U}_0\},\mathscr{V}_1=\{N_r(U)|\ U\in\mathscr{U}_1\}$. Then $\mathscr{V}_0,\mathscr{V}_1$ are r-disjoint uniformly bounded families, and $\mathscr{V}_0\cup\mathscr{V}_1$ covers $as\bigsqcup_{i=3r}^{\infty}N_r(X_{\omega+1}^{(i,n)})$, where $N_r(X_{\omega+1}^{(i,n)})$ is r-neighborhood of $X_{\omega+1}^{(i,n)}$ in $\mathbb{R}^i$. By the similar argument, we obtain the following Lemma.

Lemma 3.10. *For every $r\in\mathbb{N}$ and $r>1$, there exist $n=3^{k-1}r\in\mathbb{N}$ and r-disjoint uniformly bounded families $\mathscr{U}_0,\mathscr{U}_1,\dots,\mathscr{U}_k$ such that $\mathscr{U}_0\cup\mathscr{U}_1\cup\ldots\cup\mathscr{U}_k$ covers $as\bigsqcup_{i=n}^{\infty}X_{\omega+k}^{(i,n)}$.*

Proof. We will prove it by induction on k. By Lemma 3.8, the result is true for $k=1$. Assume that the result is true for $k=m$. Then, for every $r\in\mathbb{N}$ and $r>1$, there exist $n=3^m r\in\mathbb{N}$ and $3r$-disjoint uniformly bounded families $\mathscr{V}_0,\mathscr{V}_1,\dots,\mathscr{V}_m$ such that $\mathscr{V}_0\cup\mathscr{V}_1\cup\ldots\cup\mathscr{V}_m$ covers $as\bigsqcup_{i=n}^{\infty}X_{\omega+m}^{(i,n)}$. Now, for $k=m+1$, let

$$\mathscr{U}_0=\{N_r(V)|\ V\in\mathscr{V}_0\},\cdots,\mathscr{U}_m=\{N_r(V)|\ V\in\mathscr{V}_m\}.$$

Then $\mathscr{U}_0,\mathscr{U}_1,\cdots,\mathscr{U}_m$ are r-disjoint and uniformly bounded families such that $\mathscr{U}_0\cup\mathscr{U}_1\cup\ldots\cup\mathscr{U}_m$ covers $as\bigsqcup_{i=n}^{\infty}N_r(X_{\omega+m}^{(i,n)})$. Let

$$\begin{aligned}\mathscr{U}_{m+1}^{(i)}=\{&\{x_t\}_{t=1}^{j_1-1}\times[n_{j_1}2^n+r,(n_{j_1}+1)2^n-r]\times(x_t)_{t=j_1+1}^{j_2-1}\times[n_{j_2}2^n+r,(n_{j_2}+1)2^n-r]\\&\times\{x_t\}_{t=j_2+1}^{j_3-1}\times\cdots\times\{x_t\}_{t=j_m+1}^{j_{m+1}-1}\times[n_{j_{m+1}}2^n+r,(n_{j_{m+1}}+1)2^n-r]\times\{x_t\}_{t=j_{m+1}+1}^{i}|\ x_t\\&\in 2^n\mathbb{Z},n_{j_k}\in\mathbb{Z},1\leq k\leq m+1\}.\end{aligned}$$

It is easy to see that $\mathscr{U}_{m+1}^{(i)}$ is r-disjoint and 2^n-bounded. Note that, for every $i\geq n$,

$$X_{\omega+m+1}^{(i,n)}\setminus\bigcup\mathscr{U}_{m+1}^{(i)}\subset N_r(X_{\omega+m}^{(i,n)})$$

Indeed, for any $x=\{x_t\}_{t=1}^{i}\in X_{\omega+m+1}^{(i,n)}\setminus\bigcup\mathscr{U}_{m+1}^{(i)}$, $\{x_t\}_{t=1}^{i}\in X_{\omega+m+1}^{(i,n)}$ implies that there exists at most $m+1$ coordinates x_t such that $x_t\notin 2^n\mathbb{Z}$ and $x\notin\bigcup\mathscr{U}_{m+1}^{(i)}$ implies that, among all the x_t with $x_t\notin 2^n\mathbb{Z}$, there exists at least one x_{t_0} such that $d(x_{t_0},2^n\mathbb{Z})<r$. It follows that $x\in N_r(X_{\omega+m}^{(i,n)})$. Since

$$d(X_{\omega+m+1}^{(i,n)},X_{\omega+m+1}^{(j,n)})>r \text{ for every } i,j\geq n \text{ and } i\neq j,$$

then $\mathscr{U}_{m+1}\doteq\bigcup_{i\geq n}\mathscr{U}_{m+1}^{(i)}$ is an r-disjoint uniformly bounded family of subsets and

$$as\bigsqcup_{i=n}^{\infty}X_{\omega+m+1}^{(i,n)}\subset(\bigcup\mathscr{U}_{m+1})\cup\bigcup_{i=n}^{\infty}N_r(X_{\omega+m}^{(i,n)}).$$

Therefore, $\mathscr{U}_0\cup\mathscr{U}_1\cup\ldots\cup\mathscr{U}_{m+1}$ covers $as\bigsqcup_{i=n}^{\infty}X_{\omega+m+1}^{(i,n)}$. So the result is true for $k=m+1$. □

Proposition 3.11. *$coasdim(X_{\omega+k})\leq\omega+k$.*

Proof. For every $r>0$, by Lemma 3.10, there exist $n=n(r)\in\mathbb{N}$ and r-disjoint uniformly bounded families $\mathscr{U}_0,\mathscr{U}_1,\dots,\mathscr{U}_k$ such that $\mathscr{U}_0\cup\mathscr{U}_1\cup\ldots\cup\mathscr{U}_k$ covers $as\bigsqcup_{i=n}^{\infty}X_{\omega+k}^{(i,n)}$. Since

$X^{(i)}_{\omega+k} = X^{(i,i)}_{\omega+k} \subset X^{(i,n)}_{\omega+k}$ for $i \geq n$, $X_{\omega+k} \setminus \bigcup(\mathscr{U}_0 \cup \ldots \mathscr{U}_k) \subseteq as\bigsqcup_{i=1}^{n-1} X^{(i)}_{\omega+k}$. From Lemma 2.8, one has

$$\text{coasdim}\big(X_{\omega+k} \setminus \bigcup(\mathscr{U}_0 \cup \ldots \mathscr{U}_k)\big) \leq \text{coasdim}\big(as\bigsqcup_{i=1}^{n-1} X^{(i)}_{\omega+k}\big) \leq$$

$$\text{coasdim}\big(as\bigsqcup_{i=1}^{n-1} \mathbb{R}^i\big) \leq \text{coasdim}\big(\mathbb{R}^{n-1}\big) < \omega.$$

It follows from the definition that $\text{coasdim}(X_{\omega+k}) \leq \omega + k$. □

Theorem 3.12. *trasdim*$(X_{\omega+k}) = \omega + k$ *for every* $k \in \mathbb{N}$.

Proof. From Lemma 3.7 and Proposition 3.11, one has $\text{trasdim}(X_{\omega+k}) \leq \omega + k$. Using Proposition 3.6, we obtain that $\text{trasdim}(X_{\omega+k}) = \omega + k$. □

Theorem 3.13. *coasdim*$(X_{\omega+k}) = \omega + k$ *for every* $k \in \mathbb{N}$.

Proof. From Proposition 3.6 and Lemma 3.7, we can obtain that $\text{coasdim}(X_{\omega+k}) \leq \omega + k - 1$ is not true. Then $\text{coasdim}(X_{\omega+k}) = \omega + k$ due to Proposition 3.11. □

3.2. **A metric space with complementary-finite asymptotic dimension and transfinite asymptotic dimension** 2ω**.** In this section, we will construct a metric space $Y_{2\omega}$ by taking asymptotic union of all the metric spaces $Y_{\omega+k}$, which is coarsely equivalent to $X_{\omega+k}$ for any $k \in \mathbb{N}$.

For every $k, i \in \mathbb{N}$, let

$$Y^{(i)}_{\omega+k} = \{(x_1, \ldots, x_i) \in (2^k\mathbb{Z})^i \mid |\{j \mid x_j \notin 2^i\mathbb{Z}\}| \leq k\}, Y_{\omega+k} = \text{as}\bigsqcup_{i=1}^{\infty} Y^{(i)}_{\omega+k} \text{ and } Y_{2\omega} = \text{as}\bigsqcup_{k=1}^{\infty} Y_{\omega+k},$$

where $Y_{\omega+k}$ is a subspace of the metric space $\text{as}\bigsqcup_{j=1}^{\infty} \mathbb{R}^j$ for each $j \in \mathbb{N}$.

Proposition 3.14. *coasdim*$(Y_{2\omega}) = 2\omega$ *and trasdim*$(Y_{2\omega}) = 2\omega$.

Proof. For any $k \in \mathbb{N}$, since $Y_{\omega+k} \subseteq X_{\omega+k}$ and $X_{\omega+k} \subseteq N_{2^k}(Y_{\omega+k})$, $Y_{\omega+k}$ and $X_{\omega+k}$ are coarse equivalent. It follows that

$$\text{trasdim}(Y_{\omega+k}) = \text{trasdim}(X_{\omega+k}) = \omega + k \text{ and } \text{coasdim}(Y_{\omega+k}) = \text{coasdim}(X_{\omega+k}) = \omega + k$$

due to Lemma 2.10. It follows that $\text{coasdim}(Y_{2\omega}) \geq 2\omega$ and $\text{trasdim}(Y_{2\omega}) \geq 2\omega$.

For every $n > 0$, let

$$\mathscr{U} = \{\{x\} \mid x \in as\bigsqcup_{k=n+1}^{\infty} Y_{\omega+k}\}.$$

Then $\mathscr{U}$ is n-disjoint and uniformly bounded, and

$$Y_{2\omega} \setminus \bigcup \mathscr{U} = as\bigsqcup_{k=1}^{n} Y_{\omega+k}.$$

It follows from Lemma 2.8 that

$$\text{coasdim}(as\bigsqcup_{k=1}^{n} Y_{\omega+k}) \leq \text{coasdim}(as\bigsqcup_{k=1}^{n} X_{\omega+k}) \leq \max_{1\leq k\leq n}\{\text{coasdim}(X_{\omega+k})\} = \omega + n < 2\omega,$$

thus $\mathrm{coasdim}(Y_{2\omega} \setminus \bigcup \mathscr{U}) < 2\omega$. Then by the definition of complementary-finite asymptotic dimension, $\mathrm{coasdim}(Y_{2\omega}) \leq 2\omega$. Hence $\mathrm{trasdim}(Y_{2\omega}) \leq 2\omega$, which is due to Lemma 3.7. Therefore, $\mathrm{coasdim}(Y_{2\omega}) = 2\omega$, and $\mathrm{trasdim}(Y_{2\omega}) = 2\omega$.

□

3.3. **The decomposition dimension of metric spaces.**

Definition 3.15. ([4]) We say that the metric family $\{X_\alpha\}$ satisfies the inequality $\mathrm{asdim}(X_\alpha) \leq n$ uniformly if, for every $r > 0$, there exists $R > 0$ such that, for each α, there are r-disjoint and R-bounded families $\mathscr{U}_0^\alpha, \mathscr{U}_1^\alpha, \cdots, \mathscr{U}_n^\alpha$ of subsets of X_α such that $\bigcup_{i=0}^n \mathscr{U}_i^\alpha$ covers X_α.

Now we introduce a new dimension in coarse geometry.

Definition 3.16. Every ordinal number γ can be represented as $\gamma = \lambda(\gamma) + n(\gamma)$, where $\lambda(\gamma)$ is the limit ordinal or 0, and $n(\gamma) \in \mathbb{N} \cup \{0\}$. For a metric space X, we define *decomposition dimension* $\mathrm{decodim}(X)$ inductively as follows:

- for $\gamma = n \in \mathbb{N}$, $\mathrm{decodim}(X) \leq n \Leftrightarrow \mathrm{coadim}(X) \leq n$;
- for $\gamma = \omega$, $\mathrm{decodim}(X) \leq \omega \Leftrightarrow \mathrm{coadim}(X) \leq \omega$;
- for $\gamma > \omega$, $\mathrm{decodim}(X) \leq \gamma \Leftrightarrow$ for every $r > 0$, there is an r-disjoint family $\mathscr{U} = \{U_i\}$ of subsets of X such that $\mathrm{asdim}(U_i) \leq n(\gamma)$ uniformly and $\mathrm{decodim}(X \setminus \bigcup \mathscr{U}) \leq \alpha$ for some $\alpha < \lambda(\gamma)$;
- $\mathrm{decodim}(X) = \gamma \Leftrightarrow \mathrm{decodim}(X) \leq \gamma$ and $\mathrm{decodim}(X) \leq \beta$ is not true for any ordinal number $\beta < \gamma$.

Remark 3.17. It is easy to see that $\mathrm{coasdim}(X) \leq \omega$ implies $\mathrm{decodim}(X) = \mathrm{coasdim}(X)$.

Lemma 3.18. *Let X and Y be metric spaces, and let $\{Y_\alpha\}$ be metric family of subsets of Y and $\mathrm{asdim}(Y_\alpha) \leq n$ uniformly. Let $\phi : X \to Y$ be a coarse embedding from X to Y. Then, for the metric family $\{\phi^{-1}(Y_\alpha)\}$, $\mathrm{asdim}\big(\phi^{-1}(Y_\alpha)\big) \leq n$ uniformly.*

Proof. Since $\phi : X \to Y$ is a coarse embedding, there are nondecreasing functions $p_1, p_2 : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\lim_{x\to+\infty} p_1(x) = +\infty$ and for every $x_1, x_2 \in X$,

$$p_1(d(x_1,x_2)) \leq d(\phi(x_1), \phi(x_2)) \leq p_2(d(x_1,x_2)).$$

For every $r > 0$, $\mathrm{asdim}(Y_\alpha) \leq n$ uniformly implies there exists $R > 0$ such that, for each α, there are $\big(p_2(r)+1\big)$-disjoint and R-bounded families $\mathscr{U}_0^\alpha, \mathscr{U}_1^\alpha, \cdots, \mathscr{U}_n^\alpha$ of subsets of Y_α with $\bigcup_{i=0}^n \mathscr{U}_i^\alpha$ covers Y_α. Since $\lim_{x\to+\infty} p_1(x) = +\infty$, there exists $S > 0$ such that $p_1(S) > R$. For $i = 0, 1, 2, \cdots, n$, let

$$\mathscr{V}_i^\alpha = \{\phi^{-1}(U) \mid U \in \mathscr{U}_i^\alpha\}.$$

Then $\mathscr{V}_i^\alpha$ is r-disjoint and S-bounded families of subsets of $\phi^{-1}(Y_\alpha)$ with $\bigcup_{i=0}^n \mathscr{V}_i^\alpha$ covers $\phi^{-1}(Y_\alpha)$. So $\mathrm{asdim}\big(\phi^{-1}(Y_\alpha)\big) \leq n$ uniformly. □

Proposition 3.19. *Let X and Y be metric spaces with $\mathrm{decodim}(Y) \leq \xi$ for some countable ordinal number ξ. If there is a coarse embedding $\phi : X \to Y$ from X to Y, then $\mathrm{decodim}(X) \leq \mathrm{decodim}(Y)$. Consequently, if X and Y are coarsely equivalent, then $\mathrm{decodim}(X) = \mathrm{decodim}(Y)$.*

Proof. We will prove it by induction on ξ.

- For $\xi \le \omega$, $\operatorname{decodim}(Y) \le \xi \le \omega$ implies $\operatorname{coasdim}(Y) \le \omega$. By Lemma 2.10,
$$\operatorname{coasdim}(X) \le \operatorname{coasdim}(Y) \le \omega.$$
It follows that
$$\operatorname{decodim}(X) = \operatorname{coasdim}(X) \le \operatorname{coasdim}(Y) = \operatorname{decodim}(Y).$$
- Assume that the statement is true for every $\xi < \beta$. Now let us consider the case of $\xi = \beta$ with $\beta > \omega$. Since $\phi : X \to Y$ is a coarse embedding, there are nondecreasing functions $p_1, p_2 : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\lim_{x\to+\infty} p_1(x) = +\infty$ and for every $x_1, x_2 \in X$,
$$p_1(d(x_1,x_2)) \le d(\phi(x_1),\phi(x_2)) \le p_2(d(x_1,x_2)).$$
For every $r > 0$, there is a $\big(p_2(r)+1\big)$-disjoint family $\mathscr{U} = \{U_\alpha\}$ of subsets of Y such that
$$\operatorname{asdim}(U_\alpha) \le n(\beta) \text{ uniformly and } \operatorname{decodim}(Y \setminus \bigcup \mathscr{U}) \le \eta \text{ for some } \eta < \lambda(\beta).$$
Then $\operatorname{asdim}\big(\phi^{-1}(U_\alpha)\big) \le n(\beta)$ uniformly by Lemma 3.18. Let $\mathscr{V} = \{\phi^{-1}(U_\alpha)\}$. Since the restriction of ϕ to $X \setminus \bigcup \mathscr{V}$ is a coarse embedding into $Y \setminus \bigcup \mathscr{U}$ and $\operatorname{decodim}(Y \setminus \bigcup \mathscr{U}) \le \eta$, $\operatorname{decodim}(X \setminus \bigcup \mathscr{V}) \le \eta$ by induction hypothesis. So $\operatorname{decodim}(X) \le \beta = \operatorname{decodim}(Y)$.

$\square$

Example 3.20. For $n \in \mathbb{N}$, $\operatorname{decodim}(X_{\omega+n}) \le \omega + n$.

Proof.
- For $n = 1$, by Theorem 3.13, $\operatorname{coasdim}(X_{\omega+1}) = \omega+1$. Then $\operatorname{coasdim}(X_{\omega+1}) \le \omega$ is not true and hence $\operatorname{decodim}(X_{\omega+1}) \le \omega$ is not true. Now it suffices to show that $\operatorname{decodim}(X_{\omega+1}) \le \omega+1$. For any $k > 0$ and integer $n > 2k$, let $\mathscr{V}_0^{(n)} = \{[i2^n + k, (i+1)2^n - k] \mid i \in \mathbb{Z}\}$, and let
$$\mathscr{W}_0^{(n)} = \{\prod_{i=1}^{j-1}\{nn_i\} \times V_j \times \prod_{i=j+1}^{n}\{nn_i\} \mid n_i \in \mathbb{Z}, V_j \in \mathscr{V}_0^{(n)}, j = 1,2,3,\cdots,n\}$$
Let $\mathscr{W}_0 = \bigcup_{n>k} \mathscr{W}_0^{(n)}$. Then $\mathscr{W}_0$ is k-disjoint, and $\operatorname{asdim}(W) \le 1$ uniformly for any $W \in \mathscr{W}_0$. It is easy to see $X_{\omega+1} \setminus \bigcup \mathscr{W}_0 \subseteq N_k(X_\omega)$. Hence $\operatorname{coasdim}(X_{\omega+1} \setminus \bigcup \mathscr{W}_0) \le \omega$, which implies $\operatorname{decodim}(X_{\omega+1} \setminus \bigcup \mathscr{W}_0) \le \omega$. Therefore $\operatorname{decodim}(X_{\omega+1}) \le \omega+1$.
- Assuming $\operatorname{decodim}(X_{\omega+k}) \le \omega+k$ holds for $k \le n-1$, we have that, for any $r > n > 0$, the r-neighborhood $N_r(X_{\omega+n-1})$ of $X_{\omega+n-1}$ in $X_{\omega+n}$ has $\operatorname{decodim}(N_r(X_{\omega+n-1})) \le \omega + n - 1$ due to Proposition 3.19. Let $i_0 \in \mathbb{N}$ be the smallest number with $r \le i_0$. For any $i \ge i_0$, for any subset $F \subset \{1,\cdots,i\}$ with $|F| = n$ and $x_j \in \mathbb{Z}$ for $j \in \{1,\cdots,i\} \setminus F$, let
$$U_F^{\{x_j\}_{j\in\{1,\cdots,i\}\setminus F}} = \prod_{j\notin F}\{2^j x_j\} \times \prod_{j\in F}[2^j x_j + r, 2^j(x_j+1) - r] \text{ and}$$
$$\mathscr{A}_i = \{U_F^{\{x_j\}_{j\in\{1,\cdots,i\}\setminus F}} | F \subset \{1,\cdots,i\} \text{ with } |F| = n \text{ and } x_j \in \mathbb{Z} \text{ for } j \in \{1,\cdots,i\} \setminus F\}.$$
Then $A_i \doteq \bigcup \mathscr{A}_i = \{(x_1,\dots,x_i) \in X_{\omega+n}^{(i)} | |\{j | d(x_j, 2^i\mathbb{Z}) \ge r\}| \le n\} = X_{\omega+n} \setminus N_r(X_{\omega+n-1})$. Using the definition, $d(A_p, A_q) > r$ for $p \ne q \in \mathbb{N}$ and $p, q \ge i_0$, so $\bigcup_{i=i_0}^{\infty} \mathscr{A}_i$ is still an r-disjoint family of subsets such that $\operatorname{asdim}(V) \le n$ uniformly for every $V \in \bigcup_{i=i_0}^{\infty} \mathscr{A}_i$. So $\operatorname{decodim}(X_{\omega+n}) \le \omega + n$.

□

Example 3.21. $\mathrm{decodim}(Y_{2\omega}) \le 2\omega$.

Proof. For $R>0$, since $Y_{2\omega} = as\bigsqcup_{k=1}^{\infty} Y_{\omega+k}$, then $\mathscr{U} = \{\{x\} | x \in as\bigsqcup_{k=R+1}^{\infty} Y_{\omega+k}\}$ is a R-disjoint uniformly bounded subsets family, and $\bigcup \mathscr{U} = as\bigsqcup_{k=R+1}^{\infty} Y_{\omega+k}$. By definition $Y_{\omega+k} \subset X_{\omega+R}$ for $k \le R$, so there exists $m \in \mathbb{N}$ such that $\mathrm{decodim}(as\bigsqcup_{k=1}^{R} Y_{\omega+k}) \le \omega + m$, which implies $\mathrm{decodim}(Y_{2\omega}) \le 2\omega$. □

Definition 3.22. ([11]) A metric family $\mathscr{X}$ is *r-decomposable* over a metric family $\mathscr{Y}$ if every $X \in \mathscr{X}$ admits a decomposition

$$X = X_0 \cup X_1, X_i = \bigsqcup_{r\text{-disjoint}} X_{ij},$$

where each $X_{ij} \in \mathscr{Y}$. It is denoted by $\mathscr{X} \xrightarrow{r} \mathscr{Y}$.

Definition 3.23. ([11])
(1) Let $\mathscr{D}_0$ be the collection of bounded families: $\mathscr{D}_0 = \{\mathscr{X} : \mathscr{X}$ is uniformly bounded$\}$.
(2) Let α be an ordinal greater than 0, and let $\mathscr{D}_\alpha$ be the collection of metric families decomposable over $\bigcup_{\beta<\alpha} \mathscr{D}_\beta$:

$$\mathscr{D}_\alpha = \{\mathscr{X} : \forall\, r > 0, \exists\, \beta < \alpha, \exists\, \mathscr{Y} \in \mathscr{D}_\beta, \text{ such that } \mathscr{X} \xrightarrow{r} \mathscr{Y}\}.$$

Definition 3.24. ([11]) A metric family $\mathscr{X}$ has *finite decomposition complexity* if there exists a countable ordinal α such that $\mathscr{X} \in \mathscr{D}_\alpha$.

Remark 3.25. We view a single metric space X as a metric family with a single element $\{X\}$.

Lemma 3.26. *([11]) A metric space X has finite asymptotic dimension if and only if $\{X\} \in \mathscr{D}_n$ for some $n \in \mathbb{N}$.*

By imitating the proofs of Proposition 3.8 in [3], we easily obtain the following lemma.

Lemma 3.27. *For a metric famliy $\mathscr{X} = \{X_\alpha\}_{\alpha\in\mathfrak{A}}$, $\mathrm{asdim}(X_\alpha) \le n$ uniformly implies $\{X_\alpha\}_{\alpha\in\mathfrak{A}} \in \mathscr{D}_n$.*

Proposition 3.28. *For a metric space X and an ordinal number ξ, $\mathrm{decodim}(X) \le \xi$ implies $\{X\} \in \mathscr{D}_\xi$.*

Proof. We will prove it by an induction on ξ.

- For $\xi \in \mathbb{N}$, $\mathrm{decodim}(X) \le \xi$ implies X has finite asymptotic dimension. Then $\{X\} \in \mathscr{D}_\xi$ by Lemma 3.26.
- For $\xi = \omega$, for every $r > 0$, since $\mathrm{decodim}(X) \le \omega$, there is an r-disjoint and uniformly bounded family $\mathscr{U}$ of subsets of X such that $\mathrm{asdim}(X \setminus \bigcup \mathscr{U}) \le n$ for some $n \in \mathbb{N}$. By Lemma 3.26, $\{X \setminus \bigcup \mathscr{U}\} \in \mathscr{D}_n$. Let $\mathscr{Y} = \{U \mid U \in \mathscr{U}\} \bigcup \{X \setminus \bigcup \mathscr{U}\}$. Then $\mathscr{Y} \in \mathscr{D}_n$ and $\{X\} \xrightarrow{r} \mathscr{Y}$, which implies that $\{X\} \in \mathscr{D}_\omega$.
- For $\xi > \omega$, assume that $\mathrm{decodim}(Y) \le \gamma$ implies $\{Y\} \in \mathscr{D}_\gamma$ for every metric space Y and for every $\gamma < \xi$. Now let X be a metric space with $\mathrm{decodim}(X) \le \xi$. For every $r > 0$, there is an r-disjoint family $\mathscr{U}$ of subsets of X with $\mathrm{asdim}(U) \le n(\xi)$ uniformly for every $U \in \mathscr{U}$, such that $\mathrm{decodim}(X \setminus \bigcup \mathscr{U}) \le \beta$ for some ordinal number $\beta < \xi$. By assumption, $\{X \setminus \bigcup \mathscr{U}\} \in \mathscr{D}_\beta$. By Lemma 3.27, $\mathscr{U} \in \mathscr{D}_{n(\xi)}$. Let $\mathscr{Y} = \{U \mid U \in \mathscr{U}\} \bigcup \{X \setminus \bigcup \mathscr{U}\}$, then $\mathscr{Y} \in \mathscr{D}_\beta$ and $\mathscr{X} \xrightarrow{r} \mathscr{Y}$. So $\{X\} \in \mathscr{D}_\xi$.

□

Corollary 3.29. *For any $k \in \mathbb{N}$, $X_{\omega+k} \in \mathscr{D}_{\omega+k}$, and $Y_{2\omega} \in \mathscr{D}_{2\omega}$.*

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