J. Nonlinear Funct. Anal. 2021 (2021), Article ID 40 https://doi.org/10.23952/jnfa.2021.40

Journal of Nonlinear Functional Analysis
Available online at http://jnfa.mathres.org

CONVERGENCE THEOREMS FOR THE SPLIT VARIATIONAL INCLUSION PROBLEM IN HILBERT SPACES

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Abstract. In this paper, we propose a new iterative algorithm, which is based on the linear search method to solve a split variational inclusion problem in two Hilbert spaces. Our iteration algorithm does not involve projection operators, and the step-sizes do not depend on the operator norm. A weak convergence result on the solutions of the split variational inclusion problem is obtained. The split minimization problem and split feasibility problem are considered as two applications of our main results.

Keywords. Fixed point; Split variational inclusion problem; Split minimization problem; Strong and weak convergence.

1. Introduction

As a generalization of the classical variational inequality problem, the following variational inclusion problem (in short, VIP), first introduced by Rockefeller [1] in 1976, is to find a point

$$x^* \in H,\ such\ that\ 0 \in B(x^*), \tag{1.1}$$

where H is a real Hilbert space, and $B : H \to 2^H$ is a maximal monotone mapping (see below).

VIP (1.1) has received much attention since it includes many problems, such as, fixed point problems, equilibrium problems, saddle problems variational inequality problems, complementary problems and so on. Many authors investigated this problem due to its wide application in the real world, such as image recovery, single processing, computer tomography, and so on; see, e.g., [2, 3, 4] and the references therein.

Recently, many authors studied the relation of solutions of two variational inclusion problems in two different spaces. In 2011, Moudafi [5] proposed a so-called split variational inclusion

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Received September 5, 2021; Accepted November 20, 2021.

problem (in short, $SVIP$), which is to find a point

$$x^* \in H_1,\ such\ that\ 0 \in B_1(x^*),\ and\ 0 \in B_2(Ax^*), \tag{1.2}$$

where H_1 and H_2 are two real Hilbert spaces, $A : H_1 \to H_2$ is a bounded linear operator, and $B_1 : H_1 \to 2^{H_1}$ and $B_2 : H_2 \to 2^{H_2}$ are two maximal monotone operators. It is obvious that SVIP (1.2) reduces to VIP (1.1) when $H_1 = H_2$, $B_1 = B_2$, and A is a identity on H_1.

It is known that x^* is a solution to VIP (1.1) if and only if x^* is a fixed point of J_λ^B, the resolvent mapping of B, which is defined by

$$J_\lambda^B := (I+\lambda B)^{-1},\ \lambda > 0. \tag{1.3}$$

In 1970, Martinet [6] proposed the following so-called proximal point method (in short, PPM) to solve VIP (1.1). For given $x_1 \in H_1$ and $\lambda_n \in (0,+\infty)$, the iterative sequence $\{x_n\}$ is generated via the following scheme

$$x_{n+1} = J_{\lambda_n}^B(x_n), \tag{1.4}$$

where the $J_{\lambda_n}^B$ is the resolvent mapping of B.

In 2012, Byrne et al. [7] proposed the following iterative algorithm to solve SVIP (1.2). For any given $x_1 \in H_1$, $\lambda > 0$, they defined the following algorithm

$$x_{n+1} = J_\lambda^{B_1}(x_n + \gamma A^*(J_\lambda^{B_2} - I)Ax_n),\ \forall\, x_n \in H_1,\ \forall n \geq 1, \tag{1.5}$$

where A^* is the adjoint operator of A, L is the spectral radius of operator A^*A, $\gamma \in (0, \frac{2}{L})$. They showed that $\{x_n\}$ converges weakly to a solution of SVIP (1.2).

Similarly, it has been shown that x^* is a solution of SVIP (1.2) if and only if x^* is a fixed point of the operator $J_\lambda^{B_1}(I - \gamma A^*(I - J_\lambda^{B_2})A)$. This implies that the iterative techniques for fixed points could be used to solve the SVIP (1.2). The problem Convergence ratio of an iteration algorithm is a very important problem from the viewpoint of computation. Recently, researchers introduced various iteration algorithms to enhance the convergence speed of iterative sequences, such as viscosity iteration algorithms, subgradient extragradient iteration algorithms, inertial iteration algorithms and so on; see, e.g., [8, 9, 10, 11, 12, 13, 14] and the references therein. Inertial methods are powerful and popular when accelerating original iterative algorithms. Inertial acceleration is based on a discrete version of a second-order dissipative dynamical system. Various fast iterative algorithms with the aid of inertial extrapolation methods attached much attention of researchers recently. In [15], based on the PPM and the inertial method, Alvarez and Attouch proposed the following algorithm to solve VIP (1.1). For any given $x_0, x_1 \in H$ and $\lambda_n > 0$, their iterative sequence $\{x_n\}$ was generated via the following scheme

$$x_{n+1} = J_{\lambda_n}^B(x_n + \theta_n(x_n - x_{n-1})),\ \forall n \geq 0, \tag{1.6}$$

where $\theta_n(x_n - x_{n-1})$, which satisfies $\sum_{n=1}^\infty \theta_n \|x_n - x_{n-1}\|^2 < \infty$, is the inertial term.

In [16], Thong, Duong, and Cho introduced an iteration method by combining inertial iteration with the viscosity iteration to solve SVIP (1.2). Their iterative process reads as follows

Initialization: Set $\beta > 0$ and $x_0,\ x_1 \in H_1$.

Iterative Steps: Calculate x_{n+1} as follows:

Step 1. Given the iterates x_{n-1} and $x_n (n \geq 1)$, choose θ_n such that $0 \leq \theta_n \leq \bar{\theta}_n$, where

$$\bar{\theta}_n = \begin{cases} \min\{\theta, \frac{\tau_n}{\|x_n - x_{n-1}\|}\}, \text{ if } x_n \neq x_{n-1}; \\ \theta, \text{ otherwise.} \end{cases}$$

Step 2. Compute $w_n = x_n + \theta_n (x_n - x_{n-1})$, and

$$y_n = J_\beta^{B_1}(I - \gamma_n A^*(I - J_\beta^{B_2})Aw_n).$$

If $w_n = y_n$, then stop and y_n is a solution of SVIP (1.2). Otherwise, go to **Step 3.**

Step 3. Compute $z_n = w_n - \alpha_n D_n$, where

$$D_n = w_n - y_n - \gamma_n[A^*(I - J_\beta^{B_2})Aw_n - A^*(I - J_\beta^{B_2})Ay_n],$$

and

$$\alpha_n = \frac{\langle w_n - y_n, D_n \rangle}{\|D_n\|^2}.$$

Step 4. Compute $x_{n+1} = \sigma_n f(x_n) + (1 - \sigma_n) z_n$.
Set $n := n + 1$ and go to **Step 1**, where $\{\gamma_n\} \subset [a, b] \subset (0, \frac{1}{L})$ with $L = \|A\|^2$.

Let $f : H_1 \to H_1$ be a contraction mapping with contraction parameter $\kappa \in [0, 1)$, and $\{\tau_n\} \subset [0, \theta)$ for some $\theta > 0$ is a positive sequence such that $\tau_n = \circ(\sigma_n)$, which means $\lim_{n \to \infty} \frac{\tau_n}{\sigma_n} = 0$, where $\{\sigma_n\} \subset (0, 1)$ satisfies the following conditions:

$$\lim_{n \to \infty} \sigma_n = 0, \ \sum_{n=1}^{\infty} \sigma_n = \infty.$$

They obtained the strong convergence for SVIP (1.2). Notice that, in their algorithm, the step-sizes depend on the norm of operator A. In addition, the iteration step-sizes of many existing iteration methods heavily depend on the norm of operators when dealing with solving split feasibility problems. We have to mention that the norm of the involved operators usually is not easily to be computed. So, some authors often used the linear search method in their numerical algorithms for solving variational inequality problems, variational inclusion problems, fixed point problems, equilibrium problems, and various split feasibility problems. For the numerical algorithms incorporated with the linear search, the step-sizes of the iteration do not depend on the norm of the involved operator; see, e.g., [17].

As mentioned before, the operator norms and the projection operators are not easily to be calculated. In this paper, a new iteration algorithm which does not involve projection operators and the step-sizes do not depend on the operator norm is introduced to solve SVIP (1.2). A weak convergence theorem of solutions is obtained. Some applications to split minimization problems and split feasibility problems are also considered.

2. Preliminaries

In this section, we provide some basic definitions and lemmas, which will be used in the sequel.

Let H be a real Hilbert space, I is the identity mapping in Hilbert space H. If $Tx = x$, then the point x is called a fixed point of T, where $T : H \to H$ is a mapping. We use $Fix(T)$ to denote the fixed point set of T.

- the symbols " $\rightharpoonup$ " stands for the weak convergence.

• the symbols " $\rightarrow$ " stands for the strong convergence.

For each $x, y \in H$ and $\alpha \in \mathbb{R}$, we have the following facts

(i) $\|x+y\|^2 = \|x\|^2 + 2\langle x, y\rangle + \|y\|^2$;

(ii) $\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 + \alpha(1-\alpha)\|x-y\|^2$.

Definition 2.1. Let $T : H \to H$ be a single-valued operator. T is said to be

(i) nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \, \forall x, \, y \in H.$$

(ii) firmly nonexpansive if

$$\|Tx - Ty\|^2 \leq \langle Tx - Ty, x - y\rangle, \, \forall x, \, y \in H,$$

or, equivalently,

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 - \|(I - T)x - (I - T)y\|^2, \, \forall x, \, y \in H.$$

Furthermore, if T is firmly nonexpansive, then $I - T$ is also firmly nonexpansive.

Definition 2.2. Let $B : H \to 2^H$ be a set-valued operator. B is said to be

(i) monotone with the domain $D(B) := \{x \in H : B(x) \neq \emptyset\}$ if

$$\langle x - y, u - v\rangle \geq 0, \, \forall \, u \in Bx, \, \forall \, v \in By.$$

(ii) maximal monotone if B is a monotone mapping and $Graph(B) := \{(x, u) : x \in D(B), u \in B(x)\}$ is not properly contained in the graph of any other monotone mapping.

Lemma 2.3. *(Demiclosedness principle)* [18] *Let C be a nonempty, close, and convex subset of a real Hilbert space H. Let $T : H \to H$ be a nonexpansive mapping. If $x_n \rightharpoonup x \in C$ and* $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$, *then $x \in Fix(T)$.*

Lemma 2.4. [19] *Let H be a real Hilbert space. Let $B : H \to 2^H$ be a maximal monotone mapping and $\beta > 0$. Then the following statements hold:*

(i) *J^B_β is a single-valued and firmly nonexpansive mapping;*

(ii) *$D(J^B_\beta) = H$ and $Fix(J^B_\beta) = B^{-1}(0) = \{x \in D(B) : 0 \in B(x)\}$;*

(iii) *$\|x - J^B_\beta x\| \leq 2\|x - J^B_\gamma x\|$ for all $0 < \beta < \gamma$ and $x \in H$;*

(iv) *$(I - J^B_\beta)$ is a firmly nonexpansive mapping;*

(v) *If $B^{-1}(0) \neq \emptyset$, then*

$$\|J^B_\beta x - z\|^2 \leq \|x - z\|^2 - \|J^B_\beta x - x\|^2, \, \forall x \in H, \, z \in B^{-1}(0),$$

and

$$\langle x - J^B_\beta x, J^B_\beta x - z\rangle \geq 0, \, \forall x \in H, \, z \in B^{-1}(0).$$

Lemma 2.5. [20] *Let H_1 and H_2 be two real Hilbert space. Let $A : H_1 \to H_2$ be a linear operator, and let A^* be the adjoint of A. Let β ad λ be two positive real numbers. Let $B_1 : H_1 \to 2^{H_1}$ and $B_2 : H_2 \to 2^{H_2}$ be two maximal monotone mappings. If the solution set of SVIP (1.2) is nonempty, then, for any $x^* \in H_1$, x^* is a solution of SVIP (1.2) if and only if*

$$J^{B_1}_\beta (x^* - \lambda A^*(I - J^{B_2}_\beta)Ax^*) = x^*.$$

Lemma 2.6. [20] *Let H_1 and H_2 be two real Hilbert space. Let $A : H_1 \to H_2$ be a linear operator, and let A^* be the adjoint of A. Let β be positive real number, and let $B_2 : H_2 \to 2^{H_2}$ be a maximal monotone mapping. Then the following conclusions hold:*

(i) $\|(I-J_\beta^{B_2})Ax-(I-J_\beta^{B_2})Ay\|^2 \le \langle A^*(I-J_\beta^{B_2})Ax-A^*(I-J_\beta^{B_2})Ay, x-y\rangle$, $\forall x,\ y \in H_1$;

(ii) $\|A^*(I-J_\beta^{B_2})Ax-A^*(I-J_\beta^{B_2})Ay\| \le \|A\|^2\|x-y\|$, $\forall x,\ y \in H_1$.

Lemma 2.7. [21] *A space X is said to satisfy Opial's condition if, for any $\{x_n\}$ in X with $x_n \rightharpoonup x$, and for all $y \in X$ with $y \neq x$, the follow inequality holds:*

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|.$$

3. Main Results

In this section, we always use the following symbols and restrictions:

(i) let H_1 and H_2 be two real Hilbert spaces;
(ii) let $A : H_1 \to H_2$ be a bounded linear operator, and let A^* be the adjoint of A;
(iii) let $B_1 : H_1 \to 2^{H_1}$ and $B_2 : H_2 \to 2^{H_2}$ be two set-valued maximal monotone mappings;
(iv) let Ω denote the solution set of SVIP (1.2). In addition, Ω is assumed to be nonempty and L is borrowed to denote the spectral radius of A.

Algorithm 3.1. *Give $\beta > 0$, $\tau_1 > 0$, $\mu \in (0,1)$, $\alpha \in (1,2)$, $\alpha_n \in (0,1)$, and let $x_1 \in H_1$ be arbitrarily chosen.*

Iterative Steps: *Calculate x_{n+1} as follows:*
Step 1. *Compute*

$$y_n = J_\beta^{B_1}(I - \tau_n A^*(I-J_\beta^{B_2})A)x_n,\ \forall n \ge 1.$$

Step 2. *Compute*

$$z_n = x_n - \alpha\eta_n d(x_n,y_n),\ \forall n \ge 1,$$

where

$$d(x_n,y_n) = x_n - y_n - \tau_n[A^*(I-J_\beta^{B_2})Ax_n - A^*(I-J_\beta^{B_2})Ay_n],\ \forall n \ge 1,$$

and

$$\eta_n = \begin{cases} \frac{\langle x_n-y_n, d(x_n,y_n)\rangle}{\|d(x_n,y_n)\|^2},\ if\ d(x_n,y_n) \neq 0; \\ 0,\ if\ d(x_n,y_n) = 0. \end{cases}$$

Step 3. *Compute*

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)z_n,\ \forall n \ge 1.$$

Update

$$\tau_{n+1} = \begin{cases} min\{\tau_n,\ \mu\frac{\|x_n-y_n\|}{\|A^*(I-J_\beta^{B_2})Ax_n-A^*(I-J_\beta^{B_2})Ay_n\|}\},\ if\ d(x_n,y_n) \neq 0; \\ \tau_n,\ if\ d(x_n,y_n) = 0. \end{cases} \tag{3.1}$$

Set $n := n+1$ and go to **Step 1**.

Lemma 3.2. *The sequence $\{\tau_n\}$ generated by (3.1) is a non-increasing sequence and*

$$\lim_{n\to\infty}\tau_n=\tau\geq min\{\tau_1,\ \frac{\mu}{L^2}\}>0.$$

Proof. It is obvious that $0<\tau_n\leq\tau_1$. If $\tau_n=\tau_1$, then Lemma 3.2 is proved. Otherwise, if $\tau_n<\tau_1$, then

$$\tau_n=\mu\frac{\|x_n-y_n\|}{\|A^*(I-J_\beta^{B_2})Ax_n-A^*(I-J_\beta^{B_2})Ay_n)\|}.$$

Since A is a bounded linear operator, and $(I-J_\beta^{B_2})$ is a firmly nonexpansive mapping, for the case that $\|A^*(I-J_\beta^{B_2})Ax_n-A^*(I-J_\beta^{B_2})Ay_n\|\neq 0$, we have from Lemma 2.6 that

$$\frac{\mu\|x_n-y_n\|}{\|A^*(I-J_\beta^{B_2})Ax_n-A^*(I-J_\beta^{B_2})Ay_n\|}\geq\frac{\mu\|x_n-y_n\|}{\|A\|^2\|x_n-y_n\|}=\frac{\mu}{L^2}.$$

So, $\min(\tau_1,\frac{\mu}{L^2})\leq\tau_n\leq\tau_1$, and $\{\tau_n\}$ has the lower bounded $\min\{\tau_1,\ \frac{\mu}{L^2}\}$, where $L=\|A\|^2$. Since $\{\tau_n\}$ is non-increasing, one sees that there exists τ such that $\lim_{n\to\infty}\tau_n=\tau\geq min\{\tau_1,\ \frac{\mu}{L^2}\}$. Furthermore, one has

$$\|A^*(I-J_\beta^{B_2})Ax_n-A^*(I-J_\beta^{B_2})Ay_n\|\leq\frac{\mu}{\tau_{n+1}}\|x_n-y_n\|,\ \forall n\geq 1. \tag{3.2}$$

□

Theorem 3.3. *If $\{\alpha_n\}\in(0,1)$, $\lim_{n\to\infty}\alpha_n=0$, and $\sum_{n=0}^{\infty}\alpha_n=\infty$, then the sequence $\{x_n\}$ generated by Algorithm 3.1 converges weakly to a point $\bar{x}$ in Ω.*

Proof. We divide the proof into five steps.

Step 1. Prove that the following conclusion is true.

$$\|z_n-x^*\|^2\leq\|x_n-x^*\|^2-\frac{2-\alpha}{\alpha}\|x_n-z_n\|^2,\ \forall x^*\in\Omega,\ \forall n\geq 1. \tag{3.3}$$

Fixing $x^*\in\Omega$, one has $Ax^*\in B_2^{-1}(0)$. From Lemma 2.4, we have $J_\beta^{B_2}Ax^*=Ax^*$. It follows from Lemma 2.6 that

$$\begin{aligned}\tau_n\langle y_n-x^*,A^*(I-J_\beta^{B_2})Ay_n\rangle&=\tau_n\langle y_n-x^*,A^*(I-J_\beta^{B_2})Ay_n-A^*(I-J_\beta^{B_2})Ax^*\rangle\\&=\tau_n\langle Ay_n-Ax^*,(I-J_\beta^{B_2})Ay_n-(I-J_\beta^{B_2})Ax^*\rangle\\&\geq\tau_n\|(I-J_\beta^{B_2})Ay_n\|^2.\end{aligned} \tag{3.4}$$

By the definition of y_n, Lemma 2.4, and (3.4), we have

$$\begin{aligned}\langle y_n-x^*,d(x_n,y_n)\rangle&=\langle y_n-x^*,x_n-y_n-\tau_n[A^*(I-J_\beta^{B_2})Ax_n-A^*(I-J_\beta^{B_2})Ay_n]\rangle\\&=\langle y_n-x^*,x_n-y_n-\tau_nA^*(I-J_\beta^{B_2})Ax_n\rangle\\&\quad+\langle y_n-x^*,\tau_nA^*(I-J_\beta^{B_2})Ay_n\rangle\\&=\langle y_n-x^*,x_n-y_n-\tau_nA^*(I-J_\beta^{B_2})Ax_n\rangle\\&\quad+\tau_n\langle y_n-x^*,A^*(I-J_\beta^{B_2})Ay_n\rangle\\&\geq\tau_n\|(I-J_\beta^{B_2})Ay_n\|^2.\end{aligned} \tag{3.5}$$

In view of $\tau_n > 0$, we have

$$\begin{aligned}\langle x_n - x^*, d(x_n, y_n)\rangle &= \langle x_n - y_n, d(x_n, y_n)\rangle + \langle y_n - x^*, d(x_n, y_n)\rangle \\ &\geq \langle x_n - y_n, d(x_n, y_n)\rangle + \tau_n \|(I - J_\beta^{B_2})Ay_n\|^2 \\ &\geq \langle x_n - y_n, d(x_n, y_n)\rangle.\end{aligned} \tag{3.6}$$

From Algorithm 3.1, we have

$$\begin{aligned}\|z_n - x^*\|^2 &= \|x_n - \alpha\eta_n d(x_n, y_n) - x^*\|^2 \\ &= \|x_n - x^*\|^2 - 2\alpha\eta_n\langle x_n - x^*, d(x_n, y_n)\rangle + \alpha^2\eta_n^2\|d(x_n, y_n)\|^2.\end{aligned} \tag{3.7}$$

Substitute (3.6) into (3.7), we arrive at

$$\|z_n - x^*\|^2 \leq \|x_n - x^*\|^2 - 2\alpha\eta_n\langle x_n - y_n, d(x_n, y_n)\rangle + \alpha^2\eta_n^2\|d(x_n, y_n)\|^2.$$

When $d(x_n, y_n) \neq 0$, we know that $\eta_n = \frac{\langle x_n - y_n, d(x_n, y_n)\rangle}{\|d(x_n, y_n)\|^2}$. This shows that

$$\eta_n\|d(x_n, y_n)\|^2 = \langle x_n - y_n, d(x_n, y_n)\rangle,$$

which implies

$$\begin{aligned}\|z_n - x^*\|^2 &\leq \|x_n - x^*\|^2 - 2\alpha\eta_n\langle x_n - y_n, d(x_n, y_n)\rangle + \alpha^2\eta_n\langle x_n - y_n, d(x_n, y_n)\rangle \\ &= \|x_n - x^*\|^2 - (2 - \alpha)\alpha\eta_n^2\langle x_n - y_n, d(x_n, y_n)\rangle \\ &= \|x_n - x^*\|^2 - (2 - \alpha)\alpha\|\eta_n d(x_n, y_n)\|^2.\end{aligned} \tag{3.8}$$

From Algorithm 3.1, the definition of z_n, we have $z_n - x_n = -\alpha\eta_n d(x_n, y_n)$. In view of (3.8), we obtain

$$\|z_n - x^*\|^2 \leq \|x_n - x^*\|^2 - \frac{2-\alpha}{\alpha}\|x_n - z_n\|^2, \ \forall x^* \in \Omega, \ \forall n \geq 1. \tag{3.9}$$

Step 2. Prove that $\lim_{n\to\infty}\|x_n - x^*\|$ exists.

Since $\alpha \in (1, 2)$, we have $0 < \frac{2-\alpha}{\alpha} < 1$. It follows from (3.9) that

$$\|z_n - x^*\| \leq \|x_n - x^*\|. \tag{3.10}$$

By the definition of x_{n+1} and (3.10), we have

$$\begin{aligned}\|x_{n+1} - x^*\| &\leq \alpha_n\|x_n - x^*\| + (1 - \alpha_n)\|z_n - x^*\| \\ &\leq \alpha_n\|x_n - x^*\| + (1 - \alpha_n)\|x_n - x^*\| \\ &\leq \|x_n - x^*\|.\end{aligned}$$

So, $\{\|x_n - x^*\|\}$ is a monotone decreasing sequence with a lower bound. Hence, $\lim_{n\to\infty}\|x_{n+1} - x^*\|$ exists. This implies that $\{x_n\}$ is a bounded sequence, so is $\{z_n\}$.

Step 3. Prove that the following conclusion is true.

$$\|x_{n+1} - x^*\|^2 \leq \|x_n - x^*\|^2 - \frac{2-\alpha}{\alpha}(1 - \alpha_n)\|x_n - z_n\|^2, \ \forall x^* \in \Omega, \ \forall n \geq 1.$$

From (3.3), we have

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &= \alpha_n^2\|x_n-x^*\|^2+(1-\alpha_n)^2\|z_n-x^*\|^2+2\alpha_n(1-\alpha_n)\langle x_n-x^*,z_n-x^*\rangle\\ &\le \alpha_n^2\|x_n-x^*\|^2+(1-\alpha_n)^2\|z_n-x^*\|^2+2\alpha_n(1-\alpha_n)\|x_n-x^*\|\cdot\|z_n-x^*\|\\ &\le \alpha_n^2\|x_n-x^*\|^2+(1-\alpha_n)^2\|z_n-x^*\|^2+\alpha_n(1-\alpha_n)[\|x_n-x^*\|^2+\|z_n-x^*\|^2]\\ &\le \alpha_n\|x_n-x^*\|^2+(1-\alpha_n)\|x_n-x^*\|^2-\frac{2-\alpha}{\alpha}(1-\alpha_n)\|z_n-x_n\|^2\\ &\le \|x_n-x^*\|^2-\frac{2-\alpha}{\alpha}(1-\alpha_n)\|z_n-x_n\|^2.\end{aligned}\tag{3.11}$$

Step 4. Prove that $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$ and $\lim_{n\to\infty}\|x_n-J_\beta^{B_1}x_n\|=0$.

Since the $\lim\|x_n-x^*\|$ exits for any $x^*\in\Omega$, we have $\lim_{n\to\infty}(\|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2)=0$ holds. In view of $\lim_{n\to\infty}\alpha_n=0$, we conclude from (3.11) that

$$\lim_{n\to\infty}\|x_n-z_n\|=0.\tag{3.12}$$

From (3.2), we have

$$\begin{aligned}\langle x_n-y_n,d(x_n,y_n)\rangle &= \langle x_n-y_n,x_n-y_n-\tau_n[A^*(I-J_\beta^{B_2})Ax_n-A^*(I-J_\beta^{B_2})Ay_n]\rangle\\ &= \langle x_n-y_n,x_n-y_n\rangle-\langle x_n-y_n,\tau_n[A^*(I-J_\beta^{B_2})Ax_n-A^*(I-J_\beta^{B_2})Ay_n]\rangle\\ &= \|x_n-y_n\|^2-\tau_n\langle x_n-y_n,A^*(I-J_\beta^{B_2})Ax_n-A^*(I-J_\beta^{B_2})Ay_n\rangle\\ &\ge \|x_n-y_n\|^2-\tau_n\cdot\|x_n-y_n\|\cdot\|A^*(I-J_\beta^{B_2})Ax_n-A^*(I-J_\beta^{B_2})Ay_n\|\\ &\ge \|x_n-y_n\|^2-\tau_n\cdot\frac{\mu}{\tau_{n+1}}\|x_n-y_n\|^2\\ &= (1-\mu\cdot\frac{\tau_n}{\tau_{n+1}})\|x_n-y_n\|^2.\end{aligned}\tag{3.13}$$

Observe that

$$\begin{aligned}\|d(x_n,y_n)\|^2 &= \|x_n-y_n\|^2+\tau_n^2\|A^*(I-J_\beta^{B_2})Ax_n-A^*(I-J_\beta^{B_2})Ay_n\|^2\\ &\quad -2\tau_n\langle x_n-y_n,A^*(I-J_\beta^{B_2})Ax_n-A^*(I-J_\beta^{B_2})Ay_n\rangle\\ &\le \|x_n-y_n\|^2+\tau_n^2\cdot\frac{\mu_n^2}{\tau_{n+1}^2}\|x_n-y_n\|^2\\ &\quad +2\tau_n|\langle x_n-y_n,A^*(I-J_\beta^{B_2})Ax_n-A^*(I-J_\beta^{B_2})Ay_n\rangle|\\ &\le \|x_n-y_n\|^2+\mu_n^2\cdot\frac{\tau_n^2}{\tau_{n+1}^2}\|x_n-y_n\|^2\\ &\quad +2\tau_n\|x_n-y_n\|\cdot\|A^*(I-J_\beta^{B_2})Ax_n-A^*(I-J_\beta^{B_2})Ay_n\|\\ &\le \|x_n-y_n\|^2+\mu_n^2\cdot\frac{\tau_n^2}{\tau_{n+1}^2}\|x_n-y_n\|^2+2\tau_n\cdot\frac{\mu}{\tau_{n+1}}\|x_n-y_n\|^2\\ &= (1+\mu\cdot\frac{\tau_n}{\tau_{n+1}})^2\|x_n-y_n\|^2,\end{aligned}\tag{3.14}$$

From (3.13) and (3.14), we see that

$$\eta_n = \frac{\langle x_n - y_n, d(x_n, y_n)\rangle}{\|d(x_n, y_n)\|^2} \geq \frac{1-\mu\frac{\tau_n}{\tau_{n+1}}}{(1+\mu\frac{\tau_n}{\tau_{n+1}})^2}. \tag{3.15}$$

Further, we have

$$\eta_n\|d(x_n, y_n)\|^2 = \langle x_n - y_n, d(x_n, y_n)\rangle \geq (1-\mu\frac{\tau_n}{\tau_{n+1}})\|x_n - y_n\|^2,$$

which implies that

$$\begin{aligned} \|x_n - y_n\|^2 &\leq \frac{1}{(1-\mu\frac{\tau_n}{\tau_{n+1}})}\eta_n\|d(x_n, y_n)\|^2 \\ &= \frac{1}{(1-\mu\frac{\tau_n}{\tau_{n+1}})}\|\alpha\eta_n d(x_n, y_n)\|^2 \cdot \frac{1}{\alpha^2} \cdot \frac{1}{\eta_n} \\ &= \frac{1}{(1-\mu\frac{\tau_n}{\tau_{n+1}})}\|z_n - x_n\|^2 \cdot \frac{1}{\alpha^2} \cdot \frac{1}{\eta_n}. \end{aligned} \tag{3.16}$$

From (3.15) and (3.16), we obtain

$$\|x_n - y_n\|^2 \leq \frac{1+\mu\frac{\tau_n}{\tau_{n+1}}}{[(1-\mu\frac{\tau_n}{\tau_{n+1}})\alpha]^2} \cdot \|z_n - x_n\|^2. \tag{3.17}$$

In view of (3.12) and (3.17), we conclude

$$\lim_{n\to\infty} \|x_n - y_n\| = 0. \tag{3.18}$$

From (3.12) and (3.18), we also have $\lim_{n\to\infty}\|z_n - y_n\| = 0$. The definition of x_{n+1} and (3.12) yield that

$$\begin{aligned} \lim_{n\to\infty} \|x_{n+1} - x_n\| &= \lim_{n\to\infty} \|\alpha_n(x_n - x_n) + (1-\alpha_n)(z_n - x_n)\| \\ &\leq \lim_{n\to\infty} (1-\alpha_n)\|z_n - x_n\| \\ &= 0. \end{aligned}$$

Further, from (3.14) and (3.18), we see that

$$\lim_{n\to\infty} \|d(x_n, y_n)\| = 0. \tag{3.19}$$

Since $\|y_n - x^*\| \leq \|x_n - y_n\| + \|x_n - x^*\|$ and $\lim_{n\to\infty}\|x_n - y_n\| = 0$, $\lim_{n\to\infty}\|x_n - x^*\|$ exists, we conclude that $\{y_n\}$ is a bounded sequence. Further, from (3.5) and (3.19), we have

$$\begin{aligned} \lim_{n\to\infty} \tau_n\|(I - J_\beta^{B_2})Ay_n\| &\leq \lim_{n\to\infty} \langle y_n - x^*, d(x_n, y_n)\rangle \\ &\leq \lim_{n\to\infty} (\|y_n - x^*\| \cdot \|d(x_n, y_n)\|) \\ &\leq 0. \end{aligned} \tag{3.20}$$

Since $\lim_{n\to\infty}\|x_n - y_n\| = 0$, $\tau_n \geq 0$, and A is a bounded linear operator, we conclude (3.20) that

$$\begin{aligned} \lim_{n\to\infty} \|Ax_n - J_\beta^{B_2}Ay_n\| &\leq \lim_{n\to\infty} (\|Ax_n - Ay_n\| + \|Ay_n - J_\beta^{B_2}Ay_n\|) \\ &\leq \lim_{n\to\infty} (\|A\| \cdot \|x_n - y_n\| + \|Ay_n - J_\beta^{B_2}Ay_n\|) \\ &\leq 0. \end{aligned}$$

Since $\lim_{n\to\infty}\|x_n - y_n\| = 0$, we obtain from (3.20) that

$$\lim_{n\to\infty}\|(I - J_\beta^{B_2})Ax_n\| = 0. \tag{3.21}$$

By the definition of y_n, and the fact that $J_\beta^{B_1}$ is a firmly nonexpansive mapping, we have

$$\begin{aligned}\lim_{n\to\infty}\|y_n - J_\beta^{B_1}x_n\| &= \lim_{n\to\infty}\|J_\beta^{B_1}(x_n - \tau_n A^*(I - J_\beta^{B_2})Ax_n) - J_\beta^{B_1}x_n\| \\ &\le \tau_n\|A^*\|\cdot\lim_{n\to\infty}\|(I - J_\beta^{B_2})Ax_n\| = 0.\end{aligned} \tag{3.22}$$

Observe that $\lim_{n\to\infty}\|x_n - y_n\| = 0$. It follows from (3.22) that

$$\lim_{n\to\infty}\|x_n - J_\beta^{B_1}x_n\| = 0. \tag{3.23}$$

Step 5. Prove that the $x_n \rightharpoonup \bar{x} \in \Omega$.

Assume that there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$, which converges weakly to $\bar{x}$. Thanks to the fact that A is bounded linear oprerator, we have $Ax_{n_j} \rightharpoonup A\bar{x}$. From Lemma 2.3 and (3.23), we have $\bar{x} \in Fix(J_\beta^{B_1})$. Lemma 2.3 and (3.21) yield that $A\bar{x} \in Fix(J_\beta^{B_2})$. This implies that $\bar{x} \in \Omega$. Since $\bar{x} \in \Omega$, we have that $\lim_{n\to\infty}\|x_n - \bar{x}\|$ exists. Suppose that there exists another subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup \bar{y}$ and $\bar{x} \neq \bar{y}$. From Lemma 2.3, (3.21), and (3.23), we have $\bar{y} \in \Omega$. It follows from Lemma 2.7 that

$$\begin{aligned}\lim_{n\to\infty}\|x_n - \bar{x}\| &= \lim_{j\to\infty}\|x_{n_j} - \bar{x}\| \\ &< \lim_{j\to\infty}\|x_{n_j} - \bar{y}\| = \lim_{k\to\infty}\|x_{n_k} - \bar{y}\| \\ &< \lim_{k\to\infty}\|x_{n_k} - \bar{x}\| = \lim_{n\to\infty}\|x_n - \bar{x}\|.\end{aligned}$$

This creates a contradiction, which implies that $x_n \rightharpoonup \bar{x}$. This completes the proof. □

4. APPLICATIONS

4.1. **The application to the split minimization problem.** In this subsection, we will apply our convergence result to the following split minimization problem (In short, SMP), that is, find a point $x^* \in H_1$ such that

$$f(x^*) \in \arg\min_{x\in H_1} f(x), \ and \ g(Ax^*) \in \arg\min_{z\in H_2} g(z).$$

where $f : H_1 \to \mathbb{R}\cup\{+\infty\}$ and $g : H_2 \to \mathbb{R}\cup\{+\infty\}$ are proper, convex, and lower semi-continuous functions, and $A : H_1 \to H_2$ is a bounded linear operator. The proximal operator of f is defined by

$$prox_{\beta,f}(x) := \arg\min_{y\in H}\{f(y) + \frac{1}{2\beta}\|y - x\|^2\}, \ \forall x \in H.$$

The sub-differential ∂f of f at x is defined by

$$\partial f(x) := \{x^* \in H : f(x) + \langle y - x, x^*\rangle \le f(y), \ \forall y \in H\}.$$

It is known that

$$prox_{\beta,f}(x) = (I - \beta\partial f)^{-1}(x) = J_\beta^{\partial f}(x).$$

Because f is proper, convex, and lower semi-continuous, we have that ∂f is set-valued maximal monotone and $prox_{\beta,f}$ is firmly non-expansive.

In Algorithm 3.1, letting $J_\beta^{B_1} = prox_{\beta,f}$, and $J_\beta^{B_2} = prox_{\beta,g}$, we have the following subresult immediately.

Theorem 4.1. *Let H_1 and H_2 be two real Hilbert space. Let f, g and A be the operators defined in 3.3. Let $\beta > 0$, $\mu \in (0,1)$, $\alpha \in (1,2)$, and $\alpha_n \in (0,1)$ satisfy $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$. Then the sequence $\{x_n\}$ generated by Algorithm 3.1 converges weakly to a solution of the SMP.*

4.2. **The application to the split feasibility problem.** The SFP was firstly introduced by Censor and Elfving [22], which is to find a point

$$x^* \in C, \textit{ such that } Ax^* \in Q,$$

where C and Q are closed and convex subsets of real Hilbert spaces H_1 and H_2, respectively, and $A : H_1 \to H_2$ is a bounded linear operator with its adjoint operator A^*.

Let C be a nonempty, closed and convex subsets of a real Hilbert spaces H, and let i_C be the indicator function of C, that is,

$$i_C = \begin{cases} 0, \text{ if } x \in C; \\ \infty, \text{ if } x \notin C. \end{cases}$$

Further, the normal cone N_C of C at $x \in C$ is defined as

$$N_C x = \{z \in H : \langle z, y - x\rangle \leq 0, \ \forall y \in C\}.$$

It is known that i_C is a proper, lower semicontinuous, and convex function on H, the subdifferential $\partial(i_C)$ of i_C is a maximal monotone operator, and the fact that $\partial(i_C)x = N_c x$ holds for each $x \in C$. Then, the resolvent $J_\beta^{\partial(i_C)}$ of $\partial(i_C)$, for each $\beta > 0$, is defined as

$$J_\beta^{\partial(i_C)} x = (I + \beta\partial(i_C))^{-1} x, \textit{ for all } x \in H.$$

Therefore, for each $\beta > 0$, we have

$$\begin{aligned} x^* = J_\beta^{\partial(i_C)} x &\Longleftrightarrow x \in x^* + \beta\partial(i_C)x^* \Longleftrightarrow x - x^* \in \beta\partial(i_C)x^* \\ &\Longleftrightarrow \langle x - x^*, y - x^*\rangle \leq 0, \ \forall y \in C \\ &\Longleftrightarrow x^* = P_C x, \end{aligned}$$

where P_C is the metric projection in H onto C.

In Algorithm 3.1, letting $J_\beta^{B_1} = P_C$, and $J_\beta^{B_2} = P_Q$, we have the following subresult immediately.

Theorem 4.2. *Let H_1 and H_2 be two real Hilbert space, and let C and Q be two nonempty, closed and convex subsets of real Hilbert spaces H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator. Let $\beta > 0$, $\mu \in (0,1)$, $\alpha \in (1,2)$, and $\alpha_n \in (0,1)$ satisfy $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$. Then the sequence $\{x_n\}$ generated by Algorithm 3.1 converges weakly to a solution of the SFP.*

Acknowledgements

The authors would like to thank the associate editor and the anonymous referee for his/her comments that helped us improve this article.

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