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DYNAMICS IN A TWO SPECIES LOTKA-VOLTERRA COOPERATIVE SYSTEM WITH THE CROWLEY-MARTIN FUNCTIONAL RESPONSE

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Abstract. The dynamical behaviors of a non-autonomous two species Lotka-Volterra cooperative system with the crowley-martin functional response are studied via the comparison method and the construction of Lyapunov functions. First, the permanence and the existence of positive periodic solutions are discussed. Second, the global attractivity of the system is investigated. Finally, the theoretical results are confirmed by a special example and the numerical simulations.

Keywords. Crowley-Martin functional response; Global attractivity; Lotka-Volterra cooperative system; Permanence; Periodic solution.

1. Introduction and Preliminaries

Recently, the population dynamical systems ere extensively studied [1]-[16]. Especially, the population dynamical systems with functional responses have been extensively studied. However, most of the studies on the population dynamical predator-prey systems are based on the functional response function (the ratio dependent function) to describe the predator predation rate and conversion rate [1]-[10]. For example, in [6], Tona and Hieu considered the following predator-prey model, which has one prey and two predators with Beddington-DeAngelis

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functional responses

$$
\begin{aligned}
\dot{x}_1(t) = & \quad x_1(t)[a_1(t) - b_1(t)x_1(t)] - \frac{c_2(t)x_1(t)x_2(t)}{\alpha(t)+\beta(t)x_1(t)+\gamma(t)x_2(t)} \\
& - \frac{c_3(t)x_1(t)x_3(t)}{\alpha(t)+\beta(t)x_1(t)+\gamma(t)x_3(t)}, \\
\dot{x}_2(t) = & \quad x_2(t)\big[-a_2(t) + \frac{d_2(t)x_1(t)}{\alpha(t)+\beta(t)x_1(t)+\gamma(t)x_2(t)} - b_2(t)x_3(t)\big], \\
\dot{x}_3(t) = & \quad x_3(t)\big[-a_3(t) + \frac{d_3(t)x_1(t)}{\alpha(t)+\beta(t)x_1(t)+\gamma(t)x_3(t)} - b_3(t)x_2(t)\big],
\end{aligned}
\tag{1.1}
$$

where $x_1(t)$ is the density of the prey population at time t, $x_2(t)$ and $x_3(t)$ are the densities of two predators at time t. They assumed that there are two types of competition between the two predators. The sufficient conditions for permanence, extinction, the existence of positive periodic solution, and the global asymptotic stability were established for system (1.1). In [9], Halik and Muhammadhaji considered the following two species autonomous predator-prey system with Crowley-Martin functional response

$$
\begin{aligned}
&\dot{x}(t) = x(t)\big[1 - x(t) - \frac{c(t)y(t)}{1+a_1(t)x(t)+b_1(t)y(t)+c_1(t)x(t)y(t)}\big], \\
&\dot{y}(t) = y(t)\big[-d(t) - e(t)y(t) + \frac{f(t)x(t)}{1+a_1(t)x(t)+b_1(t)y(t)+c_1(t)x(t)y(t)}\big].
\end{aligned}
\tag{1.2}
$$

where $x(t)$ is the density of the prey population at time t, and $y(t)$ is the density of predators at time t. By means of the comparison method and Lyapunov function method, they obtained some sufficient conditions on the boundedness, permanence, extinction, the existence, and global attractivity of positive periodic solution for system 1.2.

Recently, some population dynamical cooperative systems with functional response were discussed; see, e.g., [11]-[16] and the references there. For example, in [15], Abulimiti et al. considered the following non-autonomous three species Lotka-Volterra cooperative system with functional responses and delays

$$
\begin{aligned}
\dot{y}_1(t) = & \quad y_1(t)\big[r_1(t) - a_{11}(t)y_1(t) - \frac{b_1(t)y_1(t-\tau)}{a_{12}(t)y_2(t-\tau)+a_{13}(t)y_3(t-\tau)+c_1(t)}\big], \\
\dot{y}_2(t) = & \quad y_2(t)\big[r_2(t) - a_{22}(t)y_2(t) - \frac{b_2(t)y_2(t-\tau)}{a_{21}(t)y_1(t-\tau)+a_{23}(t)y_3(t-\tau)+c_2(t)}\big], \\
\dot{y}_3(t) = & \quad y_3(t)\big[r_3(t) - a_{33}(t)y_3(t) - \frac{b_3(t)y_3(t-\tau)}{a_{31}(t)y_1(t-\tau)+a_{32}(t)y_2(t-\tau)+c_3(t)}\big].
\end{aligned}
\tag{1.3}
$$

where $y_i(t)(i=1,2,3)$ represent the density of three cooperative species $y_i(i=1,2,3)$ at time t, respectively. They obtained some sufficient conditions on the permanence, the existence, and the global attractivity of positive periodic solution for system (1.3) by means of the comparison method and Lyapunov function method.

On the other hand, to the best of our knowledge, no study has been conducted to date for the dynamics of the non-autonomous two species Lotka-Volterra cooperative system with crowley-martin functional response. Therefore, based on the above models and analysis, in this paper, we consider the following non-autonomous two species Lotka-Volterra cooperative system with crowley-martin functional response

$$\begin{aligned}\dot{x}_1(t) &= x_1(t)\big[r_1(t) - a_{11}(t)x_1(t) + \frac{f_1(t)x_2(t)}{1+a(t)x_1(t)+b(t)x_2(t)+c(t)x_1(t)x_2(t)}\big],\\ \dot{x}_2(t) &= x_2(t)\big[r_2(t) - a_{22}(t)x_2(t) + \frac{f_2(t)x_1(t)}{1+a(t)x_1(t)+b(t)x_2(t)+c(t)x_1(t)x_2(t)}\big].\end{aligned} \tag{1.4}$$

Our main purpose is to establish some new sufficient conditions on the permanence, the existence of positive periodic solutions, and the global attractivity of system (1.4). In system (1.4), $x_i(t)$ $(i=1,2)$ represents the density of two species x_i $(i=1,2)$ at time t, respectively, $r_i(t)$ $(i=1,2)$ represents the intrinsic growth rate of two species x_i $(i=1,2)$ at time t, respectively, $a_{ii}(t)$ $(i=1,2)$ represents the intrapatch restriction density of two species x_i $(i=1,2)$ at time t, respectively, $f_i(t)$ $(i=1,2)$ represents the cooperative coefficients between two species x_i $(i=1,2)$ at time t, respectively, and $\frac{f_1(t)x_2(t)}{1+a(t)x_1(t)+b(t)x_2(t)+c(t)x_1(t)x_2(t)}$ and $\frac{f_2(t)x_1(t)}{1+a(t)x_1(t)+b(t)x_2(t)+c(t)x_1(t)x_2(t)}$ describe the cooperative relations between species x_1 and x_2.

In this paper, the initial conditions for system (1.4) take the following form

$$x_1(0) > 0, \quad x_2(0) > 0. \tag{1.5}$$

For system (1.4), we introduce the following hypotheses.

(H_1) $r_i(t), a_{ii}(t), f_i(t) (i=1,2)$ and $a(t), b(t), c(t)$ are continuous, bounded and strictly positive functions on $[0,+\infty)$.

(H_2) $r_i(t), a_{ii}(t), f_i(t) (i=1,2)$ and $a(t), b(t), c(t)$ are all continuously positive ω-periodic functions on $[0,\omega]$.

Throughout this paper, for any continuous function $f(t)$, we denote

$$f^L = \inf_{t\in[0,+\infty)} f(t), \; f^M = \sup_{t\in[0,+\infty)} f(t).$$

Next, we present two useful lemmas.

Lemma 1.1. [16] *Consider the following equation $\dot{u}(t) = u(t)(d_1 - d_2 u(t))$, where $d_2 > 0$.*
(1) If $d_1 > 0$, then $\lim_{t\to+\infty} u(t) = d_1/d_2$.
(2) If $d_1 < 0$, then $\lim_{t\to+\infty} u(t) = 0$.

Lemma 1.2. [17] *Let f be a nonnegative function defined on $[0,\infty)$ such that f is integrable on $[0,\infty)$ and uniformly continuous on $[0,\infty)$. Then $\lim_{t\to\infty} f(t) = 0$.*

2. MAIN RESULTS

Theorem 2.1. *If (H_1) holds, then system (1.4) is permanent.*

Proof. Let $x(t) = (x_1(t), x_2(t))$ be a any positive solution of the system (1.4) with the initial condition (1.5). For $t \geq 0$, we have

$$\dot{x}_1(t) \leq x_1(t)[r_1^M + \frac{f_1^M}{b^L} - a_{11}^L x_1(t)],$$

and

$$\dot{x}_2(t) \leq x_2(t)[r_2^M + \frac{f_2^M}{a^L} - a_{22}^L x_2(t)].$$

In view of Lemma 1.1 and the comparison theorem, we see that there exists a constant $T_0 > 0$ such that

$$x_1(t) \leq \frac{r_1^M + \frac{f_1^M}{b^L}}{a_{11}^L} = M_1, \quad \text{and} \quad x_2(t) \leq \frac{r_2^M + \frac{f_2^M}{a^L}}{a_{22}^L} = M_2, \quad \text{as } t > T_0.$$

By using system (1.4) again, for $t \geq 0$, we have

$$\dot{x}_i(t) \geq x_i(t)[r_i(t) - a_{ii}(t)x_i(t)], \quad i = 1,2.$$

It follows Lemma 1.1 that there exists a constant $T_1 > T_0$ such that

$$x_i(t) \geq \frac{r_i^L}{a_{ii}^M} = m_i, \quad \text{as } t > T_1, \ i = 1,2.$$

This completes the proof of this theorem. □

As a direct result of [16, Lemma 2.3], we have the following result from Theorem 2.1.

Corollary 2.2. *If (H_2) holds, then system (1.4) is permanent and admits at least one positive $\omega-$periodic solution.*

Now, we will obtain the sufficient conditions for the global attractivity of system (1.4). For the sake of convenience, we denote

$$A_1(t) = \frac{f_1(t)}{G_1(t)G_2(t)}, \ A_2(t) = \frac{f_1(t)a(t)x_2(t)}{G_1(t)G_2(t)},$$
$$A_3(t) = \frac{f_1(t)a(t)x_1(t)}{G_1(t)G_2(t)}, \ A_4(t) = \frac{f_1(t)c(t)y_2(t)x_2(t)}{G_1(t)G_2(t)},$$
$$B_1(t) = \frac{f_2(t)}{G_1(t)G_2(t)}, \ B_2(t) = \frac{f_2(t)b(t)x_1(t)}{G_1(t)G_2(t)},$$
$$B_3(t) = \frac{f_2(t)b(t)x_2(t)}{G_1(t)G_2(t)}, \ B_4(t) = \frac{f_2(t)c(t)y_1(t)x_1(t)}{G_1(t)G_2(t)},$$
$$A = a_{11}^L - A_2^M - A_4^M - B_1^M - B_3^M, \ B = a_{22}^L - A_1^M - A_3^M - B_2^M - B_4^M,$$

where

$$G_1(t) = \big(1 + a(t)x_1(t) + b(t)x_2(t) + c(t)x_1(t)x_2(t)\big),$$
$$G_2(t) = \big(1 + a(t)y_1(t) + b(t)y_2(t) + c(t)y_1(t)y_2(t)\big),$$
$$A_1 = \frac{f_1^M}{G} \geq A_1(t), \ A_2 = \frac{f_1^M a^M M_2}{G} \geq A_2(t),$$
$$A_3 = \frac{f_1^M a^M M_1}{G} \geq A_3(t), \ A_4 = \frac{f_1^M c^M M_2^2}{G} \geq A_4(t),$$
$$B_1 = \frac{f_2^M}{G} \geq B_1(t), \ B_2 = \frac{f_2^M b^M M_1}{G} \geq B_2(t),$$
$$B_3 = \frac{f_2^M b^M M_2}{G} \geq B_3(t), \ B_4 = \frac{f_2^M c^M M_1^2}{G} \geq B_4(t),$$

and

$$G = \big(1 + a^L m_1 + b^L m_2 + c^L m_1 m_2\big)^2.$$

Theorem 2.3. *If H_1 holds and $C > 0$, then system (1.4) is globally attractive, where $C = \min\{A, B\}$.*

Proof. Suppose that $(x_1(t), x_2(t))$ and $(y_1(t), y_2(t))$ are any two positive solutions of system (1.4). From Theorem 2.1, we have that there exist positive constants $M_i, m_i (i = 1, 2)$ and T such that $m_i \le y_i(t), x_i(t) \le M_i (i = 1, 2)$ for all $t \ge T$. First, we let

$$V(t) = |\ln x_1(t) - \ln y_1(t)| + |\ln x_2(t) - \ln y_2(t)|.$$

Calculating the upper right derivative of $V(t)$ along system (1.4), we have

$$\begin{aligned}
D^+V(t) \le\ & \operatorname{sign}(x_1(t) - y_1(t))\Big[-a_{11}^L(x_1(t) - y_1(t)) + f_1(t)\Big(\frac{x_2(t)}{G_1(t)} - \frac{y_2(t)}{G_2(t)}\Big)\Big] \\
& + \operatorname{sign}(x_2(t) - y_2(t))\Big[-a_{22}^L(x_2(t) - y_2(t)) + f_2(t)\Big(\frac{x_1(t)}{G_1(t)} - \frac{y_1(t)}{G_2(t)}\Big)\Big] \\
=\ & \operatorname{sign}(x_1(t) - y_1(t))\big[-a_{11}^L(x_1(t) - y_1(t)) - A_1(t)(x_2(t) - y_2(t)) \\
& + A_2(t)(x_1(t) - y_1(t)) - A_3(t)(x_2(t) - y_2(t)) + A_4(t)(x_1(t) - y_1(t))\big] \\
=\ & \operatorname{sign}(x_1(t) - y_1(t))\big[-a_{11}^L(x_1(t) - y_1(t)) - A_1(t)(x_2(t) - y_2(t)) \\
& + \operatorname{sign}(x_2(t) - y_2(t))\big[-a_{22}^L(x_2(t) - y_2(t)) \\
& - B_1(t)(x_1(t) - y_1(t)) + B_2(t)(x_2(t) - y_2(t)) - B_3(t)(x_1(t) - y_1(t)) \\
=\ & \operatorname{sign}(x_1(t) - y_1(t))\big[-a_{11}^L(x_1(t) - y_1(t)) - A_1(t)(x_2(t) - y_2(t)) \\
& + B_4(t)(x_2(t) - y_2(t))\Big] \\
\le\ & -(a_{11}^L - A_2 - A_4 - B_1 - B_3)|x_1(t) - y_1(t)| \\
=\ & \operatorname{sign}(x_1(t) - y_1(t))\big[-a_{11}^L(x_1(t) - y_1(t)) - A_1(t)(x_2(t) - y_2(t)) \\
& - (a_{22}^L - A_1 - A_3 - B_2 - B_4)|x_2(t) - y_2(t)| \\
=\ & -A|x_1(t) - y_1(t)| - B|x_2(t) - y_2(t)|. \\
\le\ & -\sum_{i=1}^{2} C|x_i(t) - y_i(t)|.
\end{aligned} \tag{2.1}$$

Integrating from T to t on both sides of (2.1) produces

$$V(t) + C\int_T^t \Big(\sum_{i=1}^{2} |x_i(s) - y_i(s)|\Big)ds \le V(T) < +\infty.$$

Hence, $V(t)$ is bounded on $[T, +\infty)$ and

$$\int_T^t \Big(\sum_{i=1}^{2} |x_i(s) - y_i(s)|\Big)ds \le \frac{V(T)}{C} < +\infty.$$

It follows that

$$\sum_{i=1}^{2} |x_i(t) - y_i(t)| \in L^1(T, +\infty). \tag{2.2}$$

From the permanence of system (1.4) and (2.2), we can obtain that the derivatives $\dot{x}_i(t)$ and $\dot{y}_i(t)$ are bounded. Furthermore, we can obtain that $\sum_{i=1}^{2}(x_i(t) - y_i(t))$ and their derivatives remain bounded on $[T, \infty)$. As a consequence, $\sum_{i=1}^{2} |x_i(t) - y_i(t)|$ is uniformly continuous on $[T, \infty)$. From Lemma 1.2 (Barbalat's lemma), it follows that

$$\lim_{t\to\infty} |x_i(t) - y_i(t)| = 0, (i = 1, 2).$$

This completes the proof of this theorem. □

Corollary 2.4. *If the conditions of Corollary 2.2 hold and $C > 0$, then system (1.4) has a positive $\omega-$ periodic solution, which is globally attractive.*

3. Numerical Simulation

As an example, in this section, we consider the following Lotka-Volterra two species cooperative system with Crowley-Martin functional responses to illustrate the results obtained in this paper.

$$\begin{aligned}
&\dot{x}_1(t) \\
&= x_1(t)\big[1.85 + 0.25\cos(t) - (3.65 + 0.35\cos(t))x_1(t) \\
&\quad - \frac{(0.15 + 0.1\cos(t))x_2(t)}{1 + (2.8 + 0.15\cos(t))x_1(t) + (2.75 + 0.18\cos(t))x_2(t) + (2.5 + 0.25\cos(t))x_1(t)x_2(t)}\big], \\
&\dot{x}_2(t) \\
&= x_2(t)\big[1.75 + 0.25\cos(t) - (3.55 + 0.3\cos(t))x_2(t) \\
&\quad - \frac{(0.13 + 0.07\cos(t))x_1(t)}{1 + (2.8 + 0.15\cos(t))x_1(t) + (2.75 + 0.18\cos(t))x_2(t) + (2.5 + 0.25\cos(t))x_1(t)x_2(t)}\big].
\end{aligned} \tag{3.1}$$

We further have

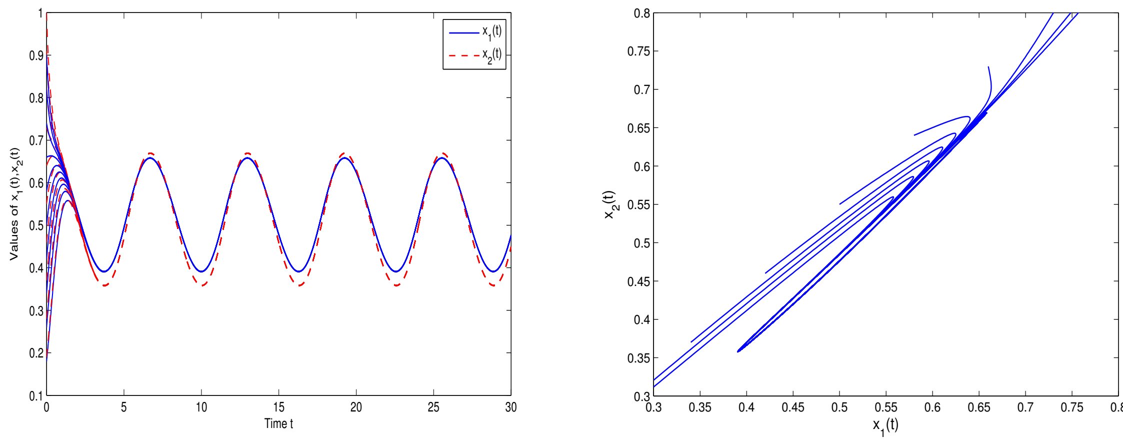
$$M_1 \approx 0.6658,\ M_2 \approx 0.6386,\ m_1 \approx 0.4000,\ m_2 \approx 0.3896,\ G \approx 11.6414,$$

$$A_1 \approx 0.0215,\ A_2 \approx 0.0405,\ A_3 \approx 0.0422,\ A_4 \approx 0.0241,\ B_1 \approx 0.0172,$$

and

$$B_2 \approx 0.0335,\ B_3 \approx 0.0321,\ B_4 \approx 0.0209,\ A \approx 3.1861,\ B \approx 3.1319,\ C = 3.1319.$$

It is easy to show that system (3.1) satisfies the conditions of Theorem 2.1, Theorem 2.3, Corollary 2.2 and Corollary 2.4. Hence, system (3.1) is permanent, globally attractive, and has a globally attractive positive periodic solution. The permanence, periodic solution, and the globally attractive of system (3.1) are shown in the following figure.



The numerical solutions of system (3.1) with different initial conditions.

From the figure, we find that system (3.1) is permanent and has a globally attractive positive periodic solution.

4. The Conclusion

In this paper, a class of non-autonomous system (1.4) was concerned. By using of the comparison method and construction of Lyapunov functions, some sufficient conditions on the permanence, the existence of positive periodic solutions, and the global attractivity of system (1.4) were obtained. Finally, one suitable example illustrated the feasibility of the main results in this paper.

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