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FRACTIONAL VISCOELASTIC EQUATION OF KIRCHHOFF TYPE WITH LOGARITHMIC NONLINEARITY

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Abstract. In this paper, we study a fractional viscoelastic equation of Kirchhoff type with logarithmic nonlinearity. Under suitable conditions, we prove the existence of global solutions and the exponential decay of the energy.

Keywords. Kirchhoff type problem; Fractional Laplacian; Logarithmic nonlinearity; Galerkin method.

1. INTRODUCTION

We consider the problem of finding $u = u(x,t)$ weak solutions to the following nonlinear heat equation of Kirchhoff type with the variable exponent of nonlinearity, viscoelastic term and a logarithmic source term, involving the fractional Laplacian

$$\begin{aligned} \left(1+a|u|^{r(x)-2}\right)u_t + M(\|u\|^2_{W_0})(-\Delta)^s u - \int_0^t g(t-\tau)(-\Delta)^s u(\tau)d\tau = f(u) \quad \text{in } \Omega\times]0,\infty[, \\ u = 0 \quad \text{in } \left(\mathbb{R}^N\backslash\Omega\right)\times[0,\infty[, \\ u(x,0) = u^0(x) \quad \text{in } \Omega, \end{aligned} \tag{1.1}$$

where $\Omega \subseteq \mathbb{R}^N$ is a smooth bounded domain, $M(t) = t^{\alpha-1}+1$, $t \geq 0$, $s \in]0,1[$, $2 < \frac{N}{s}$, $\alpha > 1$; $g : [0,\infty[\to]0,\infty[$ belongs to $C^1([0,\infty[)$, $g(0) > 0$, $l = 1 - \int_0^\infty g(\tau)d\tau > 0$, $g'(t) \leq 0$, $f(s) = s\log|s|$ and r is a given continuous function.

This type of problems without viscoelastic term (that is, $g = 0$), $r(x) = constant$, $M(t) = 1$ and $f(s)$ is a polynomial source have been considered by many authors with the standard Laplace operator $(-\triangle)^s$, $s = 1$ and can be seen as a special case of doubly nonlinear parabolic type equations

$$(\varphi(u))_t - \Delta u = f(u),$$

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which appear in the mathematical modelling of various physical processes, such as, the flows of incompressible turbulent fluids or gases in pipes, processes of filtration in porous media, glaciology; see, e.g., [1, 2, 3, 4].

The questions of the solvability and the long time behavior of solutions to the doubly nonlinear nonlocal parabolic equation

$$(\varphi(u))_t - div\sigma(\nabla u) = \int_0^t g(t-\tau)div\sigma(\nabla u(\tau))d\tau + f(x,t,u),$$

were studied in [5, 6, 7, 8, 9, 10] . This equation arises from the study of the heat conduction in materials with memory. On the other hand, many fractional and nonlocal operators were actively studied in the recent years. This type of operators arises in a quite natural way in many interesting applications, such as, finance, physics, game theory, Lévy stable diffusion processes, crystal dislocation; see, e.g., [11, 12, 13] and the references therein.

In [14], Pan, Zhang and Cao first investigated the existence of global weak solutions to the degenerate Kirchhoff-type diffusion problems involving fractional p-Laplacian by combining the Galerkin method with the potential well theory for the special function $M(t) = t$. Mingqi, Radulescu and Zhang [15] proved the local existence and blow-up of solutions for the similar equation with more general conditions on M which cover the degenerate case.

Recently, logarithmic nonlinearity frequently appears in the partial differential equations which describes important physical phenomena; see, e.g., [16, 17, 18, 19, 20] and the references therein. Especially, Din and Zhou [17] studied the following semilinear parabolic problem of Kirchhoff type with logarithmic nonlinearity

$$u_t - M([u]_s^2)\mathscr{L}_K u = |u|^{p-2}u\log|u|.$$

They obtained the existence results of global solutions and blow-up at ∞ when the initial energy is subcritical and critical by potential well method. However, in the results above, there are few about the global existence and exponential decay rate for doubly nonlinear parabolic equation, involving variable exponent conditions, with viscoelastic term in the fractional setting. Motivated by this, we study the global existence for problem (1.1) by using Galerkin's method and similar arguments as in Tartar [21]. We also give the exponential decay rate of the energy via the energy perturbation method. It is worth mentioning that we do not use the logarithmic Sobolev inequality to get our results. The organization of this paper are as follows. In Section 2, we give the preliminaries for our main results. In Section 3, by using the Galerkin approximation method, we prove the global existence, and obtain the exponential decay under certain class of initial data.

2. Preliminaries

In this section, we present some materials and assumptions needed in the rest of this paper. We denote $Q = \mathbb{R}^{2n} \setminus (\mathscr{C}\Omega \times \mathscr{C}\Omega)$ and $\mathscr{C}\Omega := \mathbb{R}^n \setminus \Omega$,

$$W = \Big\{u : \mathbb{R}^N \to \mathbb{R} : u|_\Omega \in L^2(\Omega),\ \iint_Q \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dxdy < \infty\Big\},$$

where $u|_\Omega$ represents the restriction to Ω of function $u(x)$. Also, we define the following linear subspace of W,

$$W_0 = \big\{u \in W : u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega\big\}.$$

The linear space W is endowed with the norm

$$\|u\|_W := \|u\|_{L^2(\Omega)} + \Big(\iint_Q \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy\Big)^{1/2}.$$

It is easily seen that $\|\cdot\|_W$ is a norm on W and $C_0^\infty(\Omega) \subseteq W_0$.

The functional

$$\|u\|_{W_0} = (\iint_Q \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dxdy)^{1/2},$$

is a equivalent norm on $W_0 = \{u \in W \ : \ u(x) = 0 \ a.e. \ \text{in } \mathbb{R}^N \setminus \Omega\}$, which is a closed linear subspace of W. Furthermore, $(W_0, \|\cdot\|_{W_0})$ is a Hilbert space with inner product

$$\langle u,v\rangle_{W_0} = \iint_Q \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}dxdy.$$

We review the main embedding results for the space W_0.

Lemma 2.1. [22, 23, 24] *The embedding $W_0 \hookrightarrow L^r(\Omega)$ is continuous for any $r \in [1, 2_s^*]$, and compact for any $r \in [1, 2_s^*[$.*

Lemma 2.2. [25, Lemma 2.1] *Let $N \geq 1,\ 0 < s < 1,\ p > 1,\ q \geq 1,\ \tau > 0$ and $0 < \theta < 1$ be such that $\frac{1}{\tau} = \theta\left(\frac{1}{p} - \frac{s}{N}\right) + \frac{1-\theta}{q}$. Then,*

$$\|u\|_{L^\tau(\mathbb{R}^n)} \leq \|u\|^{\theta}_{W^{s,p}(\mathbb{R}^n)} \|u\|^{1-\theta}_{L^q(\mathbb{R}^n)}, \quad \forall u \in C_0^1(\mathbb{R}^N).$$

Now, we recall some lemmas and propositions concerning the generalized Lebesgue-Sobolev spaces. We refer the reader to [26, 27, 28] for details.

Set

$$C_+(\overline{\Omega}) = \{p(x) : p(x) \in C(\overline{\Omega}),\ p(x) > 1,\ \text{for all } x \in \overline{\Omega}\}.$$

For any $p \in C_+(\overline{\Omega})$, we define

$$p^+ = \max\{p(x) : x \in \overline{\Omega}\}, \quad p^- = \min\{p(x); x \in \overline{\Omega}\},$$

and

$$L^{p(x)}(\Omega) = \big\{u : u \text{ is a measurable real-valued function}, \int_\Omega |u(x)|^{p(x)}dx < \infty\big\},$$

with the norm

$$\|u\|_{p(x)} \equiv \|u\|_{L^{p(x)}(\Omega)} = \inf\big\{\lambda > 0 : \int_\Omega |\frac{u(x)}{\lambda}|^{p(x)}dx \leq 1\big\}$$

becomes a Banach space [29]. We also define the space

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\},$$

equipped with the norm

$$\|u\|_{W^{1,p(x)}(\Omega)} = \|u(x)\|_{p(x)} + \|\nabla u(x)\|_{p(x)}.$$

We denote by $W_0^{1,p(x)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$. Of course, the norm $\|u\| = \|\nabla u\|_{p(x)}$ is an equivalent norm in $W_0^{1,p(x)}(\Omega)$.

Proposition 2.3. [27] (i) *The conjugate space of* $L^{p(x)}(\Omega)$ *is* $L^{p'(x)}(\Omega)$, *where* $\frac{1}{p(x)}+\frac{1}{p'(x)}=1$. *For any* $u\in L^{p(x)}(\Omega)$ *and* $v\in L^{p'(x)}(\Omega)$, *we have*

$$\int_\Omega |uv|dx \le \left(\frac{1}{p^-}+\frac{1}{p'^-}\right)\|u\|_{p(x)}\|v\|_{p'(x)} \le 2\|u\|_{p(x)}\|v\|_{p'(x)}.$$

(ii) *If* $p_1(x),p_2(x)\in C_+(\overline{\Omega})$ *and* $p_1(x)\le p_2(x)$ *for all* $x\in\overline{\Omega}$, *then* $L^{p_2(x)}(\Omega)\hookrightarrow L^{p_1(x)}(\Omega)$ *and the embedding is continuous.*

Proposition 2.4. [27] *Set* $\rho(u)=\int_\Omega|\nabla u(x)|^{p(x)}dx$. *Then, for* $u\in W_0^{1,p(x)}(\Omega)$ *and* $(u_k)\subset W_0^{1,p(x)}(\Omega)$, *we have*

(1) $\|u\|<1$ *(respectively* $=1;>1$*) if and only if* $\rho(u)<1$ *(respectively* $=1;>1$*);*
(2) *for* $u\neq 0$, $\|u\|=\lambda$ *if and only if* $\rho(\frac{u}{\lambda})=1$;
(3) *if* $\|u\|>1$, *then* $\|u\|^{p^-}\le\rho(u)\le\|u\|^{p^+}$;
(4) *if* $\|u\|<1$, *then* $\|u\|^{p^+}\le\rho(u)\le\|u\|^{p^-}$;
(5) $\|u_k\|\to 0$ *(respectively* $\to\infty$*) if and only if* $\rho(u_k)\to 0$ *(respectively* $\to\infty$*).*

For $x\in\Omega$, let us define

$$p^*(x)=\begin{cases}\frac{Np(x)}{N-p(x)}, & \text{if } p(x)<N,\\ \infty, & \text{if } p(x)\ge N.\end{cases}$$

Proposition 2.5. [28] *If* $q\in C_+(\overline{\Omega})$ *and* $q(x)\le p^*(x)$ $(q(x)<p^*(x))$ *for* $x\in\overline{\Omega}$, *then there is a continuous (compact) embedding* $W^{1,p(x)}(\Omega)\hookrightarrow L^{q(x)}(\Omega)$.

Lemma 2.6. *Let* $2<r<\rho<2_s^*$. *For each* $\varepsilon>0$, *there exists a positive constant* C_ε *such that*

$$\|v\|_\rho^\rho\le\varepsilon\|v\|_{W_0}^2+C_\varepsilon\|v\|_r^{kr},$$

for all $v\in W_0\bigcap L^r(\Omega)$, *where*

$$k=\frac{2\rho(1-\theta)}{r(2-\rho\theta)} \quad , \quad \theta=\left(\frac{1}{r}-\frac{1}{\rho}\right)\left(\frac{s}{N}-\frac{1}{2}+\frac{1}{r}\right)^{-1}.$$

Proof. The conclusion of this lemma immediately follows from Lemma 2.2 and Young's inequality. □

Lemma 2.7. [29, Theorem 1, pag 23] *Suppose that* $r\in L_+^\infty(\Omega)$, $r^-\ge 2$, $w\in L^{r(x)}(\Omega\times]0,T[)$ *and*

$$\frac{\partial}{\partial t}(|w|^{r(x)-2}w)\in L^{r'(x)}(\Omega\times]0,T[).$$

Then, for any $s,\tau\in[0,T]$ *with* $s<\tau$, *the following formula of integration by parts is true*

$$\int_s^\tau\int_\Omega w\left(\frac{1}{r(x)-1}|w|^{r(x)-2}w\right)dx\,dt=\int_\Omega\frac{1}{r(x)}|w(\tau)|^{r(x)}dx-\int_\Omega\frac{1}{r(x)}|w(s)|^{r(x)}dx.$$

3. Global existence and exponential decay

In this section, we focus our attention on the global existence and exponential decay of the solution to problem (1.1).

Definition 3.1. Let $T>0$. A weak solution of problem (1.1) is a function $u\in L^{\infty}(0,T;W_0)$, with $u_t\in L^2(0,T;L^2(\Omega))$ and $\left(|u|^{r(x)/2}\right)_t\in L^2\left(\Omega\times]0,T[\right)$ such that

$$\int_0^T\int_\Omega\left(1+a|u|^{r(x)-2}\right)u_t w\,dxdt+M(\|u\|_{W_0}^2)\int_0^T\langle u,w\rangle_{W_0}\,dt$$
$$-\int_0^T\int_0^t g(t-\tau)\langle u(\tau),w\rangle_{W_0}d\tau dt=\int_0^T\int_\Omega u\log|u|w\,dxdt,$$

for all $w\in L^2(0,T;W_0)$, and $u(x,0)=u^0(x)\in W_0$.

Theorem 3.2 (Local Solution)**.** *Assume $u^0\in W_0\setminus\{0\}$, $2<r^-<2_s^*$, $r^+\in]2,2_s^*[$. Then, problem* (1.1) *has a unique weak solution u for T small enough.*

Proof. We prove the local existence of weak solutions by using the Faedo–Galerkin method benefited from the ideas of [30]. We choose a sequence $\{w_\nu\}_{\nu\in\mathbb{N}}\subseteq C_0^\infty(\Omega)$ such that

$$C_0^\infty(\Omega)\subseteq\overline{\bigcup_{\nu=1}^{\infty}V_m}^{C^1(\overline{\Omega})}$$

and $\{w_\nu\}$ is a standard orthonormal basis with respect to the Hilbert space $L^2(\Omega)$ and an orthogonal basis in W_0, where $V_m=spam\{w_1,w_2,....w_m\}$.

Now, we construct approximate solutions u_m $(m=1,2,\cdots)$, of the problem (1.1), in the form

$$u_m(x,t)=\sum_{i=1}^m g_{jm}(t)w_j(x),$$

where the coefficient functions g_{jm} satisfy the system of ordinary differential equations

$$\begin{aligned}&\int_\Omega\left(1+a|u_m(t)|^{r(x)-2}\right)u_{mt}(t)w_j\,dx+M(\|u_m(t)\|_{W_0}^2)\langle u_m(t),w_j\rangle_{W_0}\\&-\int_0^t g(t-\tau)\langle u_m(\tau),w_j\rangle_{W_0}d\tau dt=\int_\Omega u_m\log|u_m|w_j\,dx,\quad j=1,2,\cdots m,\\&u_m(x,0)=u_m^0(x)\to u^0(x)\quad\text{in}\quad W_0.\end{aligned}\tag{3.1}$$

Let us show that system (3.1) is locally solvable. It is clear that (3.1) can be rewritten in the form

$$\frac{d}{dt}\Phi(g_m(t))=-M\Big(\Big\|\sum_{i=1}^m g_{jm}(t)w_j(x)\Big\|_{W_0}^2\Big)Bg_m(t)+\int_0^t g(t-\tau)Bg_m(\tau)d\tau+F(g_m(t)),\tag{3.2}$$

where

$$g_m(t)=(g_{m1}(t),g_{m2}(t),\cdots,g_{mm}(t))^t,\quad B=\left[\langle w_i,w_j\rangle\right]_{1\le i,j\le m},$$

$$\Phi(\eta)=(\Phi_1(\eta),\Phi_2(\eta),\cdots,\Phi_m(\eta))^t\qquad\text{with }\eta=(\eta_1,\eta_2,\cdots,\eta_m)\in\mathbb{R}^m,$$

$$\Phi_i(\eta)=\int_\Omega\Big\{\sum_{j=1}^m\eta_jw_j+\frac{a}{r(x)-1}\Big|\sum_{k=1}^m\eta_kw_k\Big|^{r(x)-2}\sum_{k=1}^m\eta_kw_k\Big\}w_i\,dx,\quad i=1,2,\cdots,m$$

and

$$F(\eta) = \left(\int_\Omega \left(\sum_{k=1}^m \eta_j w_j\right) \log\left(|\sum_{k=1}^m \eta_j w_j|\right) w_1\,dx, \cdots, \int_\Omega \left(\sum_{k=1}^m \eta_j w_j\right) \log\left(|\sum_{k=1}^m \eta_j w_j|\right) w_m\,dx\right)^t.$$

This system is equivalent to

$$\Phi(g_m(t)) = \Phi(g_m(0)) + \int_0^t \left[-M\Big(\left\|\sum_{i=1}^m g_{jm}(t)w_j(x)\right\|_{W_0}^2\Big)Bg_m(t) + \int_0^\xi g(\xi-\tau)Bg_m(\tau)d\tau + F(g_m(\xi))\right] d\xi.$$

Using the fact that $s \mapsto s\log s$ is increasing for large s, we get

$$(\Phi(\zeta)-\Phi(\eta), \zeta-\eta)_{\mathbb{R}^m} \geq C_m|\zeta-\eta|^2_{\mathbb{R}^m} \tag{3.3}$$

for $\zeta, \eta \in \mathbb{R}^m$, where C_m is a constant such that, for any g_m in $\mathbb{R}^m$,

$$\int_\Omega |u_m|^2\,dx \geq C_m|g_m|^2_{\mathbb{R}^m}.$$

So, by virtue of the elementary inequality $s\log \geq s-1$, $\forall s>0$, we deduce that Φ is monotone coercive. Also, it is obviously continuous. So, from the Brouwer theorem, we have that Φ is onto. In view of (3.3), Φ^{-1} is locally Lipchitz continuous. Consider the map $L : C(0,T,\mathbb{R}^m) \to C(0,T,\mathbb{R}^m)$ defined by

$$L(g_m)(t) = \Phi^{-1}\Big(\Phi(g_m(0)) + \int_0^t \left[-M\left(\left\|\sum_{i=1}^m g_{jm}(t)w_j(x)\right\|_{W_0}^2\right)Bg_m(t) + \int_0^\xi g(\xi-\tau)Bg_m(\tau)d\tau + F(g_m(\xi))\right] d\xi\Big),$$
$$t \in [0,T].$$

It is not hard to prove that L is completely continuous and also, there exist (sufficient small) $T_m > 0$ and (sufficient large) $R > 0$ such that $L(\overline{B_R}) \subseteq \overline{B_R}$, where $\overline{B_R}$ is the ball in $C(0,T_m,\mathbb{R}^m)$ with center the origin and radius R. Consequently, by Schauder's theorem, the operator L has a fixed point in $C(0,T_m,\mathbb{R}^m)$. This fixed point is a solution of (3.2). So, we can obtain an approximate solution $u_m(t)$ of (3.1) in V_m over $[0,T_m[$ and it can be extended to the whole interval $[0,T]$, for all $T>0$, as a consequence of the a priori estimates that shall be proven in the next step.

The First Estimate

Multiplying (3.1) by $g_{jm}(t)$ and adding in $j = 1;\cdots;m$, we have

$$\begin{aligned}\int_\Omega \left(1+a|u_m(t)|^{r(x)-2}\right) u_{mt}(t)u_m(t)\,dx + M(\|u_m(t)\|^2_{w_0})\langle u_m(t), u_m(t)\rangle_{W_0} \\ -\int_0^t g(t-\tau)\langle u_m(\tau), u_m(t)\rangle_{W_0} d\tau dt = \int_\Omega |u_m(t)|^2 \log|u_m(t)|\,dx,\end{aligned} \tag{3.4}$$

which implies by integrating with respect to the time variable from 0 to t on both sides, and using Lemma 2.7 that

$$S_m(t) = S_m(0) + \int_0^t d\lambda \int_0^\lambda g(\lambda-\tau)\langle u_m(\tau), u_m(\lambda)\rangle_{W_0}\, d\tau + \int_0^t \int_\Omega |u_m(\tau)|^2 \log|u_m(\tau)|\, dx\, d\tau, \tag{3.5}$$

where

$$S_m(t) = \int_\Omega |u_m(t)|^2\, dx + a\int_\Omega \frac{1}{r(x)}|u_m(t)|^{r(x)}\, dx + \int_0^t \left(\|u_m(\tau)\|_{W_0}^{2\alpha} + \|u_m(\tau)\|_{W_0}^2\right) d\tau.$$

Let us introduce the function $\Theta(\lambda) = \int_0^\lambda g(\lambda-\tau)\|u_m(\tau)\|_{W_0}$. Estimating the second term on right-hand side of (3.5), we have

$$\int_0^t d\lambda \int_0^\lambda g(\lambda-\tau)\langle u_m(\tau), u_m(\lambda)\rangle_{W_0}\, d\tau \le \frac{1}{2}\int_0^t \left(\|u_m(\tau)\|_{W_0}^{2\alpha} + \|u_m(\tau)\|_{W_0}^2\right) d\tau + \frac{1}{2}\int_0^t \Theta^2(\lambda)\, d\lambda. \tag{3.6}$$

But, using Young Inequality and noting that $\int_0^\infty g(\tau)d\tau < 1$, we get

$$\int_0^t \Theta^2(\lambda)d\lambda \le \int_0^\infty g(\tau)d\tau \int_0^t \left(\|u_m(\tau)\|_{W_0}^{2\alpha} + \|u_m(\tau)\|_{W_0}^2\right) d\tau. \tag{3.7}$$

Let $\rho := 2_s^* - 2$. Since $\log(|u|^\rho) \le |u|^\rho$, one has

$$\int_\Omega |u_m(t)|^2 \log|u_m(t)|\, dx = \frac{1}{\rho}\int_\Omega |u_m(t)|^2 \log(|u_m(t)|^\rho)\, dx \le \frac{1}{\rho}\int_\Omega |u_m(t)|^{\rho+2}\, dx. \tag{3.8}$$

Plugging (3.6)- (3.8) into (3.5), it follows that

$$S_m(t) \le S_m(0) + \frac{1}{2}\left(1 + \int_0^\infty g(\tau)d\tau\right)\int_0^t \left(\|u_m(\tau)\|_{W_0}^{2\alpha} + \|u_m(\tau)\|_{W_0}^2\right) d\tau + \frac{1}{\rho}\int_0^t \|u_m(t)\|_{\rho+2}^{\rho+2} d\tau. \tag{3.9}$$

To estimate the last term in (3.9) we use Lemma 2.6,

$$\int_0^t \|u_m(t)\|_{\rho+2}^{\rho+2} d\tau \le \varepsilon \int_0^t \left(\|u_m(\tau)\|_{W_0}^{2\alpha} + \|u_m(\tau)\|_{W_0}^2\right) d\tau + c_0\int_0^t S_m^k(\lambda)\, d\lambda, \tag{3.10}$$

where $k = \frac{2(\rho+2)(1-\theta)}{r^-[2-(\rho+2)\theta]} > 1$. Taking ε suitably small in (3.10), it follows from (3.5)-(3.10) that

$$S_m(t) \le \hat{C}_0 + \hat{C}_1\int_0^t S_m^k(\lambda)\, d\lambda. \tag{3.11}$$

Hence, by employing Bihari-Langenhop's inequality (cf. [31]), there exists a constant T_0 such that

$$S_m(t) \le C_{T_0}, \quad \forall t \in [0, T_0]. \tag{3.12}$$

The Second Estimate

Multiplying (3.1) by $g'_{jm}(t)$ and adding in $j = 1;\cdots;m$, it holds that

$$\frac{d}{dt}\Big\{\frac{1}{2\alpha}\|u_m(t)\|_{W_0}^{2\alpha} + \frac{1}{2}\left(1 - \int_0^t g(\tau)d\tau\right)\|u_m(t)\|_{W_0}^2 + \frac{1}{2}(g\diamond u)(t)$$
$$-\frac{1}{2}\int_\Omega |u_m(t)|^2\log|u_m(t)|\,dx + \frac{1}{4}\|u_m(t)\|_2^2\Big\} + \|u_{mt}(t)\|_2^2 + a\int_\Omega |u_m(t)|^{r(x)-2}|u_{mt}(t)|^2\,dx$$
$$= \frac{1}{2}(g'\diamond u)(t) - \frac{1}{2}g(t)\|u_m(t)\|_{W_0}^2, \tag{3.13}$$

where

$$(g\diamond u)(t) = \int_0^t g(t-\tau)\|u(t)-u(\tau)\|_{W_0}^2\,d\tau.$$

Integrating (3.13) on $[0,t]$ with $t \le T_0$, we get

$$\int_0^t \|u_{mt}(t)\|_2^2 + a\int_0^t\int_\Omega |u_m(t)|^{r(x)-2}|u_{mt}(t)|^2\,dx + \frac{1}{2\alpha}\|u_m(t)\|_{W_0}^{2\alpha} + \frac{l}{2}\|u_m(t)\|_{W_0}^2 \le$$
$$\frac{1}{2\alpha}\|u_m(0)\|_{W_0}^{2\alpha} + \frac{1}{2}\|u_m(0)\|\|_{W_0}^2 + \frac{1}{2}\int_\Omega |u_m(t)|^2\log|u_m(t)|\,dx + \frac{1}{4}\|u_m(0)\|_2^2$$
$$-\frac{1}{2}\int_\Omega |u_m(0)|^2\log|u_m(0)|\,dx.$$

From the assumptions on u^0, (3.8), Lemma 2.6 and (3.12), it follows that

$$\int_0^t \|u_{mt}(t)\|_2^2 + a\int_0^t\int_\Omega |u_m(t)|^{r(x)-2}|u_{mt}(t)|^2\,dx + \frac{1}{2\alpha}\|u_m(t)\|_{W_0}^{2\alpha} + \frac{l}{2}\|u_m(t)\|_{W_0}^2 \le M_1, \tag{3.14}$$

for some constant $M_1 > 0$. From (3.12) and (3.14), we have that $\{u_m\}$ have subsequences, still denoted by $\{u_m\}$, such that

$$u_m \to u \quad \text{weakly* in } L^\infty(0,T_0;W_0), \tag{3.15}$$

$$u_{mt} \to u_t \quad \text{weakly in } L^2(0,T_0;L^2(\Omega)), \tag{3.16}$$

$$\left(|u_m|^{r(x)/2}\right)_t \to \chi \quad \text{weakly in } L^2(0,T_0;L^2(\Omega)). \tag{3.17}$$

Also, reasoning as in [16], and taking into account the compact embedding of W_0 into $L^2(\Omega)$, we have

$$u_m\log|u_m| \to u\log|u| \quad \text{weakly* in } L^\infty(0,T_0;L^2(\Omega)). \tag{3.18}$$

Employing the same arguments as in [32] we arrive at

$$\chi = \left(|u|^{r(x)/2}\right)_t, \quad |u_m|^{r(x)/2}u_{mt} \to |u|^{r(x)/2}u_t \text{ weakly in } L^2(\Omega\times]0,T_0[) \tag{3.19}$$

Therefore, passing to the limit in (3.1) as $m \to +\infty$, and using (3.15)– (3.19), we can show that u satisfies the initial condition $u(0) = u^0$ and

$$\int_0^T\int_\Omega \left(1 + a|u|^{r(x)-2}\right)u_t w\,dxdt + M(\|u\|_{w_0}^2)\int_0^T \langle u,w\rangle_{W_0}\,dt$$
$$-\int_0^T\int_0^t g(t-\tau)\langle u(\tau),w\rangle_{W_0}d\tau dt = \int_0^T\int_\Omega u\log|u|w\,dxdt,$$

for all $w \in L^2(0,T_0;W_0)$.

The uniqueness property of a solutions can be derived from [3, Theorem 3, p. 1095], observing that $\left(u+\frac{a}{r(x)-1}|u|^{r(x)-2}u\right) \in L^2(\Omega\times]0,T_0[)$, $F(s)=s\log(|s|)$ is locally Lipschitz continuous and $Au=M(\|u\|^2_{w_0})(-\Delta)^s u$ is a monotone operator. We omit the details. □

Next, we consider the global existence and energy decay of solutions to problem (1.1). For this purpose, we define the energy associated with problem (1.1) by

$$\begin{aligned} E(t)=&\frac{1}{2\alpha}\|u(t)\|^{2\alpha}_{W_0}+\frac{1}{2}\left(1-\int_0^t g(\tau)d\tau\right)\|u(t)\|^2_{W_0}+\frac{1}{2}(g\diamond u)(t) \\ &-\frac{1}{2}\int_\Omega |u(t)|^2\log|u(t)|\,dx+\frac{1}{4}\|u(t)\|_2^2. \end{aligned} \tag{3.20}$$

Then, we easily can check that

$$\frac{d}{dt}E(t)=\frac{1}{2}(g'\diamond u)(t)-\frac{1}{2}g(t)\|u(t)\|^2_{W_0}-\|u_t(t)\|_2^2-a\int_\Omega |u(t)|^{r(x)-2}u_t^2(t)\,dx\le 0 \tag{3.21}$$

for any regular solution. This remains valid for weak solutions by simple density argument. This shows that $E(t)$ is a nonincreasing function. Let C_* be the optimal constant satisfying the Sobolev inequality $\|u\|_{\rho+2}\le C_*\|u\|_{W_0}$, and $B_1=\frac{C_*}{\sqrt{l}}\left(\frac{\rho+2}{\rho}\right)^{1/(\rho+2)}$. We define the function

$$h(\lambda)=\frac{1}{2}\lambda^2-\frac{B_1^{\rho+2}}{\rho+2}\lambda^{\rho+2}.$$

Then, we can verify that the function h is increasing in $]0,\lambda_1[$, decreasing in $]\lambda_1,\infty[$, $h(\lambda)\to-\infty$, as $\lambda\to\infty$ and h has a maximum at λ_1 with the maximum value

$$h(\lambda_1)=E_1=\left(\frac{1}{2}-\frac{1}{\rho+2}\right)B_1^{-\frac{2(\rho+2)}{\rho}}=\frac{\rho}{2(\rho+2)}B_1^{-\frac{2(\rho+2)}{\rho}},$$

where λ_1 is the first positive zero of the derivative function $h'(\lambda)$. Here, note that

$$\begin{aligned} E(t)&\ge\frac{l}{2}\|u(t)\|^2_{W_0}+\frac{1}{2}(g\diamond u)(t)-\frac{1}{\rho}\|u(t)\|^{\rho+2}_{\rho+2} \\ &\ge\frac{1}{2}\left(l\|u(t)\|^2_{W_0}+(g\diamond u)(t)\right)-\frac{B_1^{\rho+2}}{\rho+2}\|u(t)\|^{\rho+2}_{W_0} \\ &\ge h\left(\sqrt{l\|u(t)\|^2_{W_0}+(g\diamond u)(t)}\right),\ \forall t\ge 0. \end{aligned} \tag{3.22}$$

Now, we are ready to state our result.

Theorem 3.3. *Assume that hypotheses of Theorem 3.2 are satisfied. Consider $u_0\in W_0$ with*

$$0<l^{1/2}\|u_0\|_{W_0}<\lambda_1, \tag{3.23}$$

$$\frac{1}{2\alpha}\|u^0\|^{2\alpha}_{W_0}+\frac{1}{2}\|u^0\|^2_{W_0}-\frac{1}{2}\int_\Omega |u^0|^2\log|u^0|\,dx+\frac{1}{4}\|u^0\|_2^2<\frac{\rho}{2(\rho+2)}B_1^{-\frac{2(\rho+2)}{\rho}}. \tag{3.24}$$

Then, problem (1.1) admits a global weak solution in time. In addition, if there exists a constant $\xi_0>0$ such that $g'(t)\le-\xi_0 g(t)$, then this solution satisfies

$$E(t)\le L_0e^{-\gamma t},\forall t\ge 0, \tag{3.25}$$

where L_0 and γ are positive constants.

Proof. We will get global estimates for $u_m(t)$, the solution of the approximate system (3.1), under the conditions (3.23)–(3.24) for u^0. For this, it suffices to show that

$$E_m(t)+\int_0^t \|u_{mt}(t)\|_2^2+a\int_0^t\int_\Omega |u_m(t)|^{r(x)-2}|u_{mt}(t)|^2\,dx,$$

where $E_m(t)$ is defined in (3.20) with $u(t)$ being replaced by $u_m(t)$, which is bounded and independently of t. From (3.13) and the definition of energy, we have

$$E_m(t)+\int_0^t \|u_{mt}(t)\|_2^2+a\int_0^t\int_\Omega |u_m(t)|^{r(x)-2}|u_{mt}(t)|^2\,dx \le E_m(0). \tag{3.26}$$

Due to the convergence $u_{0m}\to u^0$ in W_0, we see that $E_m(0)<\frac{\rho}{2(\rho+2)}B_1^{-\frac{2(\rho+2)}{\rho}}$ for sufficiently large m. We claim that there exists an integer ν_0 such that

$$\sqrt{l\|u_m(t)\|_{W_0}^2+(g\diamond u_m)(t)}<\lambda_1 \qquad \forall t\in[0,T_m[, m\ge\nu_0. \tag{3.27}$$

Suppose the claim is proved. Then,

$$h\left(\sqrt{l\|u_m(t)\|_{W_0}^2+(g\diamond u_m)(t)}\right)\ge 0.$$

From (3.22) and (3.26)–(3.27), we get

$$\|u_m(t)\|_{W_0}^{2\alpha}+\|u_m(t)\|_{W_0}^2+\int_0^t \|u_{mt}(t)\|_2^2+a\int_0^t\int_\Omega |u_m(t)|^{r(x)-2}|u_{mt}(t)|^2\,dx\le C. \tag{3.28}$$

where C is a constant independent of m. Thus, we obtain the global existence.

Proof of Claim: Suppose (3.27) is not true. Thus, for each $m>\nu_0$, there exists $t_1\in[0,T_m[$ such that

$$\sqrt{l\|u_m(t_1)\|_{W_0}^2+(g\diamond u_m)(t_1)}\ge\lambda_1. \tag{3.29}$$

Here, we observe from (3.23) and the convergence $u_{0m}\to u^0$ in W_0 that there exists ν_1 such that

$$l^{1/2}\|u_m(0)\|_{W_0}<\lambda_1 \qquad \forall m>\nu_1.$$

Hence, there exists

$$t^*=\inf\{t\in[0,T_m[: \sqrt{l\|u_m(t)\|_{W_0}^2+(g\diamond u_m)(t)}\ge\lambda_1\},$$

such that

$$\sqrt{l\|u_m(t^*)\|_{W_0}^2+(g\diamond u_m)(t^*)}=\lambda_1. \tag{3.30}$$

Using (3.22), we see that

$$E_m(t^*)\ge h\left(\sqrt{l\|u_m(t^*)\|_{W_0}^2+(g\diamond u_m)(t^*)}\right)=h(\lambda_1)=E_1 \tag{3.31}$$

which contradicts $E_m(t)\le E_m(0)<E_1, \forall t\ge 0$. Therefore our claim is true. The above estimates permit us to pass to the limit in the approximate equation. To show the uniform decay of the solution, we introduce the perturbed energy functional

$$F(t)=E(t)+\varepsilon\Phi(t), \tag{3.32}$$

where ε is a positive constant, which shall be determined later, and

$$\Phi(t)=\int_\Omega\left(|u|^2+\frac{a}{r(x)}|u|^{r(x)}\right)dx. \tag{3.33}$$

It is straightforward to see that $F(t)$ and $E(t)$ are equivalent in the sense that there exist two positive constants β_1 and β_2 depending on ε such that, for $t \geq 0$,

$$\beta_1 E(t) \leq F(t) \leq \beta_2 E(t). \tag{3.34}$$

Taking the time derivative of the function F defined above in (3.32), using (3.21), and performing several integration by parts, we get

$$\begin{aligned}\frac{d}{dt}F(t) &= \frac{1}{2}(g'\diamond u)(t) - \frac{1}{2}g(t)\|u(t)\|_{W_0}^2 - \|u_t(t)\|_2^2 - a\int_\Omega |u(t)|^{r(x)-2}u_t^2(t)\,dx + \\ &-\varepsilon\|u(t)\|_{W_0}^{2\alpha} - \varepsilon\|u(t)\|_{W_0}^2 + \varepsilon\int_\Omega |u(t)|^2\log|u(t)|\,dx + \varepsilon\int_0^t g(t-\tau)\langle u(\tau),u(t)\rangle_{W_0}\,d\tau.\end{aligned} \tag{3.35}$$

On the other hand, we can easily see that the condition $E(0) < E_1$ is equivalent to the inequality

$$B_1^{\rho+2}\left(\frac{2(\rho+2)}{\rho}E(0)\right)^{\frac{\rho}{2}} < 1. \tag{3.36}$$

From the assumption (3.23)–(3.24) and (3.26), we have

$$\begin{aligned} l\|u(t)\|_{W_0}^2 &\leq \left(1 - \int_0^t g(\tau)d\tau\right)\|u(t)\|_{W_0}^2 + (g\diamond u)(t) \\ &< \lambda_1^2 = B_1^{-\frac{2(\rho+2)}{\rho}},\end{aligned}$$

which implies that

$$\begin{aligned} I(t) &= \left(1 - \int_0^t g(\tau)d\tau\right)\|u(t)\|_{W_0}^2 + (g\diamond u)(t) - \int_\Omega |u(t)|^2\log|u(t)|\,dx \\ &\geq l\|u(t)\|_{W_0}^2 + (g\diamond u)(t) - \frac{1}{\rho}\|u(t)\|_{\rho+2}^{\rho+2} \\ &\geq l\|u(t)\|_{W_0}^2 - \frac{C_*^{\rho+2}}{\rho}\|u(t)\|_{W_0}^{\rho+2} \geq 0.\end{aligned}$$

So, we have

$$\begin{aligned}\frac{1}{2\alpha}\|u(t)\|_{W_0}^{2\alpha} &\leq \frac{1}{2\alpha}\|u(t)\|_{W_0}^{2\alpha} + \frac{1}{4}\|u(t)\|_2^2 + \frac{1}{2}I(t) \\ &\leq E(t) \leq E(0),\end{aligned}$$

and then

$$\|u(t)\|_{W_0}^{\rho} \leq (2\alpha E(0))^{\frac{\rho}{2\alpha}}. \tag{3.37}$$

Using the above inequality, we can deduce that

$$\begin{aligned} \left|\int_\Omega |u|^2\log|u|\right| &\leq \frac{1}{\rho}\|u(t)\|_{\rho+2}^{\rho+2} \\ &\leq \frac{C_*^{\rho+2}}{\rho}\|u(t)\|_{W_0}^{\rho+2} \\ &\leq \left[\frac{C_*^{\rho+2}}{\rho l}(2\alpha E(0))^{\frac{\rho}{2\alpha}}\right] l\|u(t)\|_{W_0}^2 \\ &\equiv \theta l\|u(t)\|_{W_0}^2.\end{aligned} \tag{3.38}$$

From the Young inequality and the fact that $\int_0^t g(\tau)\,d\tau \le \int_0^\infty g(\tau)\,d\tau = 1-l$, it follows that

$$\begin{aligned}
&\int_0^t g(t-\tau)\langle u(\tau),u(t)\rangle_{W_0}\,d\tau \\
&\le \frac{1}{2}\|u(t)\|_{W_0}^2+\frac{1}{2}\Big\{\int_0^t g(t-\tau)\left(\|u(\tau)-u(t)\|_{W_0}+\|u(t)\|_{W_0}\right)d\tau\Big\}^2 \\
&\le \frac{1}{2}\|u(t)\|_{W_0}^2+\frac{1}{2}(1+\eta)\left(\int_0^t g(t-\tau)\|u(t)\|_{W_0}\,d\tau\right)^2 \\
&+\frac{1}{2}(1+\frac{1}{\eta})\left(\int_0^t g(t-\tau)\|u(\tau)-u(t)\|_{W_0}\,d\tau\right)^2 \\
&\le \frac{1}{2}\|u(t)\|_{W_0}^2+\frac{1}{2}(1+\eta)(1-l)^2\|u(t)\|_{W_0}^2+\frac{1}{2}(1+\frac{1}{\eta})(1-l)(g\diamond u)(t) \qquad (3.39)
\end{aligned}$$

for any $\eta>0$. Now, letting $\eta=\frac{l}{1-l}>0$, we conclude from (3.39) that

$$\int_0^t g(t-\tau)\langle u(\tau),u(t)\rangle_{W_0}\,d\tau \le \frac{2-l}{2}\|u(t)\|_{W_0}^2+\frac{1-l}{2l}(g\diamond u)(t). \qquad (3.40)$$

Substituting (3.40) into (3.35), we obtain

$$\frac{d}{dt}F(t)\le -\frac{1}{2}\left(\xi_0-\varepsilon\frac{1-l}{l}\right)(g\diamond u)(t)-\varepsilon\|u(t)\|_{W_0}^{2\alpha}-\frac{\varepsilon l}{2}\|u(t)\|_{W_0}^2+\varepsilon\int_\Omega |u(t)|^2\log|u(t)|\,dx. \qquad (3.41)$$

Using the definition of $E(t)$ and (3.38) we have, for any positive constant M,

$$\begin{aligned}
\frac{d}{dt}F(t)\le -M\varepsilon E(t)+\varepsilon\left(\frac{M}{2\alpha}-1\right)\|u(t)\|_{W_0}^{2\alpha}+\frac{\varepsilon}{2}\left[(1+\frac{c_{*s}^2}{2})M+(M+2)\theta l-l\right]\|u(t)\|_{W_0}^2 \\
+\frac{1}{2}\left[\varepsilon(\frac{1-l}{l}+\frac{M}{2})-\xi_0\right](g\diamond u)(t). \qquad (3.42)
\end{aligned}$$

At this point, we choose $1>M>0$ and $E(0)$ small sufficiently such that

$$\frac{M}{2\alpha}-1<0 \qquad \text{and} \qquad (1+\frac{c_{*s}^2}{2})M+(M+2)\theta l-l<0.$$

where c_{*s} is given by

$$\frac{1}{c_{*s}}=\inf_{u\in W_0\backslash 0}\frac{\|u\|_{W_0}}{\|u\|_2}.$$

After M is fixed, we choose ε small enough such that

$$\varepsilon(\frac{1-l}{l}+\frac{M}{2})-\xi_0<0.$$

Inequality (3.42) becomes $\frac{d}{dt}F(t)\le -M\varepsilon E(t)$. By (3.34), we have

$$\frac{d}{dt}F(t)\le -M\beta_2\varepsilon F(t).$$

So $F(t)\le Ce^{-Kt}$, where $K=M\beta_2\varepsilon>0$. Consequently, by using (3.34) once again, we conclude the result. Thus, the proof of Theorem 3.3 is completed. □

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