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A NEW RELAXED PROJECTION AND ITS APPLICATIONS

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Abstract. In this paper, we introduce a new relaxed projection onto the level sets of the convex functions. We propose new relaxed projection methods by applying the proposed relaxed projection to solve split feasibility problems and split equality problems. The weak convergence of the relaxed projection methods is established. A preliminary numerical experiment is presented to support the new relaxed projection.

Keywords. Variational inequality; Split feasibility problem; Split equality problem; Projection method; Relaxed projection.

1. Introduction

Let $\mathscr{H}$ be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$, and let C be a nonempty closed and convex set in $\mathscr{H}$.

Consider the classical variational inequality, which is to find a point $x^* \in C$ such that

$$\langle f(x^*), x - x^* \rangle \geq 0, \quad \text{for all } x \in C, \tag{1.1}$$

where $f : \mathscr{H} \to \mathscr{H}$ is a mapping. The variational inequality has various applications arising in many areas, such as, partial differential equations, optimal control, optimization, mathematical programming and some other nonlinear problems; see, [1, 2, 3, 4, 5] and the references therein.

A great deal of projection methods are proposed for solving the variational inequality (1.1) (see [6, 7, 8, 9, 10]). The simplest one is the following projection method, which can be seen an extension of the gradient projection method for optimization problems:

$$x^{k+1} = P_C(x^k - \tau_k f(x^k)), \tag{1.2}$$

where $\tau_k > 0$ and P_C is the metric projection onto C.

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An important observation concerning projection method (1.2) is the need to calculate the projection onto C per each iteration. Note that the projection onto a closed convex set C is related to a minimum distance problem. If C is a general closed and convex set, this might require a prohibitive amount of computation time. This fact might affect the efficiency and applicability of projection algorithm (1.2). To overcome this obstacle, Fukushima [11] first introduced a variable metric relaxed projection algorithm for variational inequalities in finite dimensional Euclidean spaces. Here we present Fukushima's algorithm with the symmetric positive definite matrix being the identity matrix in a Hilbert space $\mathscr{H}$.

Algorithm 1.1. (Fukushima's relaxed projection algorithm)
Step 0: Select a starting point x^0 and set $k=0$.
Step 1: Choose $\eta_k \in \partial c(x^k)$, and let

$$S_k = \left\{ x \in \mathscr{H} : c(x^k) + \langle \eta_k, x - x^k \rangle \leq 0 \right\}.$$

Step 2: Set $u^k = x^k - \tau_k f(x^k)$ and calculate

$$x^{k+1} = P_{S_k}\left(u^k\right),$$

where $\{\tau_k\}_{k\in\mathbb{N}}$ is a sequence of nonnegative numbers.
Step 3: If $x^{k+1} = x^k$, then terminate. Otherwise, set $k = k+1$ and return to **Step 1**.

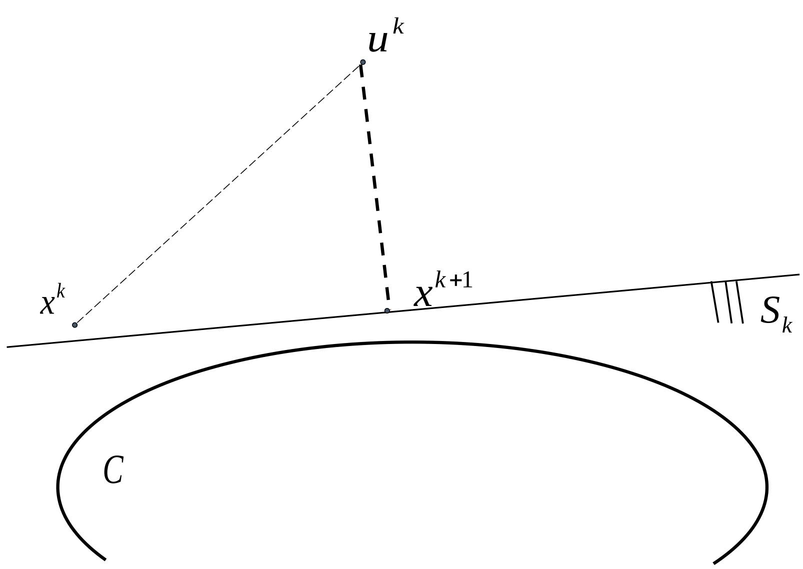

FIGURE 1. x^{k+1} is the projection of u^k onto the set S_k

By using the definition of the subgradient in (2.4), it is easy to see that $S_k \supset C$. Fukushima's idea is to replace the projection onto C by the projection onto halfspace S_k per each iteration. Since the projections onto halfspace S_k have closed forms, Fukushima's relaxed projection algorithm is implementable.

Since its inception, Fukushima's relaxed projection algorithm has received much attention and was applied to different problems, such as the convex feasibility problem, the split feasibility problem and split equality problem (see, e.g., [12, 13, 14, 15, 16, 17]). By assuming that the corresponding function to the level set of C is strongly convex, Yu et al. [18] recently replaced the projection onto C by that onto a ball containing C.

In this paper, we introduce a relaxed projection which involves a new halfspace containing C. Fukushima's algorithm with new relaxed projection is shown to converge to the solution of the variational inequality (1.1) under the same conditions with [11]. We also establish the convergence of the projection methods with the new relaxed projection for the split feasibility problem and the split equality problem. A preliminary numerical example illustrates that the iterative algorithms with the new relaxed projection behave better than that with Fukushima's relaxed projection.

After we submitted this manuscript for review, a similar relaxation projection approach for the split feasibility problem and the split equality problem has appeared very recently in a concurrent work [19]. The motivation of our work comes from the outer approximation method, which is different with that in [19]. Furthermore, a general result is given for the variational inequality, and the relaxation iterative methods for the feasibility problem and the split equality problem are two applications.

This paper is organized as follows. In Section 2, we recall some definitions and preliminary results used in main results. Section 3 gives the motivation to introduce the new relaxed projection. Section 4 focuses on the Fukushima's algorithm with new relaxed projection for the variational inequality. In section 5, we give some applications in the split feasibility problem and the split equality problem. Finally, Section 6 presents a preliminary numerical experiment to show the advantage of the new relaxed projection over Fukushima's relaxed projection.

2. Preliminaries and Assumptions

This section contains some definitions, assumptions and basic results that will be used in our subsequent analysis.

We use the following notations:

(i) $\rightharpoonup$ for weak convergence and $\to$ for strong convergence;
(ii) $\omega_w(x^k) = \{x : \exists x^{k_j} \rightharpoonup x\}$ denotes the weak ω-limit set of a sequence $\{x^k\}_{k\in\mathbb{N}}$.

Definition 2.1. Let $T : \mathscr{H} \to \mathscr{H}$ be an operator. Then

- T is called L-`Lipschitz continuous` with $L > 0$ if

$$\|Tx - Ty\| \le L\|x - y\|, \quad \forall x, y \in \mathscr{H}.$$

  if $L = 1$, then T is called `nonexpansive`, and if $L \in (0,1)$, then T is called `contractive`.
- T is called `monotone` if

$$\langle Tx - Ty, x - y\rangle \ge 0, \quad \forall x, y \in \mathscr{H}.$$

- T is called α-`strongly monotone` if there exists a constant $\alpha > 0$ such that

$$\langle Tx - Ty, x - y\rangle \ge \alpha\|x - y\|^2, \ \ \forall x, y \in \mathscr{H}.$$

The `nearest point or metric projection` from $\mathscr{H}$ onto C, denoted P_C, is defined in such a way that, for each $x \in \mathscr{H}$, $P_C x$ is the unique point in C such that

$$\|x - P_C x\| = \min\{\|x - z\| : z \in C\}. \tag{2.1}$$

The following two lemmas give the fundamental properties of the metric projection.

Lemma 2.2. *For any $x \in \mathscr{H}$ and $z \in C$, $z = P_C x$ if and only if*

$$\langle x - z, y - z\rangle \le 0, \quad \forall y \in C.$$

Lemma 2.3. *For any $x, y \in \mathscr{H}$ and $z \in C$, it holds*

$$\|P_C(x) - z\|^2 \le \|x - z\|^2 - \|P_C(x) - x\|^2.$$

Assume that a halfspace in a Hilbert space $\mathscr{H}$ has the form

$$H(a, \beta) = \{z \in \mathscr{H} : \langle a, z\rangle \le \beta\},$$

where $a \in \mathscr{H}$, $a \ne 0$, and $\beta \in \mathbb{R}$. It is clear that $H(a, \beta)$ is closed and convex. The projection of a point $x \in \mathscr{H}$ onto $H(a, \beta)$ (see [20, Section 4.1.3]) is

$$P_{H(a,\beta)}x = \begin{cases} x - \dfrac{\langle a, x\rangle - \beta}{\|a\|^2}, & \text{if } \langle a, x\rangle > \beta, \\ x, & \text{if } \langle a, x\rangle \le \beta. \end{cases} \tag{2.2}$$

Lemma 2.4. [11, Lemma 2] *Let $\{\xi_k\}_{k\in\mathbb{N}}$ and $\{\eta_k\}_{k\in\mathbb{N}}$ be sequences of nonnegative numbers, and let $\mu \in [0, 1)$ be a constant. If the inequalities*

$$\xi_{k+1} \le \mu \xi_k + \eta_k, \quad k \ge 0, \tag{2.3}$$

hold, and $\lim_{k\to\infty} \eta_k = 0$, then $\lim_{k\to\infty} \xi_k = 0$.

Lemma 2.5. [21, Theorem 5.5] *Let C be a nonempty set of $\mathscr{H}$, and let $\{x^k\}_{k\in\mathbb{N}}$ be a sequence in $\mathscr{H}$ such that the following two conditions hold:*

(i) *for all $x \in C$, $\lim_{k\to\infty} \|x^k - x\|$ exists;*

(ii) *every sequential weak cluster point of $\{x^k\}_{k\in\mathbb{N}}$ is in C.*

Then $\{x^k\}_{k\in\mathbb{N}}$ converges weakly to a point in C.

Definition 2.6. [21, Chapter 16] $s \in \mathscr{H}$ is called a `subgradient` of φ at $p \in \mathscr{H}$ if

$$\langle y - p, s\rangle + \varphi(p) \le \varphi(y), \quad \forall y \in \mathscr{H}. \tag{2.4}$$

The set of all subgradients of φ at p is denoted by $\partial\varphi(p)$. If φ is differentiable at p, this set reduces to a single vector, namely, the gradient $\nabla\varphi(p)$.

We make following assumptions for set C.

Assumption 2.7. The closed convex subset C is of the particular structure, i.e., the level set of a convex function given as follows:

$$C = \{x \in \mathscr{H} : c(x) \le 0\}, \tag{2.5}$$

where $c : \mathscr{H} \to \mathbb{R}$ is a convex function, and there exists a point x^0 such that $c(x^0) < 0$.

Assumption 2.8. The function c is subdifferentiable on $\mathscr{H}$, and ∂c is a bounded operator (i.e., bounded on bounded sets.)

It is worth noting that every convex function defined on a finite-dimensional Hilbert space is subdifferentiable and its subdifferential operator is a bounded operator (see [22, Theorem 2.5]).

3. Motivations

In this section, we state the motivation to introduce the new relaxed projection onto the level sets of the convex functions.

Consider the minimization problem as follows:

$$\min_{x\in C} F(x), \tag{3.1}$$

where C is a closed convex subset of $\mathscr{H}$ and $F: C \to \mathbb{R}$ is a convex function. If F is (Frechet) differentiable, then problem (3.1) is reduced to the variational inequality (1.1) with $f = \nabla F$.

Assume that ∇F is L-Lipschitz continuous with $L > 0$. A classical method to solve (3.1) is the gradient projection algorithm (GPA), which generates a sequence $\{x^k\}_{k\in\mathbb{N}}$ via

$$x^{k+1} = P_C(x^k - \gamma_k \nabla F(x^k)), \quad k \geq 0. \tag{3.2}$$

This is a special case of method (1.2). Combettes and Wajs [23] considered the gradient projection algorithm with perturbations as follows:

$$x^{k+1} = P_C(x^k - \gamma_k \nabla F(x^k)) + \varepsilon_k, \quad k \geq 0. \tag{3.3}$$

In [23, Theorem 3.4(i)], the following result on algorithm (3.3) was obtained, which guarantees the convergence of the iterates.

Theorem 3.1. *Suppose that problem* (3.1) *has at least one solution, let $x^0 \in \mathscr{H}$, let $\{\gamma_k\}_{k\in\mathbb{N}}$ be a sequence in $[0,+\infty]$, and let $\{\varepsilon_k\}_{k\in\mathbb{N}}$ be a sequence in $\mathscr{H}$ such that*

$$\sum_{k=0}^{\infty} \|\varepsilon_k\| < +\infty, \quad \inf_{k\in\mathbb{N}} \gamma_k > 0, \quad \sup_{k\in\mathbb{N}} \gamma_k < \frac{2}{L}.$$

Then the sequence $\{x^k\}_{k\in\mathbb{N}}$ generated by (3.3) *weakly converges to a solution of the problem* (3.1).

Based on Theorem 3.1, recently, Barlaud et al. [24] studied an outer approximation of the gradient projection method for the problem (3.1) with C given as (2.5).

Next, we recall the outer approximation method. Let $p_0 \in \mathscr{H}$. By Lemma 2.2, the projection of p_0 onto C, denoted by $P_C(p_0)$, is characterized by

$$\begin{cases} P_C(p_0) \in C, \\ \langle p - P_C(p_0), p_0 - P_C(p_0)\rangle \leq 0, \quad \forall p \in C. \end{cases} \tag{3.4}$$

Given x and y in $\mathscr{H}$, define a closed affine halfspace $H(x,y)$ by

$$H(x,y) = \{p \in \mathscr{H} : \langle p - y, x - y\rangle \leq 0\}. \tag{3.5}$$

Note that $H(x,x) = \mathscr{H}$, and if $x \neq y$, $H(x,y)$ is the closed affine halfspace, onto which the projection of x is y. According to (3.4), $C \subset H(p_0, P_C(p_0))$.

If $c(p_n) > 0$, any subgradient $\xi_n \in \partial c(p_n)$ is in the normal cone to the lower level set $\{p \in H : c(p) \leq c(p_n)\}$ at p_n, and the associated subgradient projection of p_n onto C is (see [25])

$$p_{n+1/2} = \begin{cases} p_n - \dfrac{c(p_n)}{\|\xi_n\|^2}\xi_n, & \text{if} \quad c(p_n) > 0, \\ p_n, & \text{if} \quad c(p_n) \leq 0. \end{cases} \tag{3.6}$$

In fact one can get (3.6) by using the subgradient projector relative to c (see, e.g., [26]). A subgradient projector is a cutter (also called firmly quasinonexpansive operator) which separates a point not in C and a halfspace containing C.

As noted in [27], it follows from (2.4) that the halfspace $H(p_n, p_{n+1/2})$ serves as an outer approximation to C at iteration n, i.e., $C \subset H(p_n, p_{n+1/2})$. Altogether, since we have also seen that $C \subset H(p_0, p_n)$,

$$C \subset C_n, \quad \text{where} \quad C_n = H(p_0, p_n) \cap H(p_n, p_{n+1/2}). \tag{3.7}$$

Finally, the update p_{n+1} is the projection of p_0 onto the outer approximation C_n to C.

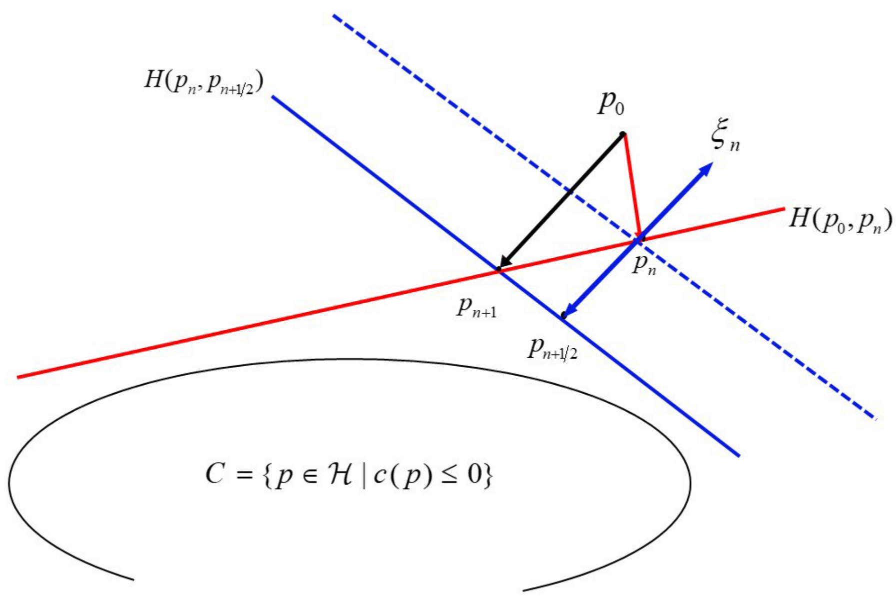

FIGURE 2. A generic iteration for computing the projection of p_0 onto C.

The principle of the method at iteration n is as follows (see Figure 2). The current iterate is p_n, and C is contained in the intersection of the halfspace $H(p_0, p_n)$ onto which p_n is the projection of p_0 and the halfspace $H(p_n, p_{n+1/2})$ onto which $p_{n+1/2}$ is the projection of p_n (see (3.5)).

As the following lemma (see [28] and [21, Corollary 29.25]) shows that the expression of the projection onto $H(p_0, p_n) \cap H(p_n, p_{n+1/2})$ is straightforward.

Lemma 3.2. *Let x, y and z be the points in $\mathscr{H}$ such that*

$$H(x,y) \cap H(y,z) \neq \emptyset.$$

Moreover, set $a = x - y$, $b = y - z$, $\chi = \langle a, b\rangle$, $\mu = \|a\|^2$, $\nu = \|b\|^2$, and $\rho = \mu\nu - \chi^2$. Then the projection of x onto $H(x,y) \cap H(y,z)$ is

$$Q(x,y,z) = \begin{cases} z, & \text{if } \rho = 0 \text{ and } \chi \geq 0, \\ x - \left(1 + \dfrac{\chi}{\nu}\right) b, & \text{if } \rho > 0 \text{ and } \chi\nu \geq \rho, \\ y + \dfrac{\nu}{\rho}\left(\chi a - \mu b\right), & \text{if } \rho > 0 \text{ and } \chi\nu < \rho. \end{cases}$$

The projection of p_0 onto the set C of (2.5) will be performed by executing the following routine, which will be chosen as an inner loop in the gradient projection algorithm.

Inner Loop (Outer approximation method) Given p and ε.
Input. Set $p_0 = p$ and $n = 0$. Given n, compute $\xi_n \in \partial c(p_n)$ and

$$\begin{cases} p_{n+1/2} = p_n - c(p_n)\xi_n/\|\xi_n\|^2 \\ p_{n+1} = Q(p_0, p_n, p_{n+1/2}). \end{cases} \tag{3.8}$$

While

$$\|p_{n+1} - P_C(p_0)\| > \varepsilon, \tag{3.9}$$

do $n = n+1$ and $p_{n+1} = p_n$.
End While
Output. p_{n+1} and n.

The following result from [29, Section 6.5] guarantees the convergence of the sequence $\{p_n\}_{n\in\mathbb{N}}$ generated by Inner Loop to the desired point.

Proposition 3.3. *Let $p_0 \in \mathscr{H}$, $c : \mathscr{H} \to \mathbb{R}$ a convex function, and $C = \{p \in \mathscr{H} : c(p) \leq 0\}$. Then either (3.8) terminates in a finite number of iterations at $P_C(p_0)$ or it generates an infinite sequence $\{p_n\}_{n\in\mathbb{N}}$ such that $p_n \to P_C(p_0)$.*

Based on Inner Loop, the outer approximation gradient method is given as follows:

$$\begin{cases} y^k = x^k - \gamma_k \nabla F(x^k), \\ x^{k+1} = \text{InnerLoop}(y^k, \|\varepsilon_k\|), \quad k \geq 0. \end{cases} \tag{3.10}$$

Combining Theorem 3.1 and Proposition 3.3, the convergence of the algorithm (3.10) is derived.

Combettes [30] compared the outer approximation method with Halpern method (also called the anchor point method) and Dykstra's algorithm, and showed that the outer approximation method has the best numerical behavior. Recently, the authors [31] combined Inner Loop with projection methods to get an outer approximation method for the split equality problems, and futhermore compared it with the relaxed projection method. It concluded that the outer approximation method needs less CPU time.

Recall that Inner Loop stops when $\|p_{n+1} - P_S(p_0)\| \leq \varepsilon$. In fact, it is difficult to compute $\|p_{n+1} - P_S(p_0)\|$ since $P_S(p_0)$ is unknown. Motivated by the work in [24], one can take $c(p_n) \leq \varepsilon$ as stopping criterions.

Now we give a numerical example and compare the number of Inner Loop for different $\|\varepsilon_n\| = 1/n^s$, $s = 1.01, 1.1, 1.2, 1.3, 1.4, 1.5, 1.7, 1.9$. In the numerical results listed in the following tables, "Iter" denotes the number of iterations and "InIt" denotes the number of total Inner Loop, respectively.

Example 3.4. Let $F : \mathbb{R}^m \to \mathbb{R}$ defined by $F(x) = \frac{1}{2}x^T Ax + bx$. where $A = Z^T Z$, $Z = (z_{ij})_{m\times m}$ and $b = (b_i) \in \mathbb{R}^m$, where $z_{ij} \in [1, 100]$ and $b_i \in [-100, 0]$ are generated randomly. We have $\nabla F(x) = Ax + b$. Let $C = \{x \in \mathbb{R}^m : c(x) \leq 0\}$, where $c(x) = -x_1 + x_2^2 + \cdots + x_m^2$. It is easy to verify that ∇F is L-Lipschitz continuous and η-strongly monotone with $L = \max(\text{eig}(A))$ and $\eta = \min(\text{eig}(A))$.

Set $m = 500$ and $\gamma_k = 1/(1.05L)$. Take the initial point $x^0 = (x_i^0) \in \mathbb{R}^m$, where $x_i^0 \in [0,1]$ is generated randomly. Although problem (3.1) has an unique solution (see [7]), it is difficult to get the exact solution. So, we use $D_k = \|x^{k+1} - x^k\| \leq \varepsilon$ as the stopping criterion.

TABLE 1. Computational results for Example 3.1 with different stopping criterions.

ε	s	1.01	1.1	1.2	1.3	1.4	1.5	1.7	1.9
10^{-4}	Iter	4216	4216	4216	4216	4216	4227	4220	4219
	InIt	4216	4216	4216	4216	4216	5955	7369	7874
10^{-6}	Iter	12454	12454	12454	12686	12484	12464	12457	12455
	InIt	12454	12454	12454	17433	20728	22429	23843	24346
10^{-8}	Iter	20755	20755	26081	20985	20783	20763	20757	20755
	InIt	20755	20755	35422	34031	37326	39027	40443	40946
10^{-10}	Iter	29057	29057	34379	29288	29083	29063	29056	29052
	InIt	29057	29057	52018	50637	53926	55627	57041	57540
10^{-12}	Iter	37358	37358	42339	37277	37045	37003	36973	37025
	InIt	37358	37358	67938	66615	69850	71507	72875	73486

From Table 1, it concludes that when s is close to 1, Iter and InIt are the same and less than those corresponding big s for very small ε. This observation motivates us to take one step in Inner Loop, which leads to the following question.

Question 1. Does the gradient projection method with one step in Inner Loop converge?

Now we deduce the expression of the projection $Q(p_0, p_0, p_{1/2})$ for one step ($n = 0$) in Inner Loop, and present the scheme of the gradient projection method with one step in Inner Loop.

Letting $n = 0$ in (3.7), we get $C_0 = H(p_0, p_0) \cap H(p_0, p_{1/2}) = H(p_0, p_{1/2}) = \{p \in \mathscr{H} : \langle p - p_{1/2}, p_0 - p_{1/2}\rangle \leq 0\}$. Therefore $p_1 = Q(p_0, p_0, p_{1/2}) = P_{C_0}(p_0)$.

The projection onto C_0 is discussed in two cases: $c(p_0) \leq 0$ and $c(p_0) > 0$.

Case 1: $c(p_0) \leq 0$. By (3.6), $p_{1/2} = p_0$, and therefore $C_0 = \mathscr{H}$. So $p_1 = P_{C_0}(p_0) = p_0$.

Case 2: $c(p_0) > 0$. Take $\xi_0 \in \partial c(p_0)$. By (3.6),

$$\begin{aligned}\langle p - p_{1/2}, p_0 - p_{1/2}\rangle \leq 0 &\Longleftrightarrow \left\langle p - \left(p_0 - \frac{c(p_0)}{\|\xi_0\|^2}\xi_0\right), \frac{c(p_0)}{\|\xi_0\|^2}\xi_0 \right\rangle \leq 0 \\ &\Longleftrightarrow \frac{c(p_0)^2}{\|\xi_0\|^2} + \frac{c(p_0)}{\|\xi_0\|^2}\langle p - p_0, \xi_0\rangle \leq 0 \\ &\Longleftrightarrow c(p_0) + \langle p - p_0, \xi_0\rangle \leq 0.\end{aligned}$$

So,

$$C_0 = \{p \in \mathscr{H} : c(p_0) + \langle p - p_0, \xi_0\rangle \leq 0\}. \tag{3.11}$$

By (2.2), one has

$$p_1 = P_{C_0}(p_0) = p_0 - \frac{c(p_0)}{\|\xi_0\|^2}\xi_0.$$

Using (2.2) again, it is easily verified that Case 1 and Case 2 can be incorporated and the projection onto C_0 is given as:

$$P_{C_0}(p_0) = \begin{cases} p_0 - \dfrac{c(p_0)}{\|\xi_0\|^2}\xi_0, & \text{if } c(p_0) > 0, \\ p_0, & \text{if } c(p_0) \le 0. \end{cases}$$

Comparing the above formula and (3.6), we know $P_{C_0}(p_0) = p_{1/2}$. By (3.11), we get the expression of the gradient method with one step in Inner Loop as follows:

$$\begin{cases} y^k = x^k - \gamma_k \nabla F(x^k), \\ x^{k+1} = P_{T_k}(y^k), \end{cases} \tag{3.12}$$

where

$$T_k = \{y \in \mathscr{H} : c(y^k) + \left\langle y - y^k, \xi_k \right\rangle \le 0\}, \xi_k \in \partial c(y^k). \tag{3.13}$$

From (3.13), it concludes that the gradient projection method with one step in Inner Loop is in fact the relaxed gradient projection algorithm (3.12), which needs to compute the projection onto a halfspace containing C per each iteration. Now, Question 1 becomes:

Question 2. Does the relaxed gradient projection algorithm (3.12) converge?

We answer this problem in next section.

It is obvious that algorithm (3.12) is very simple comparing with the outer approximation gradient algorithm (3.10). This motivates us to construct relaxed projection method by combining the projection onto halfspaces like T_k in (3.13) with other projection methods such as the method (1.2).

4. A New Relaxed Projection Algorithm

In this section, by combining projection method (1.2) and the relaxed projection proposed in Section 3, we introduce a new relaxed projection algorithm, which includes the algorithm (3.12) as a special case, for variational inequalities. We analyze the convergence of the proposed algorithm and give the convergence of algorithm (3.12) as a corollary.

Algorithm 4.1. (A new relaxed projection Algorithm)
Step 0: Select a starting point x^0 and set $k = 0$.
Step 1: Take $u^k = x^k - \tau_k f(x^k)$, choose $\xi_k \in \partial c(u^k)$ and let

$$T_k = \left\{u \in \mathscr{H} : c(u^k) + \langle \xi_k, u - u^k \rangle \le 0\right\}.$$

Step 2: Calculate

$$x^{k+1} = P_{T_k}\left(u^k\right).$$

Step 3: If $x^{k+1} = x^k$ and $c(x^k) \le 0$, then terminate. Otherwise, set $k = k+1$ and return to **Step 1**.

Remark 4.2. There are some observations on Algorithm 4.1:

(i) Algorithm (3.12) can be seen as a special case of Algorithm 4.1 when $f = \nabla F$ and $\tau_k = \gamma_k$.

(ii) It is obvious that T_k is a halfspace containing C, which is new and different with halfspace S_k.

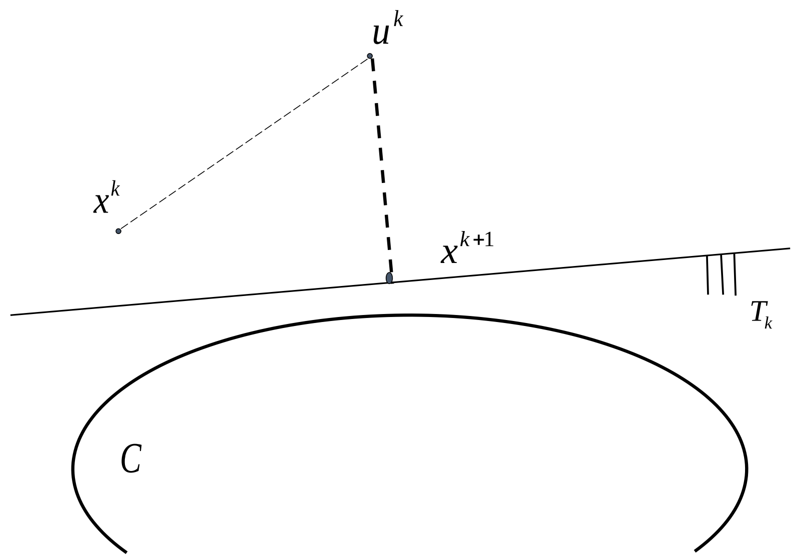

FIGURE 3. x^{k+1} is the projection of the point u^k onto the set T_k

(iii) The stopping criteria of finite iterations of Algorithm 4.1 needs $c(x^k) \leq 0$, which is obviously stronger than that of Algorithm 1.1.

Following the proof of [11, Theorem 1], it is easy to ascertain the validity of the stopping criterion used in Step 3 of Algorithm 4.1.

Lemma 4.3. *If $x^{k+1} = x^k$ and $c(x^k) \leq 0$ for some k, then x^k is a solution of problem* (1.1).

Remark 4.4. The key that determines the calculation efficiency of Algorithm 4.1 (Algorithm 1.1) is the distance of $P_C(u^k)$ and $P_{T_k}(u^k)$ (or $P_{S_k}(u^k)$) per each iteration.

We discuss three cases as follows:

Case 1. $c(u^k) \leq 0$. Note that $u^k \in C$. From $C \subseteq T_k$ and $C \subseteq S_k$, one has

$$P_{S_k}(u^k) = u^k \quad \text{and} \quad P_{T_k}(u^k) = u^k,$$

which implies that x^{k+1} generated by Algorithm 4.1 and Algorithm 1.1 is the same.

Case 2. $c(u^k) > 0$ and $c(x^k) + \langle \eta_k, u^k - x^k \rangle \leq 0$. Note that $u^k \notin T_k$ and $u^k \in S_k$. From (2.2), one ahs

$$P_{S_k}(u^k) = u^k \quad \text{and} \quad P_{T_k}(u^k) = u^k - \frac{c(u^k)}{\|\xi_k\|^2}\xi_k.$$

From Figure 4, the sequence x^{k+1} generated by Algorithm 4.1 is nearer to the projection of u^k onto C than that of Algorithm 1.1.

Case 3. $c(u^k) > 0$ and $c(x^k) + \langle \eta_k, u^k - x^k \rangle > 0$. We have $u^k \notin T_k$ and $u^k \notin S_k$. So,

$$P_{S_k}(u^k) = u^k - \frac{c(x^k)}{\|\eta_k\|^2}\eta_k \quad \text{and} \quad P_{T_k}(u^k) = u^k - \frac{c(u^k)}{\|\xi_k\|^2}\xi_k.$$

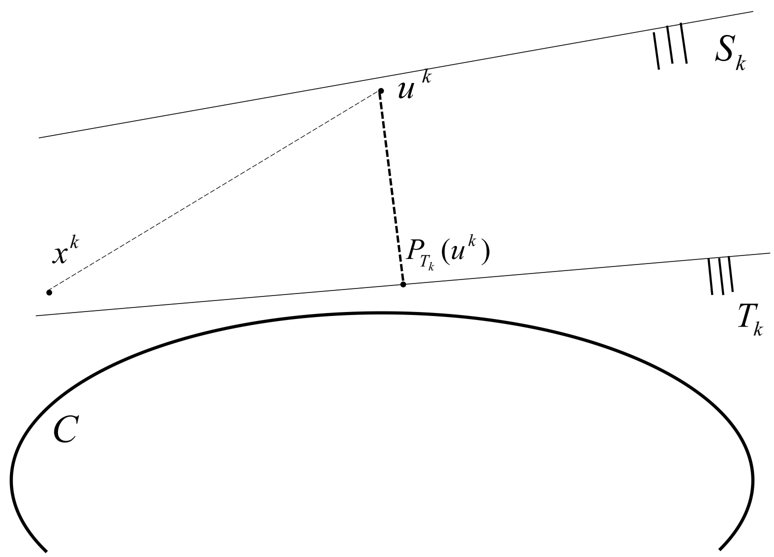

FIGURE 4. For Case 2, x^{k+1} of Algorithm 1.1 and Algorithm 4.1 are u^k and the projection of the point u^k onto the set T_k, respectively.

For this case, one cannot compare the distances of x^{k+1} generated by Algorithm 4.1 and Algorithm 1.1 with C.

Fukushima showed the convergence of Algorithm 1.1 in a finite dimensional Euclidean space $\mathbb{E}$. Next, by following his proof process, we prove the convergence of Algorithm 4.1 in $\mathbb{E}$.

Define the distance from a point $x \in \mathbb{E}$ to the set $C \subseteq \mathbb{E}$ by

$$\text{dist}[x, C] = \min\{\|x - z\| : z \in C\}.$$

Denote, for each $\delta > 0$,

$$C_\delta = \{x \in \mathbb{E} : \text{dist}[x, C] < \delta\}.$$

We make the following assumptions for the function f:
(A1) f is continuous on C_δ for some $\delta > 0$.
(A2) f is strongly monotone on C_δ for some $\delta > 0$, i.e., there exists an $\alpha > 0$ such that

$$\langle f(x) - f(y), x - y\rangle \geq \alpha\|x - y\|^2 \quad \text{for all } x, y \in C_\delta.$$

(A3) For some $z \in C$, there exist a $\beta > 0$ and a bounded set $D \subset \mathbb{E}$ such that

$$\langle f(x), x - z\rangle \geq \beta\|f(x)\| \quad \text{for all } x \notin D.$$

Let $\tau_k = \rho_k/\|f(x_k)\|^2$. We make following assumptions for the sequence of positive parameters $\{\rho_k\}_{k\in\mathbb{N}}$:

$$\text{(B1)} \lim_{k\to\infty} \rho_k = 0 \quad \text{and} \quad \text{(B2)} \sum_{k=0}^{\infty} \rho_k = \infty.$$

From Lemma 4.3, we suppose that Algorithm 4.1 generates an infinite sequence $\{x^k\}_{k\in\mathbb{N}}$. To show the convergence of Algorithm 4.1, we need the following lemmas. The proofs of some lemmas are similar with those in [11], so they are omitted.

Lemma 4.5. [11, Lemma 3] *Assume that the sequence* $\{\rho_k\}_{k\in\mathbb{N}}$ *satisfies* (B1). *Then* $\{x^k\}_{k\in\mathbb{N}}$ *is bounded.*

Lemma 4.6. *Assume that the sequence* $\{\rho_k\}_{k\in\mathbb{N}}$ *satisfies* (B1). *Then*

$$\lim_{k\to\infty} \operatorname{dist}[x^k, C] = 0.$$

Proof. By Lemma 4.5, $\{x^k\}_{k\in\mathbb{N}}$ is contained in a compact subset of $\mathbb{E}$. From Assumption 2.7, the Slater's condition is satisfied. We may therefore show, in a way similar way to [32, Lemma 4], that there exists a constant $\mu \in [0,1)$ such that

$$\operatorname{dist}[P_{T_k}(u^k), C] \le \mu \operatorname{dist}[u^k, C] \quad \text{for } k \ge 0. \tag{4.1}$$

(If $u^k \in C$, (4.1) trivially holds due to $C \subset T_k$.)

From Lemma 2.3, it follows

$$\operatorname{dist}[u^k, C] \le \|u^k - x^k\| + \operatorname{dist}[x^k, C],$$

So

$$\operatorname{dist}[x^{k+1}, C] \le \mu \operatorname{dist}[x^k, C] + \mu\|u^k - x^k\|, \quad \text{for } k \ge 0, \tag{4.2}$$

where $x^{k+1} = P_{T_k}(u^k)$ is used. It is easy to verify

$$\|u^k - x^k\| = \rho_k \|f(x^k)\| / \|f(x^k)\| \le \rho_k.$$

By the assumption on $\{\rho_k\}_{k\in\mathbb{N}}$, one has

$$\lim_{k\to\infty} \|u^k - x^k\| = 0. \tag{4.3}$$

Therefore, the desired result is obtained by Lemma 2.4, (4.2) and (4.3). immediately. □

Lemma 4.7. [11, Lemma 5] *Assume that* $\{\rho_k\}_{k\in\mathbb{N}}$ *satisfies* (B1). *Then* $\lim_{k\to\infty}\|x^{k+1} - x^k\| = 0$.

Employing the arguments which are similar to those used in the proof of [11, Theorem 2], we can easily show the following convergence result.

Theorem 4.8. *Assume that* $\{\rho_k\}_{k\in\mathbb{N}}$ *satisfies* (B1) *and* (B2). *Then the sequence* $\{x^k\}_{k\in\mathbb{N}}$ *generated by Algorithm 4.1 converges to a solution of problem* (1.1).

Remark 4.9. Yang [33] established the convergence of Algorithm 1.1 under the weakly co-coercive condition. It is easy to show the convergence of Algorithm 4.1 under Yang's condition.

By Theorem 4.8, we get the convergence of the relaxed gradient projection algorithm (3.12) in $\mathbb{E}$, which answers Question 2.

Corollary 4.10. *Assume that* $\nabla F(x^k) \neq 0$, $k \ge 0$. *Let* $\gamma_k = \rho_k / \|\nabla F(x^k)\|^2$. *Suppose that* ∇F *satisfies the assumptions* (A1)-(A3), *and the sequence* $\{\rho_k\}_{k\in\mathbb{N}}$ *satisfies* (B1) *and* (B2). *Then the sequence* $\{x^k\}_{k\in\mathbb{N}}$ *generated by* (3.12) *converges to the solution of the problem* (3.1).

Now we discuss Algorithm 4.1 in a Hilbert space, $\mathscr{H}$, and give a result which will be used in Section 5.

Theorem 4.11. *Let $\Phi \subseteq C \subseteq \mathscr{H}$ be a nonempty closed and convex set, and let the sequence $\{x^k\}_{k\in\mathbb{N}}$ be generated by Algorithm 4.1. Assume that $\{\|x^k - x^*\|\}_{k\in\mathbb{N}}$ is convergent with $x^* \in \Phi$, $\lim_{k\to\infty} \|x^{k+1} - x^k\| = 0$ and $\lim_{k\to\infty} \tau_k \|f(x^k)\| = 0$. Then $\omega_w(x^k) \subseteq C$.*

Proof. Since $\{\|x^k - x^*\|\}_{k\in\mathbb{N}}$ is convergent with $x^* \in \Phi$, we have that $\{x^k\}_{k\in\mathbb{N}}$ is bounded. It holds

$$\lim_{k\to\infty} \|x^k - u^k\| = \lim_{k\to\infty} \|\tau_k f(x^k)\| = 0, \tag{4.4}$$

which implies

$$\begin{aligned} \|x^{k+1} - u^k\| &\le \|x^{k+1} - x^k\| + \|x^k - u^k\| \\ &\to 0, \quad \text{as } k \to \infty. \end{aligned} \tag{4.5}$$

Take arbitrarily $\hat{x} \in \omega_w(x^k)$. Then there exists a subsequence of $\{x^k\}_{k\in\mathbb{N}}$ (again labeled $\{x^k\}_{k\in\mathbb{N}}$) which converges weakly to $\hat{x}$. It follows from (4.4) $u^k \rightharpoonup \hat{x}$. Since $x^{k+1} \in T_k$, we have

$$c(u^k) + \langle \xi_k, x^{k+1} - u^k \rangle \le 0,$$

where $\xi_k \in \partial c(u^k)$. Thus

$$c(u^k) \le -\langle \xi_k, x^{k+1} - u^k \rangle \le \xi \|x^{k+1} - u^k\|,$$

where ξ satisfies $\|\xi_k\| \le \xi$ for all $k \in \mathbb{N}$. The lower semicontinuity of c and (4.5) lead to

$$c(\hat{x}) \le \liminf_{k\to\infty} c(u^k) \le 0.$$

Therefore $\hat{x} \in C$ and $\omega_w(x^k) \subseteq C$. □

Remark 4.12. Theorem 4.11 still holds if f is replaced by f_k which varies with the iteration k, $k \ge 0$.

5. Some Applications

In this section, we introduce relaxed projection methods for the split feasibility problem and the split equality problem by using the proposed relaxed projection in Section 4.

5.1. The split feasibility problem.

Let C and Q be the nonempty closed convex subsets of the real Hilbert spaces $\mathscr{H}_1$ and $\mathscr{H}_2$, respectively. Censor and Elfving [34] first introduced the split feasibility problem (SFP), which is formulated as finding a point x^* with the property

$$x^* \in C \quad \text{and} \quad Ax^* \in Q, \tag{5.1}$$

where $A : \mathscr{H}_1 \to \mathscr{H}_2$ is a bounded linear operator. The SFP can be a model for many inverse problems where constraints are imposed on the solutions in the domain of a linear operator as well as in the operator's range.

Assume that the SFP (5.1) is consistent (i.e., has a solution), and denote by Ω its solution, i.e.,

$$\Omega = \{x \in C : Ax \in Q\}.$$

We make the following assumptions for sets C and Q.

Assumption 5.1. Assume that the closed convex subsets C and Q are of the particular structure, i.e., level sets of convex functions given as follows:

$$C = \{x \in \mathscr{H}_1 : c(x) \leq 0\} \quad \text{and} \quad Q = \{y \in \mathscr{H}_2 : q(y) \leq 0\}, \tag{5.2}$$

where $c : \mathscr{H}_1 \to \mathbb{R}$, and $q : \mathscr{H}_2 \to \mathbb{R}$ are convex functions. Suppose that c and q satisfy Assumption 2.8.

Let

$$F_k(x) = \frac{1}{2}\|(I - P_{Q_k})Ax\|^2, \quad k \geq 0. \tag{5.3}$$

We then have

$$\nabla F_k(x) = A^*(I - P_{Q_k})Ax. \tag{5.4}$$

Following Fukushima's idea, López et al. [16] studied the relaxed CQ algorithms for the SFP.

Algorithm 5.2. (López et al's relaxed CQ algorithm) Choose an initial guess $x^0 \in \mathscr{H}_1$ arbitrarily. Assume that the x^k has been constructed. Construct x^{k+1} via the formula:

$$x^{k+1} = P_{S_k}(x^k - \alpha_k \nabla F_k(x^k)), \tag{5.5}$$

where S_k and Q_k are respectively defined by

$$S_k = \{x \in \mathscr{H}_1 : c(x^k) + \langle \eta_k, x - x^k \rangle \leq 0\}, \tag{5.6}$$

where $\eta_k \in \partial c(u^k)$, and

$$Q_k = \{y \in \mathscr{H}_2 : q(Ax^k) + \langle \zeta_k, y - Ax^k \rangle \leq 0\}, \tag{5.7}$$

where $\zeta_k \in \partial q(v^k)$, and the stepsize α_k is chosen as follows:

$$\alpha_k = \rho_k \frac{F_k(x^k)}{\|\nabla F_k(x^k)\|^2}, \tag{5.8}$$

where $\rho_k \in (0, 4)$.

By using the new proposed halfspace to replace the halfspace S_k in Algorithm 5.2, we get the following.

Algorithm 5.3. Choose an initial guess $x^0 \in \mathscr{H}_1$ arbitrarily. Assume that the x^k has been constructed. Construct x^{k+1} via the formula:

$$x^{k+1} = P_{T_k}(x^k - \alpha_k \nabla F_k(x^k)). \tag{5.9}$$

where α_k is given as in (5.8), and T_k is defined by

$$T_k = \{x \in H_1 : c(u^k) + \langle \xi_k, x - u^k \rangle \leq 0\}, \tag{5.10}$$

where $u^k = x^k - \alpha_k \nabla F_k(x^k)$ and $\xi_k \in \partial c(u^k)$.

Remark 5.4. Note that the relaxed halfspace Q_k is the same for Algorithm 5.3 and Algorithm 5.2. This is because the definition of Q_k uses the similar idea of the definition of T_k.

Employing the arguments which are similar to those used in the proof of [16, Lemma 4.2], we can easily show the following lemma.

Lemma 5.5. *If $x^k = x^{k+1}$ and $c(x^k) \leq 0$ for some $k \geq 0$, then x^k is a solution of the SFP* (5.1).

Following the proof of [16, Lemma 3.4], it is easy to verify the the following lemma.

Lemma 5.6. *Assume $\Omega \neq \emptyset$ and let the sequence $\{x^k\}_{k\in\mathbb{N}}$ be generated by Algorithm 5.3. Then, for any $z \in \Omega$, it holds*

$$\|x^{k+1} - z\|^2 \leq \|x^k - z\|^2 - \rho_k(4-\rho_k)\frac{F_k^2(x^k)}{\|\nabla F_k(x^k)\|^2}. \tag{5.11}$$

Next we give the convergence result of Algorithm 5.3.

Theorem 5.7. *Assume that $\inf_k \rho_k(4-\rho_k) > 0$. Then the sequence $\{x^k\}_{k\in\mathbb{N}}$ generated by Algorithm 5.3 converges weakly to a solution of the SFP (5.1).*

Proof. Let $z \in \Omega$ be fixed. From Lemma 5.6, it follows that $\{\|x^k - z\|\}_{k\in\mathbb{N}}$ is convergent, and thus $\{x^k\}_{k\in\mathbb{N}}$ is bounded. From (5.11) and the assumption on ρ_k, on has

$$\sum_{k=1}^{\infty} \frac{F_k^2(x^k)}{\|\nabla F_k(x^k)\|^2} < \infty. \tag{5.12}$$

Let $\hat{x} \in \omega_w(x^k)$. Using the same technique as in [16, Theorem 4.3], we obtain $A\hat{x} \in Q$.

Next we show that $\omega_w(x^k) \subseteq C$. From (5.12), it follows that $\lim_{k\to\infty} \frac{F_k(x^k)}{\|\nabla F_k(x)\|} = 0$, which together with the formula of α_n in (5.8) and $\rho_k \in (0,4)$ implies that

$$\lim_{k\to\infty} \alpha_k \|\nabla F_k(x^k)\| = 0. \tag{5.13}$$

It holds

$$\|x^{k+1} - x^k\|^2 = \|x^k - z\|^2 - \|x^{k+1} - z\|^2 + 2\langle x^{k+1} - x^k, x^{k+1} - z\rangle. \tag{5.14}$$

Using Lemma 2.2, we have

$$\langle x^{k+1} - (x^k - \alpha_k \nabla F_k(x^k)), x^{k+1} - z\rangle \leq 0,$$

which implies

$$\begin{aligned} \langle x^{k+1} - x^k, x^{k+1} - z\rangle &\leq \langle \alpha_k \nabla F_k(x^k), x^{k+1} - z\rangle \\ &\leq \alpha_k \|\nabla F_k(x^k)\| \|x^{k+1} - z\|. \end{aligned} \tag{5.15}$$

Combining (5.13), (5.14) and (5.15), and using the fact that $\{\|x^k - z\|\}_{k\in\mathbb{N}}$ is convergent, we get

$$\lim_{k\to\infty} \|x^{k+1} - x^k\| = 0. \tag{5.16}$$

From (5.13), (5.16) and Theorem 4.11, we obtain $\omega_w(x^k) \subseteq C$. Consequently, $\omega_w(x^k) \subset \Omega$. By Lemma 2.5 and the convergence of $\{\|x^k - z\|\}_{k\in\mathbb{N}}$, the proof is completed. □

5.2. **The split equality problem.** Let $\mathscr{H}_1$, $\mathscr{H}_2$ and $\mathscr{H}_3$ be real Hilbert spaces. Let $C \subset \mathscr{H}_1$, $Q \subset \mathscr{H}_2$ be two nonempty closed convex sets, and let $A : \mathscr{H}_1 \to \mathscr{H}_3$ and $B : \mathscr{H}_2 \to \mathscr{H}_3$ be two bounded linear operators. The split equality problem (SEP) is to find

$$x \in C,\ y \in Q, \text{ such that } Ax = By, \tag{5.17}$$

which allows asymmetric and partial relations between the variables x and y. If $\mathscr{H}_2 = \mathscr{H}_3$ and $B = I$, then split equality problem (5.17) reduces to split feasibility problem (5.1). Assume SEP (5.17) is consistent (i.e., has a solution) and denote by Γ its solution, i.e.,

$$\Gamma = \{x \in C, y \in Q : Ax = By\}.$$

By using Fukushima's idea, Moudafi [35] introduced a relaxed projection method for the SEP. Recently, many authors further improved Moudafi's method by selecting the stepsizes such that the implementation of the method does not need any priori information about the operator norms [12, 31].

Here, by using the new proposed halfspaces to replace the halfspaces in [12, Algorithm 4.1], we propose the following relaxed projection method.

Algorithm 5.8. Choose an initial guess $x^0 \in \mathscr{H}_1$ and $y^0 \in \mathscr{H}_2$ arbitrarily. Assume that the (x^k, y^k) has been constructed. Construct (x^{k+1}, y^{k+1}) via the formula:

$$\begin{cases} x^{k+1} = P_{T_k}(x^k - \beta_k A^*(Ax^k - By^k)), \\ y^{k+1} = P_{Q_k}(y^k + \beta_k B^*(Ax_k - By^k)), \end{cases} \tag{5.18}$$

where T_k and Q_k are respectively defined by

$$T_k = \{x \in \mathscr{H}_1 : c(u^k) + \langle \zeta_k, x - u^k \rangle \leq 0\}, \tag{5.19}$$

where $u^k = x^k - \beta_k A^*(Ax^k - By^k)$, $\zeta_k \in \partial c(u^k)$, and

$$Q_k = \{y \in \mathscr{H}_2 : q(v^k) + \langle \varsigma_k, y - v^k \rangle \leq 0\}, \tag{5.20}$$

where $v^k = y^k + \beta_k B^*(Ax_k - By^k)$, $\varsigma_k \in \partial q(v_k)$, and the stepsize β_k is given as

$$\beta_k = \sigma_k \frac{2\|Ax^k - By^k\|^2}{\|A^*(Ax^k - By^k)\|^2 + \|B^*(Ax^k - By^k)\|^2}, \tag{5.21}$$

where $\sigma_k \in (0,1)$.

Note that the choice of the stepsize β_k in Algorithm 5.8 comes from [31, Algorithm 3.1]. Employing the arguments which are similar to those used in the proof of [12, Lemma 4.2 and Lemma 4.3], we can easily show the following lemmas.

Lemma 5.9. [12, Lemma 4.2] *Let the sequence* $\{(x^k, y^k)\}_{k\in\mathbb{N}}$ *be generated by Algorithm 5.8. Then* $\{\|x^k - x^*\|^2 + \|y^k - y^*\|^2\}_{k\in\mathbb{N}}$ *is convergent with* $(x^*, y^*) \in \Gamma$.

Lemma 5.10. [12, Lemma 4.3] *Let the sequence* $\{(x^k, y^k)\}_{k\in\mathbb{N}}$ *be generated by Algorithm 5.8. Assume* $\sigma_k \in (\varepsilon, 1-\varepsilon)$. *Then, there hold*

$$\lim_{k\to\infty} \beta_k \|A^*(Ax_k - By_k)\| = 0 \quad \textit{and} \quad \lim_{k\to\infty} \beta_k \|B^*(Ax_k - By_k)\| = 0. \tag{5.22}$$

Lemma 5.11. [12, Theorem 4.1] *Assume* $\sigma_k \in (\varepsilon, 1-\varepsilon)$. *Then the sequence* $\{(x^k, y^k)\}_{k\in\mathbb{N}}$ *generated by Algorithm 5.8 satisfies*

$$\lim_{k\to\infty} \|x^{k+1} - x^k\| = 0, \quad \textit{and} \quad \lim_{k\to\infty} \|y^{k+1} - y^k\| = 0.$$

Let $S = C \times Q$ in $\mathscr{H} = \mathscr{H}_1 \times \mathscr{H}_2$, and define $G : \mathscr{H} \to \mathscr{H}$ by $G = [A, -B]$. Then, $G^*G : \mathscr{H} \to \mathscr{H}$ has the matrix form

$$G^*G = \begin{pmatrix} A^*A & -A^*B \\ -B^*A & B^*B \end{pmatrix}. \tag{5.23}$$

The SEP can be reformulated as finding

$$w = (x,y) \in S \quad \text{with} \quad Gw = 0. \tag{5.24}$$

Denote by $\Upsilon = \{w \in C \times Q : (A, -B)w = 0\}$ the solution of the SEP (5.24). Let $w^k = (x^k, y^k)$. Then iterative formula (5.18) can be rewritten as:

$$w^{k+1} = P_{Z_k}(w^k - \beta_k G^* G w^k), \tag{5.25}$$

where $Z_k = T_k \times Q_k$.

Theorem 5.12. *Assume $\sigma_k \in (\varepsilon, 1-\varepsilon)$. Then the sequence $\{(x^k, y^k)\}_{k\in\mathbb{N}}$ generated by Algorithm 5.8 weakly converges to a solution of the SEP (5.17).*

Proof. Let $w^* \in \Upsilon$. By Lemma 5.9, $\{\|w^k - w^*\|\}_{k\in\mathbb{N}}$ is convergent. From Lemma 5.10 and (5.23), it follows $\lim_{k\to\infty} \|\beta_k G^* G w^k\| = 0$. Using Lemma 5.11, we have $\lim_{k\to\infty} \|w^{k+1} - w^k\| = 0$. Applying Theorem 4.11 to (5.25), we get $\omega_w(w^k) \subseteq \Upsilon$. Using Lemma 2.5, it follows that $\{w^k\}_{k\in\mathbb{N}}$ converges weakly to a point of Υ, that is, $\{(x^k, y^k)\}_{k\in\mathbb{N}}$ generated by Algorithm 5.8 weakly converges to a solution of the SEP (5.17). The proof is completed. □

6. NUMERICAL RESULTS

Consider the following LASSO problem [36]:

$$\min\{(1/2)\|Ax - b\|_2^2 : x \in \mathbb{R}^n, \|x\|_1 \le \tau\}, \tag{6.1}$$

where $A \in \mathbb{R}^{m\times n}$, $m < n$, $b \in \mathbb{R}^n$ and $\tau > 0$. We generate the system matrix A from a standard normal distribution with mean zero and unit variance. The true sparse signal x^* is generated from uniformly distribution in the interval $[-2, 2]$ with random K position nonzero while the rest is kept zero. The sample data $b = Ax^*$.

Under certain condition on matrix A, the solution of minimization problem (6.1) is equivalent to the ℓ_0-norm solution of the underdetermined linear system. For (5.1), we define $C = \{x | \|x\|_1 \le \tau\}$ and $Q = \{b\}$. Since the projection onto the closed convex C does not have a closed form solution, so we make use of the subgradient projection. Define a convex function $c(x) = \|x\|_1 - \tau$, and denote the level set C_k by:

$$C_k = \{x : c(x^k) + \langle \xi^k, x - x^k\rangle \le 0\}, \tag{6.2}$$

where $\xi^k \in \partial c(x^k)$. It is worth noting that the subdifferential ∂c at x^k is

$$\partial c(x^k) = \begin{cases} 1, & \text{if } x^k > 0, \\ [-1, 1], & \text{if } x^k = 0, \\ -1, & \text{if } x^k < 0. \end{cases} \tag{6.3}$$

In this example, we apply Algorithm 5.2 and Algorithm 5.3 to solve the LASSO problem [36], which aims to finding a sparse solution of an underdetermined linear system. The iteration process is stopped when the criteria "Error=$\|x^k - x^*\| \le \varepsilon$" is satisfied, where ε is a given small constant. Here, we take $\varepsilon = 0.1$. The results are reported in Table 2 and Figure 5, which illustrate that Algorithm 5.3 needs half iterative number of Algorithm 5.2 and the CPU time of Algorithm 5.3 is about a forth that of Algorithm 5.2 for big size problems. From Figure 6, it is concluded that the estimates obtained by Algorithm 5.3 are quite close to the true values.

TABLE 2. Computational results for solving the Lasso problem with Algorithm 5.1. and Algorithm 5.2.

Problem size			Iter		CPU time		Error	
m	n	K	Alg.5.2	Alg.5.1	Alg.5.2	Alg.5.1	Alg.5.2	Alg.5.1
90	320	10	131	288	0.2344	0.3750	0.0987	0.0997
180	640	20	134	299	0.3281	0.7344	0.0978	0.0994
270	960	30	144	342	1.2031	4.5156	0.0977	0.0992
360	1280	40	160	350	2.3750	9.3906	0.0987	0.0997
450	1600	50	175	391	4.6875	19.1406	0.0994	0.0998
540	1920	60	205	448	8.2500	32.1719	0.0978	0.0995
630	2240	70	249	540	14.4219	59.2500	0.0995	0.0995
720	2560	80	290	613	22.2656	93.6406	0.0998	0.0999

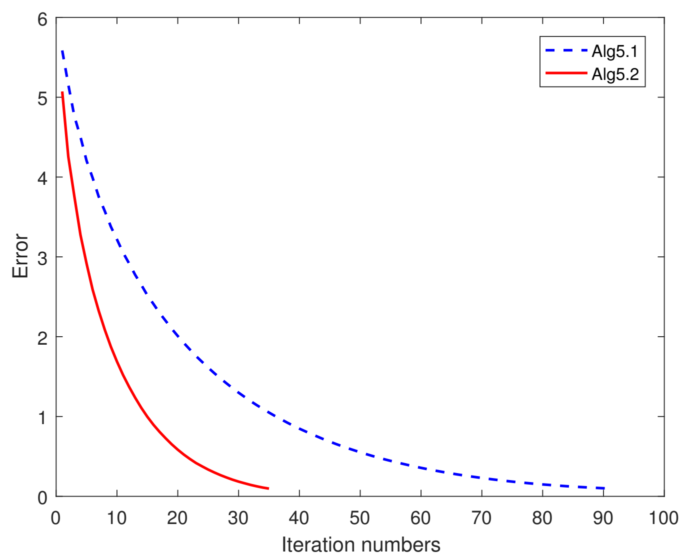

FIGURE 5. Error versus the iteration numbers for $n = 1024, m = 512, K = 20$.

7. CONCLUDING REMARKS

Inspired by the outer approximation method in [29], we introduced a new relaxed projection, which involves a projection onto a halfspace containing the given closed convex set rather than the latter set itself per each iteration. Fukushima's projection method with new relaxed projection was shown to converge to a solution of the variational inequality under the same conditions in [11]. The applications to the split feasibility problem and the split equality problem are presented. Finally, a preliminary numerical experiment was given to illustrate the advantage of the new relaxed projection over Fukushima's relaxed projection.

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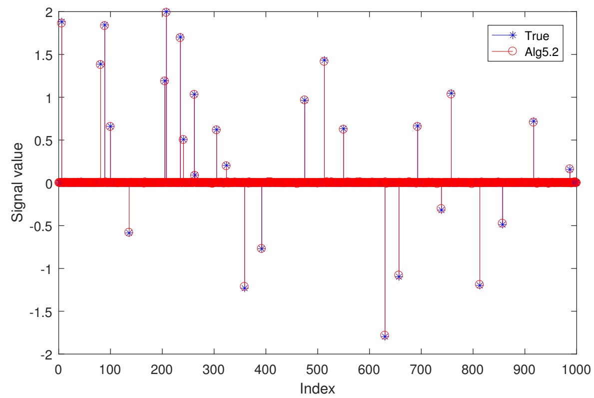

FIGURE 6. The original and recovered signal using the algorithm 5.2 where $n = 1024, m = 512, K = 30$.

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