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GLOBAL EXISTENCE AND BLOW-UP OF SOLUTIONS FOR LOGARITHMIC HIGHER-ORDER PARABOLIC EQUATIONS

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Abstract. An initial-boundary value problem of a class of higher-order parabolic equations with logarithmic nonlinearity in a bounded domain is studied. The existence of global solutions is proved by using the potential well theory, and the exponential decay of global solutions is given with the aid of an integral inequality. The blow-up of solutions in unstable sets is also obtained.

Keywords. Blow-up; Exponential decay; Global solution; Higher-order parabolic equation; Logarithmic nonlinearity.

1. Introduction

This paper is concerned with the initial-boundary problem of the higher-order parabolic equation with logarithmic nonlinearity

$$\begin{cases} u_t+(-\Delta)^m u = u\log|u|^p,\ (x,t)\in\Omega\times[0,+\infty), \\ u(x,0)=u_0(x),\ x\in\Omega, \\ u(x,t)=0,\ \frac{\partial^i}{\partial\nu^i}u(x,t)=0,\ i=1,2,\cdots,m-1,\ (x,t)\in\Gamma\times[0,+\infty), \end{cases} \tag{1.1}$$

where $m\geq 1$ is a positive integer, $p>0$ is a real number, Ω is a bounded domain in R^n with smooth boundary Γ so that the divergence theorem can be applied, ν is unit outward normal vector on Γ, and $\frac{\partial^i u}{\partial \nu^i}$ denotes the $i-$order normal derivation of u.

High-order parabolic equations arise in many physical applications, such as, the thin film theory, the convection-explosion theory, the lubrication theory, the flame and wave propagation (the Kuramoto-Sivashinskii equation and the extended Fisher-Kolmogorov equation), the phase

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transition at critical Lifschitz points, bi-stable systems and applications structural mechanics (see, e.g., [1, 2, 3]).

By applying the variational theory and the Galërkin method, Cao and Gu [4] proved the existence and uniqueness of global solutions of the following Cauchy problem

$$\begin{cases} u_t + (-\Delta)^m u = |u|^p, \ (x,t) \in R^n \times [0,+\infty), \\ u(x,0) = u_0(x), \ x \in R^n. \end{cases} \tag{1.2}$$

Recently, Miao and Zhang [5] obtained the global existence and uniqueness of solutions of problem (1.2) in time-space spaces by establishing the time-space estimates. Egorov et al. [6] studied the global non-existence of problem (1.2) for $1 < p \leq 1 + \frac{2m}{n}$, $\varphi \in L^1_{loc}(R^n/\{0\})$, and $\int_{R^n} \varphi(x)dx \geq 0$ by using the test function approach. Later, they also gave the asymptotic behavior of global solutions for the problem (1.2) with suitable initial data in the supercritical Fujita range $p > 1 + \frac{2m}{n}$ by constructing self-similar solutions of high-order parabolic operators and the stability analysis of the autonomous dynamical system [7]. Under the assumptions of the initial value and p, Galaktionov and Pohozaev [8] proved the existence and uniform decay estimate of global solutions and the blow-up of solutions in finite time of problem (1.2) with the aid of a comparison with self-similar solutions of majorizing order-preserving integral equations. When the initial data decay slowly, Caristi and Mitidieri [9] dealt with the existence and nonexistence of global solutions of the problem (1.1) with $|u|^p$ in place of the logarithmic nonlinearity based on the method presented in [9]. Galaktionov and Williams [10] considered the existence of radially symmetric self-similar solutions and the longtime behavior of solutions to the problem (1.2) with $1 < p < 1 + \frac{2m}{n}$, $m \geq 1$ according to the theory of ordinary differential equations. As $n = 1, m > 1$, Budd, Galaktionov, and Williams [11] showed the self-similar blow-up of solutions of problem (1.2). However, when the nonlinearity is a logarithmic nonlinear function $u \log |u|^p$, as far as we know, there is no result on this aspect of the global solutions and blow-up solutions of problem (1.1).

Recently, Ishige, Kawakami and Okabe [12] proved the existence of solutions of Cauchy problem (1.2) by introducing a new majorizing kernel and obtained the necessary conditions on the initial data for the existence of local solutions. In [13], Liao and Li studied fourth-order parabolic equations with logarithmic nonlinearity by applying the modified potential well method and the logarithmic Sobolev inequality. Under the different initial data, they obtained the global existence of weak solutions, decay estimates, and the blow-up in finite time, respectively. Ghoul, Nguyen and Zaag [14] considered problem (1.2) in R^n. They exhibited the nonself-similar blow-up solutions and obtained a sharp description of its asymptotic behavior. Moreover, Xiao and Li [15] further studied the initial boundary value problem of the equation in (1.2). They first proved the existence of nonzero weak solutions to the static problem with the aid of the mountain pass theorem, and then proved the existence of global weak solutions to the evolution problem by using the method of potential well.

Finally, let us mention that the following Cauchy problem of the semilinear wave equation with logarithmic nonlinearity

$$\begin{cases} u_{tt} - \Delta u = u \log |u|^p, \ (x,t) \in R^n \times [0,+\infty), \\ u(x,0) = u_0(x), \ x \in R^n. \end{cases} \tag{1.3}$$

It was studied by Cazenave and Haraux [16]. They obtained the existence and uniqueness of solutions of problem (1.3). For recent results on the logarithmic Schrödinger equation, we refer to [17, 18, 19, 20] and the references therein.

In this paper, we prove the global existence of solutions of problem (1.1) by applying the potential well theory introduced by Sattinger [21] and Payne and Sattinger [22]. Furthermore, we obtain the exponential decay and the blow-up result of solutions.

We adopt the usual notations and convention. Let $H^m(\Omega)$ denote the Sobolev space with the usual scalar products and norm. $H_0^m(\Omega)$ denotes the closure in $H^m(\Omega)$ of $C_0^\infty(\Omega)$. For simplicity of notations, hereafter we denote by $\|\cdot\|_s$ the norm of Lebesgue space $L^s(\Omega)$, and by $\|\cdot\|$ the $L^2(\Omega)$ norm. We write the equivalent norm $\|D^m\cdot\|$ instead of $H_0^m(\Omega)$ norm $\|\cdot\|_{H_0^m(\Omega)}$, where D indicates the gradient operator, that is, $Du=\nabla u=(\frac{\partial u}{\partial x_1},\frac{\partial u}{\partial x_2},\cdots,\frac{\partial u}{\partial x_n})$, and $D^m u=\Delta^j u$ if $m=2j$ and $D^m u=D\Delta^j u$ if $m=2j+1$. In addition, C_i $(i=1,2,\cdots)$ denotes various positive constant, which depends on the known constants and may be different at each appearance.

This paper is organized as follows. In Section 2, we give some definitions and lemmas, which will be used for our main results. In Section 3, we study the existence and exponential decay of global solutions of problem (1.1). Finally, Section 4 is devoted to the blow-up result of solutions in finite time.

2. Preliminaries

First, we define the following functionals:

$$J(u(t))=\frac{1}{2}\|D^m u(t)\|^2-\frac{p}{2}\int_\Omega u^2(t)\log|u(t)|dx+\frac{p}{4}\|u(t)\|^2,$$

$$K(u(t))=\|D^m u(t)\|^2-p\int_\Omega u^2(t)\log|u(t)|dx, \tag{2.1}$$

for $u\in H_0^m(\Omega)$. Then, it is obvious that

$$J(u(t))=\frac{1}{2}K(u(t))+\frac{p}{4}\|u(t)\|^2. \tag{2.2}$$

As in [22], the mountain pass value of $J(u(t))$ (also known as potential well depth) is defined as

$$d=\inf\{\sup_{\lambda\geq 0}J(\lambda u):\ u\in H_0^m(\Omega)/\{0\}\}. \tag{2.3}$$

Now, we define the so-called Nehari manifold (see, e.g., [23, 24]) as follows

$$N=\{u\in H_0^m(\Omega)/\{0\}:\ K(u(t))=0\}.$$

Thus, the stable set W and unstable set U can be defined, respectively, by

$$W=\{u\in H_0^m(\Omega):\ K(u(t))>0,\ J(u(t))<d\}\cup\{0\},$$

and

$$U=\{u\in H_0^m(\Omega):\ K(u(t))<0,\ J(u(t))<d\}.$$

It is easy to see that the potential well depth d defined in (2.3) can also be characterized as

$$d=\inf_{u\in N}J(u). \tag{2.4}$$

Next, we give the definition of weak solutions of problem (1.1) and list some known lemmas.

Definition 2.1. A function $u = u(x,t)$ is said to be a weak solution to problem (1.1) on $\Omega \times [0,T]$ if

$$u \in C([0,T], H_0^m(\Omega)) \ \ , u_t \in L^2([0,T], L^2(\Omega)),$$

and

$$\int_\Omega u_t \varphi dx + \int_\Omega D^m u D^m \varphi dx = \int_\Omega u \log |u|^p \varphi dx$$

for each test functions $\varphi \in H_0^m(\Omega)$ and for almost all $t \in [0,T]$.

Lemma 2.2. *Let r be a number with $2 \le r < +\infty$ if $n \le 2m$ and $2 \le r \le \frac{2n}{n-2m}$ if $n > 2m$. Then there exists a constant C depending on Ω and r such that*

$$\|u\|_r \le C \|D^m u\|, \ \forall u \in H_0^m(\Omega).$$

Lemma 2.3. [25, 26, 27] *If $u \in H_0^1(\Omega)$, then, for each $a > 0$,*

$$\int_\Omega |u|^2 \log |u| dx \le \|u\|^2 \log \|u\| + \frac{a^2}{2\pi} \|\nabla u\|^2 - \frac{n}{2}(1 + \log a)\|u\|^2.$$

It follows from Sobolev inequality and Lemma 2.2 that

$$\int_\Omega |u|^2 \log |u| dx \le \|u\|^2 \log \|u\| + \frac{Ba^2}{2\pi} \|D^m u\|^2 - \frac{n}{2}(1 + \log a)\|u\|^2, \tag{2.5}$$

where B is the best Sobolev constant from $H_0^m(\Omega)$ to $H_0^1(\Omega)$.

3. Global Existence and Exponential Decay

We can now proceed in the study of the existence of global solutions of problem (1.1). For this purpose, we need the following lemmas.

Lemma 3.1. *If $u \in H_0^m(\Omega)$ and $\|u\| \ne 0$, then*

$$K(\lambda u) = \lambda \frac{d}{d\lambda} J(\lambda u) \begin{cases} > 0, & 0 < \lambda < \lambda^*, \\ = 0, & \lambda = \lambda^*, \\ < 0, & \lambda^* < \lambda < +\infty, \end{cases} \tag{3.1}$$

where

$$\lambda^* = \exp\left(\frac{\|D^m u\|^2 - p\int_\Omega u^2 \log |u| dx}{p\|u\|^2} \right).$$

Proof. Since

$$J(\lambda u) = \frac{\lambda^2}{2}\|D^m u\|^2 + \frac{p\lambda^2}{4}\|u\|^2 - \frac{p}{2}\lambda^2 \log \lambda \|u\|^2 - \frac{p}{2}\lambda^2 \int_\Omega u^2 \log |u| dx,$$

we obtain

$$\frac{d}{d\lambda} J(\lambda u) = \lambda \|D^m u\|^2 - p\lambda \int_\Omega u^2 \log |u| dx - p\lambda \log \lambda \|u\|^2. \tag{3.2}$$

Letting $\frac{d}{d\lambda} J(\lambda u) = 0$, we have

$$\lambda^* = \exp\left(\frac{\|D^m u\|^2 - p\int_\Omega u^2 \log |u| dx}{p\|u\|^2} \right).$$

It follows from (2.1) that

$$K(\lambda u) = \lambda^2 \|D^m u\|^2 - p\lambda^2 \int_\Omega u^2 \log|u| dx - p\lambda^2 \log\lambda \|u\|^2. \tag{3.3}$$

From (3.2) and (3.3), we obtain (3.1) immediately. □

Lemma 3.2. *If $u \in H_0^m(\Omega)$, then*

$$d \geq \frac{p}{4}(ae)^n > 0. \tag{3.4}$$

Proof. From (2.5), we obtain that

$$\begin{aligned} K(u) &= \|D^m u\|^2 - p\int_\Omega u^2 \log|u| dx \\ &\geq \left(1 - \frac{pBa^2}{2\pi}\right)\|D^m u\|^2 + \left[\frac{np}{2}(1+\log a) - p\log\|u\|\right]\|u\|^2, \end{aligned} \tag{3.5}$$

for any $a > 0$. Taking $0 < a \leq \sqrt{\frac{2\pi}{pB}}$, we obtain from (3.5) that

$$K(u) \geq \left[\frac{np}{2}(1+\log a) - p\log\|u\|\right]\|u\|^2. \tag{3.6}$$

In view of Lemma 3.1 and (2.2), we conclude that

$$\sup_{\lambda\geq 0} J(\lambda u) = J(\lambda^* u) = \frac{1}{2}K(\lambda^* u) + \frac{p}{4}\|\lambda^* u\|^2. \tag{3.7}$$

It follows from (3.6) that

$$0 = K(\lambda^* u) \geq \left[\frac{np}{2}(1+\log a) - p\log\|\lambda^* u\|\right]\|\lambda^* u\|^2,$$

which implies

$$\|\lambda^* u\| \geq (ae)^{\frac{n}{2}}. \tag{3.8}$$

Combining (3.7) and (3.8), we arrive at

$$\sup_{\lambda\geq 0} J(\lambda u) \geq \frac{p}{4}(ae)^n,$$

which together with (2.3) yields that $d \geq \frac{p}{4}(ae)^n > 0$. □

Lemma 3.3. *If $u_0 \in W$, then $u(t) \in W$ for each $t \in [0,+\infty)$.*

Proof. Multiplying by u_t and integrating over $\Omega \times [0,t]$ on two sides of the equation in (1.1), we have

$$\int_0^t \|u_t(s)\|^2 ds + J(u(t)) = J(u_0) < d, \tag{3.9}$$

for $\forall t \in [0,+\infty)$.

Suppose that there exists a number $t^* \in [0,+\infty)$ such that $u(t) \in W$ on $[0,t^*) \cup (t^*,+\infty)$ and $u(t^*) \notin W$. Then, in virtue of the continuity of $u(t)$, we see $u(t^*) \in \partial W$. From the definition of W and the continuity of $J(u(t))$ and $K(u(t))$ with respect to t, we have

$$J(u(t^*)) = d, \tag{3.10}$$

or

$$K(u(t^*)) = 0. \tag{3.11}$$

It follows from (3.9) that

$$J(u(t^*)) < J(u_0) < d. \tag{3.12}$$

Thus, case (3.10) is impossible.

If (3.11) holds, then $u(t^*) \in N$. It follows from (2.4) that $J(u(t^*)) \geq d$, which contradicts (3.12). Consequently, case (3.11) is also impossible. Thus, we conclude that $u(t) \in W$ on $[0,+\infty)$. This completes the proof. □

Theorem 3.4. *If $u_0 \in W$, then problem (1.1) has a global solution which satisfies*

$$u \in L^\infty(0,+\infty;\ H_0^m(\Omega)),\ \ u_t \in L^2(0,+\infty; L^2(\Omega)).$$

Proof. It suffices to show that $\int_0^t \|u_t(s)\|^2 ds + \|D^m u\|^2$ is bounded and independently of t. Under the hypotheses of this theorem, we conclude from Lemma 3.3 that $u \in W$ on $[0,+\infty)$. From (2.5), we assert that the following formula holds on $[0,+\infty)$

$$\begin{aligned} J(u) &= \frac{1}{2}\|D^m u\|^2 - \frac{p}{2}\int_\Omega u^2 \log|u|dx + \frac{p}{4}\|u\|^2 \\ &\geq \frac{1}{2}\left(1 - \frac{Bpa^2}{2\pi}\right)\|D^m u\|^2 + \frac{p}{2}\left[\frac{1}{2} + \frac{n}{2}(1+\log a) - \log\|u\|\right]\|u\|^2. \end{aligned} \tag{3.13}$$

In view of (2.2) and $u \in W$, we obtain that

$$J(u(t)) = \frac{1}{2}K(u(t)) + \frac{p}{4}\|u(t)\|^2 > \frac{p}{4}\|u(t)\|^2.$$

Hence,

$$\|u\|^2 < \frac{4}{p}J(u(t)) < \frac{4d}{p}. \tag{3.14}$$

which implies that (3.13) that

$$\left(1 - \frac{Bpa^2}{2\pi}\right)\|D^m u\|^2 \leq d\log\frac{4d}{p(ae)^n}.$$

By $0 < a < \sqrt{\frac{2\pi}{Bp}}$, we obtain that

$$\|D^m u\|^2 \leq C_1, \tag{3.15}$$

where

$$C_1 = d\left(1 - \frac{Bpa^2}{2\pi}\right)^{-1}\log\frac{4d}{p(ae)^n}.$$

By (2.2) and $u \in W$, we see that $J(u(t)) > 0$. Hence, it follows from (3.9) that

$$\int_0^t \|u_t(s)\|^2 ds \leq d,$$

which together with (3.15) obtains that

$$\int_0^t \|u_t(s)\|^2 ds + \|D^m u\|^2 \leq C_2, \tag{3.16}$$

where $C_2 = d + C_1$.

On the other hand, inequality (3.16) and the continuation principle ([28, 29]) lead to the global existence of the solution u for problem (1.1). This completes the proof of Theorem 3.4. □

The following lemma plays an important role in studying the decay estimate of global solutions.

Lemma 3.5. *Let $Y(t): R^+ \to R^+$ be a nonincreasing function and assume that there exists a constant $\Lambda > 0$ such that*

$$\int_S^{+\infty} Y(t)dt \leq \frac{1}{\Lambda} Y(S),\ 0 \leq S < +\infty.$$

Then

$$Y(t) \leq Y(0)e^{1-\Lambda t},\ \forall t \geq 0.$$

The result on exponential decay of global solutions reads as follows.

Theorem 3.6. *If $u_0 \in W$, then the global solution of problem (1.1) has the following exponential decay property*

$$\|u(t)\| \leq \|u_0\| e^{\frac{1}{2} - \frac{1}{2}(p \log \frac{p(ae)^n}{4d})t}.$$

Proof. Multiplying the equation in (1.1) by u, and integrating over $\Omega \times [S,T]$, we obtain that

$$\int_S^T K(u(t))dt = \frac{1}{2}(\|u(S)\|^2 - \|u(T)\|^2) \leq \frac{1}{2}\|u(S)\|^2$$

for $0 \leq S < T < +\infty$. From (3.6) and (3.14), we have

$$K(u) \geq \frac{p}{2}\left[n(1+\log a) - \log \frac{4d}{p}\right]\|u\|^2 = \frac{p}{2}\left(\log \frac{p(ae)^n}{4d}\right)\|u\|^2,$$

which together with (3.2) yields that

$$\int_S^T \|u(t)\|^2 dt \leq \left(p \log \frac{p(ae)^n}{4d}\right)^{-1} \|u(S)\|^2. \tag{3.17}$$

Let $T \to +\infty$. It follows from (3.17) and Lemma 3.5 that

$$\|u(t)\| \leq \|u_0\| e^{\frac{1}{2} - \frac{1}{2}(p \log \frac{p(ae)^n}{4d})t}.$$

This completes the proof of the Theorem 3.6. □

4. The Blow-Up

In this section, we are concerned with the blow-up property of solutions for initial-boundary problem (1.1). For this purpose, we give the following lemma.

Lemma 4.1. *Let $u(t)$ be a solution to problem (1.1). If $u_0 \in U$, then $u(t) \in U$ for all $t \geq 0$.*

Proof. For $u_0 \in U$ and (3.9), we have

$$J(u(t)) \leq J(u_0) < d,\ \forall t \geq 0. \tag{4.1}$$

Next, let us assume by contradiction that there exists $t_0 > 0$ such that $u(t_0) \notin U$, i.e., $K(u(t_0)) = 0$ and $K(u(t)) < 0$ for $0 \leq t < t_0$. This implies that $u(t_0) \in N$. It follows from (2.4) that $J(u(t_0)) \geq d$, which is contradiction to (4.1). This completes the proof. □

In order to study the blow-up of solutions, we define the set U_e by

$$U_e = \left\{ u \in H_0^m(\Omega) : \ K(u(t)) < 0, \ J(u(t)) < \frac{p}{8}(ae)^n \right\}.$$

By (3.4), we know $\frac{p}{8}(ae)^n < d$, which implies that $U_e \subset U$.

Theorem 4.2. *If $u_0 \in U_e$, then the solution $u(t)$ of problem (1.1) blows up at $+\infty$, i.e.,*

$$\lim_{t\to+\infty} \|u(t)\|^2 = +\infty.$$

Proof. In view of $u_0 \in U_e \subset U$, we obtain from Lemma 4.1 that $u(t) \in U_e \subset U$ for all $t \geq 0$. Thus,

$$K(u(t)) = \|D^m u\|^2 - p\int_\Omega u^2 \log|u| dx < 0,$$

for all $t \geq 0$, which together with (2.5) yields that

$$(1 - \frac{Bpa^2}{2\pi})\|D^m u\|^2 + p\left[\frac{n}{2}(1+\log a) - \log\|u\|\right]\|u\|^2 < 0. \tag{4.2}$$

We conclude from $0 < a \leq \sqrt{\frac{2\pi}{Bp}}$ and (4.2) that

$$\frac{n}{2}(1+\log a) - \log\|u\| < 0.$$

Therefore,

$$\|u(t)\|^2 > (ae)^n, \ \forall t \geq 0. \tag{4.3}$$

Assume by contradiction that the solution $u(t)$ is global. Then, for any $t \geq 0$, we define $\Theta(t) : [0,\infty) \to [0,+\infty)$ by

$$\Theta(t) = \int_0^t \|u(s)\|^2 ds. \tag{4.4}$$

Observe that $\Theta(t) > 0$ for all $t \geq 0$. By the continuity of the function $\Theta(t)$, we find that there exists $\mu > 0$ such that

$$\Theta(t) \geq \mu > 0, \ \forall t \geq 0. \tag{4.5}$$

By differentiating on both sides of (4.4), we arrive at

$$\Theta'(t) = \|u(t)\|^2 > 0. \tag{4.6}$$

Taking the derivative of the function $\Theta'(t)$ in (4.6), we obtain

$$\Theta''(t) = 2\left(p\int_\Omega u^2 \log|u| dx - \|D^m u\|^2\right) = -2K(u) > 0. \tag{4.7}$$

It follows from (2.2), (3.9), and (4.7) that

$$\Theta''(t) = -4J(u) + p\|u\|^2 = -4J(u_0) + 4\int_0^t \|u_t(\tau)\|^2 d\tau + p\Theta'(t). \tag{4.8}$$

In view of

$$\begin{aligned}\left(\int_0^t \int_\Omega u(s)u_t(s) dx ds\right)^2 &= \left(\frac{1}{2}\int_0^t \frac{d}{ds}\|u(s)\|^2 ds\right)^2 = \frac{1}{4}[\Theta'(t) - \|u_0\|^2]^2 \\ &= \frac{1}{4}[\Theta'(t)^2 - 2\Theta'(t)\|u_0\|^2 + \|u_0\|^4],\end{aligned}$$

we obtain from (4.8) that

$$\begin{aligned}&\Theta(t)\Theta''(t)-\Theta'(t)^2\\&=4\left[\int_0^t\|u(s)\|^2ds\int_0^t\|u_t(s)\|^2ds-\left(\int_0^t\int_\Omega u(s)u_t(s)dxds\right)^2\right]\\&+p\Theta(t)\Theta'(t)-2\Theta'(t)\|u_0\|^2-4\Theta(t)J(u_0)+\|u_0\|^4.\end{aligned}\tag{4.9}$$

By using the Hölder inequality, we obtain

$$\left(\int_0^t\int_\Omega uu_tdxds\right)^2\leq\int_0^t\|u(s)\|^2ds\int_0^t\|u_t(s)\|^2ds,$$

which together with (4.9) yields that

$$\begin{aligned}\Theta(t)\Theta''(t)-\Theta'(t)^2&\geq p\Theta(t)\Theta'(t)-2\Theta'(t)\|u_0\|^2-4\Theta(t)J(u_0)\\&\geq 2\Theta'(t)\left(\frac{p}{4}\Theta(t)-\|u_0\|^2\right)+4\Theta(t)\left[\frac{p}{8}\Theta'(t)-J(u_0)\right].\end{aligned}\tag{4.10}$$

From (4.3), (4.6), and (4.7), we have

$$\Theta'(t)>(ae)^n,\ \ \Theta(t)>\Theta(0)+\Theta'(0)t=\|u_0\|^2t.\tag{4.11}$$

Combining (4.5), (4.10), and (4.11), we obtain

$$\Theta(t)\Theta''(t)-\Theta'(t)^2\geq 2(ae)^n\left(\frac{p}{4}t-1\right)\|u_0\|^2+4\mu\left[\frac{p}{8}(ae)^n-J(u_0)\right].\tag{4.12}$$

Choosing t sufficiently large such that $t\geq t_0=\frac{4}{p}$, and by $u_0\in U_e$, we obtain from (4.12) that

$$\Theta(t)\Theta''(t)-\Theta'(t)^2>0,\ \forall t\geq t_0,$$

which implies that

$$\Theta'(t)>\frac{\Theta'(t_0)}{\Theta(t_0)}\Theta(t),\ \Theta(t)\geq\Theta(t_0)e^{\frac{\Theta'(t_0)}{\Theta(t_0)}(t-t_0)},\ \forall t\geq t_0.$$

Hence,

$$\Theta'(t)>\|u(t_0)\|^2e^{\frac{\Theta'(t_0)}{\Theta(t_0)}(t-t_0)},\ \forall t\geq t_0.$$

It follows that

$$\lim_{t\to+\infty}\Theta'(t)=+\infty,$$

which together with (4.6) yields that

$$\lim_{t\to+\infty}\|u(t)\|^2=+\infty.$$

Hence, we can not suppose that the solution of problem (1.1) is global. This means that the solution of problem (1.1) blows up at $+\infty$. This completes the proof. □

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REFERENCES

[1] V. A.Galaktionov, J.F. Williams, Blow-up in a fouth-order semilinear parabolic equations from convection-explosion theory, European J. Appl. Math. 14 (2003), 745-764.

[2] L. Peletier, W. Troy, Spatial Patterns: Higher-order models in physics and mechanics, Birkhäuser, Boston, Berlin 2001.

[3] Y.B. Zel'dovich, G.I. Barenblatt, V.B. Librovich, G.M. Makhviladze, The mathematical theory of combustion and explosions, Consultants Bureau(Plenum), New York, London 1985.

[4] Z. Cao, L. Gu, Initial-boundary value problem for a degenerate quasilinear parabolic equation of order $2m$, J. Part. Diff. Eq. 3 (1990), 13-20.

[5] C. Miao, B. Zhang, The Cauchy problem for semilinear parabolic equations in Besov spaces, Houston J. Math. 30 (2004), 829-878.

[6] Y.V. Egorov, V.A. Galaktionov, V.A. Kondrative, S.I. Pohozaev, On the necessary conditions of global existence to a quasilinear inequality in the half-space, Comptes Rendus de l'Académie des Sciences - Series I - Mathematics 330 (2000), 93-98.

[7] Y.V. Egorov, V.A. Galaktionov, V.A. Kondratiev, S.I. Pohozaev, Global solutions of higher-order semilinear parabolic equations in the supercritical range, Adv. Differential Equations 9 (2004), 1009-1038.

[8] V.A. Galaktionov, S.I. Pohozaev, Existence and blow-up for higher-order semilinear parabolic equations: Majorizing order-preserving operators, Indiana Univ. Math. J. 51 (2002), 1321-1338.

[9] G. Caristi, E. Mitidieri, Existence and nonexistence of global solutions of higher-order parabolic problems with slow decay initial data, J. Math. Anal. Appl. 279 (2003), 710-722.

[10] V.A. Galaktionov, J.F. Williams, On very singular similarity solutions of a higher-order semilinear parabolic equations, Nonlinearity 17 (2004), 1705-1099.

[11] C.J. Budd, V.A. Galaktionov, J.F. Williams, Self-similar blow-up in higher-order semilinear parabolic equations, SIAM J. Appl. Math. 64 (2004), 1775-1809.

[12] K. Ishige, T. Kawakami, S. Okabe, Existence of solutions for a higher-order semilinear parabolic equation with singular initial data, Annales de l'Institut Henri Poincaré C, Analyse non linéaire 37 (2020), 1185-1209.

[13] M. Liao, Q. Li, A Class of Fourth-order Parabolic Equations with Logarithmic Nonlinearity, Taiwanese J. Math. 24 (2020) 975-1003.

[14] T.E. Ghoul, V.T. Nguyen, H. Zaag, Construction of type I blowup solutions for a higher order semilinear parabolic equation, Adv. Nonlinear Anal. 9 (2019), 388-412.

[15] L. Xiao, M. Li, Initial boundary value problem for a class of higher-order n-dimensional nonlinear pseudo-parabolic equations, Boundary Value Probl. 2021 (2021), 5.

[16] T.Cazenave, A. Haraux, Equations d'evolution avec non-linearite logarithmique, Ann. Fac. Sci. Toulouse Math. 2 (1980), 21-51.

[17] T. Cazenave, Stable solutions of the logarithmic Schrödinger equation, Nonlinear Anal. 7 (1983), 1127-1140.

[18] T. Cazenave, A. Haraux, Équation de Schrödinger avec non-linéarité logarithmique, C. R. Acad. Sci. Paris Sér. 288 (1979), 253-256.

[19] P. Gorka, Logarithmic quantum mechanics: Existence of the ground state, Found. Phys. Lett. 19 (2006), 591-601.

[20] P. Gorka, Convergence of logarithmic quantum mechanics to the linear one, Lett. Math. Phys. 81 (2007), 253-264.

[21] D.H. Sattinger, On global solutions for nonlinear hyperbolic equations, Arch. Rational Mech. Anal. 30 (1968), 148-172.

[22] L.E. Payne, D.H. Sattinger, Saddle points and instability of nonlinear hyperbolic equations, Israel J. Math. 22 (1975), 273-303.

[23] Z. Nehari, On a class of nonlinear second-order differential equations, Trans. Amer. Math. Soc. 95 (1960), 101-123.

[24] M. Willem, Minimax Theorems, Progress Nonlinear Differential Equations Appl. Birkhäuser Boston, Boston MA 24 (1996), 162.

[25] H. Chen, P. Luo, G. Liu, Global solution and blow-up of a semilinear heat equation with logarithmic nonlinearity, J. Math. Anal. Appl. 422 (2015), 84-98.

[26] H. Chen, S. Tian, Initial boundary value problem for a class of semilinear pseudo-parabolic equations with logarithmic nonlinearity, J. Differential Equations 258 (2015), 4424-4442.
[27] L. Gross, Logarithmic Sobolev inequalities, Amer. J. Math. 97 (1975), 1061-1083.
[28] V. Georgiev, D. Todorova, Existence of solutions of the wave equations with nonlinear damping and source terms, J. Differential Equations 109 (1994), 295-308.
[29] I. Segal, Nonlinear semigroups, Ann. Math. 78 (1963), 339-364.