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ON THE EXISTENCE OF BEST PROXIMITY POINTS OF MULTI-VALUED MAPPINGS IN CAT(0) SPACES

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Abstract. In this paper, we investigate the best proximity points of multivalued mappings via Mann and Ishikawa iteration schemes. Several convergence theorems of best proximity points are established in the framework of CAT(0) spaces. Our results extend and improve the related results in the literature.
Keywords. Best proximity point; CAT(0) space; fixed points, Multi-valued mapping; Proximally quasi-nonexpansive mapping.

1. INTRODUCTION

Let M and N be nonempty subsets of a metric space (X,d). If $M \cap N = \emptyset$, then a mapping T from M to N has no solutions for the fixed point equation $T(x) = x$. At this situation, it is natural to determine an approximate solution x such that the error $d(x,Tx)$ is minimum. The purpose of best proximity point theorems is to find sufficient conditions such that the minimization problem $\min_x d(x,Tx)$ possess the existence of solutions. In this direction, many authors investigated the best proximity point theorems for various kind of contractions. We refer the reader to [1, 2, 3, 4] for the recent theorems of best proximity points.

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There are two approaches in the investigation of fixed points for multi-valued maps. The classical method of dealing with fixed points for multi-valued maps is to use the Hausdorff metric, introduced by Nadler [5]. The other method based on sets is due to Dontchev and Hager [6]. Both the methods have their advantages. In 1973, Markin [7] obtain a fixed point theorem for set-valued contractions and set-valued nonexpansive mappings via the Hausdorff metric. The concept of multi-valued mappings plays an important role in many areas, such as game theory, control theory, differential equations, economics, and convex optimization. Recently, Dzhabarova et al. [8] showed that best proximity points have applications in the equilibrium of duopoly, and Gecheva et al. [9] showed the fixed points of multi-valued mappings have applications in economics and ecology. For more details, we refer to [10, 11, 12]. Later, a number of researchers investigated the problem of finding the approximate solutions of fixed point equations for non-self multi-valued mappings. In 2013, Abkar and Gabeleh [13] derived the existence results of best proximity points for the case of multi-valued non-self mappings; see, e.g., [14, 15, 16] for more related results. In [17], Sastry and Babu obtained some convergence results of fixed points in Hilbert spaces for various types of multi-valued mappings by using Mann and Ishikawa iterative processes. In [18], Panyanak further proved the convergence results in the framework of Banach spaces. Let K be a nonempty convex subset of a Banach space $X := (X, \|\cdot\|)$. The set K is said to be proximinal if, for each $x \in X$, there exists an element $y \in K$ such that $\|x-y\| = d(x,K)$, where $d(x,K) = \inf \|x-z\| : z \in K$. Let $CB(K), K(K)$, and $P(K)$ denote the family of nonempty closed bounded subsets, nonempty compact subsets, and nonempty proximinal bounded subsets of K, respectively.

Consider the set

$$\mathbb{P}(X) = \{A \subset X : A \neq \emptyset, bounded, closed, and convex\}.$$

The Hausdorff metric on $\mathbb{P}(X)$ is defined by

$$H(A,B) = \max\{\sup_{x\in A} d(x,B), \sup_{y\in B} d(y,A)\}$$

for $A, B \in \mathbb{P}(X)$. A single-valued mapping $T : K \to K$ is said to be nonexpansive if $\|T(x) - T(y)\| \leq \|x-y\|$ for $x, y \in K$. A multi-valued mapping $T : K \to CB(K)$ is said to be nonexpansive if $H(T(x), T(y)) \leq \|x-y\|$ for all $x, y \in K$. An element $p \in K$ is called a fixed point of $T : K \to K$ (respectively, $T : K \to CB(K)$) if $p = T(p)$ (respectively, $p \in T(p)$). The set of fixed points of T is represented by $F(T)$. The mapping $T : K \to CB(K)$ is said to be quasi-nonexpasive if $F(T) \neq \emptyset$ and $H(x, T(p)) \leq \|x-p\|$ for all $x \in K$ and all $p \in F(T)$. It is clear that every nonexpansive multi-valued map T with $F(T) \neq \emptyset$ is quasi-nonexpansive. We note that there exist quasi-nonexpansive mappings which are not nonexpansive.

The following notations are used frequently. Let M and N be nonempty subsets of a metric space (X,d).

1) $d(x,N) = \inf\{d(x,y) : y \in N\}$;
2) $d(M,N) = dist(M,N) = \inf\{d(x,y) : x \in M, y \in N\}$;
3) $M_0 = \{x \in M : d(x,y) = dist(M,N) \text{ for some } y \in N\}$;
4) $N_0 = \{y \in N : d(x,y) = dist(M,N) \text{ for some } x \in M\}$;
5) $CL(X) = \{U : U \text{ is closed in } X\}$ and
6) $2^N = \{U : U \subseteq N\}$.

Let $T : M \to 2^N$ be a mapping. Recall that a point $r \in M$ is said to be a best proximity point of T if $d(r,Tr) = d(M,N)$. The set of all best proximity point of T is denoted by $Best(T)$. A set $A \subset X$ is said to be proximinal if, for $x \in X$, there exists an $s \in A$ such that

$$d(x,s) = d(x,A) = \inf\{d(x,\eta) : \eta \in A\}.$$

We note that every closed and convex subset of a uniformly convex Banach space is proximinal.

Let $T : K \to K$ be a single-valued mapping. The Mann iteration scheme, starting from $x_0 \in K$, generates a sequence $\{x_n\}$ by

$$x_{n+1} = \alpha_n T(x_n) + (1-\alpha_n)x_n, \alpha_n \in [0,1], n \geq 0,$$

where α_n satisfies certain conditions. The Ishikawa iteration scheme, starting from $x_0 \in K$, generates a sequence $\{x_n\}$ by

$$y_n = \beta_n T(x_n) + (1-\beta_n)x_n, \beta_n \in [0,1], n \geq 0,$$
$$x_{n+1} = \alpha_n T(y_n) + (1-\alpha_n)x_n, \alpha_n \in [0,1], n \geq 0,$$

where α_n and β_n satisfy certain conditions. Iterative techniques for approximating fixed points of nonexpansive sigle-valued mappings have been investigated by various authors via the Mann iteration scheme and the Ishikawa iteration scheme; see, e.g., [19, 20, 21] and the references therein.

In 2005, Sastry and Babu [17] defined the Mann and Ishikawa iteration schemes for multi-valued mappings. Let $T : K \to P(K)$ be a multi-valued mapping and fix $p \in F(T)$.

(I) The Mann iterative sequence is generated by $x_0 \in K$,

$$x_{n+1} = \alpha_n y_n + (1-\alpha_n)x_n, \alpha_n \in [0,1], n \geq 0,$$

where $y_n \in T(x_n)$ such that $\|y_n - p\| = d(p, T(x_n))$.

(II) The Ishikawa iterative sequence is generated by $x_0 \in K$,

$$y_n = \beta_n z_n + (1-\beta_n)x_n, \beta_n \in [0,1], n \geq 0,$$

where $z_n \in T(x_n)$ such that $\|z_n - p\| = d(p, T(x_n))$,and

$$x_{n+1} = \alpha_n z'_n + (1-\alpha_n)x_n, \alpha_n \in [0,1], n \geq 0,$$

where $z'_n \in T(y_n)$ such that $\|z'_n - p\| = d(p, T(y_n))$.

They proved that the Mann and Ishikawa iterative sequences converge to a fixed point q of T under certain conditions. They also claimed that the fixed point q may be different from p. More precisely, they obtained an interesting result for nonexpansive multi-valued mappings with compact domains in Hilbert spaces (see [17, Theorem 5]). In 2007, Panyanak [18] extended the above results to uniformly convex Banach spaces with the fact that the domain of T remains compact (see [18, Theorem 3.1]). Panyanak also modified the iteration schemes of Sastry and Babu [17]. Let $T : K \to P(K)$ be a multi-valued mapping, and let $P(K)$ be a nonempty proximinal subset of K.

(III) The Mann iterative sequence is generated by $x_0 \in K$,

$$x_{n+1} = \alpha_n y_n + (1-\alpha_n)x_n, \alpha_n \in [a,b], 0 < a < b < 1, n \geq 0,$$

where $y_n \in T(x_n)$ such that $\|y_n - u_n\| = d(u_n, T(x_n))$, and $u_n \in F(T)$ such that $\|x_n - u_n\| = d(x_n, F(T))$.

(IV) The Ishikawa iterative sequence is generated by $x_0 \in K$,

$$y_n = \beta_n z_n + (1-\beta_n)x_n, \beta_n \in [a,b], 0 < a < b < 1, n \geq 0,$$

where $z_n \in T(x_n)$ such that $\|z_n - u_n\| = d(u_n, T(x_n))$, and $u_n \in F(T)$ such that $\|x_n - u_n\| = d(x_n, F(T))$, and

$$x_{n+1} = \alpha_n z_n' + (1-\alpha_n)x_n, \alpha_n \in [a,b], n \geq 0,$$

where $z_n' \in T(y_n)$ such that $\|z_n' - v_n\| = d(v_n, T(y_n))$, and $v_n \in F(T)$ such that $\|y_n - v_n\| = d(y_n, F(T))$.

With the above iterative schemes, Panyanak obtained an interesting convergence result in a uniformly convex Banach space (see [18, Theorem 1.1]). For more related results, we refer the readers to [22, 23] and the references therein. On the other hand, recently, Puttasontiphot [24] established the convergence of fixed point sequences for multi-valued mappings in a complete CAT(0) space via Ishikawa and Mann iterative process.

In this paper, inspired and motivated by the results above, we investigate the best proximity points of proximally generalized nonexpansive and proximally quasi-contractive multi-valued mappings via the Mann and Ishikawa iterative schemes in the setting of CAT(0) spaces. Our results extend and improve the related results announced by Puttasontiphot [24], Sastry and Babu [17], etc.

2. Preliminaries

The fixed point theory in a CAT(0) spaces was first studied by Kirk (see [25] and [26]). Kirk showed that every nonexpansive (single-valued) mapping defined on a bounded closed convex subset of a complete CAT(0) space always has a fixed point. Since then, the fixed point theory for single-valued and multi-valued mappings in CAT(0) spaces has been rapidly developed and a lot of papers have appeared; see, e.g., [22, 27, 28, 29, 30] and the references therein.

Let (X,d) be a metric space. A geodesic path joining $x \in X$ to $y \in X$ (or, more briefly, a geodesic from x to y) is a mapping $\eta : [0,l] \to X, [0,l] \subset R$ such that $\eta(0) = x, \eta(l) = y$, and $d(\eta(t), \eta(t')) = |t - t'|$ for all $t, t' \in [0,l]$. In particular, η is an isometry, and $d(x,y) = l$. The image α of η is called a geodesic (or metric) segment joining x and y. When it is unique, this geodesic segment is denoted by $[x,y]$. (X,d) is said to be a geodesic space if every two points of X are joined by a geodesic, and X is said to be uniquely geodesic if there is exactly one geodesic joining x and y for each $x, y \in X$. A subset $Y \subseteq X$ is said to be convex if Y includes every geodesic segment joining any two of its points. A geodesic triangle $\Delta(x_1, x_2, x_3)$ in a geodesic metric space (X,d) consists of three points x_1, x_2, x_3 in X (the vertices of Δ) and a geodesic segment between each pair of vertices (the edges of Δ). A comparison triangle for the geodesic triangle $\Delta(x_1, x_2, x_3)$ in (X,d) is a triangle $\bar{\Delta}(x_1, x_2, x_3) := \Delta(\bar{x_1}, \bar{x_2}, \bar{x_3})$ in the Euclidean plane $\mathbb{E}^2$ such that $d_{\mathbb{E}^2}(\bar{x_i}, \bar{x_j}) = d(x_i, x_j)$ for $i, j \in 1, 2, 3$.

Definition 2.1. (CAT(0) space) Let (X,d) be a geodesic space. It is a CAT(0) space if, for any geodesic triangle $\Delta \subset X$ and $x, y \in \Delta$, $d(x,y) \leq d(\bar{x}, \bar{y})$ where $\bar{x}, \bar{y} \in \bar{\Delta}$.

It is known that any complete and simply connected Riemannian manifold with nonpositive sectional curvatures is a CAT(0) space. Other examples of CAT(0) spaces include, Pre-Hilbert spaces and R-trees (see [31]), Euclidean buildings (see [32]), the complex Hilbert ball with a hyperbolic metric (see [33]), and so on. For a thorough discussion of these spaces and their fundamental roles in geometry, we refer to [31, 32].

Definition 2.2. A geodesic triangle $\Delta(p,q,r)$ in (X,d) is said to satisfy the CAT(0) inequality if, for any $u,v \in \Delta(p,q,r)$ and for their comparison points $\bar{u},\bar{v} \in \bar{\Delta}(\bar{p},\bar{q},\bar{r})$, $d(u,v) \leq d_{\mathbb{E}^2}(\bar{u},\bar{v})$.

It is well known that every CAT(0) space is uniquely geodesic. Note that if x, y_1, and y_2 are points of CAT(0) space, and if y_0 is the midpoint of the segment $[y_1,y_2]$ (we write: $y_0 = \frac{1}{2}y_1 \bigoplus \frac{1}{2}y_2$), then the CAT(0) inequality implies

$$d(x,y_0)^2 \leq \frac{1}{2}d(x,y_1)^2 + \frac{1}{2}d(x,y_2)^2 - \frac{1}{4}d(y_1,y_2)^2. \tag{2.1}$$

In fact, a geodesic metric space is a CAT(0) space if and only if it satisfies inequality (2.1). This inequality is known as the CN inequality of Bruhat and Tits (see [34]). Some interesting results in CAT(0) spaces can be found in [30, 35].

We now collect some elementary facts about CAT(0) spaces which will be used frequently in the proofs of our main results.

Lemma 2.3. *Let (X,d) be a CAT(0) space.*

(i) [31, Proposition 2.4] *Let K be a convex subset of X which is complete in the induced metric. Then, for every $x \in X$, there exists a unique point $P(x) \in K$ such that $d(x,P(x)) = \inf\{d(x,y) : y \in K\}$. Moreover, the map $x \to P(x)$ is a nonexpansive retract from X onto K.*

(ii) [30, Lemma 2.1(iv)] *For $x,y \in X$ and $t \in [0,1]$, there exists a unique point $z \in [x,y]$ such that*

$$d(x,z) = td(x,y) \text{ and } d(y,z) = (1-t)d(x,y). \tag{2.2}$$

We use the notation $(1-t)x \bigoplus ty$ for the unique point z satisfying (2.2).

(iii) [30, Lemma 2.4] *For $x,y,z \in X$ and $t \in [0,1]$, we have*

$$d\big((1-t)x \bigoplus ty, z\big) \leq (1-t)d(x,z) + td(y,z).$$

(iv) [30, Lemma 2.5] *For $x,y,z \in X$ and $t \in [0,1]$, we have*

$$d\big((1-t)x \bigoplus ty, z\big)^2 \leq (1-t)d(x,z)^2 + td(y,z)^2 - t(1-t)d(x,y)^2.$$

Lemma 2.4. [36] *Let $\{x_n\}$ be a real sequence such that $x_{n+1} \leq \alpha x_n + \beta_n$, where $x_n \geq 0, \beta_n \geq 0, 0 < \alpha < 1$, and $\lim_{n\to\infty} \beta_n = 0$. Then $\lim_{n\to\infty} x_n = 0$.*

Lemma 2.5. [17] *Let $\{\alpha_n\}$and$\{\beta_n\}$ be two sequences such that (i) $0 \leq \alpha_n, \beta_n < 1$, (ii) $\beta_n \to 0$ and $\sum \alpha_n\beta_n = \infty$. Let $\{\gamma_n\}$ be nonnegative real sequence such that $\sum \alpha_n\beta_n(1-\beta_n)\gamma_n$ is bounded. Then $\{\gamma_n\}$ has a subsequence which converges to zero.*

Definition 2.6. [13] If $M_0, N_0 \neq \emptyset$ are subsets of a CAT(0) space, then the pair (M_0,N_0) has P-property for any $x_1,x_2 \in M_0$ and $y_1,y_2 \in N_0$,

$$\left.\begin{array}{l} d(x_1,y_1) = d(M,N) \\ d(x_2,y_2) = d(M,N) \end{array}\right\} \quad \Rightarrow \quad d(x_1,x_2) = d(y_1,y_2). \tag{2.3}$$

Following [17], we have the following definition.

Definition 2.7. Let $T : M \to 2^N$ be a multi-valued mapping.

(i) T is non-expansive if $H(Tx,Ty) \le d(x,y)$ for every $x \in M$.
(ii) T is quasi-nonexpansive if r is a best proximity point of T and satisfies $d(Tx,r) \le d(x,r)$ for every $x \in M$.
(iii) T is proximally generalized non-expansive if

$$H(Tx,Ty) \le \alpha d(x,y) + \beta\big(d(x,Tx) + d(y,Ty) - 2d(M,N)\big) + \gamma\big(d(x,Ty) + d(y,Tx) - 2d(M,N)\big)$$

for all $x,y \in M$, where α, β, and γ satisfy $\alpha + 2\beta + 2\gamma \le 1$.
(iv) T is proximally quasi-contractive if $k \in [0,1)$

$$H(Tx,Ty) \le k \max\{d(x,y), d(x,Tx) - d(M,N), d(y,Ty) - d(M,N), d(x,Ty) - d(M,N), d(y,Tx) - d(M,N)\}$$

for all $x,y \in M$.

3. Main Results

In this section, we introduce the following iteration schemes in the setting of CAT(0) spaces. Let X be a CAT(0) space, and let (M,N) be a pair of convex subsets of X. Let $T : M \to 2^N$ be a multi-valued mapping, and $Tx_0 \subseteq N_0$ for every $x_0 \in M_0$. We take $P_T(u) = \{v \in Tu : d(u,v) = d(u,Tu)\}$. If Tu is a closed and convex subset of a reflexive and strictly convex space, then $P_T(u)$ contains one element. First, we give the following definitions for non-self multi-valued mappings.

Definition 3.1. The mapping $T : M \to \mathbb{P}(N)$ is said to satisfy Condition (I^*) if there is a nonincreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0, f(r) > 0$ for $r \in (0,\infty)$ such that $d(x,Tx) - d(M,N) \ge f(d(x, Best(T)))$ for all $x \in M$.

Definition 3.2. (Mann Iteration Scheme) Let M_0 and $N_0 \neq \emptyset$ be subsets of a CAT(0) space, and $Tx_0 \subseteq N_0$, $x_0 \in M_0$. Let $\zeta_0 \in P_T(x_0)$. Then there exists $u_0 \in M_0$ such that $d(u_0,\zeta_0) = d(M,N)$. Define $x_1 = (1-\alpha_0)x_0 \bigoplus \alpha_0 u_0$. Continuing this process, for each $n \ge 0, x_n \in M_0$, we have $Tx_n \subseteq N_0$. Choose $\zeta_n \in P_T(x_n)$. Then there exists $u_n \in M_0$ such that $d(u_n,\zeta_n) = d(M,N)$. Define

$$x_{n+1} = (1-\alpha_n)x_n \bigoplus \alpha_n u_n,$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$.

Definition 3.3. (Ishikawa Iteration Scheme) Let M_0 and $N_0 \neq \emptyset$ be subsets of a CAT(0) space, and $Tx_0 \subseteq N_0$, $x_0 \in M_0$. Let $\zeta_0 \in P_T(x_0)$. Then there exists $v_0 \in M_0$ such that $d(v_0,\zeta_0) = d(M,N)$. Define $y_0 = (1-\beta_0)x_0 \bigoplus \beta_0 v_0$. Then $Ty_0 \subseteq N_0$. Let $z_0 \in P_T(u_0)$. Then there exists $z'_0 \in M_0$ such that $d(z'_0,z_0) = d(M,N)$. Define $x_1 = (1-\alpha_0)x_0 \bigoplus \alpha_0 z'_0$. Continuing this process, for each $n \ge 0, x_n \in M_0$, we have $Tx_n \subseteq N_0$. Choose $\zeta_n \in P_T(x_n)$. Then there exists $z_n \in M_0$ such that $d(z_n,\zeta_n) = d(M,N)$. Define

$$y_n = (1-\beta_n)x_n \bigoplus \beta_n z_n,$$

Then $Ty_n \subseteq N_0$. Choose $\gamma_n \in P_T(y_n)$. Then there exists $z'_n \in M_0$ such that $d(z'_n,\gamma_n) = d(M,N)$. Define

$$x_{n+1} = (1-\alpha_n)x_n \bigoplus \alpha_n z'_n,$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$.

Theorem 3.4. *Let M and N be two closed convex subsets of a complete CAT(0) space, and let $T : M \to \mathbb{P}(N)$ be a multivalued mapping with $Best(T) \neq \emptyset$ such that P_T is nonexpansive. Let $\{x_n\}$ be the sequence generated by the Ishikawa iteration scheme defined by Definition 3.3. Assume that T satisfies condition (I^*), and $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy: (i) $0 \leq \alpha_n, \beta_n < 1$, (ii) $\beta_n \to 0$, (iii) $\sum \alpha_n\beta_n = \infty$, and (iv) (M_0, N_0) has the P-property. Then the sequence $\{x_n\}$ converges to a best proximity point of T.*

Proof. Let r be a best proximity point of T. From Lemma 2.3 (iv), we have

$$\begin{aligned} d(x_{n+1}, r)^2 &= d\big((1-\alpha_n)x_n \bigoplus \alpha_n z'_n, r\big)^2 \\ &\leq (1-\alpha_n)d(x_n, r)^2 + \alpha_n d(z'_n, r)^2 - \alpha_n(1-\alpha_n)d(x_n, z'_n)^2. \end{aligned}$$

Since Tr is proximinal, we have that there exists $r* \in Tr$ such that $d(r, r*) = d(M, N)$. Since r is a best proximity point of T, we obtain $r* \in P_T(r)$. From the P-property, we have

$$d(x_{n+1}, r)^2 \leq (1-\alpha_n)d(x_n, r) + \alpha_n d(\gamma_n, r*)^2 - \alpha_n(1-\alpha_n)d(x_n, z'_n)^2.$$

Since $P_T(y_n)$ is a singleton, we have $P_T(y_n) = \{\gamma_n\}$. It follows that $d(\gamma_n, r*) = d(P_T(y_n), r*)$, and

$$\begin{aligned} d(x_{n+1}, r)^2 &\leq (1-\alpha_n)d(x_n, r)^2 + \alpha_n d\big(P_T(y_n), r*\big)^2 - \alpha_n(1-\alpha_n)d(x_n, z'_n)^2 \\ &\leq (1-\alpha_n)d(x_n, r)^2 + \alpha_n H^2\big(P_T(y_n), P_T(r)\big) - \alpha_n(1-\alpha_n)d(x_n, z'_n)^2 \\ &\leq (1-\alpha_n)d(x_n, r)^2 + \alpha_n d(y_n, r)^2. \end{aligned} \tag{3.1}$$

From Lemma 2.3 (iv) and the p-property, we have

$$\begin{aligned} d(y_n, r)^2 &= d\big((1-\beta_n)x_n \bigoplus \beta_n z_n, r\big)^2 \\ &\leq (1-\beta_n)d(x_n, r)^2 + \beta_n d(z_n, r)^2 - \beta_n(1-\beta_n)d(x_n, z_n) \\ &\leq (1-\beta_n)d(x_n, r)^2 + \beta_n d(\zeta_n, r*)^2 - \beta_n(1-\beta_n)d(x_n, z_n). \end{aligned}$$

Since $P_T(x_n)$ is a singleton, we have $P_T(x_n) = \{\zeta_n\}$. It follows that $d(\zeta_n, r*) = d(P_T(x_n), r*)$, and

$$\begin{aligned} d(y_n, r)^2 &\leq (1-\beta_n)d(x_n, r)^2 + \beta_n d\big(P_T(x_n), r*\big)^2 - \beta_n(1-\beta_n)d(x_n, z_n) \\ &\leq (1-\beta_n)d(x_n, r)^2 + \beta_n H^2\big(P_T(x_n), P_T(r)\big) - \beta_n(1-\beta_n)d(x_n, z_n) \\ &\leq d(x_n, r)^2 - \beta_n(1-\beta_n)d(x_n, z_n). \end{aligned} \tag{3.2}$$

Hence,

$$d(x_{n+1}, r)^2 \leq d(x_n, r)^2 - \alpha_n\beta_n(1-\beta_n)d(x_n, z_n), \tag{3.3}$$

which is equivalent to

$$\alpha_n\beta_n(1-\beta_n)d(x_n, z_n) \leq d(x_n, r)^2 - d(x_{n+1}, r)^2.$$

We have

$$\sum_{n=1}^{\infty} \alpha_n\beta_n(1-\beta_n)d(x_n, z_n) \leq d(x_1, r)^2 < \infty.$$

From Lemma 2.5, we have that $\{d(x_n, z_n)\}$ has a subsequence $\{d(x_{n_k}, z_{n_k})\}$ such that $\{d(x_{n_k}, z_{n_k})\} \to 0$ as $k \to \infty$.

On the other hand, we have

$$\begin{aligned} d(x_{n_k},\zeta_{n_k}) &\le d(x_{n_k},z_{n_k})+d(z_{n_k},\zeta_{n_k}) \\ &\le d(x_{n_k},z_{n_k})+d(M,N). \end{aligned}$$

Since $\zeta_{n_k}\in P_T(x_{n_k})$, we have $d(x_{n_k},\zeta_{n_k})=d(x_{n_k},Tx_{n_k})$. $d(x_{n_k},z_{n_k})\to 0$ yields that $d(x_{n_k},\zeta_{n_k})-d(M,N)\to 0$. By using Condition (I^*), we obtain $d(x_{n_k},Best(T))\to 0$. Therefore, there exists a subsequence of x_{n_k} (without loss of generaliy, we assume the same sequence) such that $d(x_{n_k},r_k)<\frac{1}{2^k}$ for some $r_k\in Best(T)$ for all k. From (3.3), we have

$$d(x_{n_{k+1}},r_k)\le d(x_{n_k},r_k)<\frac{1}{2^k},$$

and

$$d(r_{k+1},r_k)\le d(r_{k+1},x_{n_{k+1}})+d(x_{n_{k+1}},r_k)\le \frac{1}{2^{k+1}}+\frac{1}{2^k}\le\frac{1}{2^{k-1}}.$$

Hence, $\{r_k\}$ is a Cauchy sequence in M_0 and thus converges to $r'\in M_0$. Note that

$$\begin{aligned} d(r_k,Tr') &\le d(r_k,Tr_k)+H(Tr_k,Tr') \\ &\le d(M,N)+d(r_k,r'). \end{aligned}$$

Letting $k\to\infty$, we have $d(r',Tr')=d(M,N)$. Then $r'\in Best(T)$. It follows that $x_{n_k}\to r'$. From (3.3), one sees that $\{(d(x_n,r)\}$ is nonincreasing. By replacing r with r', we obtain a nonincreasing sequence $\{d(x_n,r')\}$. Since $d(x_n,r')\to 0$ as $k\to\infty$, we conclude the desired conclusion immediately. □

Theorem 3.5. *Let M and N be two convex subsets of a complete CAT(0) space X. Let $T:M\to\mathbb{P}(N)$ be a multivalued mapping with $Best(T)\neq\emptyset$ such that P_T is nonexpansive. Let $\{x_n\}$ be the sequence generated by the Mann iteration scheme defined by Definition 3.2. Assume that T satisfies condition (I^*) and $\{\alpha_n\}$ satisfies (i) $0\le\alpha_n<1$, (ii) $\sum\alpha_n=\infty$, and (iii) (M_0,N_0) has the P-property. Then $\{x_n\}$ converges to a best proximity point of T.*

Proof. Let r be a best proximity point of T. By Lemma 2.3 (iv), we have

$$\begin{aligned} d(x_{n+1},r)^2 &= d\big((1-\alpha_n)x_n\bigoplus\alpha_n u_n,r\big)^2 \\ &\le (1-\alpha_n)d(x_n,r)^2+\alpha_n d(u_n,r)^2-\alpha_n(1-\alpha_n)d(x_n,u_n)^2. \end{aligned}$$

Since Tr is proximinal, there exists $r*\in Tr$ such that $d(r,r*)=d(M,N)$. Since r is a best proximity of T, we obtain $r*\in P_T(r)$. Now from $d(u_n,\zeta_n)=d(M,N)$ and the P-property, we have

$$d(x_{n+1},r)^2\le(1-\alpha_n)d(x_n,r)+\alpha_n d(\zeta_n,r*)^2-\alpha_n(1-\alpha_n)d(x_n,u_n)^2$$

Since $P_T(x_n)$ is a singleton, $P_T(x_n)=\{\zeta_n\}$. Then $d(\zeta_n,r*)=d(P_T(x_n),r*)$, and

$$\begin{aligned} d(x_{n+1},r)^2 &\le (1-\alpha_n)d(x_n,r)^2+\alpha_n d\big(P_T(x_n),r*\big)^2-\alpha_n(1-\alpha_n)d(x_n,u_n)^2 \\ &\le (1-\alpha_n)d(x_n,r)^2+\alpha_n H^2\big(P_T(x_n),P_T(r)\big)-\alpha_n(1-\alpha_n)d(x_n,u_n)^2 \\ &\le (1-\alpha_n)d(x_n,r)^2+\alpha_n d(x_n,r)^2-\alpha_n(1-\alpha_n)d(x_n,u_n)^2 \\ &= d(x_n,r)^2-\alpha_n(1-\alpha_n)d(x_n,u_n)^2. \end{aligned}$$

We can rewrite the above inequality as

$$\alpha_n(1-\alpha_n)d(x_n,u_n)^2\le d(x_n,r)^2-d(x_{n+1},r)^2.$$

Therefore

$$\sum_{n=1}^{\infty} \alpha_n(1-\alpha_n)d(x_n,u_n) \le d(x_1,r)^2 < \infty.$$

By Lemma 2.5, we find that there exists a subsequence $\{d(x_{n_k},u_{n_k})\}$ of $\{d(x_n,u_n)\}$ such that $\{d(x_{n_k},u_{n_k})\} \to 0$ as $k \to \infty$. It follows that

$$\begin{aligned} d(x_{n_k},\zeta_{n_k}) &\le d(x_{n_k},u_{n_k}) + d(u_{n_k},\zeta_{n_k}) \\ &\le d(x_{n_k},u_{n_k}) + d(M,N). \end{aligned}$$

Since $\zeta_{n_k} \in P_T(x_{n_k})$, we have $d(x_{n_k},\zeta_{n_k}) = d(x_{n_k},Tx_{n_k})$. In view of $d(x_{n_k},u_{n_k}) \to 0$, we have $d(x_{n_k},\zeta_{n_k}) - d(M,N) \to 0$. By Condition ($I^*$), we obtain $d(x_{n_k},Best(T)) \to 0$. Following the same process as in the proof of Theorem 3.4, we obtain the desired conclusion immediately. □

Theorem 3.6. *Let M and N be two convex subsets of a complete CAT(0) space. Let $T : M \to \mathbb{P}(N)$ be a multivalued mapping with $Best(T) \neq \emptyset$ such that P_T is a proximally generalized nonexpansive mapping. Let $\{x_n\}$ be the sequence generated by the Ishikawa iteration scheme defined by Definition 3.3. Assume that T satifies condition (I^*), and $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy (i) $0 \le \alpha_n,\beta_n < 1$, (ii) $\beta_n \to 0$, (iii) $\sum \alpha_n\beta_n = \infty$, and (iv) (M_0,N_0) has the P-property. Then $\{x_n\}$ converges to a best proximity point of T.*

Proof. Following the proof of Theorem 3.4, we have

$$d(x_{n+1},r)^2 \le (1-\alpha_n)d(x_n,r)^2 + \alpha_n H^2\big(P_T(y_n),P_T(r)\big) - \alpha_n(1-\alpha_n)d(x_n,z_n')^2. \tag{3.4}$$

Since P_T is a proximally generalized nonexpansive non-self mapping, we have

$$\begin{aligned} H(P_T(y_n),P_T(r)) &\le \alpha d(y_n,r) + \beta\{d(y_n,P_T(y_n)) + d(r,P_T(r)) - 2d(M,N)\} \\ &\quad + \gamma\{d(r,P_T(y_n)) + d(y_n,P_T(r)) - 2d(M,N)\} \\ &\le \alpha d(y_n,r) + \beta\{d(y_n,r) + d(r,P_T(y_n)) - d(M,N)\} \\ &\quad + \gamma\{d(r,P_T(y_n)) + d(y_n,r) + d(r,P_T(r)) - 2d(M,N)\} \\ &\le \alpha d(y_n,r) + \beta\{d(y_n,r) + d(r,P_T(r)) + H(P_Tr,P_T(y_n)) - d(M,N)\} \\ &\quad + \gamma\{d(r,P_T(r)) + H(P_T(r),P_T(y_n)) + d(y_n,r) - d(M,N)\} \\ &= (\alpha+\beta+\gamma)d(y_n,r)) + (\beta+\gamma)H(P_T(r),P_T(y_n)). \end{aligned}$$

Hence

$$H(P_T(y_n),P_T(r)) \le \frac{(\alpha+\beta+\gamma)}{(1-(\beta+\gamma))}d(y_n,r).$$

Since $\frac{(\alpha+\beta+\gamma)}{(1-(\beta+\gamma))} \le 1$, we conclude $H(P_T(y_n),P_T(r)) \le d(y_n,r)$. From (3.4), we have

$$d(x_{n+1},r)^2 \le (1-\alpha_n)d(x_n,r)^2 + \alpha_n d(y_n,r)^2,$$

which is the inequality (3.1) in the proof of Theorem 3.4. Similarly, one can show that

$$d(y_n,r)^2 \le d(x_n,r)^2 - \beta_n(1-\beta_n)d(x_n,z_n)^2,$$

which is inequality (3.2). Following the same process as in Theorem 3.4, we obtain the desired conclusion immediately. □

Theorem 3.7. *Let M and N be convex subsets of a complete CAT(0) space X. Let $T : M \to \mathbb{P}(N)$ be a multivalued mapping with $Best(T) \neq \emptyset$ such that P_T is proximally generalized nonexpansive. Let $\{x_n\}$ be the sequence generated by the Mann iteration scheme defined by Definition 3.2. Assume that T satisfies condition (I^*) and $\{\alpha_n\}$ satisfies (i) $0 \leq \alpha_n < 1$, (ii) $\sum \alpha_n = \infty$, and (iii) (M_0, N_0) has the P-property. Then $\{x_n\}$ converges to a best proximity point of T.*

From Theorem 3.6, we can obtain the desired conclusion immediately. So, we omit the proof here.

Theorem 3.8. *Let M and N be two convex subsets of a complete CAT(0) space X. Let $T : M \to \mathbb{P}(N)$ be a multivalued mapping with $Best(T) \neq \emptyset$ such that P_T is a proximally quasi-contractive map. Let $\{x_n\}$ be the sequence generated by the Ishikawa iteration scheme defined by Definition 3.3. Assume that T satisfies condition (I^*) and $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy (i) $0 \leq \alpha_n, \beta_n < 1$, (ii) $\beta_n \to 0$, and (iii) $\sum \alpha_n \beta_n = \infty$ with $c \leq \alpha_n \leq 1 - k^2$, for $k \in [0,1), c > 0$. Then $\{x_n\}$ converges to a best proximity point of T.*

Proof. Let r be a best proximity point of T. By Lemma 2.3 (iv) ([30, Lemma 2.5]), we have

$$\begin{aligned} d(x_{n+1}, r)^2 &= d\big((1-\alpha_n)x_n \bigoplus \alpha_n z'_n, r\big)^2 \\ &\leq (1-\alpha_n)d(x_n, r)^2 + \alpha_n d(z'_n, r)^2 - \alpha_n(1-\alpha_n)d(x_n, z'_n)^2. \end{aligned} \tag{3.5}$$

Since Tr is proximinal, we find that there exists $r* \in Tr$ such that $d(r, r*) = d(M,N)$. Since r is a best proximity point of T, we have $r* \in P_T(r)$. In view of $d(z'_n, \gamma_n) = d(M,N)$ and the P-property, we have

$$d(z'_n, r) = d(\gamma_n, r*). \tag{3.6}$$

Since $P_T(y_n)$ is singleton, we have $P_T(y_n) = \{\gamma_n\}$. Therefore, $d(\gamma_n, r*) = d(P_T(y_n), r*)$, and

$$d(z'_n, r) = d(P_T(y_n), r*) \leq H\big(P_T(y_n), P_T(r)\big). \tag{3.7}$$

Now, from the definition of the proximally quasi-nonexpansive mapping, we obtain

$$\begin{aligned} &H(P_T(r), P_T(y_n)) \\ &\leq k \max\{d(r, y_n), d(r, P_T(r)) - d(M,N), d(y_n, P_T(y_n)) - d(M,N), \\ &d\big(r, P_T(y_n)\big) - d(M,N), d\big(y_n, P_T(r)\big) - d(M,N)\} \\ &\leq k \max\{d(r, y_n), d\big(y_n, P_T(y_n)\big) - d(M,N), d\big(r, P_T(r)\big) + H\big(P_T(r), P_T(y_n)\big) \\ &- d(M,N), d(y_n, r) + d(r, P_T(r)) - d(M,N)\} \\ &= k \max\{d(r, y_n), d\big(y_n, P_T(y_n)\big) - d(M,N), H\big(P_T(r), P_T(y_n)\big)\}. \end{aligned}$$

We have

$$H\big(P_T(r), P_T(y_n)\big) \leq kH\big(P_T(r), P_T(y_n)\big) < H\big(P_T(r), P_T(y_n)\big),$$

which is a contradiction. Therefore,

$$H\big(P_T(r), P_T(y_n)\big) \leq k \max\{d(r, y_n), d\big(y_n, P_T(y_n)\big) - d(M,N)\},$$

and

$$\begin{aligned} H^2(P_T(r), P_T(y_n)) &\leq k^2 \max\{d(r, y_n)^2, [d\big(y_n, P_T(y_n)\big) - d(M,N)]^2\} \\ &\leq k^2 \{d(r, y_n)^2 + [d\big(y_n, P_T(y_n)\big) - d(M,N)]^2\}. \end{aligned} \tag{3.8}$$

Using Lemma 2.3 (iv), we obtain

$$\begin{aligned} d(y_n,r)^2 &= d\big((1-\beta_n)x_n \bigoplus \beta_n z_n, r\big)^2 \\ &\le (1-\beta_n)d(x_n,r)^2+\beta_n d(z_n,r)^2-\beta_n(1-\beta_n)d(x_n,z_n)^2, \end{aligned} \tag{3.9}$$

and

$$\begin{aligned} [d(y_n,P_T(y_n))-d(M,N)]^2 &\le [d(y_n,\gamma_n)-d(M,N)]^2 \\ &\le [d(y_n,z_n')+d(z_n',\gamma_n)-d(M,N)]^2 \\ &= [d((1-\beta_n)x_n \bigoplus \beta_n z_n, z_n')]^2 \\ &\le (1-\beta_n)d(x_n,z_n')^2+\beta_n d(z_n,z_n')^2-\beta_n(1-\beta_n)d(x_n,z_n)^2. \end{aligned} \tag{3.10}$$

Therefore, from (3.8), (3.9), and (3.10), we obtain

$$\begin{aligned} H^2\big(P_T(r),P_T(y_n)\big) &\le k^2\{d(r,y_n)^2+[d(y_n,P_T(y_n))-d(M,N)]^2\} \\ &\le k^2\{(1-\beta_n)d(x_n,r)^2+\beta_n d(z_n,r)^2+(1-\beta_n)d(x_n,z_n')^2 \\ &\quad +\beta_n d(z_n,z_n')^2-2\beta_n(1-\beta_n)d(x_n,z_n)^2\}, \end{aligned}$$

which together with (3.7) yields that

$$\begin{aligned} d(z_n',r)^2 \le k^2\{&(1-\beta_n)d(x_n,r)^2+\beta_n d(z_n,r)^2+(1-\beta_n)d(x_n,z_n')^2 \\ &+\beta_n d(z_n,z_n')^2-2\beta_n(1-\beta_n)d(x_n,z_n)^2\}. \end{aligned} \tag{3.11}$$

Since Tr is proximinal, there exists $r* \in Tr$ such that $d(r,r*) = d(M,N)$. Since r is a best proximity point of T, we have $r* \in P_T(r)$. From $d(z_n,\zeta_n) = d(M,N)$ and the P-property, we have $d(z_n,r) = d(\zeta_n,r*)$. Since $P_T(x_n)$ is a singleton, we have $P_T(x_n) = \{\zeta_n\}$. Then $d(\zeta_n,r*) = d(P_T(x_n),r*)$ and

$$d(z_n,r) = d\big(P_T(x_n),r*\big) \le H\big(P_T(x_n),P_T(r)\big). \tag{3.12}$$

Now, from the definition of the proximally quasi-nonexpansive mapping, we have

$$\begin{aligned} H\big(P_T(r),P_T(x_n)\big) &\le k\max\{d(r,x_n),d\big(r,P_T(r)\big)-d(M,N),d\big(x_n,P_T(x_n)\big)-d(M,N)), \\ &\qquad d\big(r,P_T(x_n)-d(M,N)\big),d\big(x_n,P_T(r)\big)-d(M,N)\} \\ &\le k\max\{d(r,x_n),d\big(x_n,P_T(x_n)\big)-d(M,N),d\big(r,P_T(r)\big) \\ &\qquad +H\big(P_T(r),P_T(x_n)\big)-d(M,N),d(x_n,r)+d\big(r,P_T(r)\big)-d(M,N))\} \\ &= k\max\{d(r,x_n),d\big(x_n,P_T(x_n)\big)-d(M,N),H\big(P_T(r),P_T(x_n)\big)\}. \end{aligned}$$

Note that

$$H\big(P_T(r),P_T(x_n)\big) \le kH\big(P_T r,P_T(x_n)\big) < H\big(P_T r,P_T(x_n)\big),$$

which is a contradiction. Therefore,

$$H\big(P_T(r),P_T(x_n)\big) \le k\max\{d(r,x_n),d\big(x_n,P_T(x_n)\big)-d(M,N)\},$$

and

$$\begin{aligned} H^2\big(P_T(r),P_T(x_n)\big) &\le k^2\max\{d(r,x_n)^2,[d\big(x_n,P_T(x_n)\big)-d(M,N)]^2\} \\ &\le k^2\{d(r,x_n)^2+[d\big(x_n,P_T(x_n)\big)-d(M,N)]^2\}. \end{aligned} \tag{3.13}$$

Note that

$$\begin{aligned}[d(x_n, P_T(x_n)) - d(M,N)]^2 &\leq [d(x_n, \zeta_n) - d(M,N)]^2 \\ &\leq [d(x_n, z_n) + d(z_n, \zeta_n) - d(M,N)]^2 \\ &= d(x_n, z_n)^2,\end{aligned}$$

which together with (3.12) and (3.13) implies that

$$d(z_n, r)^2 \leq H^2(P_T(x_n), P_T(r)) \leq k^2[d(x_n, r)^2 + d(x_n, z_n)^2].$$

From (3.11), we have

$$\begin{aligned}d(z_n', r)^2 &\leq k^2\Big[(1-\beta_n)d(x_n, r)^2 + \beta_n k^2\Big(d(x_n, r)^2 + d(x_n, z_n)^2\Big) \\ &\quad + (1-\beta_n)d(x_n, z_n')^2 + \beta_n d(z_n, z_n')^2 - 2\beta_n(1-\beta_n)d(x_n, z_n)^2\Big] \\ &= k^2\Big[(1-\beta_n + k^2\beta_n)d(x_n, r)^2 - \beta_n(2 - 2\beta_n - k^2)d(x_n, z_n)^2 \\ &\quad + (1-\beta_n)d(x_n, z_n')^2 + \beta_n d(z_n, z_n')^2\Big] \\ &\leq k^2 d(x_n, r)^2 - k^2\beta_n(2-2\beta_n - k^2)d(x_n, z_n)^2 + k^2(1-\beta_n)d(x_n, z_n')^2 + k^2\beta_n d(z_n, z_n')^2.\end{aligned}$$

Substituting the inequality above into (3.5), we obtain

$$\begin{aligned}d(x_{n+1}, r)^2 &\leq (1-\alpha_n)d(x_n, r)^2 + \alpha_n\Big[k^2 d(x_n, r)^2 - k^2\beta_n(2-2\beta_n-k^2)d(x_n, z_n)^2 \\ &\quad + k^2(1-\beta_n)d(x_n, z_n')^2 + k^2\beta_n d(z_n, z_n')^2\Big] - \alpha_n(1-\alpha_n)d(x_n, z_n')^2 \\ &= (1-\alpha_n(1-k^2))d(x_n, r)^2 - k^2\alpha_n\beta_n(2-2\beta_n-k^2)d(x_n, z_n)^2 \\ &\quad + k^2\alpha_n\beta_n d(z_n, z_n')^2 - \alpha_n(1-\alpha_n - k^2 + k^2\beta_n)d(x_n, z_n')^2.\end{aligned}$$

Since $c \leq \alpha_n \leq 1-k^2$, we have $1-\alpha_n(1-k^2) \leq 1-c(1-k^2) = \alpha$ (say) and $0 < \alpha < 1$. Since $\beta_n \to 0$, there exists N_1 such that $\beta_n \leq (2-k^2)/2, \forall n \geq N_1$. So, $2-2\beta_n - k^2 \geq 0, \forall n \geq N_1$. Also we have $1-\alpha_n - k^2 + k^2\beta_n \geq (1-k^2) - (1-k^2) + k^2\beta_n \geq 0$. Therefore, for sufficiently large n, we obtain

$$\begin{aligned}d(x_{n+1}, r)^2 &\leq \alpha d(x_n, r)^2 + k^2(1-k^2)\beta_n d(z_n, z_n')^2 \\ &\leq \alpha d(x_n, r)^2 + k^2(1-k^2)\beta_n D,\end{aligned}$$

where D is a diameter of M_0. From Lemma 2.3, we have that the sequence $x_n \to r$, which completes the proof. □

Corollary 3.9. *Let M and N be two convex bounded subsets of a Hillbert space X. Let $T : M \to \mathbb{P}(N)$ be a multivalued mapping with $Best(T) \neq \emptyset$ such that P_T is a proximally quasi-contractive mapping. Let $\{x_n\}$ be the sequence generated by the Ishikawa iteration scheme defined by Definition 3.3. Assume that T satisfies condition (I*) and $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy (i) $0 \leq \alpha_n, \beta_n < 1$, (ii) $\beta_n \to 0$, and (iii) $\sum \alpha_n\beta_n = \infty$ with $c \leq \alpha_n \leq 1-k^2$, for $k \in [0,1), c > 0$. Then $\{x_n\}$ converges to a best proximity point of T.*

Theorem 3.10. *Let M and N be two convex subsets a complete CAT(0) X space. Let $T : M \to \mathbb{P}(N)$ be a multivalued mapping with $Best(T) \neq \emptyset$ such that P_T is a proximally quasi-contractive mapping. Let $\{x_n\}$ be the sequence generated by the Mann iteration sequence defined by Definition 3.2. Assume that T satisfies condition (I^*) and $\{\alpha_n\}$ satisfies $0 \leq \alpha_n < 1$ with $c \leq \alpha_n \leq 1-k^2$, for $k \in [0,1)$, some $c > 0$. Then $\{x_n\}$ converges to a best proximity point of T.*

Corollary 3.11. *Let M and N be two convex bounded subsets of a Hilbert space X. Let $T : M \to \mathbb{P}(N)$ be a multivalued mapping with $Best(T) \neq \emptyset$ such that P_T is a proximally quasi-contractive mapping. Let $\{x_n\}$ be the sequence generated by the Mann iteration scheme defined by Definition 3.2. Assume that T satisfies condition (I^*) and $\{\alpha_n\}$ satisfies $0 \leq \alpha_n < 1$, with $c \leq \alpha_n \leq 1-k^2$, for $k \in [0,1)$, some $c > 0$. Then $\{x_n\}$ converges to a best proximity point of T.*

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