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A HYBRID PROJECTION ALGORITHM FOR A SPLIT EQUALITY PROBLEM IN BANACH SPACES

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Abstract. In this paper, a hybrid projection iterative algorithm is proposed for solving a split equality fixed point problem. A strong convergence theorem is obtained in p-uniformly convex and uniformly smooth Banach spaces. The split equality equilibrium problem and the zero point problem of maximal monotone operators are considered as the applications of our main results, and a numerical example for the proposed scheme is also provided.

Keywords. Banach space; Bregman strongly nonexpansive mapping; Fixed point; Split equality fixed point problem; Strong convergence.

1. Introduction

Let $\mathbb{C}$ and $\mathbb{Q}$ be nonempty closed convex subsets of Hilbert spaces $\mathbb{H}_1$ and $\mathbb{H}_2$, respectively. Let $\mathscr{A}:\mathbb{H}_1\to\mathbb{H}_2$ be a bounded linear operator. In 1994, Censor and Elfving [1] first introduced the following Split Feasibility Problem (SFP) in finite-dimensional spaces for modeling inverse problems which arise from phase retrievals and medical image reconstruction

$$Find\ x^*\in\mathbb{C}\ such\ that\ \mathscr{A}x^*\in\mathbb{Q}. \tag{1.1}$$

The SFP finds a number of real application, such as, image restoration, computer tomograph and radiation therapy treatment planing [2, 3, 4, 5].

Let $\mathbb{H}_1$, $\mathbb{H}_2$, and $\mathbb{H}_3$ be real Hilbert spaces. Let $\mathbb{C}$ and $\mathbb{Q}$ be two nonempty closed convex sets of $\mathbb{H}_1$ and $\mathbb{H}_2$, respectively, and let $\mathscr{A}:\mathbb{H}_1\to\mathbb{H}_3$ and $\mathscr{B}:\mathbb{H}_2\to\mathbb{H}_3$ be two bounded linear operators. In 2014, Moudafi [6] introduced the following split feasibility problem, which is

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called *split equality problem* (shortly, SEP):

$$\textit{Find } x^* \in \mathbb{C},\ y^* \in \mathbb{Q} \textit{ such that } \mathscr{A}x^* = \mathscr{B}y^*. \tag{1.2}$$

It allows asymmetric and partial relations between the variables x and y. It is easy to see that SEP (1.2) reduces to SFP (1.1) as $\mathbb{H}_2 = \mathbb{H}_3$ and $\mathscr{B} = \mathscr{I}$, where $\mathscr{I}$ stands for the identity mapping from H_2 to H_2 in (1.2). The interest of this problem is to cover many situations, for instance, in decomposition methods for partial differential equations, applications in game theory and in intensity-modulated radiation therapy.

Let $\mathscr{T} : \mathbb{H}_1 \to \mathbb{H}_1$ and $\mathscr{S} : \mathbb{H}_2 \to \mathbb{H}_2$ be nonlinear operators such that $Fix(\mathscr{T}) \neq \emptyset$ and $Fix(\mathscr{S}) \neq \emptyset$, where $Fix(\mathscr{T})$ and $Fix(\mathscr{S})$ denote the sets of fixed points of $\mathscr{T}$ and $\mathscr{S}$ and are closed and convex , respectively. If, in (1.2), $\mathbb{C} = Fix(\mathscr{T})$ and $\mathbb{Q} = Fix(\mathscr{S})$, then it called the *split equality fixed point problem* (shortly, SEFPP), considered and studied by Moudafi [7].

Recently, many authors introduced various efficient methods for solving the SEP and its extensions in real Hilbert spaces; see, e.g., [8, 9, 10, 11, 12] and the references therein. Since Hilbert spaces are too restriction, few results on the SEFPP were obtained in the frame work of Banach space. Up our knowledge, there are some results established in 2-uniformly convex and uniformly smooth real Banach spaces. Hence, the following question arise naturally.

Question: Can one obtain a strong convergence theorem for the SEFPP in p-uniformly convex and uniformly smooth real Banach spaces?

In this paper, motivated and inspired by the research going on in this direction, we introduced a hybrid projection algorithm and obtain a strong convergence theorem for the SEFPP with Bregman strongly nonexpansive mappings in p-uniformly convex and uniformly smooth real Banach spaces. As applications, we utilize our results to solve a split equality equilibrium problem and the zero point problem of maximal monotone operators. Our results mainly improve and extend some recent results in [13, 14, 15, 16, 17].

2. Preliminaries

Let $\mathbb{E}$ be a real Banach space with the dual space $\mathbb{E}^*$. $\mathbb{E}$ is said to be *strictly convex* if $\frac{\|x+y\|}{2} \leq 1$ for all $x, y \in \mathbb{U} = \{z \in \mathbb{E} : \|z\| = 1\}$ with $x \neq y$.

The *convexity modulus* of $\mathbb{E}$ is defined by

$$\delta_{\mathbb{E}}(\varepsilon) = \inf\{1 - \|\frac{1}{2}(x+y)\| : \|x\| \leq 1,\ \|y\| \leq 1,\ \|x-y\| \geq \varepsilon\}$$

for all $\varepsilon \in [0,2]$. $\mathbb{E}$ is said to be *uniformly convex* if $\delta_{\mathbb{E}}(0) = 0$ and $\delta_{\mathbb{E}}(\varepsilon) > 0$ for all $0 < \varepsilon \leq 2$. $\mathbb{E}$ is said to be p-uniformly convex if there exists $c_p > 0$ such that $\delta_{\mathbb{E}}(\varepsilon) > c_p\varepsilon^p$ for any $\varepsilon \in (0,2]$. One knows that Hilbert spaces are 2-uniformly convex, while $\mathbb{L}^p$ is $\max\{p,2\}$-uniformly convex for all $p > 1$.

The *modulus of smoothness* $\rho_{\mathbb{E}} : [0,\infty) \to [0,\infty)$ of $\mathbb{E}$ is defined by

$$\rho_{\mathbb{E}}(t) = \sup\{\frac{1}{2}(\|x+y\| + \|x-y\|) - 1 : x \in \mathbb{U},\ \|y\| \leq t\}.$$

$\mathbb{E}$ is said to be *uniformly smooth* if $\frac{\rho_{\mathbb{E}}(t)}{t} \to 0$ as $t \to 0$. Let q be a fixed real number with $q > 1$, $\mathbb{E}$ is said to be *q-uniformly smooth* if there exists a constant $c > 0$ such that $\rho_{\mathbb{E}}(t) \leq ct^q$ for all $t > 0$. It is well known that every q-uniformly smooth Banach space is uniformly smooth.

Let $p \in (1,\infty)$. Recall that the duality mapping $\mathscr{J}_{\mathbb{E}}^{P} : \mathbb{E} \to 2^{\mathbb{E}^*}$ is defined by

$$\mathscr{J}_{\mathbb{E}}^{p}(x) = \{x^* \in \mathbb{E}^* : \langle x, x^* \rangle = \|x\|^p, \|x^*\| = \|x\|^{p-1}, \forall x, y \in \mathbb{E}\}$$

If $\mathbb{E}$ be a p-uniformly convex and uniformly smooth real Banach space with $p > 1$, then its dual space $\mathbb{E}^*$ is q-uniformly smooth and uniformly convex, and the duality mapping $\mathscr{J}_{\mathbb{E}}^{p}$ is one-to-one, single-valued and satisfies $\mathscr{J}_{\mathbb{E}}^{P} = (\mathscr{J}_{\mathbb{E}^*}^{q})^{-1}$, where $\mathscr{J}_{\mathbb{E}^*}^{q}$ is the duality mapping of $\mathbb{E}^*$ and $\frac{1}{p} + \frac{1}{q} = 1$ (see [18, 19]).

One knows from [20] that, in the framework of q-uniformly smooth Banach spaces, the following inequality holds

$$\|x-y\|^q \le \|x\|^q - q\langle y, J_{\mathbb{E}}^q(x)\rangle + \mathscr{C}_q\|y\|^q, \quad \forall x, y \in \mathbb{E},$$

where $\mathscr{C}_q > 0$ is the q-uniformly smoothness constant of $\mathbb{E}$. and there is a constant $\mathscr{C}_q > 0$ such that

Given a Gateaux differentiable convex function $f : \mathbb{E} \to \mathbb{R}$, the Bregman distance of x to y with respect to f is defined by

$$\triangle_f(x,y) = f(y) - f(x) - \langle f'(x), y - x\rangle, \forall x, y \in \mathbb{E}. \tag{2.1}$$

Note that the duality mapping $\mathscr{J}_{\mathbb{E}}^{p}$ is the derivative of the function $f_p(x) = \frac{1}{p}\|x\|^p$, $1 < p < \infty$. Then the Bregman distance x to y with respect to f_p is given by

$$\triangle_p(x,y) = \frac{1}{q}(\|x\|^p - \|y\|^p) - \langle \mathscr{J}_{\mathbb{E}}^{p}(x) - \mathscr{J}_{\mathbb{E}}^{p}(y), y\rangle.$$

The Bregman distance possesses the following important properties:

$$\triangle_p(x,y) = \triangle_p(x,z) + \triangle_p(z,y) + \langle z - y, \mathscr{J}_{\mathbb{E}}^{p}(x) - \mathscr{J}_{\mathbb{E}}^{p}(z)\rangle, \forall x, y, z \in \mathbb{E}.$$

$$\triangle_p(x,y) + \triangle_p(y,x) = \langle x - y, \mathscr{J}_{\mathbb{E}}^{p}(x) - \mathscr{J}_{\mathbb{E}}^{p}(y)\rangle, \forall x, y \in \mathbb{E}.$$

In p-uniformly convex spaces, the metric and Bregman distance have the following relation (see [21]):

$$\tau\|x-y\|^p \le \triangle_p(x,y) \le \langle x - y, \mathscr{J}_{\mathbb{E}}^{p}(x) - \mathscr{J}_{\mathbb{E}}^{p}(y)\rangle, \tag{2.2}$$

where $\tau > 0$ is some fixed number.

Obviously, if $\{x_n\}$ and $\{y_n\}$ are bounded sequences in p-uniformly convex and uniformly smooth spaces, then the fact that $x_n - y_n \to 0$ as $n \to \infty$ implies that $\triangle_p(x_n, y_n) \to 0$ as $n \to \infty$.

Let $\mathbb{C}$ be a nonempty, closed and convex subset of a smooth, strictly convex and reflexive Banach space $\mathbb{E}$. The metric projection is defined as follows:

$$\mathscr{P}_{\mathbb{C}}x = \arg\min_{y\in\mathbb{C}} \|x-y\|, \ x \in \mathbb{E},$$

which can be characterized by the following variational inequality:

$$\langle \mathscr{J}_{\mathbb{E}}^{p}(x - \mathscr{P}_{\mathbb{C}}x), z - \mathscr{P}_{\mathbb{C}}x\rangle \le 0, \ \forall z \in \mathbb{C}.$$

As the metric projection, the Bregman projection is defined by

$$\Pi_{\mathbb{C}}x = \arg\min_{y\in\mathbb{C}} \triangle_p(x,y), \ x \in \mathbb{E},$$

which can also be characterized by the following variational inequality:

$$\langle \mathscr{J}_{\mathbb{E}}^{p}x - \mathscr{J}_{\mathbb{E}}^{p}\Pi_{\mathbb{C}}x, z - \Pi_{\mathbb{C}}x\rangle \le 0, \ \forall z \in \mathbb{C}.$$

Moreover this variational inequality is equivalent to the descent property:

$$\triangle_p(\Pi_{\mathbb{C}}x,z) \leq \triangle_p(x,z) - \triangle_p(x,\Pi_{\mathbb{C}}x),\ \forall z \in \mathbb{E}. \tag{2.3}$$

Following [18] and [22], we make use of the function $\mathscr{V}_p : \mathbb{E}^* \times \mathbb{E} \to [0,\infty)$, which is defined by

$$\mathscr{V}_p(\bar{x},x) = \frac{1}{q}\|\bar{x}\|^q - \langle \bar{x},x\rangle + \frac{1}{p}\|x\|^p, \forall x \in \mathbb{E}, \bar{x} \in \mathbb{E}^*.$$

Then $\mathscr{V}_p$ is nonnegative and

$$\mathscr{V}_p(\bar{x},x) = \triangle_p(\mathscr{J}_{\mathbb{E}^*}^q \bar{x},x), \forall x \in \mathbb{E}, \bar{x} \in \mathbb{E}^*. \tag{2.4}$$

Moreover, from the subdifferential inequality, we have

$$\mathscr{V}_p(\bar{x},x) + \langle \bar{y}, \mathscr{J}_{\mathbb{E}^*}^q \bar{x} - x\rangle \leq \mathscr{V}_p(\bar{x}+\bar{y},x), \forall x \in \mathbb{E}, \bar{x},\bar{y} \in \mathbb{E}^*. \tag{2.5}$$

Furthermore, $\mathscr{V}_p$ is convex in the first variable and

$$\triangle_p(\mathscr{J}_{\mathbb{E}^*}^q(\sum_{i=1}^{\mathscr{N}} t_i \mathscr{J}_{\mathbb{E}}^p(x_i)),z) \leq \sum_{i=1}^{\mathscr{N}} t_i \triangle_p(x_i,z),\ \forall z \in \mathbb{E}, \tag{2.6}$$

where $\{x_i\}_{i=1}^{\mathscr{N}} \subset \mathbb{E}$ and $\{t_i\}_{i=1}^{\mathscr{N}} \subset (0,1)$ with $\sum_{i=1}^{\mathscr{N}} t_i = 1$. For more details on $\mathscr{V}_p$, one refers to [1].

Let $\mathscr{T}$ be a self-mapping on $\mathbb{C}$. A point $p \in \mathbb{C}$ is called an asymptotic fixed point of $\mathscr{T}$ if $\mathbb{C}$ contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n\to\infty}\|x_n - \mathscr{T}x_n\| = 0$ (see [23, 24]). The set of asymptotic fixed points of $\mathscr{T}$ is denoted by $\widehat{\mathscr{F}}(\mathscr{T})$.

Recall that a mapping $\mathscr{T} : \mathbb{C} \to \mathbb{C}$ is said to be *Bregman nonexpansive* if

$$\triangle_p(\mathscr{T}x, \mathscr{T}y) \leq \triangle_p(x,y),\ \forall x,y \in \mathbb{C}.$$

Recall that a mapping $\mathscr{T}$ with a nonempty fixed point set is said to be *Bregman strongly nonexpansive* (shortly, BSNE) [25, 26] with respect to a nonempty $\widehat{\mathscr{F}}(\mathscr{T})$ if

$$\triangle_p(\mathscr{T}x, \mathscr{T}y) \leq \triangle_p(x,y),\ \forall x \in \mathbb{C}, y \in \widehat{\mathscr{F}}(\mathscr{T}),$$

and, if whenever $\{x_n\} \subset \mathbb{C}$ is bounded, $y \in \widehat{\mathscr{F}}(\mathscr{T})$, and

$$\lim_{n\to\infty}(\triangle_p(x_n,y) - \triangle_p(\mathscr{T}x_n,y)) = 0,$$

it follows from that

$$\lim_{n\to\infty}\triangle_p(x_n,\mathscr{T}x_n) = 0.$$

Recall that a mapping $\mathscr{T} : \mathbb{C} \to \mathbb{C}$ is said to be *Bregman firmly nonexpansive mapping* (shortly, BFNE) [27] if

$$\begin{aligned}\triangle_p(\mathscr{T}x, \mathscr{T}y) + \triangle_p(\mathscr{T}y, \mathscr{T}x) + \triangle_p(\mathscr{T}x,x) + \triangle_p(\mathscr{T}y,y)\\ \leq \triangle_p(\mathscr{T}x,y) + \triangle_p(\mathscr{T}y,x),\ \forall x,y \in \mathbb{C}.\end{aligned}$$

In [27], the existence and approximation of fixed points of the BFNE was studied. From the definition of $\triangle_f(x,y)$ given above in (2.1), one sees that if $\mathscr{T}$ is the BFNE and f is a Legendre function which is bounded, uniformly Fr*é*chet differentiable and totally convex on bounded subsets of a real reflexive Banach space E, then $\mathscr{F}(\mathscr{T}) = \widehat{\mathscr{F}}(\mathscr{T})$ and $\mathscr{F}(\mathscr{T})$ is closed and convex (see [27]). It also follows that every BFNE is BSNE with respect to $\mathscr{F}(\mathscr{T}) = \widehat{\mathscr{F}}(\mathscr{T})$.

3. MAIN RESULTS

First, we assume the following conditions are satisfied:

(i) $\mathbb{E}_1, \mathbb{E}_2$ are two p-uniformly convex and uniformly smooth real Banach spaces with dual space $\mathbb{E}_1^*$ and $\mathbb{E}_2^*$, respectively;

(ii) $\mathbb{E}_3$ is a smooth, reflexive and strictly convex Banach space;

(iii) $\mathscr{T}:\mathbb{E}_1\to\mathbb{E}_1$, $\mathscr{S}:\mathbb{E}_2\to\mathbb{E}_2$ are two Bregman strongly nonexpansive mappings with $Fix(\mathscr{T})=\widehat{\mathscr{F}}(\mathscr{T})\neq\emptyset$ and $Fix(\mathscr{S})=\widehat{\mathscr{F}}(\mathscr{S})\neq\emptyset$;

(iv) $\mathscr{A}:\mathbb{E}_1\to\mathbb{E}_3$, $\mathscr{B}:\mathbb{E}_2\to\mathbb{E}_3$ are two bounded linear operators with adjoints $\mathscr{A}^*$, $\mathscr{B}^*$, respectively.

Theorem 3.1. *Let $\mathbb{E}_1$, $\mathbb{E}_2$, $\mathbb{E}_3$, $\mathscr{T}$, $\mathscr{S}$, $\mathscr{A}$ and $\mathscr{B}$ be as above. Let $\mathbb{C}_1=\mathbb{E}_1$, $\mathbb{D}_1=\mathbb{E}_2$, and $(x_1,y_1)\in\mathbb{E}_1\times\mathbb{E}_2$. The iteration scheme $\{(x_n,y_n)\}$ is defined as follows:*

$$\begin{cases} u_n = \alpha_n x_n+(1-\alpha_n)\mathscr{T}(x_n-\rho_n \mathscr{J}^q_{\mathbb{E}_1^*}\mathscr{A}^*\mathscr{J}^p_{\mathbb{E}_3}(\mathscr{A}x_n-\mathscr{B}y_n)),\\ v_n = \alpha_n y_n+(1-\alpha_n)\mathscr{S}(y_n+\rho_n \mathscr{J}^q_{\mathbb{E}_2^*}\mathscr{B}^*\mathscr{J}^p_{\mathbb{E}_3}(\mathscr{A}x_n-\mathscr{B}y_n)),\\ \mathbb{C}_{n+1}=\{u\in\mathbb{C}_n:\triangle_p(u_n,u)\le\triangle_p(x_n,u)\},\\ \mathbb{D}_{n+1}=\{v\in\mathbb{D}_n:\triangle_p(v_n,v)\le\triangle_p(y_n,v)\},\\ x_{n+1}=\Pi_{\mathbb{C}_{n+1}}x_1,\ \ \forall n\ge 1,\\ y_{n+1}=\Pi_{\mathbb{D}_{n+1}}y_1,\ \ \forall n\ge 1, \end{cases} \tag{3.1}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$. If $\Omega:=\{(x,y)\in\mathscr{F}(\mathscr{T})\times\mathscr{F}(\mathscr{S}):\mathscr{A}x=\mathscr{B}y\}\neq\emptyset$, $\lim_{n\to\infty}\alpha_n=0$ and $0<\liminf_{n\to\infty}\rho_n<\frac{1}{\|\mathscr{A}\|^2+\|\mathscr{B}\|^2}$, then $\{(x_n,y_n)\}$ converges strongly to $(x^*,y^*)\in\Omega$.

Proof. We divide the proof into four steps.

Step 1. We show that $\mathbb{C}_n$ and $\mathbb{D}_n$ are closed and convex for all $n\ge 1$.

Since $\mathbb{C}_1=\mathbb{E}_1$ and $\mathbb{D}_1=\mathbb{E}_2$, one has $\mathbb{C}_1$ and $\mathbb{D}_1$ are closed and convex. Suppose that $\mathbb{C}_n$ and $\mathbb{D}_n$ are closed and convex for some $n\ge 2$. By induction, we prove that $\mathbb{C}_{n+1}$ and $\mathbb{D}_{n+1}$ are closed and convex. In fact, observe that $\triangle_p(u_n,u)+\le\triangle_p(x_n,u)$ and $\triangle_p(v_n,v)\le\triangle_p(y_n,v)$ are equivalent to

$$\frac{1}{q}(\|u_n\|^p-\|u\|^p)-\langle\mathscr{J}^p_{\mathbb{E}_1}u_n-\mathscr{J}^p_{\mathbb{E}_1}u,u\rangle \le\frac{1}{q}(\|x_n\|^p-\|u\|^p)-\langle\mathscr{J}^p_{\mathbb{E}_1}x_n-\mathscr{J}^p_{\mathbb{E}_1}u,u\rangle,$$

and

$$\frac{1}{q}(\|v_n\|^p-\|v\|^p)-\langle\mathscr{J}^p_{\mathbb{E}_2}v_n-\mathscr{J}^p_{\mathbb{E}_2}v,v\rangle\le\frac{1}{q}(\|y_n\|^p-\|v\|^p)-\langle\mathscr{J}^p_{\mathbb{E}_2}y_n-\mathscr{J}^p_{\mathbb{E}_2}v,v\rangle.$$

Furthermore

$$\langle\mathscr{J}^p_{\mathbb{E}_1}x_n-\mathscr{J}^p_{\mathbb{E}_1}u_n,u\rangle\le\frac{1}{q}(\|x_n\|^p-\|u_n\|^p),$$

and

$$\langle\mathscr{J}^p_{\mathbb{E}_2}y_n-\mathscr{J}^p_{\mathbb{E}_2}v_n,v\rangle\le\frac{1}{q}(\|y_n\|^p-\|v_n\|^p).$$

Since $\langle\mathscr{J}^p_{\mathbb{E}_1}x_n-\mathscr{J}^p_{\mathbb{E}_1}u_n,\cdot\rangle$ and $\langle\mathscr{J}^p_{\mathbb{E}_2}y_n-\mathscr{J}^p_{\mathbb{E}_2}v_n,\cdot\rangle$ are continuous and linear in $\mathbb{E}_1$ and $\mathbb{E}_2$, $\mathbb{C}_n$ and $\mathbb{D}_n$ are halfspaces, respectively. These imply that $\mathbb{C}_{n+1}$ and $\mathbb{D}_{n+1}$ are closed. In addition, it is also easy to see that $\mathbb{C}_{n+1}$ and $\mathbb{D}_{n+1}$ are a convex subset of $\mathbb{E}_1$ and $\mathbb{E}_2$, respectively.

Step 2. We show that $\Omega \subset \mathbb{C}_n \times \mathbb{D}_n$ for all $n \geq 1$.

Since $\Omega \subset \mathbb{C}_1 \times \mathbb{D}_1$, one assumes that $\Omega \subset \mathbb{C}_n \times \mathbb{D}_n$ for some $n \geq 1$. Let $(x,y) \in \Omega$. From (2.6), we obtain

$$\begin{aligned}
&\triangle_p (u_n,x)\\
=&\triangle_p (\alpha_n x_n + (1-\alpha_n)\mathscr{T}(x_n - \rho_n \mathscr{J}^q_{\mathbb{E}_1^*}\mathscr{A}^* \mathscr{J}^p_{\mathbb{E}_3}(\mathscr{A}x_n - \mathscr{B}y_n)),x)\\
\leq& \alpha_n \triangle_p (x_n,x) + (1-\alpha_n) \triangle_p (\mathscr{T}(x_n - \rho_n \mathscr{J}^q_{\mathbb{E}_1^*}\mathscr{A}^* \mathscr{J}^p_{\mathbb{E}_3}(\mathscr{A}x_n - \mathscr{B}y_n)),x)\\
\leq& \alpha_n \triangle_p (x_n,x) + (1-\alpha_n) \triangle_p (x_n - \rho_n \mathscr{J}^q_{\mathbb{E}_1^*}\mathscr{A}^* \mathscr{J}^p_{\mathbb{E}_3}(\mathscr{A}x_n - \mathscr{B}y_n),x).
\end{aligned} \tag{3.2}$$

Take $z_n = \mathscr{J}^p_{\mathbb{E}_3}(\mathscr{A}x_n - \mathscr{B}y_n)$. Since $\mathbb{E}_1$ is p-uniformly convex and uniformly smooth, one has that $\mathscr{J}^q_{\mathbb{E}_1^*}$ is norm-to-norm uniformly continuous on bounded subset of $\mathbb{E}_1^*$. It follows from (3.2), (2.4) and (2.5) that

$$\begin{aligned}
&\triangle_p (x_n - \rho_n \mathscr{J}^q_{\mathbb{E}_1^*}\mathscr{A}^* z_n,x)\\
=&\triangle_p (\mathscr{J}^q_{\mathbb{E}_1^*} \cdot \mathscr{J}^p_{\mathbb{E}_1} x_n - \rho_n \mathscr{J}^q_{\mathbb{E}_1^*}\mathscr{A}^* z_n,x)\\
=&\triangle_p (\mathscr{J}^q_{\mathbb{E}_1^*}(\mathscr{J}^p_{\mathbb{E}_1} x_n - \rho_n \mathscr{A}^* z_n),x)\\
=& \mathscr{V}_p(\mathscr{J}^p_{\mathbb{E}_1} x_n - \rho_n \mathscr{A}^* z_n,x)\\
\leq& \mathscr{V}_p(\mathscr{J}^p_{\mathbb{E}_1} x_n - \rho_n \mathscr{A}^* z_n + \rho_n \mathscr{A}^* z_n,x)\\
&- \langle \rho_n \mathscr{A}^* z_n, \mathscr{J}^q_{\mathbb{E}_1^*}(\mathscr{J}^p_{\mathbb{E}_1} x_n - \rho_n \mathscr{A}^* z_n) - x\rangle\\
\leq& \mathscr{V}_p(\mathscr{J}^p_{\mathbb{E}_1} x_n,x) - \rho_n \langle A^* z_n, \mathscr{J}^q_{\mathbb{E}_1^*}(\mathscr{J}^p_{\mathbb{E}_1} x_n - \rho_n \mathscr{A}^* z_n - x)\rangle\\
=&\triangle_p (x_n,x) - \rho_n \langle z_n, \mathscr{A}(x_n - \rho_n \mathscr{J}^q_{\mathbb{E}_1^*}\mathscr{A}^* z_n) - \mathscr{A}x\rangle\\
=&\triangle_p (x_n,x) - \rho_n \langle \mathscr{A}(x_n - \rho_n \mathscr{J}^q_{\mathbb{E}_1^*}\mathscr{A}^* z_n) - \mathscr{A}x, z_n\rangle.
\end{aligned} \tag{3.3}$$

Using (3.2) and (3.3), we have

$$\begin{aligned}
&\triangle_p (u_n,x)\\
\leq& \alpha_n \triangle_p (x_n,x) + (1-\alpha_n)(\triangle_p (x_n,x)\\
&- \rho_n \langle \mathscr{A}(x_n - \rho_n \mathscr{J}^q_{\mathbb{E}_1^*}\mathscr{A}^* z_n) - \mathscr{A}x, z_n\rangle\\
=&\triangle_p (x_n,x) - (1-\alpha_n)\rho_n \langle \mathscr{A}(x_n - \rho_n \mathscr{J}^q_{\mathbb{E}_1^*}\mathscr{A}^* z_n) - \mathscr{A}x, z_n\rangle.
\end{aligned} \tag{3.4}$$

By using similar argument, we arrive at

$$\begin{aligned}
&\triangle_p (v_n,y)\\
\leq& \alpha_n \triangle_p (y_n,y) + (1-\alpha_n)(\triangle_p (y_n,y)\\
&- \rho_n \langle \mathscr{B}(y_n - \rho_n \mathscr{J}^q_{\mathbb{E}_2^*}\mathscr{B}^* z_n) - \mathscr{B}y, z_n\rangle\\
=&\triangle_p (y_n,y) + (1-\alpha_n)\rho_n \langle \mathscr{B}(y_n - \rho_n \mathscr{J}^q_{\mathbb{E}_2^*}\mathscr{B}^* z_n) - \mathscr{B}y, z_n\rangle.
\end{aligned} \tag{3.5}$$

By adding (3.4) and (3.5), and using the fact that $\mathscr{A}x=\mathscr{B}y$, we have

$$\begin{aligned}
&\triangle_p(u_n,x)+\triangle_p(v_n,y)\\
&\leq\triangle_p(x_n,x)-(1-\alpha_n)\rho_n\langle\mathscr{A}(x_n-\rho_n\mathscr{J}^q_{\mathbb{E}_1^*}\mathscr{A}^*z_n)-\mathscr{A}x,z_n\rangle\\
&\quad+\triangle_p(y_n,y)+(1-\alpha_n)\rho_n\langle\mathscr{B}(y_n-\rho_n\mathscr{J}^q_{\mathbb{E}_2^*}\mathscr{B}^*z_n)-\mathscr{B}y,z_n\rangle\\
&=\triangle_p(x_n,x)+\triangle_p(y_n,y)-(1-\alpha_n)\rho_n\langle\mathscr{A}(x_n-\rho_n\mathscr{J}^q_{\mathbb{E}_1^*}\mathscr{A}^*z_n)\\
&\quad-\mathscr{B}(y_n-\rho_n\mathscr{J}^q_{\mathbb{E}_2^*}\mathscr{B}^*z_n),z_n\rangle\\
&=\triangle_p(x_n,x)+\triangle_p(y_n,y)-(1-\alpha_n)\rho_n\langle\mathscr{A}x_n-\mathscr{B}y_n,z_n\rangle\\
&\quad+(1-\alpha_n)\rho_n\langle\mathscr{A}(\rho_n\mathscr{J}^q_{\mathbb{E}_1^*}\mathscr{A}^*z_n)+\mathscr{B}(\rho_n\mathscr{J}^q_{\mathbb{E}_2^*}\mathscr{B}^*z_n),z_n\rangle\\
&\leq\triangle_p(x_n,x)+\triangle_p(y_n,y)-(1-\alpha_n)\rho_n\|\mathscr{A}x_n-\mathscr{B}y_n\|^2\\
&\quad+(1-\alpha_n)\rho_n^2(\|\mathscr{A}\|^2+\|\mathscr{B}\|^2)\|\mathscr{A}x_n-\mathscr{B}y_n\|^2\\
&\leq\triangle_p(x_n,x)+\triangle_p(y_n,y)\\
&\quad-(1-\alpha_n)\rho_n(1-\rho_n(\|\mathscr{A}\|^2+\|\mathscr{B}\|^2))\|\mathscr{A}x_n-\mathscr{B}y_n\|^2.
\end{aligned}\tag{3.6}$$

From the restrictions on $\{\rho_n\}$ and $\{\alpha_n\}$, we have

$$\triangle_p(u_n,x)+\triangle_p(v_n,y)\leq\triangle_p(x_n,x)+\triangle_p(y_n,y),$$

which implies that $(x,y)\in\mathbb{C}_{n+1}\times\mathbb{D}_{n+1}$. So, we have $\Omega\subset\mathbb{C}_n\times\mathbb{D}_n$.

Step 3. We show that $\{(x_n,y_n)\}$ converges strongly to a point $(x^*,y^*)\in\Omega$.

Since $x_n=\Pi_{\mathbb{C}_n}x_1$, we conclude from (2.3) that, for all $x\in\Omega$,

$$\begin{aligned}\triangle_p(x_n,x_1)&\leq\triangle_p(x,x_1)-\triangle_p(x,x_n)\\&\leq\triangle_p(x,x_1).\end{aligned}$$

This shows that $\{\triangle_p(x_n,x_1)\}$ is bounded. Hence, $\{x_n\}$ is bounded. Furthermore, since $x_n=\Pi_{\mathbb{C}_n}x_1$ and $\mathbb{C}_{n+1}\subset\mathbb{C}_n$ for all $n\geq1$, we get $\triangle_p(x_n,x_1)\leq\triangle_p(x_{n+1},x_1)$, which implies that $\{\triangle_p(x_n,x_1)\}$ is nondecreasing. Hence $\lim_{n\to\infty}\triangle_p(x_n,x_1)$ exists. Let $m>n$. From (2.3), we obtain

$$\begin{aligned}\triangle_p(x_m,x_n)&=\triangle_p(x_m,\Pi_{\mathbb{C}_n}x_1)\\&\leq\triangle_p(x_m,x_1)-\triangle_p(\Pi_{\mathbb{C}_n}x_1,x_1)\\&\leq\triangle_p(x_m,x_1)-\triangle_p(x_n,x_1).\end{aligned}$$

So, we have $\triangle_p(x_m,x_n)\to0$ as $n\to\infty$. From (2.2), we have $\|x_m-x_n\|\to0$ as $n\to\infty$, which implies that $\{x_n\}$ is a Cauchy sequence, and there exists $x^*\in E_1$ such that $\lim_{n\to\infty}x_n=x^*\in\mathbb{E}_1$. Following the similar argument, we also have $\triangle_p(y_m,y_n)\to0$ as $n\to\infty$, and $\lim_{n\to\infty}y_n=y^*\in\mathbb{E}_2$. Since $x_{n+1}=\Pi_{\mathbb{C}_{n+1}}x_1\in\mathbb{C}_{n+1}$ and $y_{n+1}=\Pi_{\mathbb{D}_{n+1}}x_1\in\mathbb{D}_{n+1}$, we conclude from (3.1) that

$$\triangle_p(u_n,x_{n+1})+\triangle_p(v_n,y_{n+1})\leq\triangle_p(x_n,x_{n+1})+\triangle_p(y_{n+1},y_n),$$

$$\lim_{n\to\infty}(\triangle_p(u_n,x_{n+1})+\triangle_p(v_n,y_{n+1}))=0.$$

It follows that $\lim_{n\to\infty}\triangle_p(u_n,x_{n+1})=0$ and $\lim_{n\to\infty}\triangle_p(v_n,y_{n+1})=0$. Thus,

$$\lim_{n\to\infty}\|u_n-x_{n+1}\|=0,\tag{3.7}$$

and

$$\lim_{n\to\infty}\|v_n-y_{n+1}\|=0. \tag{3.8}$$

From (3.6), we obtain

$$\begin{aligned}(1-\alpha_n)\rho_n(1-\rho_n(\|\mathscr{A}\|^2+\|\mathscr{B}\|^2))\|\mathscr{A}x_n-\mathscr{B}y_n\|^2\\ \leq \triangle_p(x_n,x)+\triangle_p(y_n,y)-\triangle_p(u_n,x)+\triangle_p(v_n,y).\end{aligned} \tag{3.9}$$

From the restrict conditions of $\{\alpha_n\}$ and $\{\rho_n\}$, (3.7), (3.8), and (3.9), we have

$$\lim_{n\to\infty}\|\mathscr{A}x_n-\mathscr{B}y_n\|=0,$$

which implies that

$$\mathscr{A}x^*=\mathscr{B}y^*.$$

Next, we show that $(x^*,y^*)\in\mathscr{F}(\mathscr{T})\times\mathscr{F}(\mathscr{S})$. Letting

$$e_n=x_n-\rho_n\mathscr{J}^q_{\mathbb{E}_1^*}\mathscr{A}^*\mathscr{J}^p_{\mathbb{E}_3}(\mathscr{A}x_n-\mathscr{B}y_n),$$

we get

$$\|x_n-e_n\|=\rho_n\mathscr{J}^q_{\mathbb{E}_1^*}\mathscr{A}^*\mathscr{J}^p_{\mathbb{E}_3}(\mathscr{A}x_n-\mathscr{B}y_n).$$

This implies that

$$\lim_{n\to\infty}\|x_n-e_n\|=0. \tag{3.10}$$

In addition,

$$\triangle_p(u_n,x^*)\leq\alpha_n\triangle_p(x_n,x^*)+(1-\alpha_n)\triangle_p(\mathscr{T}e_n,x^*).$$

It follows that

$$\triangle_p(u_n,x^*)\leq\triangle_p(x_n,x^*)+(1-\alpha_n)(\triangle_p(\mathscr{T}e_n,x^*)-\triangle_p(x_n,x^*)),$$

which implies that

$$(1-\alpha_n)(\triangle_p(x_n,x^*)-\triangle_p(\mathscr{T}e_n,x^*))\leq\triangle_p(x_n,x^*)-\triangle_p(u_n,x^*). \tag{3.11}$$

It follows from (3.7) and (3.11) that

$$\lim_{n\to\infty}(\triangle_p(x_n,x^*)-\triangle_p(\mathscr{T}e_n,x^*))=0.$$

Since T is Bregman strongly nonexpansive, we have

$$\lim_{n\to\infty}\triangle_p(x_n,\mathscr{T}e_n)=0.$$

This implies from (2.2) that

$$\lim_{n\to\infty}\|x_n-\mathscr{T}e_n\|=0, \tag{3.12}$$

Since $\mathscr{J}^p_{\mathbb{E}_1}$ is norm-to-norm uniformly continuous, we conclude from (3.1) that

$$\begin{aligned}\|\mathscr{J}^p_{\mathbb{E}_1}u_n-\mathscr{J}^p_{\mathbb{E}_1}x^*\|&=\|\alpha_n\mathscr{J}^p_{\mathbb{E}_1}x_n+(1-\alpha_n)\mathscr{J}^p_{\mathbb{E}_1}\mathscr{T}e_n-\mathscr{J}^p_{\mathbb{E}_1}x^*\|\\&=\|\alpha_n(\mathscr{J}^p_{\mathbb{E}_1}x_n-\mathscr{J}^p_{\mathbb{E}_1}x^*)+(1-\alpha_n)(\mathscr{J}^p_{\mathbb{E}_1}\mathscr{T}e_n-\mathscr{J}^p_{\mathbb{E}_1}x^*)\|\\&\geq(1-\alpha_n)\|\mathscr{J}^p_{\mathbb{E}_1}\mathscr{T}e_n-\mathscr{J}^p_{\mathbb{E}_1}x^*\|-\alpha_n\|\mathscr{J}^p_{\mathbb{E}_1}x_n-\mathscr{J}^p_{\mathbb{E}_1}x^*\|.\end{aligned}$$

This implies that

$$(1-\alpha_n)\|\mathscr{J}^p_{\mathbb{E}_1}\mathscr{T}e_n-\mathscr{J}^p_{\mathbb{E}_1}x^*\|\leq\|\mathscr{J}^p_{\mathbb{E}_1}u_n-\mathscr{J}^p_{\mathbb{E}_1}x^*\|+\alpha_n\|\mathscr{J}^p_{\mathbb{E}_1}x_n-\mathscr{J}^p_{\mathbb{E}_1}x^*\|. \tag{3.13}$$

Since $\mathscr{J}_{\mathbb{E}_1^*}^q$ is norm-to-norm uniformly continuous and $\lim_{n\to\infty}\alpha_n = 0$, we see from (3.7) and (3.13) that

$$\lim_{n\to\infty}\|\mathscr{T}e_n - x^*\| = 0. \tag{3.14}$$

From (3.10), (3.12) and (3.14), we have $\mathscr{T}x^* = x^*$. Following the same argument, we also obtain $\mathscr{S}y^* = y^*$. Thus, we conclude that $(x^*, y^*) \in \Omega$. This completes the proof. □

A typical example of a uniformly smooth Banach space is $\mathscr{L}^p$, where $p > 1$. Then we have the following corollary.

Corollary 3.2. *Let $\mathbb{E}_1$, $\mathbb{E}_2$ be two $\mathscr{L}^p$ spaces with $2 \le p < \infty$, and let $\mathbb{E}_3$ be a smooth, reflexive and strictly convex Banach space. Let $\mathscr{T} : \mathbb{E}_1 \to \mathbb{E}_1$ and $\mathscr{S} : \mathbb{E}_2 \to \mathbb{E}_2$ be two Bregman strongly nonexpansive mappings with Fix$(\mathscr{T}) \neq \emptyset$ and Fix$(\mathscr{S}) \neq \emptyset$. Let $\mathscr{A} : \mathbb{E}_1 \to \mathbb{E}_3$ and $\mathscr{B} : \mathbb{E}_2 \to \mathbb{E}_3$ be two bounded linear operators with adjoints $\mathscr{A}^*$ and $\mathscr{B}^*$, respectively. Let $\mathbb{C}_1 = \mathbb{E}_1$, $\mathbb{D}_1 = \mathbb{E}_2$ and $(x_1, y_1) \in \mathbb{E}_1 \times \mathbb{E}_2$ be given. Define $\{(x_n, y_n)\}$ as follows:*

$$\begin{cases} u_n = \alpha_n x_n + (1-\alpha_n)\mathscr{T}(x_n - \rho_n \mathscr{J}_{\mathbb{E}_1^*}^q \mathscr{A}^* \mathscr{J}_{\mathbb{E}_3}^p(\mathscr{A}x_n - \mathscr{B}y_n)), \\ v_n = \alpha_n y_n + (1-\alpha_n)\mathscr{S}(y_n + \rho_n \mathscr{J}_{\mathbb{E}_2^*}^q \mathscr{B}^* \mathscr{J}_{\mathbb{E}_3}^p(\mathscr{A}x_n - \mathscr{B}y_n)), \\ \mathbb{C}_{n+1} = \{u \in \mathbb{C}_n : \triangle_p(u_n, u) \le \triangle_p(x_n, u)\}, \\ \mathbb{D}_{n+1} = \{v \in \mathbb{D}_n : \triangle_p(v_n, v) \le \triangle_p(y_n, v)\}, \\ x_{n+1} = \Pi_{\mathbb{C}_{n+1}} x_1, \ \ \forall n \ge 1, \\ y_{n+1} = \Pi_{\mathbb{D}_{n+1}} y_1, \ \ \forall n \ge 1, \end{cases}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$. If $\Omega := \{(x,y) \in \mathscr{F}(\mathscr{T}) \times \mathscr{F}(\mathscr{S}) : \mathscr{A}x = \mathscr{B}y\} \neq \emptyset$, $\lim_{n\to\infty}\alpha_n = 0$, and $0 < \liminf_{n\to\infty}\rho_n < \frac{1}{\|\mathscr{A}\|^2 + \|\mathscr{B}\|^2}$, then $\{(x_n, y_n)\}$ converges strongly to $(x^*, y^*) \in \Omega$.

Corollary 3.3. *Let $\mathbb{H}_1$, $\mathbb{H}_2$ and $\mathbb{H}_3$ be Hilbert spaces. Let $\mathscr{T} : \mathbb{H}_1 \to \mathbb{H}_1$ and $\mathscr{S} : \mathbb{H}_2 \to \mathbb{H}_2$ be two nonexpansive mappings with Fix$(\mathscr{T}) \neq \emptyset$ and Fix$(\mathscr{S}) \neq \emptyset$. Let $\mathscr{A} : \mathbb{H}_1 \to \mathbb{H}_3$ and $\mathscr{B} : \mathbb{H}_2 \to \mathbb{H}_3$ be two bounded linear operators with adjoints $\mathscr{A}^*$ and $\mathscr{B}^*$, respectively. Let $\mathbb{C}_1 = \mathbb{H}_1$, $\mathbb{D}_1 = \mathbb{H}_2$, and $(x_1, y_1) \in \mathbb{H}_1 \times \mathbb{H}_2$ be given. Define $\{(x_n, y_n)\}$ as follows:*

$$\begin{cases} u_n = \alpha_n x_n + (1-\alpha_n)\mathscr{T}(x_n - \rho_n \mathscr{A}^*(\mathscr{A}x_n - \mathscr{B}y_n)), \\ v_n = \alpha_n y_n + (1-\alpha_n)\mathscr{S}(y_n + \rho_n \mathscr{B}^*(\mathscr{A}x_n - \mathscr{B}y_n)), \\ \mathbb{C}_{n+1} = \{u \in \mathbb{C}_n : \triangle_p(u_n, u) \le \triangle_p(x_n, u)\}, \\ \mathbb{D}_{n+1} = \{v \in \mathbb{D}_n : \triangle_p(v_n, v) \le \triangle_p(y_n, v)\}, \\ x_{n+1} = \Pi_{\mathbb{C}_{n+1}} x_1, \ \ \forall n \ge 1, \\ y_{n+1} = \Pi_{\mathbb{D}_{n+1}} y_1, \ \ \forall n \ge 1, \end{cases}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$. If $\Omega := \{(x,y) \in \mathscr{F}(\mathscr{T}) \times \mathscr{F}(\mathscr{S}) : \mathscr{A}x = \mathscr{B}y\} \neq \emptyset$, $\lim_{n\to\infty}\alpha_n = 0$, and $0 < \liminf_{n\to\infty}\rho_n < \frac{1}{\|\mathscr{A}\|^2 + \|\mathscr{B}\|^2}$, then $\{(x_n, y_n)\}$ converges strongly to $(x^*, y^*) \in \Omega$.

In Theorem 3.1, taking $\mathscr{B} = \mathscr{I}$, $\mathbb{E}_2 = \mathbb{E}_3$, and $\mathscr{J}_{\mathbb{E}_2} = \mathscr{J}_{\mathbb{E}_3}$, we can obtain the following convergence result for a split common fixed point problem of Bregman nonexpansive mappings in Banach spaces.

Corollary 3.4. *Let $\mathbb{E}_1$ be p-uniformly convex and uniformly smooth real Banach space, and let $\mathbb{E}_2$ be a smooth, reflexive and strictly convex Banach space. Let $\mathscr{T}:\mathbb{E}_1\to\mathbb{E}_1$ and $\mathscr{S}:\mathbb{E}_2\to\mathbb{E}_2$ be two Bregman strongly nonexpansive mappings with $Fix(\mathscr{T})\neq\emptyset$ and $Fix(\mathscr{S})\neq\emptyset$. Let $\mathscr{A}:\mathbb{E}_1\to\mathbb{E}_2$ be a bounded linear operator with adjoint $\mathscr{A}^*$. Let $\mathbb{C}_1=\mathbb{E}_1$, $\mathbb{D}_1=\mathbb{E}_2$, and $(x_1,y_1)\in\mathbb{E}_1\times\mathbb{E}_2$ be given. Define $\{(x_n,y_n)\}$ as follows:*

$$\begin{cases} u_n = \alpha_n x_n+(1-\alpha_n)\mathscr{T}(x_n-\rho_n\mathscr{J}^q_{\mathbb{E}_1^*}\mathscr{A}^*\mathscr{J}^p_{\mathbb{E}_2}(\mathscr{A}x_n-y_n)),\\ v_n = \alpha_n y_n+(1-\alpha_n)\mathscr{S}(y_n+\rho_n(\mathscr{A}x_n-y_n)),\\ \mathbb{C}_{n+1}=\{u\in\mathbb{C}_n:\triangle_p(u,u_n)\leq\triangle_p(u,x_n)\},\\ \mathbb{D}_{n+1}=\{v\in\mathbb{D}_n:\triangle_p(v,v_n)\leq\triangle_p(v,y_n)\},\\ x_{n+1}=\Pi_{\mathbb{C}_{n+1}}x_1,\ \ \forall n\geq 1,\\ y_{n+1}=\Pi_{\mathbb{D}_{n+1}}y_1,\ \ \forall n\geq 1, \end{cases}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$. If $\Omega:=\{(x,y)\in\mathscr{F}(\mathscr{T})\times\mathscr{F}(\mathscr{S}):\mathscr{A}x=\mathscr{B}y\}\neq\emptyset$, $\lim_{n\to\infty}\alpha_n=0$, and $0<\liminf_{n\to\infty}\rho_n<\frac{1}{\|\mathscr{A}\|^2+\|\mathscr{B}\|^2}$, then $\{(x_n,y_n)\}$ converges strongly to $(x^*,y^*)\in\Omega$.

4. APPLICATIONS

4.1. **Application to the split equality equilibrium problems.** In this subsection, we utilize Theorem 3.1 to study the following split equality equilibrium problems (shortly, SEEP) in Banach spaces.

Let $\mathbb{E}_1$, $\mathbb{E}_2$ and $\mathbb{E}_3$ be three Banach spaces, and let $\mathbb{C}$ and $\mathbb{Q}$ be nonempty closed convex subsets of $\mathbb{E}_1$ and $\mathbb{E}_2$, respectively. Let $\mathscr{F}:\mathbb{C}\times\mathbb{C}\to\mathbb{R}$ and $\mathscr{G}:\mathbb{Q}\times\mathbb{Q}\to\mathbb{R}$ be nonlinear bifunctions, and let $\mathscr{A}:\mathbb{E}_1\to\mathbb{E}_3$ and $\mathscr{B}:\mathbb{E}_2\to\mathbb{E}_3$ be two bounded linear operators. Then the SEEP is to find $x^*\in\mathbb{C}$ and $y^*\in\mathbb{Q}$ such that

$$\mathscr{F}(x^*,x)\geq 0,\ \forall x\in\mathbb{C},\ \mathscr{G}(y^*,y)\geq 0,\ \forall y\in\mathbb{Q},\ and\ \mathscr{A}x^*=\mathscr{B}y^*. \tag{4.1}$$

The set of solutions of (4.1) is denoted by $SEEP(\mathscr{F},\mathscr{G})$, that is,

$$\begin{aligned} &SEEP(\mathscr{F},\mathscr{G})\\ &=\{(x^*,y^*)\in\mathbb{C}\times\mathbb{Q}:\mathscr{F}(x^*,x)\geq 0,\ \forall x\in\mathbb{C},\mathscr{G}(y^*,y)\geq 0,\ \forall y\in\mathbb{Q},and\ \mathscr{A}x^*=\mathscr{B}y^*\}. \end{aligned}$$

For solving this problems, we assume that $F:\mathbb{C}\times\mathbb{C}\to\mathbb{R}$ is a bifunction satisfying the following conditions:

- (A1) $\mathscr{F}(x,x)=0,\forall x\in\mathbb{C}$;
- (A2) $\mathscr{F}(x,y)+\mathscr{F}(y,x)\leq 0,\forall x,y\in\mathbb{C}$;
- (A3) For all $x,y,z\in\mathbb{C}$, $\lim_{t\downarrow 0}\mathscr{F}(tz+(1-t)x,y)\leq\mathscr{F}(x,y)$;
- (A4) For each $x\in\mathbb{C}$, the function $y\longmapsto F(x,y)$ is convex and lower semicontinuous;

In [25], the resolvent operator of bifunction F with respect to Bregman distance $\triangle_p$ is given as

$$\mathscr{T}_p^F(x)=\{z\in\mathbb{C}:\mathscr{F}(z,y)+\frac{1}{r}\langle y-z,\mathscr{J}_p^{\mathbb{E}}z-\mathscr{J}_p^{\mathbb{E}}x\rangle\geq 0,\quad\forall y\in\mathbb{C}\}.$$

In [25], Reich and Sabach proved that $\mathscr{T}_p^F(x)$ has the following properties:

(1) $\mathscr{T}_p^F$ is single-valued;
(2) $\mathscr{T}_p^F$ is Bregman firmly nonexpansive mapping;

(3) $\mathscr{F}(\mathscr{T}_p^F) = EP(\mathscr{F})$;
(4) $EP(\mathscr{F})$ is closed and convex;
(5) for all $x \in \mathbb{E}$ and $q \in \mathscr{F}(T_p^F)$, $\triangle_p(q, \mathscr{T}_p^F(x)) + \triangle_p(\mathscr{T}_p^F(x), x) \leq \triangle_p(q, x)$.

In Theorem 3.1, setting $\mathscr{T} = \mathscr{T}_p^F$ and $\mathscr{S} = \mathscr{T}_p^G$, we can obtain the following result immediately.

Theorem 4.1. *Let $\mathbb{E}_1$ and $\mathbb{E}_2$ be two $p-$uniformly convex and uniformly smooth real Banach spaces, and let $\mathbb{E}_3$ be a smooth, reflexive and strictly convex Banach space. Let $\mathscr{F} : \mathbb{C} \times \mathbb{C} \to \mathbb{R}$ and $\mathscr{G} : \mathbb{Q} \times \mathbb{Q} \to \mathbb{R}$ be nonlinear bifunctions satisfying (A1)-(A4). Let $\mathscr{A} : \mathbb{E}_1 \to \mathbb{E}_3$ and $\mathscr{B} : \mathbb{E}_2 \to \mathbb{E}_3$ be two bounded linear operators with adjoints A^* and B^*, respectively. Let $\mathbb{C}_1 = \mathbb{E}_1$, $\mathbb{D}_1 = \mathbb{E}_2$, and $(x_1, y_1) \in \mathbb{E}_1 \times \mathbb{E}_2$ be given. The iteration scheme $\{(x_n, y_n)\}$ is defined as follows:*

$$\begin{cases} u_n = \alpha_n x_n + (1-\alpha_n)\mathscr{T}_p^F(x_n - \rho_n \mathscr{J}_{\mathbb{E}_1^*}^q \mathscr{A}^* \mathscr{J}_{\mathbb{E}_3}^p(\mathscr{A}x_n - \mathscr{B}y_n)), \\ v_n = \alpha_n y_n + (1-\alpha_n)\mathscr{T}_p^G(y_n + \rho_n \mathscr{J}_{\mathbb{E}_2^*}^q \mathscr{B}^* \mathscr{J}_{\mathbb{E}_3}^p(\mathscr{A}x_n - \mathscr{B}y_n)), \\ \mathbb{C}_{n+1} = \{u \in \mathbb{C}_n : \triangle_p(u_n, u) \leq \triangle_p(x_n, u)\}, \\ \mathbb{D}_{n+1} = \{v \in \mathbb{D}_n : \triangle_p(v_n, v) \leq \triangle_p(y_n, v)\}, \\ x_{n+1} = \Pi_{\mathbb{C}_{n+1}} x_1, \ \forall n \geq 1, \\ y_{n+1} = \Pi_{\mathbb{D}_{n+1}} y_1, \ \forall n \geq 1, \end{cases}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$. If $SEEP(\mathscr{F}, \mathscr{G}) \neq \emptyset$, $\lim_{n\to\infty} \alpha_n = 0$ and $0 < \liminf_{n\to\infty} \rho_n < \frac{1}{\|\mathscr{A}\|^2 + \|\mathscr{B}\|^2}$, then $\{(x_n, y_n)\}$ converges strongly to $(x^*, y^*) \in SEEP(\mathscr{F}, \mathscr{G})$.

4.2. **Applications to the zero point problem of maximal monotone operators.** Let $\mathbb{E}$ be a Banach space with dual $\mathbb{E}^*$. Let $\mathscr{N} : \mathbb{E} \to 2^{\mathbb{E}^*}$ be a multivalued mapping. The domain of $\mathscr{N}$ is denoted by $dom\mathscr{N} = \{x \in \mathbb{E} : \mathscr{N}x \neq \emptyset\}$ and also the graph of $\mathscr{N}$ is denote by $\mathscr{G}(\mathscr{N}) = \{(x, x^*) \in \mathbb{E} \times \mathbb{E}^* : \mathbb{X}^* \in \mathscr{N}x\}$. $\mathscr{N}$ is called a monotone operator if $\langle x^* - y^*, x - y\rangle \geq 0$ for each $(x, x^*), (y, y^*) \in \mathscr{G}(\mathscr{N})$. $\mathscr{N}$ is said to be a maximal monotone operator if the graph of $\mathscr{N}$ is not a proper subset of the graph of any other monotone operator. It is known that if $\mathscr{N}$ is maximal monotone, then the set $\mathscr{N}^{-1}(0^*) = \{z \in \mathbb{E} : 0^* \in \mathscr{N}z\}$ is closed and convex.

The Bregman resolvent operator associated with $\mathscr{N}$ is denoted by $Res_{\mathscr{N}}$ and defined by $Res_{\mathscr{N}} = (\mathscr{J}_{\mathbb{E}}^p + \mathscr{N})^{-1} \circ \mathscr{J}_{\mathbb{E}}^p$. It was proved in [25] and [28] that Res_N satisfies the following properties:

(1) $Res_{\mathscr{N}}$ is single-valued and Bregman firmly nonexpansive;
(2) $x \in \mathscr{F}(Res_{\mathscr{N}})$ *if and only if* $x \in \mathscr{N}^{-1}(0^*)$, $\forall x \in \mathbb{E}$;
(3) $\triangle_p(z, Res_{\mathscr{N}}x) + \triangle_p(Res_{\mathscr{N}}x, x) \leq \triangle_p(z, x)$.

Let $\mathbb{E}_1$, $\mathbb{E}_2$ be two $p-$uniformly convex and uniformly smooth Banach spaces, and let $\mathbb{E}_3$ be a smooth, reflexive and strictly convex Banach space. Let $\mathscr{M}_1 : \mathbb{E}_1 \to 2^{\mathbb{E}_1{}^*}$ and $\mathscr{M}_2 : \mathbb{E}_2 \to 2^{\mathbb{E}_2{}^*}$ be maximal monotone operators, and let $\mathscr{A} : \mathbb{E}_1 \to \mathbb{E}_3$ and $\mathscr{B} : \mathbb{E}_2 \to \mathbb{E}_3$ be bounded linear operators. Consider the following problem:

$$find\ x \in \mathscr{M}_1^{-1}(0), y \in \mathscr{M}_2^{-1}(0)\ such\ that\ \mathscr{A}x = \mathscr{B}y.$$

Setting $\mathscr{T} = Res_{\mathscr{M}_1}$ and $\mathscr{S} = Res_{\mathscr{M}_2}$ in Theorem 3.1, we obtain the following result immediately.

Theorem 4.2. *Let $\mathbb{E}_1$ and $\mathbb{E}_2$ be two $p-$uniformly convex and uniformly smooth Banach spaces, and let $\mathbb{E}_3$ be a smooth, reflexive and strictly convex Banach space. Let $\mathscr{M}_1 : E_1 \to 2^{\mathbb{E}_1^*}$ and*

$\mathscr{M}_2 : \mathbb{E}_2 \to 2^{\mathbb{E}_2^*}$ be maximal monotone operators. Let $\mathscr{A} : \mathbb{E}_1 \to \mathbb{E}_3$ and $\mathscr{B} : \mathbb{E}_2 \to \mathbb{E}_3$ be two bounded linear operators with adjoints $\mathscr{A}^*$ and $\mathscr{B}^*$, respectively. Let $\mathbb{C}_1 = \mathbb{E}_1$, $\mathbb{D}_1 = \mathbb{E}_2$, and $(x_1, y_1) \in \mathbb{E}_1 \times \mathbb{E}_2$ be given. The iteration scheme $\{(x_n, y_n)\}$ is defined as follows:

$$\begin{cases} u_n = \ \alpha_n x_n + (1-\alpha_n) Res_{\mathscr{M}_1}(x_n - \rho_n \mathscr{J}^q_{\mathbb{E}_1^*} \mathscr{A}^* \mathscr{J}^p_{\mathbb{E}_3}(\mathscr{A}x_n - \mathscr{B}y_n)), \\ v_n = \ \alpha_n y_n + (1-\alpha_n) Res_{\mathscr{M}_2}(y_n + \rho_n \mathscr{J}^q_{\mathbb{E}_2^*} \mathscr{B}^* J^p_{\mathbb{E}_3}(\mathscr{A}x_n - \mathscr{B}y_n)), \\ \mathbb{C}_{n+1} = \{u \in \mathbb{C}_n : \triangle_p(u_n, u) \le \triangle_p(x_n, u)\}, \\ \mathbb{D}_{n+1} = \{v \in \mathbb{D}_n : \triangle_p(v_n, v) \le \triangle_p(y_n, v)\}, \\ x_{n+1} = \Pi_{\mathbb{C}_{n+1}} x_1, \ \forall n \ge 1, \\ y_{n+1} = \Pi_{\mathbb{D}_{n+1}} y_1, \ \forall n \ge 1, \end{cases}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$. If $\Omega = \{(x,y) \in (\mathscr{M}_1^{-1}(0) \times \mathscr{M}_2^{-1}(0)) : \mathscr{A}x = \mathscr{B}y\} \neq \emptyset$, $\lim_{n\to\infty} \alpha_n = 0$, and $0 < \liminf_{n\to\infty} \rho_n < \frac{1}{\|\mathscr{A}\|^2 + \|\mathscr{B}\|^2}$, then $\{(x_n, y_n)\}$ converges strongly to $(x^*, y^*) \in \Omega$.

Finally, we present a numerical example to support our main theorem.

In Theorem 3.1, let $\mathbb{E}_1 = \mathbb{E}_2 = \mathbb{E}_3 = \mathbb{R}$ with Euclidean norms. Let $\mathscr{A} : \mathbb{R} \to \mathbb{R}$ and $\mathscr{B} : \mathbb{R} \to \mathbb{R}$ be bounded linear mappings defined by $\mathscr{A}x = 3x$ and $\mathscr{B}x = \frac{1}{2}x$ for all $x \in \mathbb{R}$. Let $\mathscr{A} = \mathscr{A}^*$ and $\mathscr{B} = \mathscr{B}^*$. Let the sequence $\{(x_n, y_n)\}$ be generated by (3.1), where $\alpha_n = \frac{1}{6}$, $\rho_n = 1$, $\mathscr{T}x = \frac{1}{4}x$, and $\mathscr{S}x = \frac{4}{5}x$ for all $x \in \mathbb{R}$. Then scheme (3.1) can be simplified as

$$\begin{cases} u_n = \ \frac{1}{6}x_n + \frac{5}{6}\mathscr{T}(x_n - \mathscr{A}^*(\mathscr{A}x_n - \mathscr{B}y_n)), \\ v_n = \ \frac{1}{6}y_n + \frac{5}{6}\mathscr{T}(y_n + \mathscr{B}^*(\mathscr{A}x_n - \mathscr{B}y_n)), \\ \mathbb{C}_{n+1} = \{u \in \mathbb{C}_n \times \mathbb{D}_n : |u_n - u| \le |x_n - u|\}, \\ \mathbb{D}_{n+1} = \{v \in \mathbb{D}_n : |v_n - v| \le |y_n - v|\}, \\ x_{n+1} = \Pi_{\mathbb{C}_{n+1}} x_1, \ \forall n \ge 1, \\ y_{n+1} = \Pi_{\mathbb{D}_{n+1}} y_1, \ \forall n \ge 1. \end{cases} \tag{4.2}$$

By choosing initial values $x_1 = 60$ and $y_1 = 40$, we obtain the following results. Table 1 and Fig 1 show that the sequence $\{(x_n, y_n)\}$ generated by the above algorithms converge to $(0,0)$.

Table 1: The values of the sequences x_n and y_n in (4.2)

n	x_n	y_n
1	60.0000000000	40.0000000000
2	-8.7500000000	10.0000000000
3	3.75000000000	14.3750000000
4	1.30859375000	12.5000000000
…	…	…
148	0.00008989307	0.0006811835
149	0.00008396165	0.0006362369

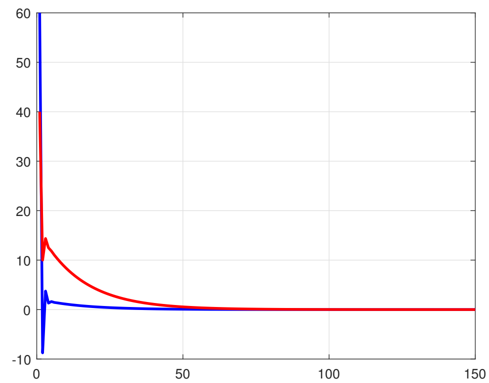

FIGURE 1. The convergence of Algorithm 4.2.

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