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DYNAMICS OF A PLANT-POLLINATOR SYSTEM WITH POLLINATION MUTUALISM

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Abstract. In this paper, one class of non-autonomous plant-pollinator system with pollination mutualism is discussed. Some new sufficient conditions on the permanence, extinction and global attractivity of the system are established by using the comparison method, inequality techniques and the construction of Lyapunov functions. In addition, some applications of the main results in this paper, we also considered two special cases of the considered system. Finally, one example with numerical simulations are presented to illustrate the obtained theoretical results.

Keywords. Extinction; Global attractivity; Pollination mutualism; Non-autonomous plant-pollinator system.

1. Introduction and Preliminaries

In nature, there are many types of interactions between species, such as competition, cooperation, and predator-prey. The close relationship between pollinators and plants is a good example of mutual benefit and cooperation between species. In recent years, many scholars studied the dynamics of population mutualism systems (cooperative systems). Among them, the dynamics of the plant-pollinator system with pollination mutualism attracted great attention of many mathematicians and biologists; see, e.g., [1, 2, 3, 4, 5] and the references therein. For example, in 2015, Wang, Deangelis and Holland [3] proposed and discussed the following autonomous

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plant-pollinator system with pollination mutualism

$$\begin{aligned}\dot{x}(t) &= x(t)\left(r_1 - d_1 x(t) + \frac{a_{12}y(t)}{1+ax(t)+by(t)}\right),\\ \dot{y}(t) &= y(t)\left(-r_2 + \frac{a_{21}x(t)}{1+ax(t)+by(t)}\right).\end{aligned} \tag{1.1}$$

In 2020, Muhammadhaji [5] further generalized system 1.1) and studied the following non-autonomous plant-pollinator system with pollination mutualism

$$\begin{aligned}\dot{x}(t) &= x(t)\left(r_1(t) - d_1(t) x(t) + \frac{a_{12}(t)y(t)}{1+a(t)x(t)+b(t)y(t)}\right),\\ \dot{y}(t) &= y(t)\left(-r_2(t) + \frac{a_{21}(t)x(t)}{1+a(t)x(t)+b(t)y(t)}\right),\end{aligned} \tag{1.2}$$

and obtained some sufficient conditions on the permanence and existence of positive periodic solution to system (1.2) by using the comparison method.

However, the above mentioned non-autonomous system mainly concerned with the permanence and the existence of positive periodic solutions. In addition, to the best of our knowledge, no study has been conducted to date for the global attractivity of the non-autonomous plant-pollinator system with pollination mutualism. Therefore, based on the above models, and analysis, we study the following non-autonomous plant-pollinator systems with pollination mutualism

$$\begin{aligned}\dot{x}(t) &= x(t)\left(r_1(t) - d_1(t) x(t) + \frac{a_{12}(t)y(t)}{c(t)+a(t)x(t)+b(t)y(t)}\right),\\ \dot{y}(t) &= y(t)\left(-r_2(t) + \frac{a_{21}(t)x(t)}{c(t)+a(t)x(t)+b(t)y(t)}\right).\end{aligned} \tag{1.3}$$

Obviously, system (1.1) and system (1.2) can be regarded as a special case of system (1.3). Our main purpose is to establish some new sufficient conditions on the permanence, extinction, and global attractivity of system (1.3).

In system (1.3), $x(t)$, and $y(t)$ represent the density of the plant and pollinator at time t, respectively; $d_1(t)$ represents the intrapatch restriction density of the plant; $r_i(t)$, $i = 1,2$, represents the intrinsic growth rate of the plant and pollinator at time t, respectively; $a_{12}(t)$ and $a_{21}(t)$ represent the pollination mutualism coefficients between the plant and pollinator at time t, respectively; and $a(t), b(t)$, and $c(t)$ are the Beddington-DeAngelis functional response coefficients, where $c(t) \le 1$.

In this paper, the initial conditions for systems (1.1), (1.2), and (1.3) take the following form

$$x(0) > 0, \quad y(0) > 0. \tag{1.4}$$

(H_1) and (H_2) are given for systems (1.2) and (1.3), respectively, as following.

(H_1) $r_i(t), d_1(t), a_{12}(t), a_{21}(t)$, and $a(t), b(t), c(t)$ are continuous, bounded, and strictly positive functions on $[0, +\infty)$.

(H_2) $r_i(t), d_1(t), a_{12}(t), a_{21}(t)$, and $a(t), b(t)$ are continuous, bounded, and strictly positive functions on $[0, +\infty)$.

Throughout this paper, for any continuous function $f(t)$, we denote

$$f^L = \inf_{t\in[0,+\infty)} f(t),\ f^M = \sup_{t\in[0,+\infty)} f(t).$$

Now, we present some useful lemmas for our main results.

Lemma 1.1. [6, 7] *Consider the following equation:*
$\dot{u}(t) = u(t)(d_1 - d_2u(t))$,
where $d_2 > 0$, we have
(1) If $d_1 > 0, then \lim_{t\to+\infty} u(t) = d_1/d_2$.
(2) If $d_1 < 0, then \lim_{t\to+\infty} u(t) = 0$.

Lemma 1.2. [6, 8] *If $a > 0, b > 0$ and $\dot{x}(t) \geq (\leq) b - ax(t)$, when $t \geq 0$ and $x(0) > 0$, we have*

$$x(t) \geq (\leq) \frac{b}{a}[1 + (\frac{ax(0)}{b} - 1)e^{-at}].$$

2. Main Results

Theorem 2.1. *Let $(x(t), y(t))$ be any positive solution of system (1.3) with the initial conditions (1.4). If (H_1) holds, then there exist positive constants M_1, M_2 and T_1 such that $x(t) \leq M_1$, and $y(t) \leq M_2$ as $t > T_1$.*

Proof. From the first equation of system (1.3), for $t \geq 0$, we have

$$\dot{x}(t) \leq x(t)[R - d_1^L x(t)],$$

where $R = r_1^M + \frac{a_{12}^M}{b^L}$. Using Lemma 1.1 and the comparison theorem, we find that there exists a constant $T_0 > 0$ such that

$$x(t) \leq \frac{R}{d_1^L} = M_1, \quad \text{as } t > T_0.$$

Next, from the second equation of system (1.3), for $t \geq T_0$, we have

$$\dot{y}(t) \leq \frac{a_{21}^M M_1}{b^L} - r_2^L y(t).$$

Using Lemma 1.2 and the comparison theorem, we see that there exists a constant $T_1 > T_0$ such that

$$y(t) \leq \frac{\frac{a_{21}^M M_1}{b^L}}{r_2^L} = M_2.$$

This completes the proof of Theorem 2.1. □

Theorem 2.2. *In system (1.3), the plant species is permanent and the pollinator species is extinct if (H_1) holds and $a_{21}^M < r_2^L a^L$.*

Proof. Let $(x(t), y(t))$ be any positive solution of system (1.3) with the initial conditions (1.4). From the first equation of system (1.3), for $t \geq 0$, we have

$$\dot{x}(t) \geq x(t)[r_1^L - d_1^M x(t)].$$

Using Lemma 1.1 and the comparison theorem, we see that there exists a constant $T_2 > 0$ such that

$$x(t) \geq \frac{r_1^L}{d_1^M} = m_1, \quad \text{as } t > T_2.$$

Next, from the second equation of system (1.3), for $t \geq 0$, we have

$$\dot{y}(t) \leq y(t)[-r_2^L + \frac{a_{21}^M}{a^L}] = -R_0 y(t). \tag{2.1}$$

where

$$R_0 = r_2^L - \frac{a_{21}^M}{a^L}.$$

By solving differential equation (2.1), we further obtain

$$y(t) \leq y(0)e^{-R_0 t}.$$

Then there exists a constant $T_3 > T_2$ such that $x(t) \geq m_1,\ y(t) \to 0$. This completes the proof. □

Next, we obtain the sufficient conditions for the global attractivity of system (1.3). First, we suppose that $(x(t), y(t))$ and $(u(t), v(t))$ are any two positive solutions of system (1.3). Next, for convenience, we denote

$$\begin{aligned} A(t) &= d_1(t) - \frac{a_{21}(t)v(t)(b(t)y(t)+c(t))}{\alpha(t)} \\ &> d_1^L - \frac{a_{21}^M M_2(b^M M_2 + c^M)}{\alpha^L} \\ &= C_1, \end{aligned}$$

$$\begin{aligned} B(t) &= r_2(t) - \frac{a_{12}(t)c(t) + a_{12}(t)a(t)x(t) + a_{21}(t)c(t)x(t) + a_{21}(t)a(t)x(t)u(t)}{\alpha(t)} \\ &> r_2^L - \frac{a_{12}^M c^M + M_1(a_{12}^M a^M + a_{21}^M c^M + a_{21}^M a^M M_1)}{\alpha^L} \\ &= C_2, \end{aligned}$$

and

$$C = \min\{C_1,\ C_2\},$$

where $\alpha(t) = \beta_1(t) * \beta_2(t)$,

$$\beta_1(t) = c(t) + a(t)x(t) + b(t)y(t),$$

$$\beta_2(t) = c(t) + a(t)u(t) + b(t)v(t),$$

and M_1, M_2 are the upper bounds of solutions $(x(t), y(t))$ and $(u(t), v(t))$.

Theorem 2.3. *If H_1 holds and $C > 0$, then system (1.3) is globally attractive.*

Proof. From Theorem 2.1 and Theorem 2.2, we find that there exist positive constants $M_i, m_1 (i = 1,2)$ and T^M such that $m_1 \leq x(t) \leq M_1$, and $y(t) \leq M_2$ for all $t \geq T^M$. Let

$$V(t) = |\ln x(t) - \ln u(t)| + |y(t) - v(t)|.$$

Calculating the upper right derivative of $V(t)$ along system (1.3), we have

$$
\begin{aligned}
D^{+}V(t) = \;& \operatorname{sign}(x(t)-u(t))\big[-d_1(t)(x(t)-u(t))+a_{12}(t)\big(\frac{y(t)}{\beta_1(t)}-\frac{v(t)}{\beta_2(t)}\big)\big] \\
& +\operatorname{sign}(y(t)-v(t))\big[-r_2(t)(y(t)-v(t))+a_{12}(t)\big(\frac{x(t)y(t)}{\beta_1(t)}-\frac{u(t)v(t)}{\beta_2(t)}\big)\big] \\
= \;& \operatorname{sign}(x(t)-u(t))\big[-d_1(t)(x(t)-u(t))-\frac{a_{12}(t)u(t)a(t)-a_{12}(t)}{\alpha(t)}(y(t)-v(t)) \\
& -\frac{a_{12}(t)a(t)v(t)}{\alpha(t)}(x(t)-u(t))+\operatorname{sign}(y(t)-v(t))\big[-r_2(t)(y(t)-v(t)) \\
& +\frac{a_{12}(t)a(t)x(t)u(t)+a_{12}(t)x(t)}{\alpha(t)}(y(t)-v(t)) \\
& +\frac{a_{12}(t)b(t)v(t)y(t)+a_{12}(t)v(t)}{\alpha(t)}(x(t)-u(t))\big] \\
\leq \;& -A(t)|x(t)-u(t)|-B(t)|y(t)-v(t)| \\
\leq \;& -C(|x(t)-u(t)|+|y(t)-v(t)|).
\end{aligned}
\tag{2.2}
$$

Integrating from T^M to t on both sides of (2.2) produces

$$
V(t)+C\int_{T^M}^{t}\big(|x(s)-u(s)|+|y(s)-v(s)|\big)ds \leq V(T^M), \quad t \geq T^M. \tag{2.3}
$$

Hence, $V(t)$ is bounded on $[T^M,+\infty)$, and

$$
\int_{T^*}^{t}\big(|x(s)-u(s)|+|y(s)-v(s)|\big)ds \leq \frac{V(T^M)}{C}, \quad t \geq T^M. \tag{2.4}
$$

It follows that

$$
|\ln x(t)-\ln u(t)|+|y(t)-v(t)| \leq V(t) \leq V(T^M), \quad t \geq T^M. \tag{2.5}
$$

From the boundedness of the solutions of system (1.3), we can obtain that $|x(t)-u(t)|$ and $|y(t)-v(t)|$ is uniformly continuous on $[T^M,+\infty)$. By Barbalat's lemma, we conclude that

$$
\lim_{t\to+\infty}(|x(t)-u(t)|+|y(t)-v(t)|)=0.
$$

Hence,

$$
\lim_{t\to+\infty}(x(t)-u(t))=0, \quad \lim_{t\to+\infty}(y(t)-v(t))=0.
$$

This completes the proof of Theorem 2.3. □

3. Applications

In this section, as the application of the main results in this paper, we consider system (1.1) and system (1.2).

Corollary 3.1. *Let $(x(t),y(t))$ be any positive solution of system (1.1) with initial conditions (1.4). Let $r_i>0(i=1,2), d_1>0, a_{12}>0, a_{21}>0, a>0, b>0$ holds, then there exist positive constants M_1', M_2', and T_1' such that $x(t)\leq M_1'$, and $y(t)\leq M_2'$ as $t>T_1'$.*

Corollary 3.2. *Let $(x(t), y(t))$ be any positive solution of system (1.2) with the initial conditions (1.4). If (H_2) holds, then there exist positive constants M_1, M_2, and T_1 such that $x(t) \leq M_1$, and $y(t) \leq M_2$ as $t > T_1$.*

Corollary 3.3. *In system (1.1), the plant species is permanent, and the pollinator species is extinct if the conditions of Corollary 3.1 hold ,and $a_{21} < r_2 a$.*

Corollary 3.4. *In system (1.2), the plant species is permanent and the pollinator species is extinct if (H_2) holds, and $a_{21}^M < r_2^L a^L$.*

In order to obtain the sufficient conditions for the global attractivity of system (1.1), we denote

$$\begin{aligned} D(t) &= d_1 - \frac{a_{21}v(t)(by(t)+1)}{\delta(t)} \\ &> d_1 - \frac{a_{21}M_2'(bM_2+1)}{\delta^L} \\ &= D_1, \end{aligned}$$

$$\begin{aligned} E(t) &= r_2 - \frac{a_{12} + a_{12}ax(t) + a_{21}x(t) + a_{21}ax(t)u(t)}{\delta(t)} \\ &> r_2 - \frac{a_{12} + M_1'(a_{12}a + a_{21} + a_{21}aM_1')}{\delta^L} \\ &= D_2, \end{aligned}$$

and

$$D = \min\{D_1,\ D_2\},$$

where

$$\delta(t) = \sigma_1(t) * \sigma_2(t), \quad \sigma_1(t) = 1 + ax(t) + by(t), \quad \sigma_2(t) = 1 + au(t) + bv(t).$$

Corollary 3.5. *If the conditions of Corollary 3.1 hold and $D > 0$, then system (1.1) is globally attractive.*

Next, we will obtain the sufficient conditions for the global attractivity of system (1.2) .Denote

$$\begin{aligned} F_1(t) &= d_1(t) - \frac{a_{21}(t)v(t)(b(t)y(t)+1)}{\eta(t)} \\ &> d_1^L - \frac{a_{21}^M M_2(b^M M_2+1)}{\eta^L} \\ &= E_1, \end{aligned}$$

$$\begin{aligned} F_2(t) =& r_2(t) - \frac{a_{12}(t) + a_{12}(t)a(t)x(t) + a_{21}(t)x(t) + a_{21}(t)a(t)x(t)u(t)}{\eta(t)} \\ &> r_2^L - \frac{a_{12}^M + M_1(a_{12}^M a^M + a_{21}^M + a_{21}^M a^M M_1)}{\eta^L} \\ &= E_2, \end{aligned}$$

and

$$E = \min\{E_1,\ E_2\},$$

where

$$\eta(t)=\tau_1(t)*\tau_2(t),\quad \tau_1(t)=1+a(t)x(t)+b(t)y(t),\quad \tau_2(t)=1+a(t)u(t)+b(t)v(t).$$

Corollary 3.6. *If the conditions of Corollary 3.2 hold and $E>0$, then system (1.2) is globally attractive.*

4. Numerical Simulation

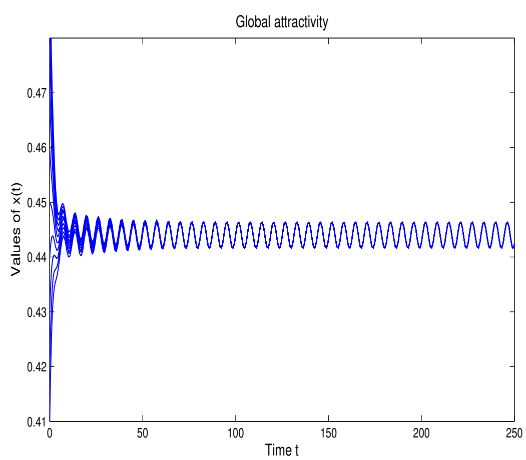
Example 4.1. Consider the following system

$$\begin{aligned}\dot{x}(t)= & \; x(t)\big[0.6+0.01\cos(t)-(1.37+0.01\cos(t))x(t)\\ & +\frac{(0.13+0.01\cos(t))y(t)}{0.51+0.01\cos(t)+(0.13+0.01\cos(t))x(t)+(1.81+0.01\cos(t))y(t)}\big],\\ \dot{y}(t)= & \; y(t)\big[-(0.5+0.001\sin(t))\\ & +\frac{(0.65+0.001\sin(t))x(t)}{0.51+0.01\cos(t)+(0.13+0.01\cos(t))x(t)+(1.81+0.01\cos(t))y(t)}\big].\end{aligned}\tag{4.1}$$

We have

$$R\approx 0.68,\ M_1\approx 0.5,\ M_2\approx 0.36,\ m_1\approx 0.43,\ \alpha^L\approx 0.56,\ C_1\approx 0.88,\ C_2\approx 0.02.$$

It is easy to show that system (4.1) satisfies the conditions of Theorem 2.3. Hence, system (4.1) is globally attractive. The globally attractivity of system (4.1) is shown in Figure 4.1.



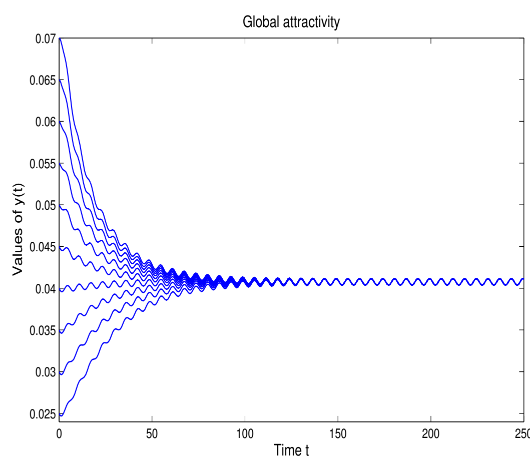


Fig 4.1. The numerical solutions of system (4.1) with different initial conditions.

From Fig 4.1, we see that system (4.1) is globally attractive.

5. Conclusion

In this paper, we investigated system (1.3). First, using the related inequalities, and the comparison method, we obtained a set of conditions that ensure that the system is bounded, plant species permanent, and the pollinator species extinct. Second, using the Lyapunov function method, we derived sufficient conditions on the global attractivity of the system. Moreover, to show the generality of the main results in this paper, we also obtained some sufficient conditions on the above mentioned dynamical behaviors of system (1.1) and system (1.2). Finally, we provide a suitable example to illustrate the feasibility of our main results. Since we extended systems (1.1) and (1.2) to system (1.3), we also obtained some sufficient conditions for the permanence, extinction, and global attractivity of system (1.1), system (1.2), and system (1.3). Hence, the results obtained in this paper can be seen as the supplements and extensions of previously known results presented in [1, 2, 3, 4, 5].

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