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A NEW APPROACH ON THE WELL-POSEDNESS OF NONAUTONOMOUS CAUCHY PROBLEMS: AN APPLICATION IN POPULATION GROWTH

S. SUTRIMA[1,*], M. MARDIYANA[2], R. SETIYOWATI[1], R. RESPATIWULAN[3]

[1]Department of Mathematics, Sebelas Maret University, Indonesia
[2]Department of Mathematics Education, Sebelas Maret University, Indonesia
[3]Department of Statistics, Sebelas Maret University, Indonesia

Abstract. This paper addresses the well-posedness of nonautonomous Cauchy problems by using strongly continuous quasi-semigroups (C_0-quasi-semigroups) on Banach spaces. Necessary and sufficient conditions for the nonautonomous Cauchy problem $\dot{u}(t) = A(t)u(t)$ to be well-posed are that $A(t)$ is the infinitesimal generator of a C_0-quasi-semigroup. This paper also verifies the sufficiency for an operator to be the infinitesimal generator of a C_0-quasi semigroup. A simple application in predicting that the persistence or extinction of a population growth model with diffusion and intrinsic growth rate are time-dependent is considered.

Keywords. Infinitesimal generator; Population growth; Nonautonomous Cauchy problem; Strongly continuous quasi-semigroup; Well-posed.

1. Introduction

Nonautonomous Cauchy problems (NCP) usually represent some phenomena of the transport-reaction arising in physical and biological systems [1, 2, 3, 4]. The NCP on a Banach space X takes a general form

$$\begin{aligned} \dot{u}(t) &= A(t)u(t), \qquad t \geq 0, \\ u(0) &= u_0, \qquad\quad u_0 \in X, \end{aligned} \tag{1.1}$$

where u is an unknown function from $[0,\infty)$ into X, and each $A(t)$ is a densely defined closed linear operator in X with domain $\mathscr{D}(A(t)) = \mathscr{D}$, which is independent of t.

The main concern for (1.1) is the conditions for well-posedness. The NCP (1.1) is said to be well-posed if its solution obeys the principles of existence, uniqueness, and continuous dependence. Thus, the well-posedness plays an important role in applications. There is a prominent

*Correspoinding author.

E-mail addresses: sutrima@mipa.uns.ac.id (S. Sutrima), mardiyana@staff.uns.ac.id (M. Mardiyana), ririnsetiyowati@staff.uns.ac.id (R. Setiyowati), respatiwulan@staff.uns.ac.id (R. Respatiwulan).

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difference between autonomous and nonautonomous Cauchy problems. For the autonomous cases, the uses of the C_0-semigroups are well-understood in the framework. Based on the Hille-Yosida theorem and its generalizations, many results on the autonomous problems that were established; see, e.g., [5, 6, 7, 8, 9] and the references therein. A family of evolution operators is an existing approach that was used to characterize the well-posedness of NCP; see, e.g., (1.1) [10, 11, 12, 13, 14, 15]. Unfortunately, by the approach for both parabolic and hyperbolic types, the well-posedness of NCP (1.1) requires strongly sufficient conditions of $A(t)$. Although, by following the semigroups theory, the family can be reduced to be an evolution semigroup on the appropriate space for the well-posedness [11, 12, 14, 15].

The results in the autonomous Cauchy problems motivate a generalization to the NCP (1.1). It is reasonable to take over some assumptions on the infinitesimal generator A in the Hille-Yosida theorem into $A(t)$ for each t. Related to the assertions, it has been initiated a construction of a strongly continuous quasi semigroup (C_0-quasi semigroup) [16]. This construction follows the C_0-semigroup theory. The properties of C_0-quasi-semigroups and its applications can be founded in [16, 17, 18, 19]. In general, there are different properties between C_0-semigroups and C_0-quasi-semigroups [20]. A sufficient condition for NCP (1.1) to admit a unique solution is that $\{A(t)\}_{t\geq 0}$ generates a C_0-quasi-semigroup, regardless of the parabolic or the hyperbolic. This implies the importance of the investigations of the sufficient conditions for $\{A(t)\}_{t\geq 0}$ to be the infinitesimal generator of a C_0-quasi-semigroup. Moreover, this provides an opportunity to investigate the well-posedness of (1.1).

In this paper, we consider using the C_0-quasi-semigroups to investigate the well-posedness of (1.1). In Section 2, we focus on the sufficiency for $\{A(t)\}_{t\geq 0}$ to be the infinitesimal generator of a C_0-quasi-semigroup. The main results, which are presented in Section 3, include the sufficient and necessary conditions for the well-posedness of (1.1) and its applications in population growth model with diffusion and intrinsic growth rate, which are time-dependent.

2. PRELIMINARIES

A strongly continuous quasi-semigroup initiated by Leiva and Barcenas [16] is a generalization of strongly continuous semigroups for the nonautonomous problems. The following definition is the weaker version.

Definition 2.1. Let $\mathscr{L}(X)$ be the set of all bounded linear operators on a Banach space X. A two-parameter commutative family $\{R(t,s)\}_{s,t\geq 0}$ in $\mathscr{L}(X)$ is called a strongly continuous quasi-semigroup (in short C_0-quasi-semigroup) on X if

(a) $R(t,0) = I$, the identity operator on X;
(b) $R(t,s+r) = R(t+r,s)R(t,r)$;
(c) $\lim_{s\to 0^+} \|R(t,s)x - x\| = 0$;
(d) there is a continuous increasing function $M : [0,\infty) \to [1,\infty)$ such that

$$\|R(t,s)\| \leq M(s), \tag{2.1}$$

for all $r,s,t \geq 0$ and $x \in X$.

A contraction quasi-semigroup is a C_0-quasi semigroup $\{R(t,s)\}_{s,t\geq 0}$ such that the function M in (2.1) satisfies $M(s) \leq 1$ for all $s \geq 0$. The condition (c) of Definition 2.1 implies that, for each $t \geq 0$, $R(t,\cdot)$ is strongly continuous on $[0,\infty)$. Analogously, for each $s \geq 0$, $R(\cdot,s)$ is also strongly continuous on $[0,\infty)$.

Let $\mathscr{D}$ denote a set of all $x \in X$ such that the following limits exist

$$\lim_{s\to 0^+} \frac{R(t,s)x - x}{s}, \quad t \geq 0.$$

The infinitesimal generator of the C_0-quasi-semigroup $\{R(t,s)\}_{s,t\geq 0}$ is defined to be a family of operators $\{A(t)\}_{t\geq 0}$ on $\mathscr{D}$, where

$$A(t)x = \lim_{s\to 0^+} \frac{R(t,s)x - x}{s}.$$

The examples and properties of the C_0-quasi-semigroups can be found in [16, 17, 20]. If A is the infinitesimal generator of the C_0-semigroup $T(t)$ on the domain $\mathscr{D}(A)$, then A is a closed set, and $\mathscr{D}(A)$ is dense in X. These are not applicable for any C_0-quasi-semigroup, as shown by Example 3.2 and Example 3.3 of [20]. Henceforth, the quasi-semigroup $\{R(t,s)\}_{s,t\geq 0}$ and the infinitesimal generator $\{A(t)\}_{t\geq 0}$ are simplified by $R(t,s)$ and $A(t)$, respectively. Due to applications, we only focus on the C_0-quasi-semigroups with the infinitesimal generator, wich have a dense domain in Banach spaces. This assumption implies that every C_0-quasi-semigroup has a unique infinitesimal generator.

Lemma 2.2. *Let $R_1(t,s)$ and $R_2(t,s)$ be C_0-quasi-semigroups on a Banach space X with the infinitesimal generators $A_1(t)$ and $A_2(t)$, respectively. If $A_1(t) = A_2(t)$ for all $t \geq 0$, then $R_1(t,s) = R_2(t,s)$ for all $t,s \geq 0$.*

Proof. This is a special case of Lemma 1 of [21] for the C_0-quasi-groups. □

Lemma 2.2 implicitly suggests the importance of sufficient conditions for $A(t)$ to be the infinitesimal generator of a C_0-quasi-semigroup. The requirement is parallel with the sufficiency of Hille-Yosida Theorem for C_0-semigroups. Therefore, we call the following theorem as the version of Hille-Yosida Theorem for C_0-quasi-semigroups. We denote the resolvent operator of $A(t)$ by $\mathscr{R}(\lambda, A(t)) := (\lambda - A(t))^{-1}$, $\lambda \in \rho(A(t))$, where $\rho(A(t))$ is the resolvent set of $A(t)$.

Theorem 2.3. *For each $t \geq 0$, let $A(t)$ be a closed and densely defined operator on $\mathscr{D}$, and the map $t \mapsto A(t)y$ be a continuous function from $\mathbb{R}^+$ to X for all $y \in \mathscr{D}$. If $\mathscr{R}(\lambda, A(\cdot))$ is locally integrable, and there exist constants $N, \omega \geq 0$ such that $[\omega, \infty) \subseteq \rho(A(t))$ and*

$$\|\mathscr{R}(\lambda, A(t))^r\| \leq \frac{N}{(\lambda - \omega)^r}, \quad \lambda > \omega, \quad r \in \mathbb{N},$$

then $A(t)$ is the infinitesimal generator of a C_0-quasi semigroup.

Proof. For each $t \geq 0$, we define the Yosida's approximation of $A(t)$ by

$$A_n(t) := nA(t)\mathscr{R}(n, A(t)) = n^2\mathscr{R}(n, A(t)) - nI, \quad n > \omega, \quad n \in \mathbb{N}.$$

From Lemma 3.6 of [20], we have

$$\lim_{n\to\infty} A_n(t)y = A(t)y, \quad y \in \mathscr{D}. \tag{2.2}$$

We further define

$$G_n(t) := \int_0^t A_n(s)\,ds, \quad n \in \mathbb{N}, \quad t \geq 0.$$

Since $G_n(t) \in \mathscr{L}(X)$, we can define a C_0-quasi-semigroup

$$R_n(t,s)x := e^{G_n(t+s) - G_n(t)}x, \quad t,s \geq 0, \quad x \in X.$$

We see that $A_n(t)$ is the infinitesimal generator of $R_n(t,s)$ and

$$\|R_n(t,s)\| \leq e^{-ns}\sum_{k=0}^{\infty}\frac{\left(\int_t^{t+s} n^2\|\mathscr{R}(n,A(v))\|dv\right)^k}{k!}$$

$$\leq Ne^{-ns}\sum_{k=0}^{\infty}\left(\frac{n^2 s}{n-\omega}\right)^k = Ne^{\frac{n\omega}{n-\omega}s}, \quad t,s\geq 0, \quad n\in\mathbb{N}. \tag{2.3}$$

Since $\frac{n\omega s}{n-\omega}\to\omega s$ as $n\to\infty$, we find that there exists $\omega_1>\omega$ such that $\|R_n(t,s)\|\leq Ne^{\omega_1 s}$ for sufficiently large n. Further, we see $A_m(t)A_n(t)=A_n(t)A_m(t)$ and $A_n(t)R_m(t,s)=R_m(t,s)A_n(t)$ for all $m,n\in\mathbb{N}$ and $t,s\geq 0$. Therefore, for $u\in X$,

$$\begin{aligned} R_m(t,s)x - R_n(t,s)x &= \int_0^s \frac{\partial}{\partial r}\left[R_m(t,r)R_n(t+r,s-r)x\right]dr \\ &= \int_0^s R_m(t,r)R_n(t+r,s-r)[A_m(t+r)x - A_n(t+r)x]dr. \end{aligned}$$

By the uniform continuity of the map $r\mapsto A(r)y$ on $[0,s]$, for $t\geq 0$ fixed, (2.2) provides

$$\lim_{m,n\to\infty}\sup_{r\in[0,s]}\|A_m(t+r)y - A_n(t+r)y\| = 0, \quad y\in\mathscr{D}. \tag{2.4}$$

Therefore, for each $y\in\mathscr{D}$, estimation (2.3) gives

$$\|R_m(t,s)y - R_n(t,s)y\| \leq N^2 s e^{\omega_1 s}\sup_{r\in[0,s]}\|A_m(t+r)y - A_n(t+r)y\|. \tag{2.5}$$

By (2.4), the right hand of (2.5) converges to 0 as $m,n\to\infty$. Thus, $(R_n(t,s)y)$ is a Cauchy sequence in X for all $t,s\geq 0$, and so it converges. Moreover, (2.3) implies that, for each $x\in X$, $\{R_n(t,s)x : n\in\mathbb{N}\}$ is a bounded set. The density of $\mathscr{D}$ in X and Theorem 18 of Chapter II of [22] imply that the convergence can be extended for each $x\in X$. Therefore, we can define

$$R(t,s)x := \lim_{n\to\infty} R_n(t,s)x, \quad s,t\geq 0, \quad x\in X. \tag{2.6}$$

This definition gives $R(t,0)=I$ and $R(t,s+r)=R(t+r,s)R(t,r)$ for all $r,s,t\geq 0$. Equation (2.3) also implies that

$$\|R(t,s)x\| \leq \liminf_{n\to\infty}\|R_n(t,s)x\| \leq Ne^{\omega_1 s}\|x\|, \quad x\in X, \quad t,s\geq 0.$$

Thus, there exists an increasingly continuous function $M_1(s)=Ne^{\omega_1 s}$ such that

$$\|R(t,s)\| \leq M_1(s), \quad t,s\geq 0.$$

The proofs that $R(t,s)$ is strongly continuous and $A(t)$ in (2.2) is the infinitesimal generator of $R(t,s)$ are similar with the proof of Theorem 1 of [23], and the fact that, as $m\to\infty$, (2.5) gives

$$\|R(t,s)y - R_n(t,s)y\| \leq N^2 s e^{\omega_1 s}\sup_{r\in[0,s]}\|A(t+r)y - A_n(t+r)y\|.$$

□

The converse of Theorem 2.3 is not true for any C_0-quasi-semigroups (see Example 3.2 and Example 3.3 of [20]). In addition, the construction of the necessary condition is also constrained by the lack of a standard form of the resolvent operator of $A(t)$. This is different from the C_0-semigroups, and the resolvent operator of the infinitesimal generator of a C_0-semigroup can be expressed as a Laplace integral of the semigroup. Although, the resolvent operator of $A(t)$

can be determined [24]. For the contraction quasi-semigroups, from Theorem 2.3, we have the following result.

Corollary 2.4. *Let, for $t \geq 0$, $A(t)$ be a closed and densely defined in a Banach space X with domain $\mathscr{D}$, $[0,\infty) \subset \rho(A(t))$, and the mapping $t \mapsto A(t)y$ be continuous from $\mathbb{R}^+$ to X for all $y \in \mathscr{D}$. If $\mathscr{R}(\lambda, A(\cdot))$ is locally integrable and $\|\mathscr{R}(\lambda, A(t))\| \leq \frac{1}{\lambda}$, for all $\lambda > 0$, then $A(t)$ generates a contraction quasi-semigroup.*

3. Main Results

The main results in this section are split into two parts. The first part discusses the well-posedness of Cauchy problem (1.1). The second part confirms an application of the well-posedness in predicting the persistence or the extinction of population growth model, which has time-dependency of diffusion and intrinsic growth rate.

3.1. **Well-posedness of nonautonomous Cauchy problems.** In this subsection, we investigate the sufficient and necessary conditions for well-posedness of the Cauchy problem (1.1). The investigation uses the quasi-semigroup approach. We consider the inhomogeneous form of Cauchy problem (1.1)

$$\begin{aligned} \dot{u}(t) &= A(t)u(t) + f(t), \qquad t \geq 0, \\ u(0) &= u_0, \qquad\qquad\qquad u_0 \in X, \end{aligned} \tag{3.1}$$

where f is a continuous function from $[0,\infty)$ to a Banach space X. We recall the definition of a classical solution of nonautonomous Cauchy problem (3.1) [25]. We denote that $\mathscr{C}(\Omega, X)$ and $\mathscr{C}^1(\Omega, X)$ are the set of all continuous functions on Ω, and the set of all functions whose continuous derivative on Ω, respectively.

Definition 3.1. A function u is called a classical solution of (3.1) on $[0,\tau]$ if $u \in \mathscr{C}^1([0,\tau], X)$, $u(t) \in \mathscr{D}$ for all $t \in [0,\tau]$, and $u(t)$ satisfies (3.1) for all $t \in [0,\tau]$. The function u is called a classical solution on $[0,\infty)$ if u is a classical solution on $[0,\tau]$ for each $\tau > 0$.

Definition 3.1 is also valid for nonautonomous abstract Cauchy problem (1.1), that is, $f = 0$. We have the following results.

Lemma 3.2. *Let $A(t)$ be the infinitesimal generator of a C_0-quasi-semigroup $R(t,s)$ on a Banach space X and $u_0 \in \mathscr{D}$. If $f \in \mathscr{C}([0,\tau], X)$ and u is a classical solution of (3.1), then $A(\cdot)u(\cdot) \in \mathscr{C}([0,\tau], X)$ and*

$$u(t) = R(0,t)u_0 + \int_0^t R(s, t-s) f(s) ds. \tag{3.2}$$

Proof. Since $A(t)u(t) = \dot{u}(t) - f(t)$ and $\dot{u} \in \mathscr{C}([0,\tau], X)$, we have $A(\cdot)u(\cdot) \in \mathscr{C}([0,\tau], X)$. From Theorem 3.2 (b) of [20], we have

$$\begin{aligned} \frac{\partial}{\partial s}[R(s,t-s)u(s)] &= -A(s)R(s,t-s)u(s) + R(s,t-s)[A(s)u(s) + f(s)] \\ &= R(s,t-s)f(s). \end{aligned} \tag{3.3}$$

Integrating (3.3) respect to s over $[0,t]$, we obtain (3.2). Therefore, a classical solution of (3.1) must have the form (3.2). ∎

Theorem 3.3. *(Theorem 4,[25]) If $A(t)$ is the infinitesimal generator of a C_0-quasi-semigroup $R(t,s)$ on a Banach space X, $f \in \mathscr{C}^1([0,\tau],X)$, and $u_0 \in \mathscr{D}$, then the function u defined by (3.2) has a continuous derivative on $[0,\tau]$. Moreover, u is the unique classical solution of problem (3.1).*

Remark 3.4. Theorem 3.3 remains valid if f is Hölder continuous on $[0,\tau]$ and $u_0 \in X$. In facts, $R(0,t)u_0$ is the uniquely classical solution of the homogeneous Cauchy problem of (3.1) and

$$v(t) := \int_0^t R(s,t-s)f(s)\,ds$$

is the solution of the Cauchy problem

$$\begin{aligned} \dot{v}(t) &= A(t)v(t) + f(t), \\ v(0) &= 0 \end{aligned}$$

which guarantees the existence of solutions of (3.1).

Next, we investigate the sufficient and necessary conditions for nonautonomous abstract Cauchy problem (1.1) to have a solution. Related to this, we recall the terminology of *well-posedness* of the nonautonomous abstract Cauchy problem (1.1) that follows the definition for the autonomous case. Lemma 3.2 and Theorem 3.3 lead us that the infinitesimal generator has an important role in the well-posedness.

Definition 3.5. The nonautonomous abstract Cauchy problem (1.1) is said to be well-posed if it satisfies the following conditions:

(WP1) *Existence*. For each $u_0 \in \mathscr{D}$, there exists a classical solution u of (1.1) on $[0,\infty)$.

(WP2) *Uniqueness*. If $u, v : [0,\tau] \to X$ are the classical solutions of (1.1), then $u(t) = v(t)$ for all $t \in [0,\tau]$, $\tau > 0$.

(WP3) *Continuous dependence*. The classical solution x continuously depends on $t \in [0,\infty)$ and $u_0 \in \mathscr{D}$, i.e. the map $\phi : [0,\infty) \times \mathscr{D} \to X$ with $\phi(t,u_0) = u(t)$ is continuous.

Condition (WP3) implies that if the map $\phi : [0,\infty) \times \mathscr{D} \to X$ with $\phi(t,u_0) = u(t)$ for $t \in [0,\infty)$ and $u_0 \in \mathscr{D}$, where u is a classical solution of (1.1), then, for each $\tau > 0$, there exists an $N \geq 1$ such that $\|\phi(t,u_0)\| \leq N\|u_0\|$ for all $t \in [0,\tau]$ and $u_0 \in \mathscr{D}$.

To investigate the well-posedness of nonautonomous Cauchy problem (1.1) by the quasi-semigroup approach, we consider the Cauchy problem

$$\begin{aligned} \dot{u}(t) &= A(t+r)u(t), \quad & t,r \geq 0 \\ u(0) &= u_0, & u_0 \in X. \end{aligned} \tag{3.4}$$

Thus, nonautonomous Cauchy problem (1.1) is a special case of (3.4) with $r = 0$.

The following result characterizes the well-posedness of nonautonomous Cauchy problem (3.4) by the existence and uniqueness of the infinitesimal generator as indicated in Theorem 3.3.

Theorem 3.6. *For each $t \geq 0$, let $A(t) : \mathscr{D} \to X$ be a closed and densely defined operator in a Banach space X. The followings are equivalent*

(a) $A(t)$ is the infinitesimal generator of a C_0-quasi-semigroup on X;
(b) nonautonomous abstract Cauchy problem (3.4) is well-posed.

Proof. $(a) \Rightarrow (b)$. Theorem 2.2 of [16] guarantees the existence and uniqueness of the classical solution u of (3.4), where $u(t) = R(r,t)u_0$ and $R(t,s)$ is the C_0-quasi-semigroup generated by $A(t)$ on X. By condition (d) of Definition 2.1, for each $\tau > 0$, there exists an increasing continuous function $M : [0,\infty) \to [1,\infty)$ such that

$$\|u(t)\| \leq \|R(r,t)\| \|u_0\| \leq M(\tau)\|u_0\|, \quad t \in [0,\tau].$$

This shows that the classical solution u depends on t and u_0 continuously.

$(b) \Rightarrow (a)$. By the well-posedness, for $r \geq 0$ fixed, there exists a map

$$\phi_r : [0,\infty) \times \mathscr{D} \to \mathscr{D},$$

which uniquely assigns each $u_0 \in \mathscr{D}$ with the classical solution $u_r(t) = \phi_r(t,u_0)$, $t \geq 0$, of (3.4). The set of all the classical solutions u_r of (3.4) is a subspace of the space of all functions from $[0,\infty)$ to X. If we define $T_r : \mathscr{D} \to X$, where $T_r(u_0) = \phi_r(t,u_0)$ for $u_0 \in \mathscr{D}$, then (WP2) guarantees that T_r is a linear operator. Moreover, (WP3) gives that T_r is bounded. Next, the density of $\mathscr{D}$ in X implies that, for any $u \in X$, there exists a sequence $(u_n) \subset \mathscr{D}$, which converges to u. Hypothesis (WP3) gives that $(\phi_r(t,u_n))$ is a Cauchy sequence, so it converges in X. Theorem 18 of Chapter II of [22] implies that T_r can be extended uniquely on X. Therefore, for each $t \geq 0$, we can define a bounded linear operator $R(r,t)$ on X by

$$R(r,t)u := \lim_{n\to\infty} \phi_r(t,u_n).$$

By the uniqueness of limit, this definition does not depend on the choice of sequence $(u_n) \subset \mathscr{D}$. In this case, we have

$$R(r,t)u_0 = \phi_r(t,u_0), \quad u_0 \in \mathscr{D}. \tag{3.5}$$

From (3.5), the function $u_r(t) := R(r,t)u_0 = \phi_r(t,u_0)$ is a classical solution of (3.4) as $u_0 \in \mathscr{D}$. Next, for any $s \geq 0$, $u_r(t+s) = R(r,t+s)u_0 = \phi_r(t+s,u_0)$ is a classical solution of (3.4), where u_0 is replaced by $R(r,s)u_0 = \phi_r(s,u_0)$. Consequently,

$$R(r,t+s)u_0 = \phi_r(t+s,u_0) = \phi_{r+s}(t,R(r,s)u_0 =)R(r+s,t)R(r,s)u_0.$$

The density of $\mathscr{D}$ in X and the continuity of $R(r,t+s)$ and $R(r+s,t)R(r,s)$ give

$$R(r,t+s)u = R(r+s,t)R(r,s)u, \quad u \in X, \quad r,s,t \geq 0.$$

Thus, $R(t,s)$ is a quasi-semigroup on X. We have to show that $R(r,t)$ is strongly continuous. Given $u \in X$ and $\varepsilon > 0$, by hypothesis (WP3), there exists an $N \geq 1$ such that $\sup_{0\leq t\leq 1} \|\phi_r(t,u_0)\| \leq N\|u_0\|$ for all $u_0 \in \mathscr{D}$, i.e., $\sup_{0\leq t\leq 1} \|R(r,t)\| \leq N$. Choose $u_0 \in \mathscr{D}$ such that

$$\|u - u_0\| \leq \tfrac{\varepsilon}{2(N+1)}.$$

We have $0 < \delta \leq 1$ such that, for $0 \leq t < \delta$,

$$\|\phi_r(t,u_0) - u_0\| < \frac{\varepsilon}{2}.$$

Consequently, for $0 \leq t < \delta$,

$$\begin{aligned}\|R(r,t)u - u\| &\leq \|R(r,t)u - R(r,t)u_0\| + \|R(r,t)u_0 - u_0\| + \|u - u_0\| \\ &\leq (N+1)\|u - u_0\| + \|\phi_r(t,u_0) - u_0\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

This provides that $R(r,t)$ is strongly continuous on X.

Finally, $A(t)$ is the infinitesimal generator of $R(t,s)$. In fact, for any $u_0 \in \mathscr{D}$, $u_r(t) = R(r,t)u_0 = \phi_r(t,u_0)$ is continuously differentiable with its range in $\mathscr{D}$ and satisfies (3.4) for all $t \geq 0$. This completes the proof. □

Remark 3.7. The uniqueness of the infinitesimal generator in Lemma 2.2 is a consequence of the well-posedness. In fact, if both $u(t) = R_1(r,t)u_0$ and $v(t) = R_2(r,t)u_0$ satisfy (3.4), then (WP2) holds. The density of $\mathscr{D}$ forces that $R_1(r,t)u_0 = R_2(r,t)u_0$ for all $u_0 \in X$ and $r,t \geq 0$.

We consider a simple example that demonstrates how Theorem 3.6 characterizes the well-posedly nonautonomous Cauchy problem.

Example 3.8. Let X be the Hilbert space $L_2(0,1)$. The nonautonomous Cauchy problem given by

$$\dot{u}(t) = A(t)u(t), \quad u(0) = u_0, \tag{3.6}$$

where, for $x \in [0,1]$,

$$A(t)u(x) = \begin{cases} -\frac{1}{(x+t)^2}u(x), & 0 < t \leq 1, \\ -u(x), & t = 0 \quad \text{or} \quad t > 1, \end{cases}$$

is well-posed on X.

First, we show that $A(t)$ is the infinitesimal generator of a C_0-quasi-semigroup on X. We note that the domain of each $A(t)$ is $\mathscr{D} = X$. Thus, $\mathscr{D}$ is dense in X. Moreover, since each $A(t)$ is self-adjoint, then it is closed.

Next, we have $\langle A(t)u,u\rangle \leq -\frac{1}{4}\|u\|^2$ for all $u \in \mathscr{D}$. Therefore, $\sigma(A(t)) \subset (-\infty, -\frac{1}{4}]$ for all $t \geq 0$. For $\lambda \notin (-\infty, -\frac{1}{4}]$, we obtain

$$\mathscr{R}(\lambda, A(t))u(x) = \begin{cases} \frac{(x+t)^2}{\lambda(x+t)^2+1}u(x), & 0 < t \leq 1, \\ \frac{1}{\lambda+1}u(x), & t = 0 \quad \text{or} \quad t > 1. \end{cases}$$

For each $r \geq 1$, we verify that

$$\|\mathscr{R}(\lambda, A(t))^r\| \leq \frac{1}{\lambda^r}, \quad \lambda > 0, \quad t \geq 0.$$

Therefore, each operator $A(t)$ satisfies the sufficiency of Theorem 2.3 with $N = 1$ and $\omega = 0$. Thus, $A(t)$ generates a C_0-quasi-semigroup $R(t,s)$ on X, where

$$R(t,s)u(x) = \begin{cases} \exp\left(\frac{1}{x+t+s} - \frac{1}{x+t)}\right)u(x), & 0 < t,s \leq 1 \\ \exp(-s)u(x), & t,s \quad \text{others}, \end{cases}$$

Theorem 3.6 implies that nonautonomous Cauchy problem (3.6) is well-posed. In this case, $u(t) = R(0,t)u_0$ for some $u_0 \in X$ is a classical solution of the Cauchy problem.

Remark 3.9. (a) In fact, Example 3.8 is the nonautonomous Cauchy problem of the parabolic type. We see that the quasi-semigroup approach is simpler than the evolution operator approach in solving the Cauchy problem; see, e.g., [11, 12]. In this case, we do not need to investigate all parabolic conditions $(P1-P3)$ [11]. Moreover, the quasi-semigroup approach is also applicable for the hyperbolic type.

(b) In case that the C_0-quasi-semigroup in Theorem 3.6 is a contraction quasi semigroup, each $A(t)$ is the maximal dissipative operator (see [26]). This is a consequence of Theorem 2 of [23]. We see that the quasi-semigroup $R(t,s)$ in Example 3.8 is a contraction quasi-semigroup on X.

In applications, for example, in control systems, the condition $f \in \mathscr{C}([0,\tau],X)$ in problem (3.1) is too strong. In this case, we need a mild solution that requires a weaker condition than the classical solution. From Definition 2.1 of [17], a mild solution of the nonautonomous abstract Cauchy problem (3.1) on $[0,\tau]$ is defined to be the function u given by (3.2), where $f \in L^p([0,\tau],X)$, $1 \leq p < \infty$.

Theorem 3.10. *If $f \in L^p([0,\tau],X)$, $1 \leq p < \infty$, and $u_0 \in X$, then the mild solution u given by (3.2) is strongly continuous on $[0,\tau]$.*

Proof. Theorem 3.3 remains valid for $f \in L^p([0,\tau],X)$, so the integral in (3.2) is well-defined. In facts, for $0 \leq s \leq t \leq \tau$ and $p = 1$, we have

$$\left\| \int_0^t R(s,t-s)f(s)ds \right\| \leq \int_0^t M(t)\|f(s)\|ds \leq M(t)\|f\|_{L^1} < \infty.$$

For $1 < p < \infty$, the Holder's inequality gives

$$\begin{aligned}\left\| \int_0^t R(s,t-s)f(s)ds \right\| &\leq \left[\int_0^t \|R(s,t-s)\|^q ds \right]^{1/q} \left[\int_0^t \|f\|^p ds \right]^{1/p} \\ &\leq t^{1/q} M(t)\|f\|_{L^p}, \quad \tfrac{1}{p} + \tfrac{1}{q} = 1.\end{aligned}$$

For $h \geq 0$ small enough and $t \geq 0$, we obtain

$$\begin{aligned}u(t+h) - u(t) =& R(0,t+h)u_0 - R(0,t)u_0 + (R(t,h) - I)\int_0^t R(s,t-s)f(s)ds \\ &+ \int_t^{t+h} R(s,t+h-s)f(s)ds,\end{aligned} \tag{3.7}$$

and

$$\begin{aligned}u(t-h) - u(t) =& R(0,t-h)u_0 - R(0,t)u_0 + (I - R(t-h,h))\int_0^{t-h} R(s,t-h-s)f(s)ds \\ &+ \int_{t-h}^{t} R(s,t-s)f(s)ds.\end{aligned} \tag{3.8}$$

The right hands in (3.7) and (3.8) converge to 0 as $h \to 0$, respectively. These imply that u is continuous at t. □

Theorem 3.10 provides us the leeway to replace the classical solution with the mild solution in the well-posedness of the inhomogeneous nonautonomous Cauchy problem (3.1) with $f \in L^p([0,\tau],X)$. In particular, we consider

$$\begin{aligned}\dot{u}(t) &= A(t)u(t) + f(t), \qquad t \geq 0 \\ u(0) &= u_0, \qquad\qquad\qquad u_0 \in X,\end{aligned} \tag{3.9}$$

where $f \in W^{1,p}([0,\infty),X)$, $1 \leq p < \infty$. Recall that $W^{1,p}([0,\infty),X)$ denote the space of all functions f such that f, f' are absolutely continuous on $[0,\infty)$ and $f, f' \in L^p([0,\infty),X)$. This is a Sobolev space equipped with the norm

$$\|f\|_{1,p} := \left(\int_0^\infty ||f(s)||^p\, ds + \int_0^\infty ||f'(s)||^p\, ds\right)^{1/p}.$$

Theorem 3.11. *If $A(t)$ is the infinitesimal generator of a C_0-quasi-semigroup on X, then the inhomogeneous nonautonomous abstract Cauchy problem (3.9) is well-posed.*

Proof. Set a product space $\mathscr{X} = X \times L^p([0,\infty), X)$. Let $R(t,s)$ be the C_0-quasi-semigroup generated by $A(t)$. We define the operator matrices $\mathscr{K} t,s)$ on $\mathscr{X}$ by

$$\mathscr{K}(t,s) := \begin{bmatrix} R(t,s) & S(t,s) \\ 0 & Q(t,s) \end{bmatrix}, \quad s,t \geq 0,$$

where $S(t,s)f = \int_0^s R(t+\alpha, s-\alpha) f(\alpha) d\alpha$ and $Q(t,s)f(\cdot) = f(\cdot + s)$ for all $f \in L_p([0,\infty), X)$. We consider the operator matrices

$$\mathscr{A}(t) := \begin{bmatrix} A(t) & I \\ 0 & \frac{d}{d(\cdot)} \end{bmatrix}, \quad t \geq 0,$$

defined on $\mathfrak{D} := \mathscr{D} \times W^{1,p}([0,\infty), X)$, where $\mathscr{D}$ is the domain of $A(t)$. The definition of $S(t,s)$ and the transformation of variable $\upsilon = r + \alpha$ give

$$\begin{aligned} S(t+r,s)f + R(t+r,s)S(t,r)f & \\ &= \int_0^s R(t+r+\alpha, s-\alpha) f(\alpha) d\alpha + \int_0^r R(t+\upsilon, r+s-\upsilon) f(\upsilon) d\upsilon \\ &= \int_0^{r+s} R(t+\upsilon, r+s-\upsilon) f(\upsilon) d\upsilon = S(t, r+s) f. \end{aligned}$$

This implies

$$\begin{aligned} \mathscr{K}(t, r+s)w &= \begin{bmatrix} R(t+r,s) & S(t+r,s) \\ 0 & Q(t+r,s) \end{bmatrix} \begin{bmatrix} R(t,r) & S(t,r) \\ 0 & Q(t,r) \end{bmatrix} w \\ &= \mathscr{K}(t+r,s)\mathscr{K}(t,r)w, \end{aligned}$$

where $w = \begin{bmatrix} u \\ f \end{bmatrix} \in \mathscr{X}$. This concludes that $\mathscr{K}(t,s)$ is a C_0-quasi-semigroup on $\mathscr{X}$. Further, we can verify that $\mathscr{K}(t,s)$ is generated by $\mathscr{A}(t)$. Therefore, we can formulate the homogeneous nonautonomous Cauchy problem on $\mathscr{X}$,

$$\begin{aligned} \dot{w}(t) &= \mathscr{A}(t) w(t), \quad t \geq 0 \\ w(0) &= w_0, \quad w_0 \in \mathscr{X}, \end{aligned} \tag{3.10}$$

where $w(0) = \begin{bmatrix} u_0 \\ f_0 \end{bmatrix}$ with $f_0 = f(t)$, $t \geq 0$. By Theorem 3.6, problem (3.10) is well-posed in $\mathscr{X}$ and the classical solution is given by

$$w(t) = \mathscr{K}(0,t) w_0, \quad t \geq 0. \tag{3.11}$$

Taking its first coordinate of (3.11), we obtain the mild solution of the inhomogeneous nonautonomous Cauchy problem (3.9) given by (3.2). □

Remark 3.12. Theorem 3.11 remains valid for $f \in \mathscr{C}^1([0,\infty),X)$. In this case, if $u_0 \in \mathscr{D}$, then Theorem 3.11 and Theorem 3.3 are identic. In fact, both are hyperbolic cases (see, e.g., [11, 13]). If $f:[0,\tau]\to X$ is Hölder continuous, then Cauchy problem (3.9) is parabolic type and therefore it is well-posed due to Remark 3.4.

3.2. **An application in population growth with time-dependent diffusion.** In the subsection, we simulate the C_0-quasi-semigroups to analyze the simple predictions about the persistence, extinction, or stability of a population or community inhabiting a spatially heterogeneous environment. The predictions refer to "average" of spatially varying demographic parameters over the environment [27]. In the context of spatially explicit population models, the principal eigenvalues of differential operators or matrices describing the dispersal and demographics of populations are defined to be the averages.

A model of population growth with time-dependent diffusion assumes that the population inhabit a finite region $\Omega \subset \mathbb{R}^2$ with a lethal exterior. The model takes the form

$$\begin{aligned} &u_t(x,t) = d(t)\Delta u(x,t) + r(x,t)u(x,t), \quad (x,t) \in \Omega \times (0,\tau),\\ &u(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,\tau), \end{aligned} \tag{3.12}$$

where u is the population density, Δ is the Laplace operator, d is the time-dependent diffusion of the population, and r is the intrinsic growth rate function of the population. We also assume that d and r are τ-periodic in time t. In addition, d and r are also positively continuous functions on $[0,\infty)$ and on $\Omega \times [0,\infty)$, respectively. Due to the boundary condition, problem (3.12) is spatially heterogeneous. For d and r are constants, the persistence, extinction and stability of the population had been analyzed in [27]. The persistence of the population depends on a relation of d, r, and the principal eigenvalue of the Laplace operator. For $\Omega \subset \mathbb{R}^2$, we set a Banach space $X = L^p(\Omega)$, $2 < p < \infty$, as the state space of system (3.12). The τ-periodicity in t of $r(t) = r(\cdot,t)$ implies that $r(t)$ is bounded. We define the operators $A(t) := d(t)\Delta + r(t)$ on

$$\mathscr{D} = W_0^{2,p}(\Omega) := \{u \in W^{2,p}(\Omega) : u = 0 \text{ on } \partial\Omega\},$$

where $u(x) = u(x,\cdot)$. We verify that $d(t)\Delta$ is the infinitesimal generator of a C_0-quasi-semigroup on X (see [28]). If necessary, first we assume that Ω is a rectangle. From the perturbation result of Theorem 3 [29], the family of operators $A(t)$ is the infinitesimal generator of the C_0-quasi-semigroup $R(t,s)$ on X. Therefore, Theorem 3.6 guarantees that problem (3.12) is well-posed. Next, we give a simple analysis of the quasi-semigroups in persistence or extinction of the population.

Let $C^{i+\alpha,j+\alpha/2}(\overline{\Omega},\mathbb{R})$ be a class of all functions $u(x,t)$, where u has continuous derivatives up to order i in x with the ith order derivatives Hölder continuous with exponent α, and continuous and uniformly bounded derivatives up to order j in t, with $\partial^j u/\partial t^j$ uniformly Hölder continuous with exponent $\alpha/2$. We define

$$\begin{aligned} &F := \{w \in C^{\theta,\theta/2}(\overline{\Omega},\mathbb{R}) : w \text{ is } \tau\text{-periodic in } t\}.\\ &F_1 := \{w \in C^{2+\theta,1+\theta/2}(\overline{\Omega},\mathbb{R}) : w = 0 \text{ on } \partial\Omega \times \mathbb{R},\ w \text{ is } \tau\text{-periodic in } t\}. \end{aligned}$$

Let L be the operator in F with domain F_1 defined by

$$L := \partial_t - A(t). \tag{3.13}$$

We consider the periodic eigenvalue problem

$$
\begin{aligned}
Lu = \lambda u, &\quad \text{in } \Omega \times \mathbb{R}, \\
u = 0, &\quad \text{on } \partial\Omega \times \mathbb{R}, \\
u \ \tau\text{-periodic in } t. &
\end{aligned} \tag{3.14}
$$

We recall that $\lambda \in \mathbb{C}$ is an eigenvalue if there is a nontrivial solution (eigenfunction) u satisfying (3.14). In particular, we concern on a principal eigenvalue (an eigenvalue $\lambda \in \mathbb{R}$) with a positive eigenfunction. We set $K := R(0,\tau)$. The τ-periodicity of u requires that $R(t+\tau, s+\tau) = R(t,s)$ for all $t, s \geq 0$. We derive some properties of L.

Lemma 3.13. *If $u_0 \in X$, $u_0 > 0$, then $R(0,t)u_0 > 0$ in $\mathscr{D}$ for all $0 \leq t \leq \tau$.*

Proof. Let $t < \tau$ be fixed. Theorem 3.2 (a) of [20] gives $R(0,t)u \in \mathscr{D}$ for all $u \in \mathscr{D}$. Thus, $R(0,t)$ is a positive operator on $\mathscr{D}$ (see [30]). Since $\mathscr{D}$ is dense in X, the positive operator $R(0,t)|_{\mathscr{D}} : \mathscr{D} \to \mathscr{D}$ can be extended to the positive operator $R(0,t) \in \mathscr{L}(X, \mathscr{D})$. Hence $R(0,t)u_0 \geq 0$ in $\mathscr{D}$. Since $s \mapsto R(0,s)u_0$ is continuous from $[0,t]$ to X, we have $R(0,s)u_0 \neq 0$ in X. So, $R(0,s)u_0 > 0$ in $\mathscr{D}$ for s, which is close to t. By $R(0,t) = R(s,t-s)R(0,s)u_0$ and $R(0,s)u_0 > 0$, we obtain $R(0,t)u_0 > 0$ in $\mathscr{D}$. □

Lemma 3.14. *If $r_0 := \operatorname{spr}(K)$, the spectral radius of K, then $0 < r_0 < 1$.*

Proof. The Krein-Rutman Theorem (Theorem 7.2) of [30] implies that $r_0 > 0$ and r_0 is the unique eigenvalue of K. To prove $r_0 < 1$, let $u_0 \in \mathscr{D}$, $u_0 > 0$, be the principal eigenfunction of K, i.e., $Ku_0 = r_0 u_0$. We see that $u = R(0,\cdot)u_0$ satisfies

$$
\begin{aligned}
Lu = 0, &\quad \text{in } \Omega \times (0,\tau] \\
u(x,t) = 0, &\quad \text{on } \partial\Omega \times (0,\tau] \\
u(\cdot,0) = u_0, &\quad \text{in } \Omega.
\end{aligned}
$$

Lemma 3.13 gives $u \geq 0$. Setting $v := u - \|u_0\|_{C(\overline{\Omega})}$ on $\Omega \times [0,\tau]$, we obtain

$$
\begin{aligned}
Lv = r(t)\|u_0\|_{C(\overline{\Omega})} \geq 0, &\quad \text{in } \Omega \times (0,\tau] \\
v(x,t) \geq 0, &\quad \text{on } \partial\Omega \times (0,\tau] \\
v(\cdot,0) \geq 0, &\quad \text{in } \Omega.
\end{aligned}
$$

This shows that $v(t) \geq 0$ for all $0 \leq t \leq \tau$. Moreover,

$$
r_0\|u_0\|_{C(\overline{\Omega})} = \|Ku_0\| = \|u(\tau)\|_{C(\overline{\Omega})} < \|u_0\|_{C(\overline{\Omega})},
$$

which implies that $r_0 < 1$. □

Lemma 3.15. *The operator L given in (3.13) is closed in F. Further, it has a compact inverse $L^{-1} \in \mathscr{L}(F)$.*

Proof. Observe $L \in \mathscr{L}(F_1, F)$. We need to show that L is a closed operator in F. First, we show that $L : F_1 \to F$ is bijective. Given any $f \in F$, we define a function $u(t) = u(\cdot,t)$ by

$$
\begin{aligned}
u(t) &= R(0,t)u_0 + \int_0^t R(s,t-s)f(s)\,ds, \quad 0 < t < \tau, \\
u(\tau) &= u(0).
\end{aligned} \tag{3.15}
$$

We verify that $Lu = f$. Further, if u satisfies (3.15), then

$$\begin{aligned}
u(t+\tau) &= R(0,t+\tau)u_0 + \int_0^{t+\tau} R(s,t+\tau-s)f(s)\,ds \\
&= R(0,t)R(0,\tau)u_0 + \int_0^{\tau} R(s,t+\tau-s)f(s)\,ds + \int_{\tau}^{t+\tau} R(s,t+\tau-s)f(s)\,ds \\
&= R(0,t)[R(0,\tau)u_0 + \int_0^{\tau} R(s,\tau-s)f(s)\,ds] + \int_0^{t} R(s,t-s)f(s)\,ds \\
&= R(0,t)u(\tau) + \int_0^{t} R(s,t-s)f(s)\,ds = u(t).
\end{aligned}$$

This gives $u \in F_1$. Hence, L is onto. Assume that $Lu = 0$. This equation has a solution $u(t) = R(0,t)u_0$, $0 \le t \le \tau$. Since $u(\tau) = u(0)$ and $0 < r_0 < 1$, we have

$$u_0 = R(0,\tau)u_0 \quad \text{or} \quad (1-K)u_0 = 0,$$

which implies $u_0 = 0$. This gives $u(t) = 0$, $0 \le t \le \tau$, that is, L is injective. Therefore, L is invertible and the Theorem 4.2-B of [31] implies that L is closed. Further, the Open Mapping Theorem gives $L^{-1} : F \to F_1 \hookrightarrow F$ is continuous. Thus, $L^{-1} \in \mathscr{L}(F)$ is compact. □

Theorem 3.16. *The constant $r_0 = \operatorname{spr}(K)$ is the principal eigenvalue of K with the eigenfunction u_0, $u_0 \in \mathscr{D}$, $u_0 > 0$, if and only if $\lambda = -\frac{1}{\tau}\ln r_0$ is the eigenvalue of L with a positive eigenfunction $u(t) := e^{\lambda t}R(0,t)u_0$, $0 \le t \le \tau$.*

Proof. Let $Ku_0 = r_0u_0$ in $\mathscr{D}$. We verify that $u(t) = e^{\lambda t}R(0,t)u_0$ satisfies

$$\begin{aligned}
&\dot{u}(t) - A(t)u(t) = \lambda u(t), \quad 0 < t \le \tau, \quad \text{in} \quad X, \\
&u(0) = u_0 = \frac{1}{r_0}Ku_0 = e^{\lambda\tau}Ku_0 = u(\tau).
\end{aligned}$$

This provides that $u \in F_1$ and $Lu = \lambda u$, $u > 0$ in F_1. Conversely, if $Lu = \lambda u$ with $u > 0$, then $v(t) := e^{-\lambda t}u(t)$ solves

$$\begin{aligned}
&\dot{v}(t) - A(t)v(t) = 0, \quad 0 < t \le \tau, \quad \text{in} \quad X, \\
&v(0) = u(0) = u_0 \in \mathscr{D}.
\end{aligned}$$

We note that $v(t) = R(0,t)u_0$ with $u_0 > 0$ in $\mathscr{D}$. Consequently,

$$Ku_0 = R(0,\tau)u_0 = v(\tau) = e^{-\lambda\tau}u(\tau) = r_0u_0,$$

i.e., r_0 is the principal eigenvalue of K with the eigenfunction u_0. By Krein-Rutman Theorem of [30], one has $r_0 = \operatorname{spr}(K)$. This completes the proof. □

We next illustrate the persistence or extinction of the population growth of (3.12) in the one-dimensional state space $[0,\ell]$. In this case, $\Delta = \frac{d^2}{dx^2}$ has the eigenvalues and eigenfunctions

$$\lambda_n = -\left(\frac{n\pi}{\ell}\right)^2, \qquad \phi_n(x) = \sqrt{2/\ell}\sin\left(\frac{n\pi x}{\ell}\right), \quad 0 \le x \le \ell, \quad n \ge 1, \tag{3.16}$$

respectively. Following the Example 1 of [28], we obtain a C_0-quasi-semigroup $R(t,s)$ on X given by

$$R(t,s)u = \sum_{n=1}^{\infty} e^{\lambda_n[D(t+s)-D(t)]}\langle u,\phi_n\rangle\phi_n + e^{E(t+s)-E(t)}u, \quad s,t \in [0,\tau],$$

where $D(t) = \int_0^t d(s)\,ds$, $E(t) = \int_0^t r(s)\,ds$, and $\langle u,v\rangle = \int_0^\ell u(x)v(x)\,dx$. From Theorem 3.16, the solution of (3.12) is given by

$$\begin{aligned} u(x,t) &= R(0,t)u_0(x) \\ &= \sum_{n=1}^{\infty} e^{\lambda_n D(t)}\langle u_0,\phi_n\rangle\phi_n(x) + e^{E(t)}u_0(x), \quad t\in[0,\tau], \quad x\in[0,\ell]. \end{aligned} \tag{3.17}$$

From (3.16), the solution given in (3.17) will grow exponentially if $D(t) < 0$ and $E(t) > 0$ on $[0,\tau]$ but decay exponentially if $D(t) > 0$ or $E(t) < 0$ on $[0,\tau]$. Therefore, population (3.12) predicts persistence or extinction depending on the sign of the diffusion function and the intrinsic growth rate function of the population. For instance, if we specify $d(t) = 1/(t+1)^2$ and $r(x,t) = 0.2xt$, the population will grow exponentially.

Remark 3.17. (a) We note that if d and r are constants, then the persistence or extinction depends on the sign of the principal eigenvalue of the operator $d\Delta + r$, which can be conditioned; see [27]. In our illustration, the principal eigenvalue λ_1 of Δ is surely negative, which can not be conditioned.

(b) Theorem 3.16 shows that the classical solution of problem (3.12) can be determined from the eigenfunction of the operator L. Further, if λ is the eigenvalue of L, then $r_0 = e^{-\lambda\tau}$ is the spectral radius of the operator $K = R(0,\tau)$.

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