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CHAOTIC PROPERTIES OF A CLASS OF COUPLED MAPPING LATTICES IN NON-AUTONOMOUS DISCRETE DYNAMICAL SYSTEMS

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Abstract. This paper discusses the chaotic characteristics of coupled mapping lattices (CMLs). Unlike the discussions of CMLs in autonomous discrete dynamical systems, this paper considers the non-autonomous case. The relationship between the original map sequence and the coupled system in $(\mathscr{F}_1,\mathscr{F}_2)$-chaos (or sensitivity, spatio-temporal chaos, densely chaos, transitivity, accessibility, exact, Ruelle-Takens chaos, and Kato chaos) is studied.
Keywords. Chaos; Coupled map lattices; Non-autonomous discrete dynamical systems; Sensitivity.

1. INTRODUCTION

In 1983, Kaneko [1] proposed coupled mapping lattices (shortly, CMLs). Since then, CMLs have been studied in physics, communication, and image processing; see, e.g., [2–6]. Using coupled mapping lattice models, one can imitate Rayleigh-Denard convective structure, observe the dynamic behavior of the spatiotemporal circuit system, and recover the initial value of the signal, realize image coding and compression, etc. In particular, CMLs have been widely used in chaotic encryption algorithms; see, e.g., [7–10]. The study of the chaotic properties of CMLs can provide theoretical support to various applications. In 2013, Lu, Zhu, and Wu [11] showed the dense-chaos, Spatio-temporal chaos, sensitivity, and Li-Yorke sensitivity of a class of CMLs in autonomous systems. In 2015, Liu, Lu, and Li [12] obtained the Li-Yorke chaoticity, distributional chaoticity, and ω-chaoticity of the above CMLs. In 2017, Lu and Li [13] demonstrated a class of CMLs in the autonomous system, which is $(\mathscr{F}_1,\mathscr{F}_2)$-chaotic, ω-chaotic, and topologically chaotic. In 2020, for the stronger form of transitivity and weak mixing, Li and Zhao [14] gave a sufficient and necessary condition for the CML with zero coupled constants. It deserves mentioning that there are many results on CMLs in autonomous systems, however, the research

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on CMLs in non-autonomous systems is still limited. In this paper, we focuses on the chaotic properties of CMLs in non-autonomous systems.

Denote by $\mathbb{I}$ the subinterval of $\mathbb{R}$ with compactness. Consider a sequence continuous mappings $f_n : \mathbb{I} \mapsto \mathbb{I}, n \in \mathbb{N}$. Let $f_{1,\infty} = (f_1, f_2, \ldots)$. $(\mathbb{I}, f_{1,\infty})$ is called a non-autonomous discrete system. For any $a \in \mathbb{I}$, the orbit of a under $f_{1,\infty}$ is

$$Orb(a, f_{1,\infty}) = \{a, f_1(a), f_2 \circ f_1(a), \ldots, f_1^n(a), \ldots\},$$

where $f_1^n = f_n \circ \ldots \circ f_1$, and f_1^0 is the identity mapping.

In this paper, the following CML is considered

$$x_{m+1,n} = (1-\varepsilon) f_{m+1}(x_{m,n}) + \frac{1}{2}\varepsilon[f_{m+1}(x_{m,n-1}) + f_{m+1}(x_{m,n+1})], \tag{1.1}$$

where $x_{m,n} \in \mathbb{I}$, $m \in \mathbb{N}_0 = \{0,1,2,\ldots\}, n \in \mathbb{Z} = \{\ldots,-1,0,1,\ldots\}$, $\mathbb{I}$ is a non-degenerate compact interval, and $\varepsilon \in [0,1]$ is a constant.

For $t \in \mathbb{Z}$, let $\mathbb{N}_t = \{t, t+1, \ldots\}$ and $\Omega = \{(0,n) : n \in \mathbb{Z}\} = \{\ldots,(0,-1),(0,0),(0,1),\ldots\}$. For any sequence $\phi = \{\phi_{0,n}\}_{\infty}^{\infty}$ on Ω, by induction, one can obtain a double-indexed sequence $x = \{x_{m,n} : m = 0,1,2,\ldots; n = \ldots,-1,0,1,\ldots\}$, which is said to be a solution of system (1.1) with initial condition ϕ. Write

$$\mathbb{I}_{\infty}^{\infty} = \{\{a_n\}_{n=-\infty}^{\infty} = (\ldots, a_{-1}, a_0, a_1, \ldots) : a_n \in \mathbb{I}, n \in \mathbb{Z}\}$$

and

$$\triangle_{\infty}^{\infty} = \{(\ldots, a_{-1}, a_0, a_1, \ldots) : a_i = a_j \in \mathbb{I}, i, j \in \mathbb{Z}\},$$

which is called the diagonal set of $\mathbb{I}_{\infty}^{\infty}$. For arbitrary two sequences $x_1 = \{x_{1,n}\}_{n=-\infty}^{\infty}, x_2 = \{x_{2,n}\}_{n=-\infty}^{\infty} \in \mathbb{I}_{\infty}^{\infty}$, it is easy to prove that

$$d(x_1, x_2) = \sum_{n=-\infty}^{\infty} \frac{|x_{1,n} - x_{2,n}|}{2^{|n|}} \tag{1.2}$$

is a metric on $\mathbb{I}_{\infty}^{\infty}$.

Let $x = \{x_{m,n} : m \in \mathbb{N}_0, n \in \mathbb{Z}\}$ be a solution of system (1.1) with initial condition $\phi = \{\phi_{0,n}\}_{\infty}^{\infty} \in \mathbb{I}_{\infty}^{\infty}$. Let

$$x_m = \{x_{m,n}\}_{n=-\infty}^{\infty} = (\ldots, x_{m,-1}, x_{m,0}, x_{m,1}, \ldots), \quad \forall m \in \mathbb{N}_0,$$

and let

$$x_{m+1} = \{x_{m+1,n}\}_{n=-\infty}^{\infty} = (\ldots, x_{m+1,-1}, x_{m+1,0}, x_{m+1,1}, \ldots) = F_1^{m+1}(x_m), \quad \forall m \in \mathbb{N}_0,$$

where

$$x_0 = \phi = \{x_{0,n} = \phi_{0,n}\}_{n=-\infty}^{\infty}$$

and

$$x_{m+1,n} = (1-\varepsilon) f_{m+1}(x_{m,n}) + \frac{1}{2}\varepsilon[f_{m+1}(x_{m,n-1}) + f_{m+1}(x_{m,n+1})], \forall m \in \mathbb{N}_0, n \in \mathbb{Z}.$$

Then, one can see that system (1.1) is equivalent to the following system

$$x_{m+1} = F_{m+1}(x_m), x_m \in \mathbb{I}_{\infty}^{\infty}, \quad \forall m \geq 0. \tag{1.3}$$

In system (1.3), $F_{m+1} (m = 0,1,2,\ldots)$ is said to be induced by system (1.1). Obviously, a double-indexed sequence $\{x_{m,n} : m \in \mathbb{N}_0, n \in \mathbb{Z}\}$ is a solution of system (1.1) if and only if

$\{x_m = \{x_{m,n}\}_{n=-\infty}^{\infty} : m \in \mathbb{N}_0\}_{m=0}^{\infty}$ is a solution of system (1.3). If mapping sequence $F_{1,\infty}$ is chaotic on $\mathbb{I}_{\infty}^{\infty}$, it is said that system (1.3) is chaotic on $\mathbb{I}_{\infty}^{\infty}$.

2. Preliminaries

In 1975, for the first time, Li and Yorke [15] expressed the chaos mathematically. Subsequently, Schweizer and Smital [16] gave an extended form of Li-Yorke chaos, named distributional chaos. In 2009, Tan and Xiong [17] described the chaos with Furstenberg families and defined $(\mathscr{F}_1, \mathscr{F}_2)$-chaos, which makes that Li-Yorke chaos and distributional chaos are the special cases of $(\mathscr{F}_1, \mathscr{F}_2)$-chaos. Subsequently, the definitions of Li-Yorke chaos, distributional chaos, and $(\mathscr{F}_1, \mathscr{F}_2)$-chaos in the non-autonomous discrete systems are given.

Definition 2.1. The sequence mapping $f_{1,\infty}$ is said to be Li-Yorke chaotic if there is an uncountable set $S \subset X$ such that, for any $a, b \in S$ with $a \neq b$,

$$\limsup_{n\mapsto\infty} d(f_1^n(a), f_1^n(b)) > 0 \quad and \quad \liminf_{n\mapsto\infty} d(f_1^n(a), f_1^n(b)) = 0,$$

where the uncountable set S is called the Li-Yorke scrambled set (or Li-Yorke irregular set) of $f_{1,\infty}$.

Definition 2.2. The sequence mapping $f_{1,\infty}$ is said to be distributional chaotic if there is an uncountable set $S \subset X$ such that, for any $a, b \in S$ with $a \neq b$,

(i) $\forall t > 0, F_{ab}^*(t, f_{1,\infty}) = \limsup\limits_{n\mapsto\infty} \frac{1}{n} card\{j \in \{1,2,...,n\} : d(f_1^n(a), f_1^n(b)) < t\} = 1;$

(ii) $\exists t_0 > 0, F_{ab}(t, f_{1,\infty}) = \liminf\limits_{n\mapsto\infty} \frac{1}{n} card\{j \in \{1,2,...,n\} : d(f_1^n(a), f_1^n(b)) < t\} = 0,$

where $F_{ab}^*(t, f_{1,\infty})$ and $F_{ab}(t, f_{1,\infty})$ are upper and lower distributional functions, respectively. The uncountable set S is called the distributional scrambled set (or the irregularly distributional set) of $f_{1,\infty}$.

Definition 2.3. Let $\mathscr{F}_1$ and $\mathscr{F}_2$ be Furstenberg families, and let $f_{1,\infty}$ be a sequence mapping. A subset $D \subset X$ is called a $(\mathscr{F}_1, \mathscr{F}_2)$-scrambled set of $f_{1,\infty}$ if, for any $a, b \in D$ with $a \neq b$, the following two conditions are satisfied:

(i) $\{n \in \mathbb{N} : d(f_1^n(a), f_1^n(b) < t\} \in \mathscr{F}_1 \quad for \quad any \quad t > 0;$

(ii) $\{n \in \mathbb{N} : d(f_1^n(a), f_1^n(b) > \delta\} \in \mathscr{F}_2 \quad for \quad some \quad \delta > 0.$

The pair (a, b), which satisfies the above two conditions, is called a $(\mathscr{F}_1, \mathscr{F}_2)$-scrambled pair of $f_{1,\infty}$.

After Li-Yorke chaos, various definitions of chaos appeared recently, for example, Devaney chaos, infinite sensitivity, transitivity, accessibility, densely chaotic, densely δ-chaotic, Li-Yorke sensitivity, Spatio-temporal chaos, Kato's chaotic, and Ruelle-Takens chaos. The definitions of those chaos in non-autonomous discrete systems are given as follows.

In general, the set of all Li-Yorke scrambled pairs of system $(X, f_{1,\infty})$ is denoted by $LY_d(f_{1,\infty})$, that is,

$$LY_d(f_{1,\infty}) = \{(a,b) \in X \times X : \limsup_{n\mapsto\infty} d(f_1^n(a), f_1^n(b)) > 0 \quad and \quad \liminf_{n\mapsto\infty} d(f_1^n(a), f_1^n(b)) = 0\}$$

and the set of all Li-Yorke-δ scrambled pairs of system $(X, f_{1,\infty})$ for some $\delta > 0$ is denoted by $LY_d(f_{1,\infty}, \delta)$, that is,

$$LY_d(f_{1,\infty}, \delta) = \{(a,b) \in X \times X : \limsup_{n\mapsto\infty} d(f_1^n(a), f_1^n(b)) > \delta \quad and \quad \liminf_{n\mapsto\infty} d(f_1^n(a), f_1^n(b)) = 0\}.$$

Definition 2.4. Let (X,d) be a compact metric space, and let $f_n : X \mapsto X$ be a sequence mapping.
(1) $f_{1,\infty}$ is said to be densely chaotic if $\overline{LY_d(f_{1,\infty})} = X \times X$;
(2) $f_{1,\infty}$ is said to be densely δ-chaotic if there exist $\delta > 0$ such that $\overline{LY_d(f_{1,\infty},\delta)} = X \times X$. Obviously, if $\delta = 0$, then the densely δ-chaotic is densely chaotic.

The following notations are needed in the sequel. For any $\varepsilon > 0$,

$$Prox(f_{1,\infty})(a) = \{b \in X : \liminf_{n\to\infty} d(f_1^n(a), f_1^n(b)) = 0\},$$

$$Asym_\varepsilon(f_{1,\infty})(a) = \{b \in X : \limsup_{n\to\infty} d(f_1^n(a), f_1^n(b)) < \varepsilon\},$$

$$Asym(f_{1,\infty})(a) = \cap_{\delta>0} Asym_\delta(f_{1,\infty}) = \{b \in X : \lim_{n\to\infty} d(f_1^n(a), f_1^n(b)) = 0\},$$

and

$$Q_\delta(f_{1,\infty}) = \{a \in X : \forall \varepsilon > 0, \exists b \in B(x,\varepsilon) \textit{ such that } \quad (a,b) \in LY_d(f_{1,\infty},\delta)\}.$$

Definition 2.5. Let (X,d) be a compact metric space, and let $f_n : X \mapsto X$ be a sequence mapping.
(1) $f_{1,\infty}$ is spatio-temporal chaotic if, for any $a \in X$ and $\delta > 0$,

$$B(a,\delta) \cap (Prox(f_{1,\infty})(a) \setminus Asym(f_{1,\infty})(a)) \neq \emptyset;$$

(2) $f_{1,\infty}$ is Li-Yorke sensitivity if there is a $\varepsilon > 0$ such that, for any $a \in X$ and $\delta > 0$,

$$B(a,\delta) \cap (Prox(f_{1,\infty})(a) \setminus Asym_\varepsilon(f_{1,\infty})(a)) \neq \emptyset;$$

(3) $f_{1,\infty}$ is densely Li-Yorke sensitive if $Q_\delta(f_{1,\infty})$ is dense in X for some $\delta > 0$;
(4) $f_{1,\infty}$ is sensitive if there exists an $\eta > 0$ such that, for any $a \in X$ and $\varepsilon > 0$, there exist a $b \in B(a,\varepsilon)$, and $n \in \mathbb{N}$ such that $\rho(f_1^n(a), f_1^n(b)) > \eta$;
(5) $f_{1,\infty}$ is infinitely sensitive if there exists an $\eta > 0$ such that, for any $a \in X$ and $\varepsilon > 0$, there exist $b \in B(a,\varepsilon)$ and $n \in \mathbb{N}$ such that $\limsup_{n\to\infty} \rho(f_1^n(a), f_1^n(b)) \geq \eta$;
(6) $f_{1,\infty}$ is transitive if, for any nonempty open subsets $U_1, U_2 \subset Y$, $f_1^n(U_1) \cap U_2 \neq \emptyset$ for some integer $n > 0$;
(7) $f_{1,\infty}$ is accessible if, for any $\varepsilon > 0$ and any two nonempty open subsets $U_1, U_2 \subset X$, there are two points $a \in U_1$ and $b \in U_2$ such that $\rho(f_1^n(a), f_1^n(b))) < \varepsilon$ for some integer $n > 0$;
(8) $f_{1,\infty}$ is exact if, for any open subset $U \subset X$, there is a $n \in \mathbb{N}$ such that $f_1^n(U) = X$.

Remark 2.6. Here we present another equivalent definition of the transitivity. $f_{1,\infty}$ is said to be transitivity if there is an $a \in X$ such that $\overline{Orb(a, f_{1,\infty})} = X$.

Definition 2.7. (1) A dynamic system $(X, f_{1,\infty})$ (or the sequence mapping $f_{1,\infty} : X \to X$) is said to be Kato chaotic if it is both sensitive and accessible;
(2) A dynamic system $(X, f_{1,\infty})$ (or $f_{1,\infty} : X \to X$) is said to be chaotic in the sense of Ruelle and Takens if it is both transitive and sensitive.

The definitions in the non-autonomous systems mentioned above are based on the corresponding definitions in the autonomous systems which are in references [16–24] and the references therein.

Proposition 2.8. *A dynamical system $(X, f_{1,\infty})$ is infinitely sensitive if and only if it is sensitive.*

Proof. Necessity is obvious. We next only give the sufficiency. It is similar to the proof of [18, Theorem 2.1]. If $(X, f_{1,\infty})$ is sensitive, then there exists an $\eta > 0$ such that, for any $a \in X$ and $\varepsilon > 0$, there exist a $b \in B(a, \varepsilon)$ and $n \in \mathbb{N}$ such that $\rho(f_1^n(a), f_1^n(b)) > \eta$. Given any $N \in \mathbb{N}$, set $\mathscr{D}_N = \{(a,b) : \rho(f_1^n(a), f_1^n(b)) \le \frac{\eta}{4}\}$. It is clear that $\mathscr{D}_N$ is a closed set.

Now, we assert that, for any $N \in \mathbb{N}$, $\text{int}\mathscr{D}_N = \varnothing$. In fact, if there exists some $N \in \mathbb{N}$ such that $\text{int}\mathscr{D}_N \ne \varnothing$, and there exist nonempty open sets $U, V \in X$ such that $U \times V \subset \mathscr{D}_N$, then, for any pair $(a,b) \in U \times V$, $\rho(f_1^n(a), f_1^n(b)) \le \frac{\eta}{4}$ holds for any $n > \mathbb{N}$. For all points $a_1, a_2 \in U$ and any $n > \mathbb{N}$,

$$\rho(f_1^n(a_1), f_1^n(a_2)) \le \rho(f_1^n(a_1), f_1^n(b)) + \rho(f_1^n(b), f_1^n(a_2)) \le \frac{\eta}{2}.$$

Note that there exists a nonempty open set $U^* \subset U$ such that, for any pair $a_1, a_2 \in U^*$ and any $0 \le m \le N$,

$$\rho(f_1^m(a_1), f_1^m(a_2)) \le \frac{\eta}{2}.$$

So, for all points $a_1, a_2 \in U^*$ and any $n \in \mathbb{N}$, $\rho(f_1^m(a_1), f_1^m(a_2)) \le \frac{\eta}{2}$, which contradicts the sensitivity of $(X, f_{1,\infty})$. It follows that set $\mathscr{D} = \cup_{N \in \mathbb{N}} \mathscr{D}_N$ is a first category set in $X \times X$. Then, set $(X \times X) \setminus \mathscr{D} = \{(a,b) : \forall N \in \mathbb{N}, \exists n > N \textit{ such that } \rho(f_1^n(a), f_1^n(b)) > \frac{\eta}{4}\}$ is residual in $X \times X$.

on the other hand, if $(X, f_{1,\infty})$ is not infinitely sensitive, then there exist an $a_0 \in X$ and $\xi > 0$ such that, for any $b \in B(a_0, \xi)$,

$$\limsup_{n \to \infty} \rho(f_1^n(a_0), f_1^n(b)) \le \frac{\eta}{16}.$$

Note the fact that $(X \times X) \setminus \mathscr{D}$ is residual in $X \times X$. It follows that there exist a pair $(b_1, b_2) \in [B(a_0, \xi) \times B(a_0, \xi)] \cap [(X \times X) \setminus \mathscr{D}]$. Then, for any $n \in \mathbb{N}$,

$$\rho(f_1^n(b_1), f_1^n(b_2)) \le \rho(f_1^n(b_1), f_1^n(a_0)) + \rho(f_1^n(a_0), f_1^n(b_2)) \le \frac{\eta}{8},$$

one has

$$\limsup_{n \to \infty} \rho(f_1^n(b_1), f_1^n(b_2)) \le \frac{\eta}{8},$$

which contradicts $(b_1, b_2) \in X \times X \setminus \mathscr{D}$. So, $(X, f_{1,\infty})$ is infinitely sensitive. □

3. Chaotic Properties of Coupled Map Lattice (1.1)

In this section, one always assumes that $X = \mathbb{I} = [0,1]$. The metric ρ in $\mathbb{I}$ is defined by $\rho(a,b) = \mid a - b \mid$, $\forall a, b \in \mathbb{I}$. The metric d in $\mathbb{I}_\infty^\infty$ is defined by (1.2), and $f_n : \mathbb{I} \mapsto \mathbb{I}$ is a sequence of mappings.

Theorem 3.1. *If $f_{1,\infty}$ is $(\mathscr{F}_1, \mathscr{F}_2)$-chaotic, then system $(\Delta_\infty^\infty, d, F_{1,\infty})$ is $(\mathscr{F}_1, \mathscr{F}_2)$-chaotic.*

Proof. Since $f_{1,\infty}$ is $(\mathscr{F}_1, \mathscr{F}_2)$-chaotic, then there exists an uncountable set $D \subset \mathbb{I}$ such that, for any $a \ne b \in D$,

(i) $\forall t > 0, \{n \in N : d(f_1^n(a), f_1^n(b)) < t\} \in \mathscr{F}_1$;

(ii) $\exists \delta > 0, \{n \in \mathbb{N} : d(f_1^n(a), f_1^n(b)) > \delta\} \in \mathscr{F}_2$.

Take

$$(\Delta^*)_\infty^\infty = \{(\dots, x_1, x_0, x_1, \dots) : x_n = a \in D, n \in \mathbb{Z}\} \subset \triangle_\infty^\infty \subset \mathbb{I}_\infty^\infty.$$

Clearly, $(\Delta^*)^\infty_\infty$ is uncountable. For any $x=\{...,a,a,a,...\}, y=\{...,b,b,b,...\}\in(\Delta^*)^\infty_\infty, x\neq y$, since $x_{0,n}=a(n\in\mathbb{Z})$, then

$$x_{1,n}=(1-\varepsilon)f_1(a)+\frac{1}{2}\varepsilon[f_1(a)+f_1(a)]=f_1(a)\quad(\forall n\in\mathbb{Z}),$$

that is,

$$F_1(x)=x_1=\{...,f_1(a),f_1(a),f_1(a),...\}.$$

Similarly,

$$F_2\circ F_1(x)=x_2=\{...,f_2\circ f_1(a),f_2\circ f_1(a),f_2\circ f_1(a),...\};$$

$$......$$

$$F_1^m(x)=x_m=\{...,f_1^m(a),f_1^m(a),f_1^m(a),...\}, m\in\mathbb{N}.$$

It is obvious that, for any $k\in\mathbb{N}$,

$$F_1^k(x)=\{f_1^k(a)\}_{n=-\infty}^{\infty},\qquad F_1^k(y)=\{f_1^k(b)\}_{n=-\infty}^{\infty}.$$

Thus, for any $k\in\mathbb{N}$,

$$\begin{aligned}d(F_1^k(x),F_1^k(y))&=d(\{f_1^k(a)\}_{n=-\infty}^{\infty},\{f_1^k(b)\}_{n=-\infty}^{\infty})\\&=\sum_{n=-\infty}^{\infty}\frac{\mid f_1^k(a)-f_1^k(b)\mid}{2^{|n|}}\\&=3\mid f_1^k(a)-f_1^k(b)\mid.\end{aligned}\tag{3.1}$$

According to (3.1), (i), and (ii), one has that, for any $t>0$,

$$\{n\in\mathbb{N}:d(F_1^n(x),F_1^n(y))<t\}=\{n\in\mathbb{N}:d(f_1^n(a),f_1^n(b))<\frac{t}{3}\}\in\mathscr{F}_1$$

and

$$\{n\in\mathbb{N}:d(F_1^n(x),F_1^n(y))>3\delta\}=\{n\in\mathbb{N}:d(f_1^n(a),f_1^n(b))>\delta\}\in\mathscr{F}_2.$$

This implies that $F_{1,\infty}$ is $(\mathscr{F}_1,\mathscr{F}_2)$-chaotic. □

The following corollaries can be obtained easily.

Corollary 3.2. *If $f_{1,\infty}$ is Li-Yorke chaotic, then system $(\Delta^\infty_\infty,d,F_{1,\infty})$ is Li-Yorke chaotic.*

Corollary 3.3. *If $f_{1,\infty}$ is distributional chaotic, then system $(\Delta^\infty_\infty,d,F_{1,\infty})$ is distributional chaotic.*

Next, we restrict the space to the diagonal Δ^∞_∞ of $\mathbb{I}^\infty_\infty$, and discuss some chaotic properties.

Theorem 3.4. *(1) If $f_{1,\infty}$ is Li-Yorke sensitive, then system $(\Delta^\infty_\infty,d,F_{1,\infty}\mid_{\Delta^\infty_\infty})$ is Li-Yorke sensitive;*

(2) If $f_{1,\infty}$ is densely Li-Yorke sensitive, then system $(\Delta^\infty_\infty,d,F_{1,\infty}\mid_{\Delta^\infty_\infty})$ is densely Li-Yorke sensitive;

(3) If $f_{1,\infty}$ is spatio-temporal chaotic, then system $(\Delta^\infty_\infty,d,F_{1,\infty}\mid_{\Delta^\infty_\infty})$ is spatio-temporal chaotic.

Proof. (1) Fix $x=(...,a,a,a,...)\in\triangle^\infty_\infty$. Since $f_{1,\infty}$ is Li-Yorke sensitive, there exist $\varepsilon>0$ and $\delta>0$ such that

$$B(a,\frac{\varepsilon}{3})\cap(Prox(f_{1,\infty})(a)\setminus Asym_\delta(f_{1,\infty})(a))\neq\varnothing.$$

Let $b\in B(a,\frac{\varepsilon}{3})\cap(Prox(f_{1,\infty})(a)\setminus Asym_\delta(f_{1,\infty})(a))$. Setting $y=(...,b,b,b,...)\in\Delta^\infty_\infty$, one has

$$d(x,y)=\sum_{n=-\infty}^{\infty}\frac{|a-b|}{2^{|n|}}=3|a-b|<\varepsilon.$$

So, $y \in B(x,\varepsilon)$,

$$\begin{aligned}\liminf_{n\mapsto\infty} d(F_1^n(x),F_1^n(y)) &= \liminf_{n\to\infty}\sum_{n=-\infty}^{\infty}\frac{|f_1^n(a)-f_1^n(b)|}{2^{|n|}}\\ &= 3\liminf_{n\mapsto\infty}|f_1^n(a)-f_1^n(b)|\\ &= 0,\end{aligned}$$

and

$$\limsup_{n\mapsto\infty}(d(F_1^n(x),F_1^n(y)) = 3\limsup_{n\mapsto\infty}|f_1^n(a)-f_1^n(b)| \geq 3\delta.$$

That is,

$$y \in B(x,\varepsilon)\cap(Prox(F_{1,\infty})(x)\backslash Asym_{3\delta}(F_{1,\infty})(x)) \neq \varnothing.$$

Hence, $F_{1,\infty}|_{\triangle_\infty^\infty}$ is Li-Yorke sensitive.

(2) Since $f_{1,\infty}$ is densely Li-Yorke sensitive, then, for any $a \in Q_\delta(f_{1,\infty})$ and any $\varepsilon > 0$, there exists $b \in B(a,\frac{\varepsilon}{3})$ such that $(a,b) \in LY_\rho(f_{1,\infty},\delta)$. Take $x^* = \{x_n = a\}_{n=-\infty}^{\infty}$ and $y^* = \{y_n = b\}_{n=-\infty}^{\infty}$. One has

$$\limsup_{n\to\infty} d(F_1^n(x^*),F_1^n(y^*)) = \limsup_{n\to\infty} d(f_1^n(a),f_1^n(b)) = 3\limsup_{n\to\infty} \mid f_1^n(a)-f_1^n(b)\mid > \delta$$

and

$$\liminf_{n\to\infty} d(F_1^n(x^*),F_1^n(y^*)) = \liminf_{n\to\infty} d(f_1^n(a),f_1^n(b)) = 3\liminf_{n\to\infty} \mid f_1^n(a)-f_1^n(b)\mid = 0.$$

Thus, there exists $x^* \in Q_\delta(F_{1,\infty})$. For any fixed $x \in \Delta_\infty^\infty$, let $x = (\ldots,x_{m,-1},x_{m,0},x_{m,1},\ldots)$, where $x_{m,p} = x_{m,p+1}, p \in \mathbb{Z}$. Since $f_{1,\infty}$ is densely Li-Yorke sensitive, then, for any $\varepsilon > 0$ and the above $x_{m,0}$, $B(x_{m,0},\frac{\varepsilon}{3})\cap Q_\delta(f_{1,\infty}) \neq \emptyset$. Taking $a \in B(x_{m,0},\frac{\varepsilon}{3})\cap Q_\delta(f_{1,\infty})$, one has

$$d(x,x^*) = \sum_{n=-\infty}^{\infty}\frac{|x_{m,p}-a|}{2^{|n|}} = 3|x_{m,p}-a| < \varepsilon.$$

So $x^* \in B(x,\varepsilon)$. This indicates that $\overline{Q_\delta(F_{1,\infty})} = \Delta_\infty^\infty$. Thus, system $(\Delta_\infty^\infty,d,F_{1,\infty}\mid_{\Delta_\infty^\infty})$ is densely Li-Yorke sensitive.

(3) The proof is similar to (1). $F_{1,\infty}|_{\triangle_\infty^\infty}$ is spatio-temporal chaotic. □

Theorem 3.5. *(1) If $f_{1,\infty}$ is densely δ-chaotic, then system $(\Delta_\infty^\infty,d,F_{1,\infty}\mid_{\Delta_\infty^\infty})$ is densely δ-chaotic;*
(2) If $f_{1,\infty}$ is densely chaotic, then system $(\Delta_\infty^\infty,d,F_{1,\infty}\mid_{\Delta_\infty^\infty})$ is densely chaotic.

Proof. (1) Fix $(x,y) \in \triangle_\infty^\infty\times\triangle_\infty^\infty$, and let $x = (\ldots,x_{m,-1},x_{m,0},x_{m,1},\ldots)$, $y = (\ldots,y_{n,-1},y_{n,0},y_{n,1},\ldots)$, where $x_{m,p} = x_{m,p+1}$ and $y_{n,q} = y_{n,q+1}, p,q \in \mathbb{Z}$. Since $f_{1,\infty} : \mathbb{I}\mapsto\mathbb{I}$ is densely δ-chaotic, then, for any $\varepsilon > 0$,

$$B((x_{m,0},y_{n,0}),\frac{\varepsilon}{3})\cap LY_d(f_{1,\infty},\delta) \neq \varnothing.$$

Take $(a,b) \in B((x_{m,0},y_{n,0}),\frac{\varepsilon}{3})\cap LY_d(f_{1,\infty},\delta)$. Letting $x^* = \{\ldots,a,a,a,\ldots\}$ and $y^* = \{\ldots,b,b,b\ldots\}$, one has

$$d(F_1^k(x^*),F_1^k(y^*)) = 3\mid f_1^k(a)-f_1^k(b)\mid < 3\delta,$$
$$(x^*,y^*) \in LY_d(F_{1,\infty},3\delta) \subset LY_d(F_{1,\infty},\delta),$$

and

$$d(x,x^*) = \sum_{p=-\infty}^{\infty}\frac{|x_{m,p}-a|}{2^{|p|}} = 3|x_{m,0}-a| < \varepsilon,$$

$$d(y,y^*) = \sum_{p=-\infty}^{\infty} \frac{|y_{n,p}-b|}{2^{|p|}} = 3|y_{n,0}-b| < \varepsilon.$$

By the arbitrariness of (x,y), one concludes that $\overline{LY_d(F_{1,\infty},\delta)} = \triangle_\infty^\infty \times \triangle_\infty^\infty$. So $(\Delta_\infty^\infty, d, F_{1,\infty}|_{\Delta_\infty^\infty})$ is densely δ-chaotic.

(2) Obviously, if $\delta = 0$, then the densely δ-chaotic is densely chaotic. □

Theorem 3.6. *(1) If $f_{1,\infty}$ is transitive, then system $(\Delta_\infty^\infty, d, F_{1,\infty}|_{\Delta_\infty^\infty})$ is transitive;*

(2) If $f_{1,\infty}$ is sensitivity, then system $(\Delta_\infty^\infty, d, F_{1,\infty}|_{\Delta_\infty^\infty})$ is sensitivity;

(3) If $f_{1,\infty}$ is accessible, then system $(\Delta_\infty^\infty, d, F_{1,\infty}|_{\Delta_\infty^\infty})$ is accessible;

(4) If $f_{1,\infty}$ is exact, then system $(\Delta_\infty^\infty, d, F_{1,\infty}|_{\Delta_\infty^\infty})$ is exact.

Proof. (1) Since $f_{1,\infty}$ is transitive, there exist an $a \in \mathbb{I}$ satisfying $\overline{Orb(a,f_{1,\infty})} = \mathbb{I}$. Then, for any $b \in \mathbb{I}$ and any $\varepsilon > 0$, $B(b,\varepsilon) \cap Orb(a,f_{1,\infty}) \neq \emptyset$. That is, there exists a $k_0 > 0$ such that $\rho(f_1^{k_0}(a),b) = |f_1^{k_0}(a)-b| < \frac{\varepsilon}{3}$. Take $x_0 = (...,a,a,a,...) \in \Delta_\infty^\infty$. Since $F_1^k(x_0) = \{f_1^k(a)\}_{n=-\infty}^\infty$ for any $k \in \mathbb{N}$, then $Orb(x_0,F_{1,\infty}) = \{\{f_1^k(a)\}_{n=-\infty}^\infty |_{k\in\mathbb{N}}\}$. For $y = (...,b,b,b,...) \in \triangle_\infty^\infty$ and above $k_0 > 0$,

$$d(F_1^{k_0}(x_0),y) = \sum_{n=-\infty}^{\infty} \frac{|f_1^{k_0}(a)-b|}{2^{|n|}} = 3|f_1^{k_0}(a)-b| < \varepsilon.$$

So, $B(y,\varepsilon) \cap Orb(x_0,F_{1,\infty}) \neq \emptyset$. By the arbitrariness of b, system $(\Delta_\infty^\infty, d, F_{1,\infty}|_{\Delta_\infty^\infty})$ is transitive.

(2) Since $f_{1,\infty}$ is sensitivity, then there exists a $\delta > 0$ such that, for any $a \in \mathbb{I}$ and $\varepsilon > 0$, there exist $b_{a,\varepsilon} \in B(a,\frac{\varepsilon}{3})$ and $n_{a,\varepsilon} \in \mathbb{N}$ such that $|f_1^{n_{a,\varepsilon}}(a) - f_1^{n_{a,\varepsilon}}(b_{a,\varepsilon})| > \delta$. So, for any fixed $x = (...,a,a,a,...) \in \triangle_\infty^\infty$ and any $\varepsilon > 0$, taking $y = (...,b_{a,\varepsilon},b_{a,\varepsilon},b_{a,\varepsilon},...) \in \triangle_\infty^\infty$, one has that

$$d(x,y) = \sum_{n=-\infty}^{\infty} \frac{|a-b_{a,\varepsilon}|}{2^{|n|}} = 3|a-b_{a,\varepsilon}| < \varepsilon,$$

that is, $y \in B(x,\varepsilon)$. In view of $d(F_1^{n_{a,\varepsilon}}(x),F_1^{n_{a,\varepsilon}}(y)) = 3|f_1^{n_{a,\varepsilon}}(a) - f_1^{n_{a,\varepsilon}}(b)| > 3\delta > \delta$, one concludes that $F_{1,\infty}|_{\triangle_\infty^\infty}$ is sensitivity.

(3) For any open subsets U_1 and U_2, let

$$(\Delta_1)_\infty^\infty = \{(...,x_{-1},x_0,x_1,...), x_n = a \in U_1 \subset \mathbb{I}, n \in \mathbb{Z}\} \subset \mathbb{I}_\infty^\infty$$

and

$$(\Delta_2)_\infty^\infty = \{(...,y_{-1},y_0,y_1,...), y_n = b \in U_2 \subset \mathbb{I}, n \in \mathbb{Z}\} \subset \mathbb{I}_\infty^\infty.$$

Since $f_{1,\infty}$ is accessible, then there exist $a \in U_1$ and $b \in U_2$ such that

$$\rho(f_1^k(a), f_1^k(b)) = | f_1^k(a) - f_1^k(b) | < \frac{\varepsilon}{3}$$

for some $k > 0$. Letting $x = (...,a,a,a,...) \in (\Delta_1)_\infty^\infty$ and $y = (...,b,b,b,...) \in (\Delta_2)_\infty^\infty$, one has $d(F_1^k(x),F_1^k(y)) = 3 \mid f_1^k(a) - f_1^k(b) \mid < \varepsilon$. So, system $(\Delta_\infty^\infty, d, F_{1,\infty}|_{\Delta_\infty^\infty})$ is accessible.

(4) Since $f_{1,\infty}$ is exact, for any open subset $D \subset \mathbb{I}$, there exists $m \in \mathbb{N}$ such that $f_1^m(D) = \mathbb{I}$, that is, for any $a \in D$, there exists $m > 0$ such that $B(f_1^m(a),\frac{\varepsilon}{3}) \cap \mathbb{I} \neq \emptyset$ for any $\varepsilon > 0$. Hence, there is a $b \in X$ such that $\rho(f_1^m(a),b) = \mid f_1^m(a) - b \mid < \frac{\varepsilon}{3}$. Let $(\Delta^*)_\infty^\infty$ be an arbitrary open subset of $(\triangle)_\infty^\infty$ and $x_0 = (...,a,a,a,...) \in (\Delta^*)_\infty^\infty$. Clearly, for any $k \in \mathbb{N}$, $F_1^k(x_0) = \{f_1^k(a)\}_{n=-\infty}^\infty$. For any $y_0 = (...,b,b,b,...) \in (\triangle)_\infty^\infty$, $d(F_1^m(x_0),y_o) = 3|f_1^m(a)-b| < \varepsilon$. That is, there exists an $m \in \mathbb{N}$, such that $F_1^m((\Delta^*)_\infty^\infty) = \triangle_\infty^\infty$. So, system $(\Delta_\infty^\infty, d, F_{1,\infty}|_{\Delta_\infty^\infty})$ is exact. □

Theorem 3.7. *If $f_{1,\infty}$ is chaotic in the sense of Ruelle and Takens, then system $(\Delta_\infty^\infty, d, F_{1,\infty}|_{\Delta_\infty^\infty})$ is chaotic in the sense of Ruelle and Takens.*

Proof. From Theorem 3.6 (1), Theorem 3.6 (2), and the definition of Ruelle-Takens chaos, the conclusion can be derived easily. □

Theorem 3.8. *If $f_{1,\infty}$ is Kato chaotic, system $(\Delta_\infty^\infty, d, F_{1,\infty}|_{\Delta_\infty^\infty})$ is Kato chaotic.*

Proof. From Theorem 3.6 (2), and Theorem 3.6 (3), one concludes the desired conclusion immediately. □

According to Proposition 2.8 and Theorem 3.6 (3), one obtains the following corollary immediately.

Corollary 3.9. *If system $(\triangle_\infty^\infty, d, F_{1,\infty}|_{\triangle_\infty^\infty})$ is infinitely sensitive, then it is sensitive.*

Indeed, we have a stronger conclusion than the corollary above.

Theorem 3.10. *If $f_{1,\infty}$ is infinitely sensitive, then system $(\triangle_\infty^\infty, d, F_{1,\infty}|_{\triangle_\infty^\infty})$ is infinitely sensitive.*

Proof. Since $f_{1,\infty}$ is infinitely sensitive, then there exists a $\delta > 0$ such that, for any $a \in \mathbb{I}$ and any ε, there exist $b_{a,\varepsilon} \in B(a, \frac{\varepsilon}{3})$ and $n_{a,\varepsilon} \in \mathbb{N}$ such that $\limsup_{n_{a,\varepsilon}\to\infty} \rho(f_1^{n_{a,\varepsilon}}(a), f_1^{n_{a,\varepsilon}}(b_{a,\varepsilon})) \geq \delta$. So, for any $x = (\ldots, a, a, a, \ldots) \in \triangle_\infty^\infty$, and any $\varepsilon > 0$, taking $x = (\ldots, b_{a,\varepsilon}, b_{a,\varepsilon}, b_{a,\varepsilon}, \ldots) \in \triangle_\infty^\infty$, one has that

$$d(x,y) = \sum_{n=-\infty}^{\infty} \frac{|f_1^{n_{a,\varepsilon}}(a) - f_1^{n_{a,\varepsilon}}(b_{a,\varepsilon})|}{2^{|n|}} = 3|f_1^{n_{a,\varepsilon}}(a) - f_1^{n_{a,\varepsilon}}(b_{a,\varepsilon})| < \varepsilon$$

that is, $y \in B(x, \varepsilon)$. In view of

$$\limsup_{n_{a,\varepsilon}\to\infty} d(F_1^{n_{a,\varepsilon}}(x), F_1^{n_{a,\varepsilon}}(y)) = \limsup_{n_{a,\varepsilon}\to\infty} d(f_1^{n_{a,\varepsilon}}(a), f_1^{n_{a,\varepsilon}}(b_{a,\varepsilon})) \geq \delta,$$

one concludes that $F_{1,\infty}|_{\Delta_\infty^\infty}$ is infinitely sensitive. □

Remark 3.11. From [18], one sees that, on $(\mathbb{I}_\infty^\infty)$ (or its subsystem), the mapping F induced by chaotic map f is chaotic or not, which is related to the measurement on $(\mathbb{I}_\infty^\infty, d)$. In fact, in the non-autonomous discrete system, the chaotic of $F_{1,\infty}$ induced by chaotic mapping $f_{1,\infty}$ is also related to the measurement on $(\mathbb{I}_\infty^\infty, d)$. This can be obtained from the following example. $d_1(x_1,x_2) = 0$, $x_1 = x_2$ and $d_1(x_1,x_2) = 1$, $x_1 \neq x_2$. It can be seen that the measurement d_1 and d defined by (1.2) are not equivalent, and system (1.1) does not chaotic under metric d_1.

4. Some Examples

Let $\mathbb{I}$ be a compact subinterval of $\mathbb{R}$. Let $\mathbb{I}_\infty^\infty$ be the metric space induced by $\mathbb{I}$, and let $\triangle_\infty^\infty$ be the diagonal set of $\mathbb{I}_\infty^\infty$. From system (1.1), one has the following inducted system

$$x_{m+1} = F_{m+1}(x_m), x_m = (\ldots, a_m, a_m, a_m, \ldots) \in \triangle_\infty^\infty, m \in \mathbb{N}^+, a_m \in \mathbb{I}. \quad (4.1)$$

It is easy to see that $a_{m+1} = f_{m+1}(a_m)$. Let

$$f(x) = 19saw(x) + sin(x(1-x)), x \in \mathbb{I},$$

where $saw(x)$ is the sawtooth function defined by

$$saw(x) = (-1)^m(x - 2m), 2m - 1 \leq x \leq 2m + 1, m \in \mathbb{Z}.$$

One can prove that map f satisfies the definitions of chaos in Section 2. The simulation with explanations of chaotic behavior is provided by Figure 1. The red dots and the green dots, respectively, represent the trajectory of initial value $x_1 = 0.8751$ and $x_2 = 0.8752$ iterations for 7000 times. It is clear that, after iterations, the trajectory of x_1 (or x_2) is ergodic and disorderly (see red dots or green dots). And with little difference between initial values x_1 and x_2, there is a big gap between the iteration values after 5999 iterations (see $f^n(x_1) = -12.2681$ and $f^n(x_2) = 13.7032$). This means that f is sensitive dependence on initial condition.

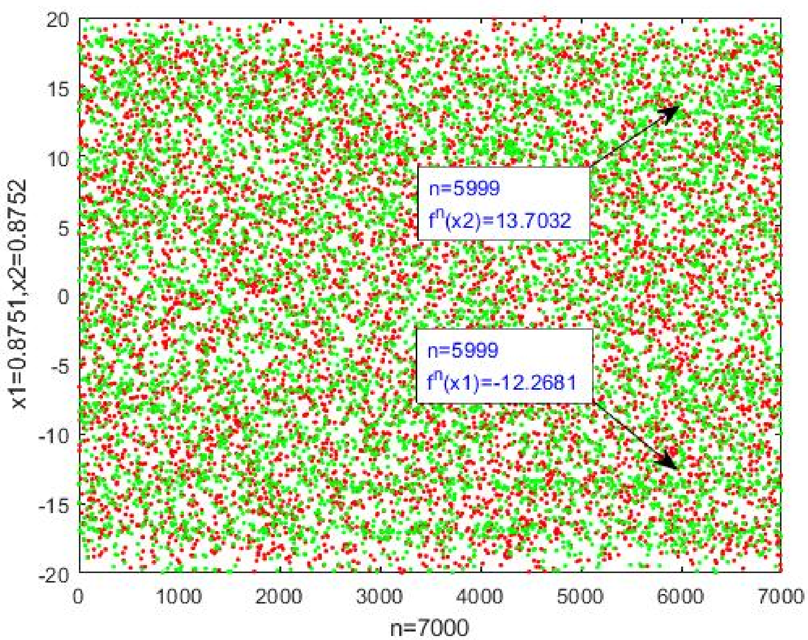


FIGURE 1. Chaotic behaviors of f with the initial data $x_1 = 0.8751, x_2 = 0.8752$, and $n = 7000$.

Now, let $f_m = f$, for any $m \in \mathbb{N}$. One has that

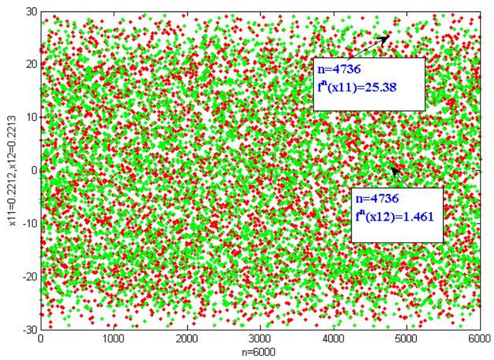
$$a_{m+1} = f(a_m) = 19saw(a_m) + sin(a_m(1 - a_m)),$$

for any $a_m \in \mathbb{I}$. Since map f is chaotic, $x_m = (\ldots, a_m, a_m, a_m, \ldots)$, and $x_{m+1} = (\ldots, a_{m+1}, a_{m+1}, a_{m+1}, \ldots)$, then coupled system (4.1) (or coupled map F_{m+1}) is chaotic too.

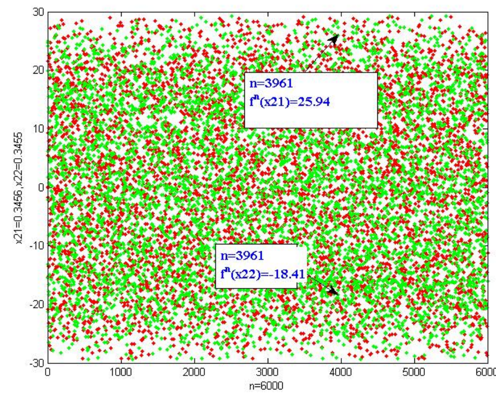
On the other hand, the results presented in Section 3 indicate that if the original mapping sequence $f_{1,\infty}$ is chaotic, then the coupled system induced by $f_{1,\infty}$ and coupling model (1.1) is chaotic. This is consistent with the above conclusion.

Remark 4.1. By appropriately changing the coefficients of the function $f(x)$ in the above examples, many examples that satisfy the above conclusions can be obtained. For example $f_1(x) = 7saw(x) + 18sin(x(1-x)), x \in \mathbb{I},\ f_2(x) = 5saw(x) + 24sin(x(1-x)), x \in \mathbb{I},\ f_3(x) = 10saw(x) + 20sin(x(1-x)), x \in \mathbb{I},,\ f_4(x) = 25saw(x) + \frac{9}{2}sin(x(1-x)), x \in \mathbb{I}$, and so on. For any given initial value, the simulation of these functions shows the ergodicity. We give the initial values of the four functions as $x_{11} = 0.2212, x_{12} = 0.2213$; $x_{21} = 0.3456, x_{22} = 0.3455$;

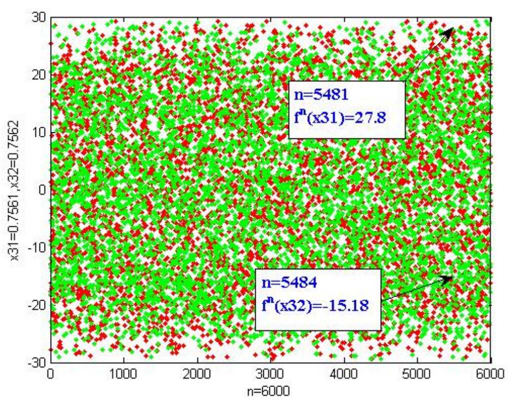
$x_{31} = 0.7561, x_{32} = 0.7562$; and $x_{41} = 0.8651, x_{42} = 0.8652$, respectively, the number of iterations is 6000 times. Then, the simulations are shown in Figure 2, Figure 3, Figure 4, and Figure 5, respectively.



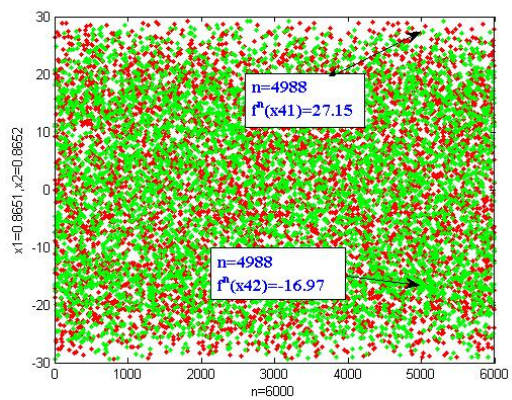
(a) Figure 2



(b) Figure 3



(c) Figure 4



(d) Figure 5

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