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A NEW ITERATION SCHEME FOR MIXED TYPE ASYMPTOTICALLY NONEXPANSIVE MAPPINGS IN HYPERBOLIC SPACES

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Abstract. In this paper, we study the strong convergence of a new iteration scheme for two asymptotically nonexpansive self-mappings and two asymptotically nonexpansive nonself-mappings in the setting of uniformly convex hyperbolic spaces. The results presented in this paper extend and improve some recent results announced in the literature.

Keywords. Mixed type asymptotically nonexpansive mapping; Strong convergence; Common fixed points; Uniformly convex hyperbolic space.

1. Introduction and Preliminaries

Fixed point problems of nonlinear mappings play an important role in many real world problems, such as, image recovery and signal processing and iterative methods are efficient for studying fixed points of nonlinear mappings; see, e.g, [1, 2, 3, 4, 5] and the references therein. Recently, many new convergence theorems of fixed points were established in hyperbolic spaces, which are general in nature and inherit rich geometrical structures; see, e.g, [6, 7, 8, 9, 10] and the references therein.

Throughout this paper, we work in the setting of hyperbolic spaces, which was introduced by Kohlenbach [11]. Recall that a hyperbolic space (X,d,H) is a metric space (X,d) together with a mapping $H: X\times X\times[0,1]\to X$ satisfying

(H1): $d(z,H(x,y,\beta))\leq(1-\beta)d(z,x)+\beta d(z,y)$,

(H2): $d(H(x,y,\beta),H(x,y,\gamma))=|\beta-\gamma|\,d(x,y)$,

(H3): $H(x,y,\beta)=H(y,x,(1-\beta))$,

(H4): $d(H(x,z,\beta),H(y,w,\beta))\leq(1-\beta)d(x,y)+\beta d(z,w)$

for all $x,y,w,z\in X$ and $\beta,\gamma\in[0,1]$.

A subset K of a hyperbolic space X is convex if $H(x,y,\beta)\in K$ for all $x,y\in K$ and $\beta\in[0,1]$. If a space satisfies $(H1)$ only, it coincides with the convex metric space introduced by Takahashi

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[12]. The concept of hyperbolic spaces in [11] is more restrictive than the hyperbolic type introduced by Goebel et al. [13]. On the other hand, it is general than the hyperbolic space defined by Reich et al. [14].

Recall that a hyperbolic space (X,d,H) is said to be
(i) strictly convex [12] if, for any $x,y \in X$ and $\beta \in [0,1]$, there exists a unique element $z \in X$ such that $d(z,x) = \beta d(x,y)$ and $d(z,y) = (1-\beta)d(x,y)$;
(ii) uniformly convex [15] if, for all $x,y,w \in X$, $r > 0$ and $\varepsilon \in (0,2]$, there exists $\delta \in (0,1]$ such that $d(H(x,y,\frac{1}{2}),x) \leq (1-\delta)r$ whenever $d(x,w) \leq r, d(y,w) \leq r$ and $d(x,y) \geq \varepsilon r$.

Recall that a mapping $\eta : (0,\infty) \times (0,2] \to (0,1]$ is sid to be the modulus of uniform convexity if $\delta = \eta(r,\varepsilon)$ for given $r > 0$ and $\varepsilon \in (0,2]$. It is said to be η-monotone if it decreases with r (for a fixed ε). A uniformly convex hyperbolic space is strictly convex (see [16]).

In the sequel, let (X,d) be a metric space, and let K be a nonempty subset of X. We denote the fixed point set of a mapping T by

$$F(T) = \{x \in K : Tx = x\}$$

and

$$d(x,F(T)) = \inf\{d(x,p) : p \in F(T)\}.$$

Recall that a self-mapping $T : K \to K$ is said to be nonexpansive if $d(Tx,Ty) \leq d(x,y)$ for all $x,y \in K$. Recently, various fixed-point iteration processes for nonexpansive mappings have been studied extensively by many authors [17, 18, 19, 20, 21].

Recall that T is said to be asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [1,\infty)$ with $k_n \to 1$ such that $d(T^n x, T^n y) \leq k_n d(x,y)$ for all $x,y \in K$ and $n \geq 1$.

In 1972, Goebel and Kirk [22] introduced the class of asymptotically nonexpansive self-mappings. They proved that if K is nonempty closed convex subset of a real uniformly convex Banach space and T is an asymptotically nonexpansive self-mapping on K, then T has a fixed point.

Recall that T is said to be uniformly L-Lipschitzian if there exists a constant $L > 0$ such that $d(T^n x, T^n y) \leq L d(x,y)$ for all $x,y \in K$ and $n \geq 1$. From the above definitions, one clearly sees that each nonexpansive mapping is an asymptotically nonexpansive mapping with $k_n = 1, \forall n \geq 1$. Both nonexpansive mappings and asymptotically nonexpansive mappings are Lipschitzian continuous. To be more precise, each nonexpansive mapping is 1-Lipschitzian and each asymptotically nonexpansive mapping is uniformly L-Lipschitzian mapping with $L = \sup_{n\in\mathbb{N}}\{k_n\}$.

In 1991, Schu [23] introduced the following modified Mann iteration process

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n x_n, \quad n \geq 1, \tag{1.1}$$

to approximate fixed points of asymptotically nonexpansive self-mappings in a Hilbert space. Since then, Schu's iteration process (1.1) has been widely used to approximate fixed points of asymptotically nonexpansive self-mappings in Hilbert spaces or Banach spaces; see, e.g., [24, 25, 26] and the references therein.

Recall that a subset K of space X is said to be a retract if there exists a continuous mapping $P : X \to K$ such that $Px = x$, $\forall x \in K$. $P : X \to K$ is said to be a retraction if $P^2 = P$. If P is a retraction, then $x = Px$ for all x in the range of P. We refer to [27, 28, 29] for more details.

For any nonempty subset K of a real metric space (X,d), let $P : X \to K$ be a nonexpansive retraction of X onto K. $T : K \to X$ is said to be an asymptotically nonexpansive nonself-mapping (see [30]) if there exists a sequence $\{k_n\} \subset [1,\infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$d(T(PT)^{n-1}x, T(PT)^{n-1}y) \leq k_n d(x,y) \tag{1.2}$$

for all $x,y \in K$ and $n \geq 1$. We denote by $(PT)^0$ the identity mapping from K onto itself. We see that if T is a self-mapping, then P becomes the identity mapping.

For asymptotically nonexpansive nonself-mappings Chidume, Ofoedu, and Zegeye [30] studied the following iterative sequence

$$x_{n+1} = P((1-\alpha_n)x_n + \alpha_n T(PT)^{n-1}x_n) \tag{1.3}$$

to approximate some fixed point of mapping T. They obtained a convergence theorem under suitable conditions in real uniformly convex Banach spaces. If T is a self-mapping, then P becomes the identity mapping. Hence, (1.3) reduces to (1.1).

In 2006, Wang [31] considered the following iteration process which is a generalization of (1.3),

$$\begin{aligned} y_n &= P((1-\beta_n)x_n + \beta_n T_2(PT_2)^{n-1}x_n), \\ x_{n+1} &= P((1-\alpha_n)x_n + \alpha_n T_1(PT_1)^{n-1}y_n),\ n \geq 1, \end{aligned} \tag{1.4}$$

where $T_1, T_2 : K \to E$ are asymptotically nonexpansive nonself-mappings, and $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[0,1)$. They obtain a strong convergence theorem under weak restrictions imposed on the control parameters.

In 2009, Thianwan [32] studied the following modified Ishikawa iterative scheme for a pari of asymptotically nonexpansive nonself-mappings T_1 and T_2

$$\begin{aligned} y_n &= P((1-\beta_n)x_n + \beta_n T_2(PT_2)^{n-1}x_n), \\ x_{n+1} &= P((1-\alpha_n)y_n + \alpha_n T_1(PT_1)^{n-1}y_n), n \geq 1, \end{aligned} \tag{1.5}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are appropriate real sequences in $[0,1)$. They obtain several strong convergence theorem under various conditions. A weak convergence theorem was also obtained in uniformly convex Banach space which satisfies Opial's condition.

In 2012, Guo, Cho and Guo [33] further studied the following iteration scheme

$$\begin{aligned} y_n &= P((1-\beta_n)S_2^n x_n + \beta_n T_2(PT_2)^{n-1}x_n), \\ x_{n+1} &= P((1-\alpha_n)S_1^n x_n + \alpha_n T_1(PT_1)^{n-1}y_n),\ n \geq 1, \end{aligned} \tag{1.6}$$

where $S_1, S_2 : K \to K$ are asymptotically nonexpansive self-mappings, $T_1, T_2 : K \to E$ are asymptotically nonexpansive nonself-mappings, and $\{\alpha_n\}$, $\{\beta_n\}$ are two sequences in $[0,1)$. Weak and strong convergence theorems of common fixed points of S_1, S_2, T_1 and T_2 were obtained.

Let K be a nonempty closed convex subset of a real uniformly convex hyperbolic space (X,d,H) and $P : X \to K$ be a nonexpansive retraction of X onto K. Let $S_1, S_2 : K \to K$ be two asymptotically nonexpasive self-mappings, and let $T_1, T_2 : K \to K$ be two asymptotically nonexpasive nonself-mappings. For an arbitrary $x_1 \in K$, we suggest the following new iterative scheme for mixed type asypmtotically nonexpansive mappings

$$\begin{aligned} x_{n+1} &= P(H(S_1^n y_n, T_1(PT_1)^{n-1}y_n, \alpha_n)), \\ y_n &= P(H(S_2^n x_n, T_2(PT_2)^{n-1}x_n, \beta_n)), \end{aligned} \tag{1.7}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $[0,1)$.

In this paper, we, motivated by the above recent results, study the strong convergence of the new iteration scheme for two asymptotically nonexpansive self-mappings S_1 and S_2, and two asymptotically nonexpansive nonself-mappings T_1 and T_2 in the setting of uniformly convex hyperbolic spaces. The results presented in this paper extend and improve some recent results announced in the literature. To show our main convergence theorems, we shall need the following useful lemmas.

Lemma 1.1. [34] *Let $\{a_n\}, \{b_n\}$ and $\{c_n\}$ be sequences of non-negative real numbers such that $a_{n+1} \leq (1+b_n)a_n + c_n$, $\forall n \geq 1$. If $\sum_{n=1}^{\infty} b_n < \infty$ and $\sum_{n=1}^{\infty} c_n < \infty$, then $\lim_{n\to\infty} a_n$ exists.*

Lemma 1.2. [7] *Let $\{x_n\}$ and $\{y_n\}$ be two sequences of a uniformly convex hyperbolic space (X,d,H) such that, for $r \in [0,\infty)$, $\limsup_{n\to\infty} d(x_n,a) \leq r$, $\limsup_{n\to\infty} d(y_n,a) \leq r$, and*

$$\lim_{n\to\infty} d(H(x_n,y_n,\alpha_n),a) = r,$$

where $\alpha_n \in [a,b]$ with $0 < a \leq b < 1$. Then, $\lim_{n\to\infty} d(x_n,y_n) = 0$.

2. Main Results

In this section, we denote the set of common fixed points of S_1, S_2, T_1 and T_2 by F, that is, $F := F(S_1) \cap F(S_2) \cap F(T_1) \cap F(T_2)$. The following lemmas are needed to prove our main results.

Lemma 2.1. *Let (X,d,H) be a real uniformly convex hyperbolic space and let K be a nonempty closed convex subset of X. Let $S_1, S_2 : K \to K$ be two asymptotically nonexpasive self-mappings with $\{k_n^{(1)}\}, \{k_n^{(2)}\} \subset [1,\infty)$ and let $T_1, T_2 : K \to X$ be two asymptotically nonexpasive nonself-mappings with $\{l_n^{(1)}\}, \{l_n^{(2)}\} \subset [1,\infty)$ such that $\sum_{n=1}^{\infty}(k_n^{(i)} - 1) < \infty$ and $\sum_{n=1}^{\infty}(l_n^{(i)} - 1) < \infty$ for $i = 1,2$, respectively and $F \neq \emptyset$. Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[0,1)$. From an arbitrary $x_1 \in K$, let $\{x_n\}$ be the sequence defined by (1.7). Then $\lim_{n\to\infty} d(x_n,q)$ exists for any $q \in F$.*

Proof. Let $q \in F$ and set $h_n = max\{k_n^{(1)}, k_n^{(2)}, l_n^{(1)}, l_n^{(2)}\}$. Using (1.7), we have

$$\begin{aligned} d(y_n,q) &= d\left(P(H\left(S_2^n x_n, T_2(PT_2)^{n-1}x_n, \beta_n)\right), q\right) \\ &\leq d\left((H\left(S_2^n x_n, T_2(PT_2)^{n-1}x_n, \beta_n)\right), q\right) \\ &\leq (1-\beta_n)\, d(S_2^n x_n, q) + \beta_n d\left(T_2(PT_2)^{n-1}x_n, q\right) \\ &\leq h_n d(x_n,q), \end{aligned} \tag{2.1}$$

which implies that

$$\begin{aligned} d(x_{n+1},q) &= d\left(P(H\left(S_1^n y_n, T_1(PT_1)^{n-1}y_n, \alpha_n)\right), q\right) \\ &\leq d\left(H\left(S_1^n y_n, T_1(PT_1)^{n-1}y_n, \alpha_n\right), q\right) \\ &\leq (1-\alpha_n)\, d(S_1^n y_n, q) + \alpha_n d\left(T_1(PT_1)^{n-1}y_n, q\right) \\ &\leq h_n d(y_n,q) \\ &\leq h_n^2 d(x_n,q) \\ &\leq \left(1 + (h_n^2 - 1)\right) d(x_n,q). \end{aligned} \tag{2.2}$$

Since $\sum_{n=1}^{\infty}(k_n^{(i)}-1)<\infty$ and $\sum_{n=1}^{\infty}(l_n^{(i)}-1)<\infty$ for $i=1,2$, we have $\sum_{n=1}^{\infty}(h_n^2-1)<\infty$. It follows from Lemma 1.1 that $\lim_{n\to\infty}d(x_n,q)$ exists. □

Lemma 2.2. *Let (X,d,W) be a real uniformly convex hyperbolic space and let K be a nonempty closed convex subset of X. Let $S_1,S_2:K\rightharpoonup K$ be two asymptotically nonexpansive self-mappings with $\{k_n^{(1)}\},\{k_n^{(2)}\}\subset[1,\infty)$ and let $T_1,T_2:K\to X$ be two asymptotically nonexpansive nonself-mappings with $\{l_n^{(1)}\},\{l_n^{(2)}\}\subset[1,\infty)$ such that $\sum_{n=1}^{\infty}(k_n^{(i)}-1)<\infty$ and $\sum_{n=1}^{\infty}(l_n^{(i)}-1)<\infty$ for $i=1,2$, and $F\neq\emptyset$. From an arbitrary $x_1\in K$, let $\{x_n\}$ be the sequence defined by (1.7). Assume that the following conditions hold:*

(i) $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[\varepsilon,1-\varepsilon]$ for some $\varepsilon\in(0,1)$;

(ii) $d(x,T_iy)\le d(S_ix,T_iy)$ for all $x,\ y\in K$ and $i=1,\ 2$.

Then, $\lim_{n\to\infty}d(x_n,S_ix_n)=\lim_{n\to\infty}d(x_n,T_ix_n)=0$ for $i=1,\ 2$.

Proof. Let $q\in F$ and set $h_n=\max\{k_n^{(1)},k_n^{(2)},l_n^{(1)},l_n^{(2)}\}$. From Lemma 2.1, we have $\lim_{n\to\infty}d(x_n,q)$ exists. Assume $\lim_{n\to\infty}d(x_n,q)=c$. Letting $n\to\infty$ in (2.2), we have

$$\lim_{n\to\infty}d(H(S_1^ny_n,T_1(PT_1)^{n-1}y_n,\alpha_n),q)=c. \tag{2.3}$$

In addition, using (2.1), we have $d(S_1^ny_n,q)\le h_n^2d(x_n,q)$. Taking the lim sup on both sides in this inequality, we have

$$\lim_{n\to\infty}\sup d(S_1^ny_n,q)\le c. \tag{2.4}$$

Taking the lim sup on both sides of (2.1), we obtain $\lim_{n\to\infty}\sup d(y_n,q)\le c$, and then

$$\lim_{n\to\infty}\sup d(T_1(PT_1)^{n-1}y_n,q)\le\lim_{n\to\infty}\sup h_nd(y_n,q)=c. \tag{2.5}$$

Using (2.3), (2.4), (2.5), and Lemma 1.2, we have

$$\lim_{n\to\infty}d(S_1^ny_n,T_1(PT_1)^{n-1}y_n)=0. \tag{2.6}$$

From condition (ii), we have

$$d(y_n,T_1(PT_1)^{n-1}y_n)\le d(S_1^ny_n,T_1(PT_1)^{n-1}y_n). \tag{2.7}$$

Letting $n\to\infty$ in (2.7), we conclude from (2.6) that

$$\lim_{n\to\infty}d(y_n,T_1(PT_1)^{n-1}y_n)=0. \tag{2.8}$$

Using (2.2), we have

$$\begin{aligned}d(x_{n+1},q)&\le(1-\alpha_n)d(S_1^ny_n,q)+\alpha_nd(T_1(PT_1)^{n-1}y_n,q)\\&\le(1-\alpha_n)d(S_1^ny_n,q)+\alpha_nd(S_1^ny_n,T_1(PT_1)^{n-1}y_n)+\alpha_nd(S_1^ny_n,q)\\&=d(S_1^ny_n,q)+\alpha_nd(S_1^ny_n,T_1(PT_1)^{n-1}y_n)\\&\le h_nd(y_n,q)+\alpha_nd(S_1^ny_n,T_1(PT_1)^{n-1}y_n).\end{aligned} \tag{2.9}$$

Taking the liminf on both sides of (2.9), and $\sum_{n=1}^{\infty}(h_n-1)<\infty$ and $\lim_{n\to\infty}d(x_{n+1},q)=c$, we conclude from (2.6) that

$$\lim_{n\to\infty}\inf d(y_n,q)\ge c. \tag{2.10}$$

Since $\lim_{n\to\infty}\sup d(y_n,q)\le c$, we conclude from (2.10) that $\lim_{n\to\infty} d(y_n,q)=c$. It follows from (2.1) that

$$\begin{aligned} c=\lim_{n\to\infty} d(y_n,q) &\le \lim_{n\to\infty} d(H(S_2^n x_n, T_2(PT_2)^{n-1}x_n,\beta_n),q)\\ &\le \lim_{n\to\infty} d(x_n,q)=c. \end{aligned}$$

This further yields that

$$\lim_{n\to\infty} d(H(S_2^n x_n, T_2(PT_2)^{n-1}x_n,\beta_n),q)=c. \tag{2.11}$$

In addition, we have

$$\lim_{n\to\infty}\sup d(S_2^n x_n,q)\le \lim_{n\to\infty}\sup h_n d(x_n,q)=c \tag{2.12}$$

and

$$\lim_{n\to\infty}\sup d(T_2(PT_2)^{n-1}x_n,q)\le \lim_{n\to\infty}\sup h_n d(x_n,q)=c. \tag{2.13}$$

It follows from (2.11), (2.12), (2.13), and Lemma 1.2 that

$$\lim_{n\to\infty} d(S_2^n x_n, T_2(PT_2)^{n-1}x_n)=0. \tag{2.14}$$

Now, we prove that $\lim_{n\to\infty} d(x_n,T_1x_n)=0=\lim_{n\to\infty} d(x_n,T_2x_n)$. Indeed, by using condition (ii), we have

$$d(x_n,T_2(PT_2)^{n-1}x_n)\le d(S_2^n x_n, T_2(PT_2)^{n-1}x_n). \tag{2.15}$$

Using (2.14) and (2.15), we have

$$\lim_{n\to\infty} d(x_n,T_2(PT_2)^{n-1}x_n)=0. \tag{2.16}$$

From (1.7), we have

$$\begin{aligned} d(y_n,S_2^n x_n) &\le (1-\beta_n)d(S_2^n x_n,S_2^n x_n)+\beta_n d(S_2^n x_n,T_2(PT_2)^{n-1}x_n)\\ &=\beta_n d(S_2^n x_n,T_2(PT_2)^{n-1}x_n). \end{aligned}$$

It follows from (2.14) that

$$\lim_{n\to\infty} d(y_n,S_2^n x_n)=0. \tag{2.17}$$

Furthermore, we have

$$\begin{aligned} d(y_n,x_n) &\le d(y_n,S_2^n x_n)+d(S_2^n x_n,T_2(PT_2)^{n-1}x_n)\\ &\quad +d(T_2(PT_2)^{n-1},x_n). \end{aligned} \tag{2.18}$$

It follows from (2.14), (2.16), (2.17), and (2.18) that

$$\lim_{n\to\infty} d(x_n,y_n)=0. \tag{2.19}$$

From condition (ii), we have

$$d(x_n,T_1(PT_1)^{n-1}x_n)\le d(S_1^n x_n,T_1(PT_1)^{n-1}x_n).$$

In view of

$$\begin{aligned} d(S_1^n x_n, T_1(PT_1)^{n-1}x_n) &\le d(S_1^n x_n, S_1^n y_n) + d(S_1^n y_n, T_1(PT_1)^{n-1}y_n) \\ &\quad + d(T_1(PT_1)^{n-1}y_n, T_1(PT_1)^{n-1}x_n) \\ &\le h_n d(x_n, y_n) + d(S_1^n y_n, T_1(PT_1)^{n-1}y_n) \\ &\quad + h_n d(y_n, x_n), \end{aligned} \tag{2.20}$$

(2.6), (2.19), and (2.20), we have

$$\lim_{n\to\infty} d(S_1^n x_n, T_1(PT_1)^{n-1}x_n) = 0, \tag{2.21}$$

and

$$\lim_{n\to\infty} d(x_n, T_1(PT_1)^{n-1}x_n) = 0. \tag{2.22}$$

Note that

$$\begin{aligned} d(x_{n+1}, S_1^n y_n) &= d((H(S_1^n y_n, T_1(PT_1)^{n-1}y_n, \alpha_n)), S_1^n y_n) \\ &\le (1-\alpha_n) d(S_1^n y_n, S_1^n y_n) + \alpha_n d(T_1(PT_1)^{n-1}y_n, S_1^n y_n) \\ &= \alpha_n d(T_1(PT_1)^{n-1}y_n, S_1^n y_n). \end{aligned}$$

Thus, it follows from (2.6) that

$$\lim_{n\to\infty} d(x_{n+1}, S_1^n y_n) = 0. \tag{2.23}$$

In addition, we have

$$d(x_{n+1}, T_1(PT_1)^{n-1}y_n) \le d(x_{n+1}, S_1^n y_n) + d(S_1^n y_n, T_1(PT_1)^{n-1}y_n).$$

Using (2.6) and (2.23), we have

$$\lim_{n\to\infty} d(x_{n+1}, T_1(PT_1)^{n-1}y_n) = 0. \tag{2.24}$$

It follows from (2.21) and (2.22) that

$$\begin{aligned} d(S_1^n x_n, x_n) &\le d(S_1^n x_n, T_1(PT_1)^{n-1}x_n) + d(T_1(PT_1)^{n-1}x_n, x_n) \\ &\to 0 \text{ (as } n\to\infty). \end{aligned} \tag{2.25}$$

On the other hand, we have

$$d(S_1^n x_n, T_2(PT_2)^{n-1}x_n) \le d(S_1^n x_n, x_n) + d(x_n, T_2(PT_2)^{n-1}x_n).$$

Thus, it follows from (2.16) and (2.25) that

$$\lim_{n\to\infty} d(S_1^n x_n, T_2(PT_2)^{n-1}x_n) = 0, \tag{2.26}$$

and

$$\begin{aligned} d(S_1^n y_n, T_2(PT_2)^{n-1}x_n) &\le d(S_1^n y_n, S_1^n x_n) + d(S_1^n x_n, T_2(PT_2)^{n-1}x_n) \\ &\le h_n d(y_n, x_n) + d(S_1^n x_n, T_2(PT_2)^{n-1}x_n). \end{aligned}$$

Using (2.19) and (2.26), we have

$$\lim_{n\to\infty} d(S_1^n y_n, T_2(PT_2)^{n-1}x_n) = 0. \tag{2.27}$$

It follows from (2.19), (2.23) and (2.27) that

$$\begin{aligned} d(x_{n+1}, T_2(PT_2)^{n-1}y_n) &\le d(x_{n+1}, S_1^n y_n) + d(S_1^n y_n, T_2(PT_2)^{n-1}x_n) \\ &\quad + d(T_2(PT_2)^{n-1}x_n, T_2(PT_2)^{n-1}y_n) \\ &\le d(x_{n+1}, S_1^n y_n) + d(S_1^n y_n, T_2(PT_2)^{n-1}x_n) + h_n d(x_n, y_n) \\ &\quad \to 0 \text{ (as } n \to \infty). \end{aligned} \tag{2.28}$$

Again, since $(PT_i)(PT_i)^{n-2}y_{n-1}, x_n \in K$ for $i = 1,2$ and T_1, T_2 are two asymptotically nonexpansive nonself-mappings, we have

$$\begin{aligned} d(T_i(PT_i)^{n-1}y_{n-1}, T_i x_n) &= d(T_i(PT_i)(PT_i)^{n-2}y_{n-1}, T_i(Px_n)) \\ &\le \max\{l_1^{(1)}, l_1^{(2)}\} d((PT_i)(PT_i)^{n-2}y_{n-1}, Px_n) \\ &\le \max\{l_1^{(1)}, l_1^{(2)}\} d(T_i(PT_i)^{n-2}y_{n-1}, x_n). \end{aligned} \tag{2.29}$$

Using (2.24), (2.28), and (2.29), for $i = 1,2$, we have

$$\lim_{n\to\infty} d(T_i(PT_i)^{n-1}y_{n-1}, T_i x_n) = 0. \tag{2.30}$$

Moreover, we have

$$d(x_{n+1}, y_n) \le d(x_{n+1}, T_1(PT_1)^{n-1}y_n) + d(T_1(PT_1)^{n-1}y_n, y_n).$$

Using (2.8) and (2.24), we have

$$\lim_{n\to\infty} d(x_{n+1}, y_n) = 0. \tag{2.31}$$

In addition, for $i = 1,2$, we have

$$\begin{aligned} d(x_n, T_i x_n) &\le d(x_n, T_i(PT_i)^{n-1}x_n) + d(T_i(PT_i)^{n-1}x_n, T_i(PT_i)^{n-1}y_{n-1}) \\ &\quad + d(T_i(PT_i)^{n-1}y_{n-1}, T_i x_n) \\ &\le d(x_n, T_i(PT_i)^{n-1}x_n) + max\{sup_{n\ge1} l_n^{(1)}, sup_{n\ge1} l_n^{(2)}\} d(x_n, y_{n-1}) \\ &\quad + d(T_i(PT_i)^{n-1}y_{n-1}, T_i x_n). \end{aligned}$$

Thus, it follows from (2.16), (2.22), (2.30), and (2.31) that

$$\lim_{n\to\infty} d(x_n, T_1 x_n) = \lim_{n\to\infty} d(x_n, T_2 x_n) = 0.$$

Finally, we prove that

$$\lim_{n\to\infty} d(x_n, S_1 x_n) = \lim_{n\to\infty} d(x_n, S_2 x_n) = 0.$$

In fact, for $i = 1,2$, we have

$$\begin{aligned} d(x_n, S_i x_n) &\le d(x_n, T_i(PT_i)^{n-1}x_n) + d(S_i x_n, T_i(PT_i)^{n-1}x_n) \\ &\le d(x_n, T_i(PT_i)^{n-1}x_n) + d(S_i^n x_n, T_i(PT_i)^{n-1}x_n). \end{aligned}$$

Thus, it follows from (2.14), (2.16), (2.21), and (2.22) that

$$\lim_{n\to\infty} d(x_n, S_1 x_n) = \lim_{n\to\infty} d(x_n, S_2 x_n) = 0.$$

The proof is completed. □

Let $\{a_n\}$ be a sequence that converges to a, where $a_n \neq a$ for all n. If positive constants λ and ϑ exist with $\lim_{n\to\infty} \frac{|a_{n+1}-a|}{|a_n-a|^{\vartheta}} = \lambda$, then $\{a_n\}$ converges to a of order ϑ, with asymptotic error constant λ. If $\vartheta = 1$ (and $\lambda < 1$), the sequence is linearly convergent and if $\vartheta = 2$, the sequence is quadratically convergent (see [35].

The following example presents the condition (ii) in Lemma 2.2.

Example 2.3. [25] Let X be the real line with metric $d(x,y) = |x-y|$ and $K = [-1,1]$. Define $H : X \times X \times [0,1] \to X$ by $H(x,y,\alpha) := \alpha x + (1-\alpha)y$ for all $x,y \in X$ and $\alpha \in [0,1]$. Then (X,d,H) is a complete uniformly hyperbolic space with the monotone modulus of uniform convexity and K is a nonempty closed convex subset of X. Define two mappings $S,T : K \to K$ by

$$Tx = \begin{cases} -2\sin\frac{x}{2}, & \text{if } x \in [0,1], \\ 2\sin\frac{x}{2}, & \text{if } x \in [-1,0) \end{cases}$$

and

$$Sx = \begin{cases} x, & \text{if } x \in [0,1], \\ -x, & \text{if } x \in [-1,0). \end{cases}$$

Clearly, $F(T) = \{0\}$ and $F(S) = \{x \in K; 0 \le x \le 1\}$. Now, we show that T is nonexpansive. In fact, if $x,y \in [0,1]$ or $x,y \in [-1,0)$, then

$$d(Tx,Ty) = |Tx-Ty| = 2|\sin\frac{x}{2} - \sin\frac{y}{2}| \le |x-y| = d(x,y).$$

If $x \in [0,1]$ and $y \in [-1,0)$ or $x \in [-1,0)$ and $y \in [0,1]$, then

$$\begin{aligned} d(Tx,Ty) &= |Tx-Ty| \\ &= 2|\sin\frac{x}{2} + \sin\frac{y}{2}| \\ &= 4|\sin\frac{x+y}{4}\cos\frac{x-y}{4}| \\ &\le |x+y| \\ &\le |x-y| \\ &= d(x,y). \end{aligned}$$

That is, T is nonexpansive. It follows that T is an asymptotically nonexpansive mapping with $k_n = 1$ for each $n \ge 1$. Similarly, we can show that S is an asymptotically nonexpansive mapping with $l_n = 1$ for each $n \ge 1$. Next, to show that S and T satisfy the condition (ii) in Lemma 2.2, we have to consider the following cases:

Case 1. Let $x,y \in [0,1]$. It follows that

$d(x,Ty) = |x-Ty| = |x+2\sin\frac{y}{2}| = |Sx-Ty| = d(Sx,Ty).$

Case 2. Let $x,y \in [-1,0)$. It follows that

$d(x,Ty) = |x-Ty| = |x-2\sin\frac{y}{2}| \le |-x-2\sin\frac{y}{2}| = |Sx-Ty| = d(Sx,Ty).$

Case 3. Let $x \in [-1,0)$ and $y \in [0,1]$. It follows that

$d(x,Ty) = |x-Ty| = |x+2\sin\frac{y}{2}| \le |-x+2\sin\frac{y}{2}| = |Sx-Ty| = d(Sx,Ty).$

Case 4. Let $x \in [0,1]$ and $y \in [-1,0]$. It follows that

$d(x,Ty) = |x-Ty| = |x-2\sin\frac{y}{2}| = |Sx-Ty| = d(Sx,Ty).$

Hence the condition (ii) in Lemma 2.2 is satisfied. Additionally, let $\alpha_n = \frac{n}{2n+1}$ and $\beta_n = \frac{n}{3n+1}$, $\forall n \ge 1$. Therefore, the conditions of Lemma 2.2 are fulfilled. So, the convergence of the sequence $\{x_n\}$ generated by (1.7) to a point $0 \in F(T) \cap F(S)$ can be received.

We choose $x_1 = 1$ and run our process within 100 iterations. All codes were written in Matlab 2019b. We obtain the iteration steps and its amplification factor of the proposed algorithms as shown in Table 1. For convenience, we call the iteration (1.7) the proposed iteration process.

TABLE 1. Numerical experiment of the proposed method for Example 2.3.

The Proposed Iteration Process		
Iteration Number (n)	$\lvert x_n\rvert$	$\frac{\lvert x_{n+1}\rvert}{\lvert x_n\rvert}$
1	1.0000e+00	1.7194e-01
2	1.7194e-01	7.7131e-02
3	1.3261e-02	5.1022e-02
4	6.7663e-04	3.8549e-02
5	2.6083e-05	3.1100e-02
$\vdots$	$\vdots$	$\vdots$
10	1.6717e-13	1.5978e-02
$\vdots$	$\vdots$	$\vdots$
20	6.0065e-33	8.1436e-03
$\vdots$	$\vdots$	$\vdots$
40	7.0265e-78	4.1169e-03
$\vdots$	$\vdots$	$\vdots$
60	2.3187e-127	2.7553e-03
$\vdots$	$\vdots$	$\vdots$
80	8.5336e-180	2.0706e-03
$\vdots$	$\vdots$	$\vdots$
100	2.0039e-234	1.6585e-03

Table 1 shows that the proposed method converges to the solutions of Example 2.3. It can be concluded that the proposed method is linearly convergent and its amplification factor less than 0.002.

We now discuss the strong convergence of iteration process (1.7) for mixed type asymptotically nonexpansive mappings in hyperbolic spaces.

Theorem 2.4. *Let K, X, S_1, S_2, T_1 and T_2 satisfy the hypotheses of Lemma 2.2. Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[\varepsilon, 1-\varepsilon]$ for some $\varepsilon \in (0,1)$, and S_i and T_i, for all $i = 1,2$, satisfy the condition (ii) in Lemma 2.2. If there is a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0,\infty)$ such that*

$$f(d(x,F)) \leq d(x,S_1x) + d(x,S_2x) + d(x,T_1x) + d(x,T_2x)$$

for all $x \in K$, where $d(x,F) = \inf\{d(x,q) : q \in F\}$, then the sequence $\{x_n\}$ defined by (1.7) converges strongly to a common fixed point of S_1, S_2, T_1 and T_2.

Proof. From Lemma 2.2, we have $\lim_{n\to\infty} d(x_n, S_i x_n) = \lim_{n\to\infty} d(x_n, T_i x_n) = 0$ for $i = 1,2$. It follows from the hypothesis that

$$\begin{aligned}\lim_{n\to\infty} f(d(x_n,F)) \leq \lim_{n\to\infty} (&d(x_n,S_1x_n) + d(x_n,S_2x_n) \\ &+ d(x_n,T_1x_n) + d(x_n,T_2x_n) = 0.\end{aligned}$$

Thus $\lim_{n\to\infty} f(d(x_n,F)) = 0$. Since $f : [0,\infty) \to [0,\infty)$ is a nondecreasing function satisfying $f(0) = 0, f(r) > 0$ for all $r \in (0,\infty)$, we conclude from Lemma 2.1 that $\lim_{n\to\infty} d(x_n,F)$ exists. This implies that $\lim_{n\to\infty} d(x_n,F) = 0$. Now, we show that $\{x_n\}$ is a Cauchy sequence in K. In fact, from (2.2), we have

$$d(x_{n+1},q) \leq (1 + (h_n^2 - 1))d(x_n,q)$$

for each $n \geq 1$, where $h_n = \max\{k_n^{(1)},\ k_n^{(2)},\ l_n^{(1)},\ l_n^{(2)}\}$ and $q \in F$. For any $m,n,m > n \geq 1$, we have

$$\begin{aligned}d(x_m,q) &\leq (1 + (h_{m-1}^2 - 1))d(x_{m-1},q) \\ &\leq e^{h^2_{m-1}-1} d(x_{m-1},q) \\ &\leq e^{h^2_{m-1}-1} e^{h^2_{m-2}-1} d(x_{m-2},q) \\ &\ \ \vdots \\ &\leq M d(x_n,q),\end{aligned}$$

where $M = e^{\Sigma_{i=1}^{\infty}(h_i^2 - 1)}$. Thus, for any $q \in F$, we have

$$d(x_n,x_m) \leq d(x_n,q) + d(x_m,q) \leq (1+M)d(x_n,q).$$

Taking the infimum over all $q \in F$, we have $d(x_n,x_m) \leq (1+M)d(x_n,F)$. Thus it follows from $\lim_{n\to\infty} d(x_n,F) = 0$ that $\{x_n\}$ is a Cauchy sequence. Since K is a closed subset in complete hyperbolic space X, we have that $\{x_n\}$ converges strongly to some $q^* \in K$. It is easy to prove that $F(S_1), F(S_2), F(T_1)$ and $F(T_2)$ are all closed, that is, F is a closed subset of K. Since $\lim_{n\to\infty} d(x_n,F) = 0$ gives that $d(q^*,F) = 0$, we have $q^* \in F$. This completes the proof. □

Theorem 2.5. *Let K, X, S_1, S_2, T_1 and T_2 satisfy the hypotheses of Lemma 2.2. Suppose that $\{\alpha_n\}, \{\beta_n\}$ are real sequences in $[\varepsilon, 1-\varepsilon]$ for some $\varepsilon \in (0,1)$ and S_i, T_i for all $i = 1,2$ satisfy the condition (ii) in Lemma 2.2. If one of S_1, S_2, T_1 and T_2 is completely continuous, then the sequence $\{x_n\}$ defined by (1.7) converges strongly to a common fixed point of S_1, S_2, T_1 and T_2.*

Proof. Without loss of generality, we can assume that S_1 is completely continuous. Since $\{x_n\}$ is bounded by Lemma 2.1, we see that there exists a subsequence $\{S_1 x_{n_j}\}$ of $\{S_1 x_n\}$ such that $\{S_1 x_{n_j}\}$ converges strongly to some $q^* \in K$. From Lemma 2.2, we have

$$\lim_{j\to\infty} d(x_{n_j}, S_1 x_{n_j}) = \lim_{j\to\infty} d(x_{n_j}, S_2 x_{n_j}) = 0,$$

and

$$\lim_{j\to\infty} d(x_{n_j}, T_1 x_{n_j}) = \lim_{j\to\infty} \|x_{n_j} - T_2 x_{n_j}) = 0.$$

Thus $d(x_{n_j},q^*) \leq d(x_{n_j},S_1x_{n_j}) + d(S_1x_{n_j},q^*) \to 0$ as $j \to \infty$ implies that $x_{n_j} \to q^* \in K$ as $j \to \infty$. From the continuity of S_1, S_2, T_1 and T_2, for $i = 1,2$, we have

$$d(q^*,S_iq^*) = \lim_{j\to\infty} d(x_{n_j},S_ix_{n_j}) = 0,$$

and

$$d(q^*,T_iq^*) = \lim_{j\to\infty} d(x_{n_j},T_ix_{n_j}) = 0,$$

which imply that $q^* \in F(S_1)\cap F(S_2)\cap F(T_1)\cap F(T_2)$. From Lemma 2.1, we have $\lim_{n\to\infty} d(x_n,q^*)$ exists. So $\lim_{n\to\infty} d(x_n,q^*) = 0$. Thus $\{x_n\}$ converges strongly to a common fixed point of S_1,S_2,T_1 and T_2. This completes the proof. □

Theorem 2.6. *Let K,X,S_1,S_2,T_1 and T_2 satisfy the hypotheses of Lemma 2.2. Suppose that $\{\alpha_n\},\{\beta_n\}$ are real sequences in $[\varepsilon,1-\varepsilon]$ for some $\varepsilon \in (0,1)$ and S_i,T_i for all $i = 1,2$ satisfy the condition (ii) in Lemma 2.2. If one of S_1, S_2, T_1 and T_2 is semi-compact, then the sequence $\{x_n\}$ defined by (1.7) converges strongly to a common fixed point of S_1, S_2, T_1 and T_2.*

Proof. Suppose that one of S_1,S_2,T_1 and T_2 is semi-compact. From Lemma 2.2, one has $\lim_{n\to\infty} d(x_n, S_ix_n) = \lim_{n\to\infty} d(x_n,T_ix_n) = 0$ for $i = 1,2$. Then, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\{x_{n_j}\}$ converges strongly to some $q^* \in K$. Moreover, from the continuity of S_1,S_2,T_1 and T_2, we have $T_iq^* = q^* = S_iq^*$ for $i = 1,2$. Thus it follows that $q^* \in F(S_1)\cap F(S_2)\cap F(T_1)\cap F(T_2)$. Using Lemma 2.1, we have $\lim_{n\to\infty} d(x_n,q^*)$ exists and $\lim_{n\to\infty} d(x_n,q^*) = 0$. This completes the proof. □

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