J. Nonlinear Funct. Anal. 2021 (2021), Article ID 3 https://doi.org/10.23952/jnfa.2021.3

MATHRES

Journal of Nonlinear Functional Analysis

Available online at http://jnfa.mathres.org

MOUNTAIN PASS TYPE SOLUTIONS FOR A NONLACAL FRACTIONAL $a(.)$-KIRCHHOFF TYPE PROBLEMS

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Abstract. In this paper, we investigate the existence of a weak solution of a fractional Kirchhoff type problem driven by a nonlocal operator of elliptic type in a fractional Orlicz-Sobolev space with homogeneous Dirichlet boundary conditions. The approach is based on the mountain pass theorem and some variational methods.

Keywords. Fractional $a(.)$-laplacian; Fractional Orlicz-Sobolev spaces; Nonlocal problem; Mountain pass theorem.

1. Introduction

In the last decade, much attention has been devoted to the study of the nonlinear problems involving non-local operators. These types of operator come up in a quite natural way in several applications such as phase transition phenomena, crystal dislocation, soft thin films, minimal surfaces and finance; see, e.g., [1, 2, 3] and the references therein. We also refer the interested reader to [4, 5], where a more extensive bibliography and an introduction to the subject are given.

In this paper, we are interested to study the existence of a weak solution of the following problem

$$(P_a)\quad \begin{cases} M\left(\int_{\Omega\times\Omega}\Phi\left(\frac{|u(x)-u(y)|}{|x-y|^s}\right)\frac{dxdy}{|x-y|^N}\right)(-\Delta)^s_{a(.)}u &= f(x,u) \quad \text{in} \quad \Omega, \\ u &= 0 \quad \text{in} \quad \mathbb{R}^N\setminus\Omega, \end{cases}$$

where Ω is an open bounded subset in $\mathbb{R}^N$, $N\geqslant 1$, with Lipschitz boundary $\partial\Omega$, $0<s<1$, Φ is an N-function, and $M:[0,\infty)\to(0,\infty)$ is a continuous function satisfying the following

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Received June 10, 2020; Accepted January 8, 2021.

conditions:

$$\text{there exists } m_0 > 0 \text{ such that } M(t) \geqslant m_0 \text{ for all } t \geqslant 0, \tag{M_1}$$

$$\text{there exists } \theta > 0 \text{ such that } \widehat{M}(t) \geqslant \theta M(t)t \text{ for all } t \geqslant 0, \tag{M_2}$$

where $\widehat{M}(t) = \int_0^t M(\tau)d\tau$. Let $f : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ be a Carathéodory function and $(-\Delta)^s_{a(.)}$ be the nonlocal integro-differential operator of elliptic type defined as:

$$(-\Delta)^s_{a(.)}u(x) = 2\lim_{\varepsilon \searrow 0} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} a\left(\frac{|u(x)-u(y)|}{|x-y|^s}\right)\frac{u(x)-u(y)}{|x-y|^s}\frac{dy}{|x-y|^{N+s}},$$

for all $x \in \mathbb{R}^N$, where $a : \mathbb{R} \longrightarrow \mathbb{R}$, which will be specified later.

A typical prototype for M, due to Kirchhoff in 1883 [6], is given by

$$M(t) = a + bt^{\alpha-1}, \quad a,b \geqslant 0,\ a+b > 0,\ t \geqslant 0, \tag{1.1}$$

and

$$\begin{cases} \alpha \in (1,+\infty), & if \quad b > 0, \\ \alpha = 1, & if \quad b = 0, \end{cases}$$

when $M(t) > 0$ for all $t \geqslant 0$. The Kirchhoff problems are said to be nondegenerate and this happens, for example, if $a > 0$ and $b \geqslant 0$ in the model case (1.1). Otherwise, if $M(0) = 0$ and $M(t) > 0$ for all $t > 0$, then the Kirchhoff problems are said to be degenerate and this occurs in the model case (1.1) when $a = 0$ and $b > 0$.

The problem (P_a) is related to the stationary version of the Kirchhoff equation

$$\rho\frac{\partial^2 u}{\partial t^2} - \left(\frac{P_0}{h} + \frac{E}{2L}\int_0^L |\frac{\partial u}{\partial x}|^2 dx\right)\frac{\partial^2 u}{\partial x^2} \tag{1.2}$$

presented by Kirchhoff [6] in 1883 is an extension of the classical d'Alembert's wave equation by considering the changes in the length of the string during vibrations. In (1.2), L is the length of string, h is the area of the cross section, E is the Young modulus of the material, ρ is the mass density, and P_0 is the initial tension. The Kirchhoff's model takes into account the length changes of the string produced by transverse vibrations. Some interesting results can be found in, for example, [7]. On the other hand, Kirchhoff-type boundary value problems model several physical and biological systems where u describes a process which depend on the average of itself, for instance, the population density. We refer the reader to [8, 9] for some related works. In [10], Azroul *et al.* showed the existence and multiplicity of solutions to a class of $p(x)$-Kirchhoff type equations via variational methods. In [11], Azroul, Benkirane and Srati obtained the existence of three weak solutions for a p-Kirchhoff type problem via the three critical points theorem. In [12], Azroul *et al.* obtained the existence of three weak solutions for a p-Kirchhoff type elliptic system involving the nonlocal fractional p-Laplacian by using the three point critique theorem. In [13], Colasuonno and Pucci investigated higher order $p(x)$-Kirchhoff type equations via symmetric mountain pass Theorem, even in the degenerate case. In the very recent paper [14], Fiscella and Valdinoci first provided a detailed discussion about the physical meaning underlying the fractional Kirchhoff models and their applications; see [14] for further details.

For the problems involving fractional Kirchhoff type, we refer the reader to the works [15, 16, 17].

Notice that if $a(t) = t^{p-2}$, where $1 < p < \infty$, then problem (P_a) is reduced to the fractional p-Laplacian problem

$$(P_p) \quad \begin{cases} M([u]_{s,p})(-\Delta)_p^s u = f(x,u) & \text{in} \quad \Omega, \\ \qquad\qquad u = 0 & \text{in} \quad \mathbb{R}^N \setminus \Omega, \end{cases}$$

where $(-\Delta)_p^s$ is the fractional p-Laplacian operator define by

$$(-\Delta)_p^s u(x) = 2 \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+sp}} dy.$$

In recent years, problem (P_p) has been studied in many papers, we refer to [11, 18, 19, 20]. If $M = 1$ and $p = 2$, then problem (P_p) is reduced to the fractional Laplacian problem

$$(P) \quad \begin{cases} (-\Delta)^s u = f(x,u) & \text{in} \quad \Omega, \\ \qquad u = 0 & \text{in} \quad \mathbb{R}^N \setminus \Omega. \end{cases}$$

One typical feature of the problem is the nonlocality in the sense that the value of $(-\Delta)^s u(x)$ at any point $x \in \Omega$ depends not only on Ω, but actually on the entire space $\mathbb{R}^N$. We refer also to [21, 22, 23] for further details on the functional framework and its applications to the existence of solutions. Motivated by the above results, the aim of this paper to study the existence of a weak solution for problem (P_a). This paper is organized as follows. In Section 2, we recall some properties on fractional Orlicz-Sobolev spaces. In Section 3, using the mountain pass theorem, we obtain the existence of a weak solution of problem (P_a). In Section 4, the last section is devoted to giving an example which illustrates the mains abstracts results.

2. Variatoinal setting and preliminaries results

To deal with this situation, we introduce the fractional Orlicz-Sobolev space to investigate problem (P_a). Let us recall the definitions and some elementary properties of this spaces. We refer the reader to [24, 25, 26] for further reference and for some of the proofs of the results in this subsection.

Let Ω be an open subset of $\mathbb{R}^N$ with $N \geqslant 1$. We assume that $a : \mathbb{R} \longrightarrow \mathbb{R}$ in (P_a) is such that $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$, defined by

$$\varphi(t) = \begin{cases} a(|t|)t & \text{for} \quad t \neq 0, \\ 0 & \text{for} \quad t = 0, \end{cases}$$

is increasing homeomorphism from $\mathbb{R}$ onto itself. Let

$$\Phi(t) = \int_0^t \varphi(\tau) d\tau. \tag{2.1}$$

Then, Φ is a N-function (see [27]), i.e. $\Phi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is a continuous, convex and increasing function, with $\dfrac{\Phi(t)}{t} \to 0$ as $t \to 0$ and $\dfrac{\Phi(t)}{t} \to \infty$ as $t \to \infty$.

For the function Φ introduced above, we define the Orlicz space:

$$L_\Phi(\Omega) = \left\{ u : \Omega \longrightarrow \mathbb{R} \text{ mesurable} : \int_\Omega \Phi(\lambda |u(x)|) dx < \infty \text{ for some } \lambda > 0 \right\}.$$

The space $L_\Phi(\Omega)$ is a Banach space endowed with the Luxemburg norm

$$||u||_\Phi=\inf\left\{\lambda>0:\ \int_\Omega\Phi\left(\frac{|u(x)|}{\lambda}\right)dx\leqslant 1\right\}.$$

The conjugate N-function of Φ is defined by $\overline{\Phi}(t)=\int_0^t\overline{\varphi}(\tau)d\tau$, where $\overline{\varphi}:\mathbb{R}\longrightarrow\mathbb{R}$ is given by $\overline{\varphi}(t)=\sup\{s:\ \varphi(s)\leqslant t\}$. Furthermore, it is possible to prove a Hölder type inequality, that is,

$$\left|\int_\Omega uvdx\right|\leqslant 2||u||_\Phi||v||_{\overline{\Phi}}\qquad\forall u\in L_\Phi(\Omega)\text{ and }v\in L_{\overline{\Phi}}(\Omega).\tag{2.2}$$

Throughout this paper, we assume that

$$1<\varphi^-:=\inf_{t\geqslant 0}\frac{t\varphi(t)}{\Phi(t)}\leqslant\varphi^+:=\sup_{t\geqslant 0}\frac{t\varphi(t)}{\Phi(t)}<+\infty.\tag{2.3}$$

The above relation implies that $\Phi\in\Delta_2$ i.e., Φ satisfies the global Δ_2-condition (see [28]):

$$\Phi(2t)\leqslant K\Phi(t)\text{ for all }t\geqslant 0,$$

where K is a positive constant.

Furthermore, we assume that Φ satisfies the following condition

$$\text{the function }[0,\infty)\ni t\mapsto\Phi(\sqrt{t})\text{ is convex.}\tag{2.4}$$

The above relation assures that $L_\Phi(\Omega)$ is an uniformly convex space (see [28]).

Definition 2.1. Let A, B be two N-functions. A is said to be stronger (resp essentially stronger) than B, $A\succ B$ (resp $A\succ\succ B$) in symbols if

$$B(x)\leqslant A(ax),\ ,x\geqslant x_0\geqslant 0,$$

for some (resp for each) $a>0$ and x_0 (depending on a).

Remark 2.1. [27, Section 8.5] $A\succ\succ B$ is equivalent to the condition

$$\lim_{x\to\infty}\frac{B(\lambda x)}{A(x)}=0,$$

for all $\lambda>0$.

Now, we defined the fractional Orlicz-Sobolev space $W^sL_\Phi(\Omega)$ as follows:

$$W^sL_\Phi(\Omega)=\left\{u\in L_\Phi(\Omega)\ :\ \int_\Omega\int_\Omega\Phi\left(\frac{\lambda|u(x)-u(y)|}{|x-y|^s}\right)\frac{dxdy}{|x-y|^N}<\infty\ \text{ for some }\lambda>0\right\}.$$

This space is equipped with the norm,

$$||u||_{s,\Phi}=||u||_\Phi+[u]_{s,\Phi},\tag{2.5}$$

where $[.]_{s,\Phi}$ is the Gagliardo seminorm, defined by

$$[u]_{s,\Phi}=\inf\left\{\lambda>0:\int_\Omega\int_\Omega\Phi\left(\frac{|u(x)-u(y)|}{\lambda|x-y|^s}\right)\frac{dxdy}{|x-y|^N}\leqslant 1\right\}.$$

To deal with the problem under consideration, we choose

$$W_0^sL_\Phi(\Omega)=\left\{u\in W^sL_\Phi(\mathbb{R}^N)\ :\ u=0\text{ a.e. }\mathbb{R}^N\setminus\Omega\right\}.$$

which can be equivalently renormed by setting $\|.\| = [.]_{s,\Phi}$. By [26], $W_0^s L_\Phi(\Omega)$ is Banach space. Indeed, it is separable (resp. reflexive) space if and only if $\Phi \in \Delta_2$ (resp. $\Phi \in \Delta_2$ and $\overline{\Phi} \in \Delta_2$). Furthermore if $\Phi \in \Delta_2$ and $\Phi(\sqrt{t})$ is convex, then the space $W^s L_\Phi(\Omega)$ is uniformly convex. The dual space of $(W^s L_\Phi(\Omega), ||.||)$ is denoted by $((W^s L_\Phi(\Omega))^*, ||.||_*)$.

Lemma 2.1. [26] Let Φ be an N-function as defined in (2.1). Assume condition (2.3) is satisfied. Then we have,

$$\overline{\Phi}(\varphi(t)) \leqslant c\Phi(t) \text{ for all } t \geqslant 0 \tag{2.6}$$

where $c > 0$.

Lemma 2.2. [28] Assume condition (2.3) is satisfied. Then, for every $b > 1$ and $t \geqslant 0$, we have

$$\Phi(bt) \leqslant b^{\varphi^+} \Phi(t). \tag{2.7}$$

Proposition 2.1. [24] Assume condition (2.3) is satisfied. Then the following relations hold true,

$$[u]_{s,\Phi}^{\varphi^-} \leqslant \int_\Omega \int_\Omega \Phi\left(\frac{|u(x)-u(y)|}{|x-y|^s}\right) \frac{dxdy}{|x-y|^N} \leqslant [u]_{s,\Phi}^{\varphi^+},\ \forall u \in W^s L_\Phi(\Omega) \text{ with } [u]_{s,\Phi} > 1, \tag{2.8}$$

$$[u]_{s,\Phi}^{\varphi^+} \leqslant \int_\Omega \int_\Omega \Phi\left(\frac{|u(x)-u(y)|}{|x-y|^s}\right) \frac{dxdy}{|x-y|^N} \leqslant [u]_{s,\Phi}^{\varphi^-},\ \forall u \in W^s L_\Phi(\Omega) \text{ with } [u]_{s,\Phi} < 1. \tag{2.9}$$

In this paper, we assume the following conditions :

$$\int_0^1 \frac{\Phi^{-1}(\tau)}{\tau^{\frac{N+s}{N}}} d\tau < \infty, \tag{2.10}$$

and

$$\int_1^\infty \frac{\Phi^{-1}(\tau)}{\tau^{\frac{N+s}{N}}} d\tau = \infty. \tag{2.11}$$

We define the inverse Sobolev conjugate N-function of Φ as follows

$$\Phi_*^{-1}(t) = \int_0^t \frac{\Phi^{-1}(\tau)}{\tau^{\frac{N+s}{N}}} d\tau. \tag{2.12}$$

Theorem 2.1. [24] Let Ω be a bounded open subset of $\mathbb{R}^N$ with $C^{0,1}$-regularity and bounded boundary. If (2.3), (2.11) and (2.12) hold, then

$$W^s L_\Phi(\Omega) \hookrightarrow L_{\Phi_*}(\Omega). \tag{2.13}$$

Theorem 2.2. [24] Let Ω be a bounded open subset of $\mathbb{R}^N$ with $C^{0,1}$-regularity and bounded boundary. If (2.3), (2.11) and (2.12) hold, then

$$W^s L_\Phi(\Omega) \hookrightarrow L_B(\Omega), \tag{2.14}$$

is compact for all $B \prec\prec \Phi_*$.

Theorem 2.3. [24] Let Ω be a bounded open subset of $\mathbb{R}^N$. Then,

$$C_0^\infty(\Omega) \subset C_0^2(\Omega) \subset W^s L_\Phi(\Omega).$$

Lemma 2.3. Let Ω be a bounded open subset of $\mathbb{R}^N$, and let $s \in (0,1)$. Assume that condition (2.3) is satisfied. Then, there exists a positive constant λ_1 such that

$$\int_\Omega \Phi(|u(x)|)dx \leqslant \lambda_1 \int_\Omega \int_\Omega \Phi\left(\frac{|u(x)-u(y)|}{|x-y|^s}\right)\frac{dxdy}{|x-y|^N}, \tag{2.15}$$

for all $u \in W_0^s L_\Phi(\Omega)$.

Proof. Let $u \in W_0^s L_\Phi(\Omega)$ and $B_R \subset \mathbb{R}^N \setminus \Omega$, that is, the ball of radius $R>0$ in the complement of Ω. Then, for all $x \in \Omega$, $y \in B_R$ and all $\lambda > 0$ we have,

$$\Phi(|u(x)|) = \Phi\left(\frac{|u(x)-u(y)|}{|x-y|^s}|x-y|^s\right)\frac{|x-y|^N}{|x-y|^N},$$

which implies that

$$\Phi(|u(x)|) \leqslant \Phi\left(\frac{|u(x)-u(y)|}{|x-y|^s} diam(\Omega \cup B_R)^s\right)\frac{diam(\Omega \cup B_R)^N}{|x-y|^N}.$$

Putting $\alpha = diam(\Omega \cup B_R)^s$ and using estimation (2.7), we get

$$\Phi(|u(x)|) \leqslant \max\left\{\alpha, \alpha^{\varphi^+}\right\}\Phi\left(\frac{|u(x)-u(y)|}{|x-y|^s}\right)\frac{diam(\Omega \cup B_R)^N}{|x-y|^N}.$$

Therefore

$$|B_R|\Phi(|u(x)|) \leqslant \max\left\{\alpha, \alpha^{\varphi^+}\right\} diam(\Omega \cup B_R)^N \int_{B_R} \Phi\left(\frac{|u(x)-u(y)|}{|x-y|^s}\right)\frac{dy}{|x-y|^N}.$$

Then

$$\int_\Omega \Phi(|u(x)|)dx \leqslant \lambda_1 \int_\Omega \int_\Omega \Phi\left(\frac{|u(x)-u(y)|}{|x-y|^s}\right)\frac{dxdy}{|x-y|^N},$$

where

$$\lambda_1 = \frac{\max\left\{\alpha, \alpha^{\varphi^+}\right\} diam(\Omega \cup B_R)^N}{|B_R|}.$$

□

We conclude this section by recalling the version of the mountain pass theorem [29] as given in [30] (see also [31]).

Theorem 2.4. Let X be a real Banach space and $I \in C^1(X,\mathbb{R})$ with $I(0)=0$. Suppose that the following conditions hold:

(G_1) there exists $\rho > 0$ and $r > 0$ such that $I(u) \geqslant r$ for $||u|| = \rho$.

(G_2) there exists $e \in X$ with $||e|| > \rho$ such that $I(e) \leqslant 0$.

Let

$$c := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t)) \text{ with } \Gamma = \{\gamma \in C([0,1],X); \gamma(0)=0, \gamma(1)=e\}.$$

Then there exists a sequence $\{u_n\}$ in X such that

$$I(u_n) \to c \quad \text{and} \quad I'(u_n) \to 0.$$

3. Main results

In this section, we prove the existence of a weak solution in fractional Orlicz-Sobolev spaces applying the mountain pass theorem. For this, we suppose that $f:\Omega\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function satisfying:

$$|f(x,t)|\leqslant c_1(1+g(|t|)). \tag{f_1}$$

where c_1 is a nonnegative constant and $g:\mathbb{R}^+\to\mathbb{R}^+$ is a non-decreasing, right continuous function with $g(0)=0$, $g(t)>0$, $\forall t>0$ and $g(t)\to\infty$ as $t\to\infty$. By setting

$$G(t)=\int_0^t g(\tau)d\tau, \qquad \overline{G}(t)=\int_0^t \overline{g}(\tau)d\tau, \tag{3.1}$$

where $\overline{g}(t)=\sup\{s:g(s)\leqslant t\}$, we obtain complementary N-functions which define corresponding Orlicz spaces L_G and $L_{\overline{G}}$. We will also assume that

$$1<q^-=\inf_{t\geqslant 0}\frac{tg(t)}{G(t)}\leqslant q^+=\sup_{t\geqslant 0}\frac{tg(t)}{G(t)}<+\infty. \tag{f_2}$$

$$\lim_{t\to\infty}\frac{G(kt)}{\Phi_*(t)}=0 \text{ for all } k>0. \tag{f_3}$$

$$tf(x,t)\geqslant \mu F(x,t)\geqslant 0 \ \text{ for all } |t|\geqslant r \text{ and a.e. } x\in\Omega, \tag{f_4}$$

where $F(x,t):=\int_0^t f(x,\tau)d\tau$, $r>0$, $\mu>\frac{\varphi^+}{\theta}$, and θ is given in assumption (M_2),

$$\limsup_{t\to 0}\frac{F(x,t)}{\Phi(t)}<\frac{1}{\lambda_1} \text{ uniformly for a.e. } x\in\Omega, \tag{f_5}$$

where λ_1 is as in (2.15).

Remark 3.1. The assumption (f_4) is not the usual Ambrosetti-Rabinowitz condition since we here suppose that $\mu>\frac{\varphi^+}{\theta}$. This difference is caused by the function M in problem (P_a).

To simplify the notation, we let

$$D^s u:=\frac{u(x)-u(y)}{|x-y|^s} \text{ and } \Psi(u):=\int_\Omega\int_\Omega \Phi\left(\frac{|u(x)-u(y)|}{|x-y|^s}\right)\frac{dxdy}{|x-y|^N}.$$

Now, we give the definition of weak solutions of problem (P_a).

Definition 3.1. $u\in W_0^sL_\Phi(\Omega)$ is called a weak solution of problem (P_a) if,

$$M(\Psi(u))\int_\Omega\int_\Omega a(|D^s u|)D^s uD^s v\frac{dxdy}{|x-y|^N}=\int_\Omega f(x,u)vdx, \tag{3.2}$$

for all $v\in W_0^sL_\Phi(\Omega)$.

The main result of this paper is as follows.

Theorem 3.1. Suppose that M satisfies (M_1) and (M_2), and f satisfies (f_1)-(f_5). Then, problem (P_a) has a nontrivial solution $u\in W_0^sL_\Phi(\Omega)$, which is a critical point of mountain pass type for the functional

$$I(u)=\widehat{M}\left(\int_\Omega\int_\Omega \Phi\left(\frac{|u(x)-u(y)|}{|x-y|^s}\right)\frac{dxdy}{|x-y|^N}\right)-\int_\Omega F(x,u)dx. \tag{3.3}$$

Let us denote by $I_i : W_0^s L_\Phi(\Omega) \longrightarrow \mathbb{R}$, i = 1, 2, the functionals

$$I_1(u) = \widehat{M}\left(\int_\Omega\int_\Omega \Phi\left(\frac{|u(x)-u(y)|}{|x-y|^s}\right)\frac{dxdy}{|x-y|^N}\right) \text{ and } I_2(u) = \int_\Omega F(x,u)dx.$$

Remark 3.2. We note that the functional $I : W_0^s L_\Phi(\Omega) \longrightarrow \mathbb{R}$ in (3.3) is well defined. Indeed, if $u \in W_0^s L_\Phi(\Omega)$, then we have from condition (f_3) that $u \in L_G(\Omega)$ and thus $u \in L^1(\Omega)$. Hence, it follows from the condition (f_1) that

$$|F(x,u)| \leqslant \int_0^u |f(x,t)|dt = c_1(|u| + G(|u|)).$$

Thus

$$\int_\Omega |F(x,u)|dx < \infty.$$

3.1. **Auxiliary results.**

Lemma 3.1. If f satisfies assumption (f_1) . Then $I_2 : L_G(\Omega) \longrightarrow \mathbb{R}$ is of class C^1 with Frechet derivative given by

$$\langle I_2'(u), v\rangle = \int_\Omega f(x,u(x))v(x)dx \tag{3.4}$$

for all $u, v \in W_0^s L_\Phi(\Omega)$.

Proof. First, from the assumption of (f_1) and the embedding theorem, we have that I_2 is well-defined on $W_0^s L_\Phi(\Omega)$. Usual arguments shown that I_2 is Gâteaux-differentiable on $W_0^s L_\Phi(\Omega)$ with the derivative given by (3.4). Actually, let $\{u_n\} \subset W_0^s L_\Phi(\Omega)$ be a sequence converging strongly to $u \in W_0^s L_\Phi(\Omega)$. Since $W_0^s L_\Phi(\Omega)$ is embedded in $L_G(\Omega)$, then $\{u_n\}$ converges strongly to u in $L_G(\Omega)$. So, there exists a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$, and a function $\overline{u} \in L_G(\Omega)$ such that $\{u_n\}$ converges to u almost everywhere in Ω and $|u_n| \leqslant |\overline{u}|$ for all $n \in \mathbb{N}$ and almost everywhere in Ω. Since f satisfies the assumption (f_1), we have that, for all measurable functions $u : \Omega \longrightarrow \mathbb{R}$, the operator defined by $u \longmapsto f(.,u(.))$ maps $L_G(\Omega)$ into $L_{\overline{G}}(\Omega)$. Fixing $v \in W_0^s L_\Phi(\Omega)$ with $||v|| \leqslant 1$, we use the Hölder inequality and the embedding of $W_0^s L_\Phi(\Omega)$ into $L_G(\Omega)$, we have

$$\begin{aligned} |\langle I_2'(u_n) - I_2'(u), v\rangle| &= \left|\int_\Omega (f(x,u_n(x)) - f(x,u(x)))\, v(x)dx\right|, \\ &\leqslant ||f(x,u_n(x)) - f(x,u(x))||_{\overline{G}} ||v||_G, \\ &\leqslant c_2 ||f(x,u_n(x)) - f(x,u(x))||_{\overline{G}} ||v||, \end{aligned}$$

for some $c_2 > 0$. Thus, passing to the supremum for $||v|| \leqslant 1$, we get

$$||I_2'(u_n) - I_2'(u)||_* \leqslant ||f(.,u_n(.)) - f(.,u(.))||_{\overline{G}}.$$

By (f_1), we deduce

$$f(x,u_n(x)) - f(x,u(x)) \longrightarrow 0 \text{ as } n \to \infty$$

and

$$|f(x,u_n(x)) - f(x,u(x))| \leqslant c_1(2 + g(|\overline{u}(x)|) + g(|u(x)|)$$

for almost everywhere $x \in \Omega$. Note that the majorant function in the previous relation is in $L_{\overline{G}}(\Omega)$. Hence, by applying the dominate convergence theorem, we get that $||f(x,u_n(x)) - f(x,u(x))||_{\overline{G}} \to 0$ as $n \to \infty$. This proves that I_2' is continuous. □

Lemma 3.2. The functional $I_1 : W_0^s L_\Phi(\Omega) \longrightarrow \mathbb{R}$ is of class C^1 and

$$\langle I_1'(u), v\rangle = M(\Psi(u)) \int_\Omega \int_\Omega a\left(\frac{|u(x)-u(y)|}{|x-y|^s}\right) \frac{u(x)-u(y)}{|x-y|^s} \frac{v(x)-v(y)}{|x-y|^s} \frac{dxdy}{|x-y|^N},$$

for all $u, v \in W_0^s L_\Phi(\Omega)$.

Proof. First, it is easy to see that

$$\langle I_1'(u), v\rangle = M(\Psi(u)) \int_\Omega \int_\Omega a\left(\frac{|u(x)-u(y)|}{|x-y|^s}\right) \frac{u(x)-u(y)}{|x-y|^s} \frac{v(x)-v(y)}{|x-y|^s} \frac{dxdy}{|x-y|^N}, \tag{3.5}$$

for all $u, v \in W_0^s L_\Phi(\Omega)$. It follows from (3.5) that $I_1'(u) \in (W_0^s L_\Phi(\Omega))^*$ for each $u \in W_0^s L_\Phi(\Omega)$.

Next, we prove that $I_1 \in C^1(W_0^s L_\Phi(\Omega), \mathbb{R})$. Let $\{u_n\} \subset W_0^s L_\Phi(\Omega)$ with $u_n \longrightarrow u$ strongly in $W_0^s L_\Phi(\Omega)$. Then $D^s u_n \longrightarrow D^s u$ in $L_\Phi(\Omega\times\Omega, d\mu)$, where

$$d\mu := |x-y|^{-N} dxdy$$

is a regular Borel measure on the set $\Omega\times\Omega$. So, by dominated convergence theorem, there exists a subsequence $\{D^s u_{n_k}\}$ and a function h in $L_\Phi(\Omega\times\Omega, d\mu)$ such that

$$a(|D^s u_{n_k}|)D^s u_{n_k} \longrightarrow a(|D^s u|)D^s u \text{ a.e. } (x,y) \in \Omega\times\Omega.$$

Using Lemma 2.1, we have

$$|a(|D^s u_{n_k}|)D^s u_{n_k}| \leqslant |a(|h|)h| \in L_{\overline{\Phi}}(\Omega\times\Omega, d\mu) \text{ a.e in } \Omega\times\Omega.$$

So, for $w \in W^s L_\Phi(\Omega)$ we have $D^s w \in L_\Phi(\Omega\times\Omega, d\mu)$. In view of the Hölder's inequality, we have

$$\begin{aligned} &\left|\int_\Omega \int_\Omega (a(|D^s u_{n_k}|)D^s u_{n_k} - a(|D^s u|)D^s u) D^s w d\mu\right| \\ &\leqslant 2\left[a(|D^s u_{n_k}|)D^s u_{n_k} - a(|D^s u|)D^s u\right]_{s,\overline{\Phi}} [w]_{s,\Phi} \\ &\leqslant 2\left[a(|D^s u_{n_k}|)D^s u_{n_k} - a(|D^s u|)D^s u\right]_{s,\overline{\Phi}} ||w||. \end{aligned}$$

It follows from the dominated convergence theorem that

$$\sup_{||w||\leqslant 1} \left|\int_\Omega \int_\Omega (a(|D^s u_{n_k}|)D^s u_{n_k} - a(|D^s u|)D^s u) D^s w d\mu\right| \longrightarrow 0. \tag{3.6}$$

Then, we have

$$||\Psi(u_n) - \Psi(u)||_* = \sup_{||v||\leqslant 1} |\langle \Psi(u_n) - \Psi(u), v\rangle| \longrightarrow 0.$$

On the other hand, the continuity of M implies that

$$M(\Psi(u_n)) \longrightarrow M(\Psi(u)). \tag{3.7}$$

Combining (3.7) with the Hölder inequality, we have

$$||I_1'(u_n) - I_1'(u)||_* = \sup_{||v||\leqslant 1} |\langle I_1'(u_n) - I_1'(u), v\rangle| \longrightarrow 0.$$

The proof of Lemma 3.2 is completed. □

Next, we show an important lemma. That is, if the functional I of (3.3) satisfies the conclusion of Theorem 2.4, then it has a critical point.

Lemma 3.3. Let (f_1)-(f_3) hold true. Let I be the functional defined in (3.3), and let $\{u_n\}$ be a sequence in $W_0^sL_\Phi(\Omega)$ such that

$$(i) \quad I(u_n) \longrightarrow c_3 > 0\,, \qquad ||I'(u_n)||_* \longrightarrow 0.$$

Then there exists $u \in W_0^sL_\Phi(\Omega)$ such that

$$I(u) = c_3, \quad I'(u) = 0.$$

Proof. It follows from (i) that there exists $c_4 > 0$ such that $|I(u_n)| \leqslant c_4$ and $|\langle I'(u_n), u_n\rangle| \leqslant c_4||u_n||$. By assumption (2.3) and (f_1)-(f_3), we have

$$0 < t\varphi(t) \leqslant \varphi^+\Phi(t) \text{ for all } t > 0, \tag{3.8}$$

$$0 < tg(t) \leqslant q^+G(t) \text{ for all } t > 0 \tag{3.9}$$

and

$$\begin{aligned}\left|\int_{\Omega\cap\{|u_n|\leqslant r\}}(F(x,u_n) - \mu^{-1}f(x,u_n)u_n)dx\right| &\leqslant c_1\left[(1+\mu^{-1})r + (1+\mu^{-1}q^+)G(r)\right] \\ &\leqslant c_5.\end{aligned} \tag{3.10}$$

Thus, by (M_1), (M_2), and (3.8)-(3.10), we get

$$\begin{aligned} c_4 + c_4||u_n|| &\geqslant I(u_n) - \frac{1}{\mu}\langle I'(u_n), u_n\rangle \\ &\geqslant \widehat{M}(\Psi(u_n)) - \frac{1}{\mu}M(\Psi(u_n))\int_\Omega\int_\Omega \varphi(h(u_n))h(u_n)\frac{dxdy}{|x-y|^N} \\ &\quad - \left|\int_{\Omega\cap\{|u_n|\leqslant r\}}(F(x,u_n) - \mu^{-1}f(x,u_n)u_n)dx\right| \\ &\geqslant \widehat{M}(\Psi(u_n)) - \frac{\varphi^+}{\mu}M(\Psi(u_n))\Psi(u_n) - c_5 \\ &\geqslant \left(1 - \frac{\varphi^+}{\theta\mu}\right)\widehat{M}(\Psi(u_n)) - c_5 \\ &\geqslant m_0\left(1 - \frac{\varphi^+}{\theta\mu}\right)||u_n||^{\varphi^\mp} - c_5. \end{aligned} \tag{3.11}$$

Hence, $\{u_n\}$ is bounded in $W_0^sL_\Phi(\Omega)$. Since $W_0^sL_\Phi(\Omega)$ is a reflexive space, we may assume that u_n converges weakly to u in $W_0^sL_\Phi(\Omega)$. Further, since the embedding of $W_0^sL_\Phi(\Omega)$ into $L_G(\Omega)$ is compact, we obtain that $u_n \longrightarrow u$ in $L_G(\Omega)$. It follows from Lemma 3.1 that $\lim\limits_{n\to\infty} I_2(u_n) = I_2(u)$ and $\lim\limits_{n\to\infty} I_2'(u_n) = I_2'(u)$ in $(W_0^sL_\Phi(\Omega))^*$, and as $I'(u_n) \to 0$ in $(W_0^sL_\Phi(\Omega))^*$, we have

$$I_1'(u_n) \longrightarrow I_2'(u) \text{ in } (W_0^sL_\Phi(\Omega))^* \tag{3.12}$$

since Ψ is a convex function and $\widehat{M}$ is a convex non-decreasing function. So, I_1 is convex and then

$$I_1(u_n) \leqslant I_1(u) + \langle I_1'(u_n), u_n - u\rangle.$$

Using (3.12), we deduce that

$$\limsup_{n\to\infty} I_1(u_n) \leqslant I_1(u).$$

It further follows from the convexity of I_1 that it is weakly lower semicontinuous. Hence

$$\liminf_{n\to\infty} I_1(u_n) \geqslant I_1(u),$$

which implies that

$$\lim_{n\to\infty} I_1(u_n) = I_1(u)$$

and

$$\lim_{n\to\infty} I(u_n) = I(u).$$

We finally show that $I'(u) = 0$. The convexity of I_1 implies that I_1' is monotone. Hence

$$\left\langle I_1'(u_n), u_n - v\right\rangle \geqslant \left\langle I_1'(v), u_n - v\right\rangle, \text{ for all } v \in W_0^s L_\Phi(\Omega).$$

From (3.12), we have

$$\left\langle I_2'(u) - I_1'(v), u - v\right\rangle \geqslant 0 \text{ for all } v \in W_0^s L_\Phi(\Omega).$$

Setting $v = u - th$, $h \in W_0^s L_\Phi(\Omega)$, $t \in \mathbb{R}^+$, we get

$$\left\langle I_2'(u) - I_1'(u - th), h\right\rangle \geqslant 0$$

for all $h \in W_0^s L_\Phi(\Omega)$. Letting $t \to 0$, and using the fact that h is arbitrary in $W_0^s L_\Phi(\Omega)$, we find that

$$I'(u) = I_1'(u) - I_2'(u) = 0.$$

Therefore, u is a critical point of I. □

3.2. **On the geometry of the functional** I**.** In this subsection, we will show that, under the conditions we have imposed on the functions a and f, the geometric conditions (G_1) and (G_2) of Theorem 2.4 will hold.

Lemma 3.4. Under the assumptions of Theorem 3.1, the geometric condition (G_1) of the mountain pass Theorem 2.4 hold for the functional I defined in (3.3).

Proof. For all $u \in W_0^s L_\Phi(\Omega) \setminus \{0\}$, the functional I is satisfied:

$$\begin{aligned} I(u) &= \widehat{M}(\Psi(u)) - \int_\Omega F(x,u)dx \\ &\geqslant m_0 \Psi(u) - \int_\Omega F(x,u)dx \\ &= m_0 \Psi(u) \left[1 - \frac{\int_\Omega F(x,u)dx}{m_0 \Psi(u)}\right]. \end{aligned} \tag{3.13}$$

Using condition (f_5), we have that there exist $\varepsilon \in (0,1)$ and $t_0 > 0$ such that

$$F(x,t) \leqslant \frac{1-\varepsilon}{\lambda_1} \Phi(t) \text{ for all } |t| \leqslant t_0 \text{ and all } x \in \overline{\Omega}.$$

We pose $\Omega_0 := \{x \in \Omega : |u(x)| \geqslant t_0\}$ to have

$$\int_\Omega F(x,u(x))dx \leqslant \frac{1-\varepsilon}{\lambda_1} \int_{\Omega\setminus\Omega_0} \Phi(u(x))dx + \int_{\Omega_0} F(x,u(x))dx. \tag{3.14}$$

By (2.15), we have

$$\frac{(1-\varepsilon)\int_{\Omega\setminus\Omega_0} \Phi(|u(x)|)dx}{\lambda_1 \Psi(u)} \leqslant 1 - \varepsilon. \tag{3.15}$$

Next, from (f_1), we have

$$F(x,t) \leqslant c_1(|t| + G(|t|)),$$

for all $|t| \geqslant t_0$ and for a.e. $x \in \Omega$. Then

$$\begin{aligned}\int_{\Omega_0} F(x,u)dx &\leqslant c_1 \left(||u||_{L^1} + \int_{\Omega} G(|u|)dx \right) \\ &\leqslant c_1 \left(||u||_{L^1} + ||u||_G^{q^-} + ||u||_G^{q^+} \right).\end{aligned}$$

From the embedding $W_0^s L_\Phi(\Omega) \hookrightarrow L_G(\Omega)$ and $W_0^s L_\Phi(\Omega) \hookrightarrow L^1(\Omega)$, we have

$$\int_{\Omega_0} F(x,u(x))dx \leqslant c_6 c_1(||u|| + ||u||^{q^-} + ||u||^{q^+}). \tag{3.16}$$

Then, for $||u|| \leqslant 1$,

$$\int_{\Omega_0} F(x,u(x))dx \leqslant 3c_6 c_1 ||u||. \tag{3.17}$$

By Proposition 2.1, we have

$$\frac{\int_{\Omega_0} F(x,u)dx}{m_0 \Psi(u)} \leqslant \frac{3c_6 c_1}{m_0} ||u||^{1-\varphi^+}. \tag{3.18}$$

Now, using (3.13), (3.14), (3.15) and (3.18), we obtain that

$$\begin{aligned}I(u) &\geqslant m_0 \Psi(u) \left(\varepsilon - \frac{3c_6 c_1}{m_0} ||u||^{1-\varphi^+} \right) \\ &\geqslant \frac{\varepsilon}{2} \Psi(u)\end{aligned}$$

whenever

$$\rho \leqslant \min \left\{ 1, \left(\frac{m_0 \varepsilon}{6c_6 c_1} \right)^{\frac{1}{1-\varphi^+}} \right\}. \tag{3.19}$$

Finally, by Proposition 2.1, we get

$$||u|| \longrightarrow 0 \Longleftrightarrow \Psi(u) \longrightarrow 0.$$

Hence, for $\rho > 0$ as given in (3.19), we see that there exists a $\alpha = \alpha(\rho) > 0$ such that, for all u with $||u|| = \rho$,

$$\Psi(u) \geqslant \alpha.$$

It follows that

$$I(u) \geqslant \alpha \frac{\varepsilon}{2}.$$

Setting $r = \alpha \frac{\varepsilon}{2}$, we obtain that (G_1) is satisfied. □

Lemma 3.5. Under the assumptions of Theorem 3.1, the geometric condition (G_2) of the mountain pass Theorem 2.4 hold for the functional I defined in (3.3).

Proof. First, by assumption (M_2), we get that

$$\widehat{M}(t) \leqslant \widehat{M}(1)t^{\frac{1}{\theta}} \tag{3.20}$$

for any $t \geqslant 1$. From (f_4), it follows that

$$F(x,\xi) \geqslant r^{-\mu} \min\{F(x,r),F(x,-r)\}|\xi|^{\mu} \tag{3.21}$$

for all $|\xi| > r$ and a.e. $x \in \Omega$. Thus by (3.21) and $F(x,\xi) \leqslant \max\limits_{|\xi|\leqslant r} F(x,\xi)$ for all $|\xi| \leqslant r$, we obtain

$$F(x,\xi) \geqslant r^{-\mu} \min\{F(x,r),F(x,-r)\}|\xi|^{\mu} - \max_{|\xi|\leqslant r} F(x,\xi) - \min\{F(x,r),F(x,-r)\} \tag{3.22}$$

for any $\xi \in \mathbb{R}$ and a.e. $x \in \Omega$. From Theorem 2.3, we can fix $u_0 \in C_0^{\infty}(\Omega)$ such that $||u_0|| = 1$ and let $t \geqslant 1$. Combining (3.20) with (3.22), we have

$$\begin{aligned} I(tu_0) &= \widehat{M}(\Psi(tu_0)) - \int_\Omega F(x,tu_0)dx \\ &\leqslant \widehat{M}(||tu_0||^{\varphi^+}) - \int_\Omega F(x,tu_0)dx \\ &\leqslant \widehat{M}(1)t^{\frac{\varphi^+}{\theta}} - r^{-\mu}|t|^{\mu} \int_\Omega \min\{F(x,r),F(x,-r)\}|u_0(x)|^{\mu}dx \\ &\quad + \int_\Omega \max_{|\xi|\leqslant r} F(x,\xi) + \min\{F(x,r),F(x,-r)\}dx. \end{aligned}$$

From assumptions (f_1) and (f_5), we get that $0 < F(x,\xi) \leqslant c_1(|r| + G(|r|))$ for $|\xi| \leqslant r$ a.e. $x \in \Omega$. Thus, $0 < \min\{F(x,r),F(x,-r)\} < c_1(|r| + G(|r|))$, a.e. $x \in \Omega$. Observe that $\mu > \dfrac{\varphi^+}{\theta}$. Using assumption (f_4) and passing to the limit as $t \to \infty$, we obtain that $I(tu_0) \to -\infty$. Thus, the assertion follows by taking $e = Tu_0$ with T sufficiently large. □

3.3. **Proof of Theorem 3.1.** It follows from Lemma 3.4 and Lemma 3.5 that the hypotheses of Theorem 2.4 are satisfied. So, Lemma 3.3 implies the existence of a nontrivial critical point of the functional I, which is a weak solution to problem (P_a).

4. The example

We present in this section an example of functions that satisfies the conditions of Theorem 3.1. Take

$$M(t) = a + bt^{\alpha-1}, \tag{4.1}$$

$$\varphi(t) = \log(1+|t|)|t|^{p-2}t, \tag{4.2}$$

$$f(x,t) = f(t) = |t|^{\delta-1}t, \tag{4.3}$$

where $p \in (1, N-1)$, $\alpha \geqslant 1$ and $\delta + 1 > \alpha(p+1)$. We considerer the problem

$$(P_{log}) \quad \begin{cases} \left(a + b(\Psi(u))^{\alpha-1}\right)(-\Delta)^s_{\log} u = |u|^{\delta-1}u & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where

$$(-\Delta)^s_{\log} u = 2\,\text{p.v} \int_{\mathbb{R}^N} \log(1+|D^s u|)|D^s u|^{p-2} D^s u \frac{dy}{|x-y|^{N+s}}.$$

So, from (4.1) – (4.3), we have

$$\Phi(t)=\frac{1}{p}\log(1+|t|)|t|^p-\frac{1}{p}\int_0^{|t|}\frac{\tau^p}{1+\tau}d\tau, \tag{4.4}$$

$$\widehat{M}(t)=at+\frac{b}{\alpha}t^{\alpha}, \tag{4.5}$$

$$F(x,t)=F(t)=\frac{|t|^{\delta+1}}{\delta+1}. \tag{4.6}$$

We will next show that all the hypotheses of Theorem 3.1 are satisfied.

• First, we verify that (2.3) holds. By [32, Example 2], we find that

$$\varphi^+=p+1 \quad \text{and} \quad \varphi^-=p. \tag{4.7}$$

Then (2.3) hold true.

• On the other hand, it is easy to see that

$$M(t)=a+bt^{\alpha-1}\geqslant a>0 \text{ for all } t\geqslant 0$$

and

$$\widehat{M}(t)=\int_0^t M(\tau)d\tau\geqslant\frac{1}{\alpha}M(t)t \text{ for all } t\geqslant 0.$$

So, for $m_0=a$ and $\theta=\dfrac{1}{\alpha}$, we find that (M_1) and (M_2) holds true.

• By L' Hôpital' s rule, we have

$$\begin{aligned}\lim_{t\to 0}\frac{\Phi(t)}{t^{p+1}}&=\lim_{t\to 0}\frac{\varphi(t)}{(p+1)t^p}\\&=\frac{1}{p+1}\lim_{t\to 0}\frac{\log(1+t)}{t}\\&=\frac{1}{p+1}\lim_{t\to 0}\frac{1}{1+t}\\&=\frac{1}{p+1}.\end{aligned}$$

We deduce that Φ is equivalent to t^{p+1} near zero. Using that fact and the remarks in [27, p.248], we infer that the condition (2.11) holds true if and only if

$$\int_0^1\frac{\tau^{\frac{1}{p+1}}}{\tau^{\frac{s+N}{N}}}d\tau<\infty,$$

or

$$s(p+1)<N. \tag{4.8}$$

The last inequality holds since $p<N-1$. On the other hand, by the change of variable $\tau=\Phi(t)$, we have

$$\int_1^t\frac{\Phi^{-1}(\tau)}{\tau^{\frac{N+s}{N}}}d\tau=\int_{\Phi^{-1}(1)}^{\Phi^{-1}(t)}\frac{t\varphi(t)}{\Phi(t)}(\Phi(t))^{-s/N}dt. \tag{4.9}$$

A simple calculation yields

$$0\leqslant\lim_{t\to\infty}\frac{\displaystyle\int_0^t\frac{\tau^p}{1+\tau}d\tau}{\log(1+t)t^p}\leqslant\lim_{t\to\infty}\frac{\displaystyle\int_0^t\frac{\tau^p}{\tau}d\tau}{\log(1+t)t^p}\leqslant\lim_{t\to\infty}\frac{\frac{1}{p}t^p}{\log(1+t)t^p}=0.$$

Thus

$$\lim_{t\to\infty}\frac{\displaystyle\int_0^t \frac{\tau^p}{1+\tau}d\tau}{\log(1+t)t^p}=0. \tag{4.10}$$

A first consequence of the above relation is that

$$\lim_{t\to\infty}\frac{\Phi(t)}{\log(1+t)t^p}=\frac{1}{p}. \tag{4.11}$$

On the other hand, by (4.10), we have

$$\lim_{t\to\infty}\frac{t\varphi(t)}{\Phi(t)}=\lim_{t\to\infty}p\left(1-\frac{\displaystyle\int_0^t \frac{\tau^p}{1+\tau}d\tau}{\log(1+t)t^p}\right)^{-1}=p \tag{4.12}$$

and

$$\lim_{t\to\infty}\Phi(t)=\lim_{t\to\infty}\frac{1}{p}\log(1+t)t^p\left[1-\frac{\displaystyle\int_0^{|t|} \frac{\tau^p}{1+\tau}d\tau}{\log(1+t)|t|^p}\right]=\infty. \tag{4.13}$$

Relations (4.9), (4.12) and (4.13) yield

$$\int_1^\infty \frac{\Phi^{-1}(\tau)}{\tau^{\frac{N+s}{N}}}d\tau=\infty.$$

Equivalently, we can write

$$\int_{\Phi^{-1}(1)}^\infty \frac{d\tau}{[\Phi(\tau)]^{s/N}}=\infty$$

or, by (4.11),

$$\int_{\Phi^{-1}(1)}^\infty \frac{d\tau}{[\log(1+\tau)]^{s/N}\,\tau^{sp/N}}=\infty. \tag{4.14}$$

Since

$$\log(1+x)\leqslant x,\ \text{for all } x>0,$$

we deduce that,

$$\frac{1}{[\log(1+\tau)]^{s/N}\,\tau^{sp/N}}\geqslant\frac{1}{\tau^{s(p+1)/N}}.$$

Since $s(p+1)<N$, we find

$$\int_{\Phi^{-1}(1)}^\infty \frac{d\tau}{\tau^{s(p+1)/N}}=\infty,$$

which conclude that

$$\int_1^\infty \frac{\Phi^{-1}(\tau)}{\tau^{\frac{N+s}{N}}}d\tau=\infty.$$

So, (2.12) is satisfied.

• We next check that, for any given $\delta>0$, conditions (f_1) and (f_4) are satisfied. Indeed, (f_1) is trivially satisfied with $c_1=1$ and $g(t)=|t|^{\delta-1}t$. Also, since $\dfrac{tf(t)}{F(t)}=\delta+1>\alpha(p+1)=\dfrac{\varphi^+}{\theta}$, (f_4) is satisfied. On the other hand, it is easy to see that (f_3) is satisfied.

• By Adams [27], we have

$$\lim_{t\to\infty}\frac{t^{1+\delta}}{\Phi_*(kt)}=0 \text{ for all } k>0$$

if and only if

$$\lim_{t\to\infty}\frac{\Phi_*^{-1}(t)}{t^{\frac{1}{1+\delta}}}=0. \tag{4.15}$$

Using L' Hôpital' s rule, we deduce that

$$\limsup_{t\to\infty}\frac{\Phi_*^{-1}(t)}{t^{\frac{1}{1+\delta}}}\leqslant(\delta+1)\limsup_{t\to\infty}\frac{\Phi^{-1}(t)}{t^{\frac{1}{\delta+1}+\frac{s}{N}}}.$$

Setting $\tau=\Phi^{-1}(t)$, we obtain

$$\limsup_{t\to\infty}\frac{\Phi_*^{-1}(kt)}{t^{\frac{1}{1+\delta}}}\leqslant(\delta+1)\limsup_{\tau\to\infty}\frac{\tau}{\Phi(\tau)^{\frac{1}{\delta+1}+\frac{s}{N}}}.$$

Now, since

$$\lim_{t\to\infty}\frac{t^{\frac{N(\delta+1)}{N+s(\delta+1)}}}{\Phi(t)}=\lim_{t\to\infty}\frac{t^{\frac{N(\delta+1)}{N+s(\delta+1)}}}{\log(1+t)t^p}\left(1-\frac{\int_0^t\frac{\tau^p}{1+\tau}d\tau}{\log(1+t)t^p}\right)^{-1}, \tag{4.16}$$

by using (4.10) and (4.16), we see that condition (f_3) is satisfied if

$$p-\frac{N(\delta+1)}{N+s(\delta+1)}=\frac{Np-(\delta+1)(N-ps)}{N+s(\delta+1)}\geqslant 0,$$

which is equivalent to

$$\delta+1\leqslant\frac{Np}{N-ps}. \tag{4.17}$$

• Now we verify that (f_5) holds. By (4.10), We have that

$$\lim_{t\to0}\frac{F(t)}{\Phi(t)}=\frac{p}{\delta+1}\lim_{t\to0}\frac{t^{\delta+1}}{\log(1+t)t^p}\left(1-\frac{\int_0^t\frac{\tau^p}{1+\tau}d\tau}{\log(1+t)t^p}\right)^{-1}.$$

So, $\lim\limits_{t\to0}\dfrac{F(t)}{\Phi(t)}=0$ if and only if $p<\delta$.

In conclusion, if

$$1<p<N-1 \text{ and } \alpha(p+1)<\delta+1\leqslant\frac{Np}{N-ps},$$

we find from Theorem 3.1 that problem (P_{log}) has a nontrivial nonnegative weak solution $u\in W_0^sL_\Phi(\Omega)$.

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