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ENTROPY SOLUTIONS OF NONLINEAR ELLIPTIC EQUATIONS WITH L^1-DATA AND WITHOUT STRICT MONOTONOCITY CONDITIONS IN WEIGHTED ORLICZ-SOBOLEV SPACES

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Abstract. In this paper, we study the existence of entropy solutions for a class of nonlinear elliptic problems in weighted Orlicz-Sobolev spaces of with the form $Au+g(x,u)=f$, where $A(u)=-\text{div }(\rho(x)a(x,u,\nabla u))$ is a Leray-Lions operator defined from the weighted Orlicz-sobolev spaces $W_0^1L_M(\rho,\Omega)$ into its dual. The right hand side $f\in L^1(\Omega)$, and the function $a(x,s,\xi)$ satisfies only the large monotonicity instead of the monotonicity strict.

Keywords. Elliptic problem; Entropy solutions; Weighted Orlicz-Sobolev spaces; Compact imbedding; Δ_2-condition.

1. INTRODUCTION

The aim of this paper is to study the existence of entropy solutions to the following nonlinear Dirichlet problem

$$\begin{cases} A(u)+g(x,u)=f(x) & \text{in} \quad \Omega, \\ u=0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.1}$$

where $A(u)=-\text{div }(\rho(x)a(x,u,\nabla u))$, Ω is a bounded domain of $\mathbb{R}^N$, $N\geq 2$. $a(x,u,\nabla u)=(a_i(x,u,\nabla u))_{1\leq i\leq N}$, $a_i:\Omega\times\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}$ is a Carathéodory functions (that is measurable with respect to x in Ω for every $(s,\xi)\in\mathbb{R}\times\mathbb{R}^N$, and continuous with respect to $(s,\xi)\in\mathbb{R}\times\mathbb{R}^N$ for almost every $x\in\Omega$) such that, for all ξ, ξ' in $\mathbb{R}^N$,

$$|a_i(x,s,\xi)|\leq|\phi_i(x)|+K_i\overline{P}^{-1}(\rho^{-1}(x)M(c_2|s|))+K_i(\overline{M}^{-1}M(c_1|\xi|)), \tag{1.2}$$

$$\Big(a(x,s,\xi)-a(x,s,\xi')\Big)(\xi-\xi')\geq 0, \tag{1.3}$$

$$a(x,s,\xi)\xi\geq M(\lambda_1|\xi|), \tag{1.4}$$

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where c_1, c_2, λ_1 and K_i belongs to $\mathbb{R}_+$ and M, P are two N-functions such that $P \prec\prec M$. Moreover $\overline{M}$, $\overline{P}$ are the complementary functions of M and P, respectively, ρ is a weight function on Ω (that is, measurable and positive a.e. on Ω) and $\phi_i \in E_{\overline{M}}(\Omega, \rho)$.

Moreover, the function $g(x,s)$ is a Carathéodory function satisfying

$$g(x,s)s \geq 0, \tag{1.5}$$

$$\sup_{|s|\leq n} |g(x,s)| = h_n(x) \in L^1(\Omega), \tag{1.6}$$

$$f \in L^1(\Omega), \tag{1.7}$$

The notion of entropy solutions, used in [1], allows us to give a meaning to a possible solution of (1.1). In fact, in [1], Boccardo proved, for $\rho(x) = 1$ and p such that $2 - 1/N < p < N$, the existence and the regularity of an entropy solution u of problem (1.1). For the case that $\rho(x) = 1$, $\phi = 0$ and f is a bounded measure, Bénilan et al. [2] proved the existence and the uniqueness of entropy solutions. The same problem is treated by using the notion of entropy solutions introduced in [3], where $\rho(x) = 1, f \in L^1(\Omega)$, and $F \in L^{p'}(\Omega)^N$. In the framework of weighted Sobolev spaces, Akdim, Azroul and Rhoudaf [4] proved the existence of T-solutions for the elliptic problem

$$\begin{cases} -\operatorname{div} a(x,u,\nabla u) = F & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

where $F \in W^{-1,p'}(\Omega, \omega^*)$, and only used the large monotonicity on the Carathéodory function $a(x,u,\nabla u)$. For the case of Orlicz spaces, Gossez and Mustonen [5] studied the following strongly nonlinear elliptic problem

$$A(u) + g(x,u) = f \quad \text{in} \quad \Omega. \tag{1.8}$$

They proved the existence of solutions for the unilateral elliptic problem (1.8). We refer to [6] for the anisotropic case, and [7] for the case of variable exponent.

In the general framework of weighted Orlicz-Sobolev spaces, the authors in [8] investigated the equation

$$-\operatorname{div}(\rho(x)a(x,u,\nabla u)) - \operatorname{div}\Phi(u) + g(x,u,\nabla u) = f$$

with $\Phi = 0$ and the fact that g is a nonlinear with the following (natural) growth condition:

$$|g(x,s,\xi)| \leq b(|s|)(c(x) + M(|\xi|)\rho(x)),$$

which satisfies the sign condition

$$g(x,s,\xi)s \geq 0,$$

and the right-hand side $f \in W^{-1}E_M(\Omega,\rho)$, in the case $f \in L^1(\Omega)$, $\phi = 0$ and $g = 0$. The existence theorem was proved by the authors in [9]. Recently, much attention has been paid to the existence of solutions of elliptic and parabolic problems under various assumptions; see, e.g., [9, 10, 11, 12, 13, 14, 15] and the references therein.

It is well known that the setting of weighted Orlicz–Sobolev spaces includes many spaces as special spaces, such as, Lebesgue spaces, weighted Lebesgue spaces and Orlicz spaces; see [16]. These spaces have many applications in various fields, such as, PDE, electrorheological fluids, and image restoration; see [17, 18, 19]. The feature of this paper is to treat a class of problems for which the classical monotone operator methods (developed by Minty [20], Browder [21], Brézis [22] and Lions [23] in $W_0^{1,p}(\Omega)$ case) do not apply. The reason for this is that $a(x,u,\nabla u)$ does not need to satisfy the strict monotonicity condition, that is, $\big(a(x,s,\xi)-a(x,s,\xi')\big)(\xi-\xi')>0$, for all $\xi,\xi'\in\mathbb{R}^N$, $(\xi\neq\xi')$, of a typical Leray–Lions operator but only a large monotonicity, that is,

$$\Big(a(x,s,\xi)-a(x,s,\xi')\Big)(\xi-\xi')\geq 0, \text{ for all } \xi,\xi'\in\mathbb{R}^N,$$

The aim of this paper is to demonstrate the existence of solutions of (1.1) under the weaker assumption, the large monotonicity condition, without using the almost everywhere convergence of the gradients of the approximate equations, since this is impossible to prove in our setting. The main tools in our proof are a version of Minty's Lemma (Now we follow the idea in Minty [24]). The paper is organized as follows: We introduce some basic definitions and properties in the setting of weighted Orlicz–Sobolev spaces as well as an abstract theorem, and prepare some auxiliary results to prove our theorem in the following section, Section 2. In Section 3, the last section, we state the main result and its proof.

2. Preliminaries

This section presents some definitions and well-known results about N-functions,and weighted Orlicz-Sobolev spaces (see, e.g., [25] and [26].

2.1. **N-function.** Let $M:\mathbb{R}^+\to\mathbb{R}^+$ be an N-function, i.e. M is continuous and convex with

$$M(t)>0 \text{ for } t>0, \quad \frac{M(t)}{t}\to 0 \text{ as } t\to 0 \text{ and } \frac{M(t)}{t}\to\infty \text{ as } t\to\infty.$$

Equivalently, M admits the representation $M(t)=\int_0^t m(\tau)d\tau$, where $m:\mathbb{R}^+\to\mathbb{R}^+$ is non-decreasing right continuous with $m(0)=0$, $m(t)>0$ for $t>0$ and $m(t)\to\infty$ as $t\to\infty$. The N-function $\overline{M}$ conjugates to M defined by $\overline{M}(t)=\int_0^t\overline{m}(\tau)d\tau$, where $\overline{m}:\mathbb{R}^+\to\mathbb{R}^+$ is given by $\overline{m}(t)=\sup\{s:m(s)\leq t\}$. It is well known that we can assume that m and $\overline{m}$ are continuous and strictly increasing. We will extend the N-functions into even function on all $\mathbb{R}^+$. Clearly $\overline{\overline{M}}=M$ and has Young's inequality

$$st\leq M(t)+\overline{M}(s) \text{ for all } s,t\geq 0.$$

The N-function M is said to satisfy the Δ_2-condition every-where (resp. infinity) if there exist $k>0$ (resp. $t_0>0$) such that, $M(2t)\leq kM(t)$ for all $t\geq 0$ (resp. $t\geq t_0$).

Let P and Q be two N-functions. The notation $P\prec\prec Q$ means that P grows essentially less rapidly than Q, that is, for all $\varepsilon>0$, $\frac{P(t)}{Q(\varepsilon t)}\to 0$ as $t\to+\infty$. That is the case if and only if $\frac{Q^{-1}(t)}{P^{-1}(t)}\to 0$ as $t\to\infty$.

2.2. **Orlicz-Sobolev space.** Let Ω be a domain in $\mathbb{R}^N$, M be an N-function and $\rho(x)$ be a weight function on Ω, i.e. measurable positive a.e. on Ω such that

$$\rho\in L^1(\Omega). \tag{2.1}$$

The weighted Orlicz classe $K_M(\Omega,\rho)$ (resp. the weighted Orlicz space $L_M(\Omega,\rho)$) is the set of all (equivalence classes modulo equality a.e. in Ω) of real-valued measurable functions u

defined in Ω satisfying $m_\rho(u,M)=\int_\Omega M(|u(x)|)\rho(x)dx<\infty$

$$\Big(\text{resp. } m_\rho\Big(\frac{u}{\lambda},M\Big)=\int_\Omega M\Big(\frac{|u(x)|}{\lambda}\Big)\rho(x)dx<\infty \text{ for some } \lambda>0\Big).$$

$L_M(\Omega,\rho)$ is a Banach space under the norm

$$\|u\|_{M,\rho}=\inf\Big\{\lambda>0; m\Big(\frac{u}{\lambda},M\Big)\leq 1\Big\}. \tag{2.2}$$

The closure in $L_M(\Omega,\rho)$ of the set of bounded measurable function with compact support in $\overline{\Omega}$ is denoted by $E_M(\Omega,\rho)$ (we have usual $E_M(\Omega,\rho)\subset K_M(\Omega,\rho)\subset L_M(\Omega,\rho)$). The equality $L_M(\Omega,\rho)=E_M(\Omega,\rho)$ holds if and only if M satisfies the Δ_2-condition, for all t or for t large according to whether Ω has a infinite measure or not. The dual of $E_M(\Omega,\rho)$ can be identified with $L_{\overline{M}}(\Omega,\rho)$ by means of the pairing $\int_\Omega u(x)v(x)\rho(x)dx$, where $u\in L_M(\Omega,\rho)$ and $v\in L_{\overline{M}}(\Omega,\rho)$ and the dual norm on $L_{\overline{M}}(\Omega,\rho)$, which is equivalent to $\|.\|_{\overline{M},\Omega}$ (see [8]). The space $L_M(\Omega,\rho)$ is reflexive if and only if M and $\overline{M}$ satisfy the Δ_2-condition for all t or for t large according to whether Ω is infinite measure or note. We return now to the weighted Orlicz-Sobolev spaces $W^1L_M(\Omega,\rho)$ (resp. $W^1E_M(\Omega,\rho)$), the spaces of all functions u such that $u\in L_M(\Omega)$ (resp. $u\in E_M(\Omega)$) and their distributional derivatives up to order 1 in $L_M(\Omega,\rho)$ (resp. $E_M(\Omega,\rho)$). It is a Banach space under the norm

$$\|u\|_{1,M,\rho}=\|u\|_M+\|\nabla u\|_{M,\rho}, \tag{2.3}$$

where $\|u\|_M=\|u\|_{M,\Omega}$. Thus, $W^1L_M(\Omega,\rho)$ and $W^1E_M(\Omega,\rho))$ can be identified with the subspaces of $\prod L_{M,\rho}=L_M\times\prod L_M(\Omega,\rho)$. Te have the weak topology $\sigma(\prod L_{M,\rho},\prod E_{\overline{M},\rho})$ and $\sigma(\prod L_{M,\rho},\prod L_{\overline{M},\rho})$. The space $W_0^1E_M(\Omega,\rho)$ (resp. $W_0^1L_M(\Omega,\rho)$) is defined by the closure of $D(\Omega)$ in $W^1E_M(\Omega,\rho))$ (resp. $W^1L_M(\Omega,\rho))$ for norm (2.3) (resp. for the topology $\sigma(\prod L_{M,\rho}$, $\prod E_{\overline{M},\rho}$)).

Definition 2.1. The sequence u_n converges to u in $L_M(\Omega,\rho)$ for the modular convergence (denoted by $u_n\to u$ (mod) $L_M(\Omega,\rho)$) if $\int_\Omega M\Big(\frac{|u_n-u|}{\lambda}\Big)\rho(x)dx\to 0$ as $n\to\infty$ for some $\lambda>0$.

Definition 2.2. The sequence u_n converges to u in $W^1L_M(\Omega,\rho)$ for the modular convergence (denoted by $u_n\to u$ (mod) $W^1L_M(\Omega,\rho)$) if, for some $\lambda>0$, $\int_\Omega M\Big(\frac{|u_n-u|}{\lambda}\Big)dx\to 0$ as $n\to\infty$ and $\int_\Omega M\Big(\frac{|D^\alpha(u_n-u)|}{\lambda}\Big)\rho(x)dx\to 0$ as $n\to\infty$ for $|\alpha|=1$.

Lemma 2.3. [27] *Let M be an N-function. If $u_n\in L_M(\Omega)$ converges a.e. to u and u_n is bounded in $L_M(\Omega)$, then $u\in L_M(\Omega)$ and $u_n\to u$ for the topology $\sigma(L_M(\Omega),E_{\overline{M}}(\Omega))$.*

Lemma 2.4. [27] *If the sequence $u_n\in L_M(\Omega,\rho)$ converges to u a.e. is bounded in $L_M(\Omega,\rho)$, then $u\in L_M(\Omega,\rho)$ and $u_n\to u$ for the topology $\sigma(L_M(\Omega,\rho),E_{\overline{M}}(\Omega,\rho))$.*

2.3. **Compactness results.** Let Ω be an open bounded locally-border lipschitzian in $\mathbb{R}^N$, ρ the weight function and M an N-function. Let the following integrability assumptions hold: There exists a real $s>0$ such that

$$(M(t))^{\frac{s}{s+1}} \text{ be } N\text{-function and that } \rho^{-s}\in L^1(\Omega), \tag{2.4}$$

$$\int_1^\infty \frac{t}{M(t)^{1+\frac{s}{N(s+1)}}}dM(t)=\infty, \tag{2.5}$$

$$\lim_{t\to\infty}\frac{1}{M^{-1}(t)}\int_0^{t^{\frac{s+1}{s}}}\frac{M^{-1}(u)}{u^{1+\frac{s}{N(s+1)}}}du=0. \tag{2.6}$$

Remark 2.5. In the particular case where $M(t)=\frac{|t|^p}{p}$ $(1<p<\infty)$, the first part of (2.4) is satisfied if $s>\frac{1}{p-1}$.

Theorem 2.6. [28, Theorem 9-5] *Let Ω be an open bounded subset of $\mathbb{R}^N$ with locally-lipschitzian, and M an N-function. Suppose that assumptions (2.4)–(2.6) are satisfied. So, we have the following compact injection :*

$$W^1L_M(\Omega,\rho)\hookrightarrow\hookrightarrow E_M(\Omega).$$

Theorem 2.7. *(Weighted Poincaré inequality) Let Ω be a bounded open subset of $\mathbb{R}^N$ with locally Lipschitzian boundary, ρ a weight function, and M an N -function. If $u\in W_0^1L_M(\Omega)$, then*

$$\|u\|_M\le c\|\nabla u\|_{M,\rho}$$

where c is a positive constant, which implies that $\|\nabla u\|_{M,\rho}$ and $\|u\|_{1,M}$ are equivalent norms on $W_0^1L_{M,\rho}$.

Proof. Under the assumptions (2.4)-(2.6), the Sobolev conjugate N -function M_s^* of M_s is well defined by

$$M_s^{*-1}=\int_0^s\frac{M^{-1}(t)}{t^{1+\frac{1}{N}}}dt$$

and we have $W_0^1L_{M_s}\subset L_{M_s^*}$. Since $M\ll M_s^*$, we have $L_{M_s^*}\subset L_M$. Hence

$$\|u\|_M\le c_1\|u\|_{M_s^*}\le c_2\|u\|_{1,M_s},$$

where c_1 and c_2 are two positive constants. Then, by using the Poincaré inequality in the non-weighted Orlicz-Sobolev space, we find that there exists a positive constant c' such that $\|u\|_{1,M_s}\le c'\|\nabla u\|_{M_s}$. We will show that

$$\|\nabla u\|_{M_s}\le c\|\nabla u\|_{M,\rho}.$$

Note that

$$\begin{aligned}\|v\|_{M_s}\le\int_\Omega M_s(v(x))dx+1&=\int_\Omega M_s(v(x))\frac{1}{\rho(x)}\rho(x)dx+1\\&\le\int_\Omega S(M_s(v(x)))\,\rho(x)dx+\int_\Omega\frac{1}{\rho(x)}\rho(x)dx+2\\&=\int_\Omega M(v(x))\rho(x)dx+\int_\Omega\rho^{-s}(x)dx+1,\end{aligned}$$

which implies that $\|v\|_{M_s}\le c\|v\|_{M,\rho}$, for some positive constant c. In fact, if this is not true, then there exists a sequence v_n such that $\|v_n\|_{M_s}\to\infty$ and for n large, $\|v_n\|_{M,\rho}\le 1$. Hence, for n sufficiently large, we get

$$\int_\Omega M(v_n(x))\,\rho(x)dx\le\|v_n\|_{M,\rho}\le 1.$$

Then

$$\|v_n\|_{M_s} \leq \int_\Omega M(v(x))\rho(x)dx + \int_\Omega \rho^{-s}(x)dx + 1$$
$$\leq \|v_n\|_{M,\rho} + \int_\Omega \rho^{-s}(x)dx + 1,$$

which is a contradiction since the left hand-side tends to infinity while the right hand-side is bounded. Finally, by taking $v = \nabla u$, we conclude the result immediately. □

We will also use the following technical lemmas.

2.4. **Some technical lemmas.**

Lemma 2.8. *Let f_n and f be in $L^1(\Omega)$ such that $f_n \geq 0$ a.e. in Ω, and let $f_n \to f$ a.e. in Ω such that $\int_\Omega f_n(x)dx \to \int_\Omega f(x)dx$. Then $f_n \to f$ strongly in $L^1(\Omega)$.*

Lemma 2.9. *Let $F : \mathbb{R} \to \mathbb{R}$ be uniformly Lipschitzian, with $F(0) = 0$. Let $u \in W_0^1 L_M(\Omega, \rho)$. Then $F(u) \in W_0^1 L_M(\Omega, \rho)$. Moreover, if the set D of discontinuity points of F' is finite, then*

$$\frac{\partial}{\partial x_i} F(u) = \begin{cases} F'(u)\frac{\partial u}{\partial x_i} & a.e. \quad in \{x \in \Omega; u(x) \notin D\}, \\ 0 & a.e. \quad in \{x \in \Omega; u(x) \in D\}. \end{cases}$$

Proof. We suppose for the moment that F is also C^1. Then, there exist a sequence $u_n \in D(\Omega)$ such that $u_n \to u$ (mod) $W^1 L_M(\Omega, \rho)$). Passing to subsequence, we can assume that

$$D^\alpha u_n \to D^\alpha u \quad \forall |\alpha| \leq 1 \text{ a.e. in } \Omega.$$

From the relation $|F(s)| \leq k|s|$, where k denote the Lipschitz constant for F, and $\frac{\partial}{\partial x_i} F(u_n) = F'(u_n)\frac{\partial u_n}{\partial x_i}$, we deduce that $F(u_n)$ remains bounded in $W_0^1 L_M(\Omega, \rho)$. It follows that

$$F(u_n) \to w \in W_0^1 L_M(\Omega, \rho) \text{ for } \sigma(\prod L_{M,\rho}, \prod E_{\overline{M},\rho}).$$

By a local application of the compact imbedding theorem, we have that $F(u_n) \to w$ a.e. in Ω. Consequently, $w = F(u)$, and $F(u) \in W_0^1 L_M(\Omega, \rho)$. Finally, by the usual chain rule for weak derivatives, we have

$$\frac{\partial}{\partial x_i} F(u) = F'(u)\frac{\partial u}{\partial x_i} \text{ a.e. in } \Omega. \tag{2.7}$$

For the general case, taking the convolution with the mollifiers, we get a sequence $F_n \in C^\infty(\mathbb{R})$ such that $F_n \to F$ uniformly on each compact, $F_n(0) = 0$ and $\left|F_n'\right| \leq k$. For each n, $F_n(u) \in W_0^1 L_M(\Omega, \rho)$, we have (2.7) with F replaced by F_n. Finally, (2.7) follows from the generalized chain rule for weak derivatives. This completes the proof. □

The following lemmas follow from the previous lemma.

Lemma 2.10. *Let $u, v \in W_0^1 L_M(\Omega, \rho)$ and let $w = min(u, v)$. Then, $w \in W_0^1 L_M(\Omega, \rho)$ and*

$$\frac{\partial w}{\partial x_i} = \begin{cases} \frac{\partial u}{\partial x_i} \text{ a.e. in } \{x \in \Omega; u(x) \leq v(x)\}, \\ \frac{\partial v}{\partial x_i} \text{ a.e. in } \{x \in \Omega; u(x) > v(x)\}. \end{cases}$$

Proof. Note that $\min(u, v) = u - (u - v)^+$. Using Lemma 2.9 with $F(s) = s^+$, we get the desired conclusion immediately. □

We introduce the truncate operator. For a given constant $k > 0$, we define the function $T_k : \mathbb{R} \to \mathbb{R}$ as

$$T_k(s) = \begin{cases} s, & |s| \le k, \\ k\frac{s}{|s|}, & |s| > k. \end{cases}$$

Lemma 2.11. *Assume that (2.4)–(2.6) holds. Let $u \in W_0^1 L_M(\Omega,\rho)$, and let $T_k(u), k \in \mathbb{R}^+$, be the usual truncation. Then, $T_k(u) \in W_0^1 L_M(\Omega,\rho)$. Moreover,*

$$T_k(u) \to u \text{ strongly in } W_0^1 L_M(\Omega,\rho).$$

Lemma 2.12. *Let u_n be a sequence in $W_0^1 L_M(\Omega,\rho)$ such that $u_n \rightharpoonup u$ for the topology $\sigma(\prod L_{M,\rho}, \prod E_{\overline{M},\rho})$. Then, $T_k(u_n) \rightharpoonup T_k(u)$ for $\sigma(\prod L_{M,\rho}, \prod E_{\overline{M},\rho})$.*

Proof. Since $u_n \rightharpoonup u$ and $W_0^1 L_M(\Omega,\rho) \hookrightarrow\hookrightarrow E_M(\Omega)$, we have $u_n \to u$ strongly in $E_M(\Omega)$ and a.e. in Ω. Then, $T_k(u_n) \to T_k(u)$ a.e. in Ω. On the other hand, for some $\lambda > 0$,

$$\int_\Omega M\Big(\frac{|T_k(u_n)|}{\lambda}\Big)dx \le \int_\Omega M\Big(\frac{|u_n|}{\lambda}\Big)dx$$

and

$$\int_\Omega M\Big(\frac{|\nabla T_k(u_n)|}{\lambda}\Big)\rho(x)dx = \int_\Omega M\Big(\frac{|T_k^{'}(u_n)|\,|\nabla u_n|}{\lambda}\Big)\rho(x)dx \le \int_\Omega M\Big(\frac{|\nabla u_n|}{\lambda}\Big)\rho(x)dx$$

which imply that

$$\|T_k(u_n)\|_{1,M,\rho} \le \|u_n\|_{1,M,\rho}\,.$$

Then, $(T_k(u))_n$ is bounded in $W_0^1 L_M(\Omega,\rho)$. It follows that $T_k(u_n) \rightharpoonup T_k(u)$ in $W_0^1 L_M(\Omega,\rho)$ for $\sigma(L_M(\Omega,\rho), E_{\overline{M}}(\Omega,\rho))$. This completes the proof. □

Lemma 2.13. *If the sequence $u_n \in E_M(\Omega,\rho)$ converges a.e. in Ω with $\rho \in L^1(\Omega)$, then it converges in norm in $E_M(\Omega,\rho)$ if and only if the norms are uniformly absolutely continuous, i.e., for each $\varepsilon > 0$, there exist $\delta > 0$ such that $\|u_n\chi_E\|_{M,\rho} < \varepsilon$, for all n and $E \subset \Omega$ with $|E| < \delta$.*

Proof. By the same argument introduced in the proof of Lemma 11.2 in [29], we find $E_{n,m} = \{x \in \Omega : |u_n(x) - u_m(x)| > \alpha\}$, where $\alpha = M^{-1}\Big(\frac{\varepsilon}{3\|\rho\|_1}\Big)$ such that

$$\|u_n\chi_E\|_{M,\rho} < \frac{\varepsilon}{3}.$$

We denote by $H(E_M(\Omega), r)$ the set of functions $u \in L_M(\Omega)$ whose distance to $E_M(\Omega)$ (with respect to the Orlicz norm) is strictly less than r and by $B_{L_M}(\Omega)(0,r)$ the ball in $L_M(\Omega)$ (with respect to the Orlicz norm) of radius r and center 0. □

Lemma 2.14. *Let Ω be a bounded subset of $\mathbb{R}^N$ with finite mesure. Let M, R and Q be N-functions such that $Q << R$, and let f be a Carathéodory function such that, for a.e. $x \in \Omega$ and all $s \in \mathbb{R}$,*

$$|f(x,s)| \le b(x) + k_1 R^{-1}\left(\rho^{-1}(x) Q(k_2\,|s|)\right), \tag{2.8}$$

where $0 \le b(x) \in E_M(\Omega,\rho), \rho \in L^1(\Omega)$ and $k_1, k_2 \in \mathbb{R}^+$. Then, the Nemytskii operator $N_f(u)(x) = f(x,u(x))$ satisfies:

(1) it sends $\Big(H(E_Q(\Omega), \frac{1}{k_2})\Big)^p$ into $L_R(\Omega,\rho)$ and is continuous from $\Big(H(E_Q(\Omega), \frac{1}{k_2})\Big)^p$ to the norm topology of $(L_Q(\Omega))^p$ into $L_R(\Omega,\rho)$ to the modular convergence;

(2) it is uniformly bounded on $\left(B_{L_Q(\Omega)}(0,\frac{1}{k_2})\right)^p$;

(3) If $b(x)\in E_{R_1}(\Omega,\rho)$ *with* $R_1<<R$*, then* N_f *is continuous to the norm topology of* $E_{R_1}(\Omega,\rho)$.

Proof. (1) Let $u=(u_1,u_2,...,u_p)\in\left(H(E_Q(\Omega),\frac{1}{k_2})\right)^p$. Since $d(u_i,E_Q(\Omega))<\frac{1}{k_2}$ $(1\le i\le p)$, we have $\int_\Omega Q(k_2|u|)dx\le 1$, (see Theorem 10.1 [29]). Let $\lambda\ge 2k_1$ such that $\frac{2b(x)}{\lambda}\in K_R(\Omega,\rho)$. By the growth condition (2.8) and the convexity of R, we get

$$\int_\Omega R\left(\frac{|f(x,s)|}{\lambda}\right)\rho(x)dx\le\frac{1}{2}\int_\Omega R\left(\frac{2b(x)}{\lambda}\right)\rho(x)dx+\frac{1}{2}\int_\Omega R\left(k_2|u(x)|\right)dx<\infty.$$

On the other hand, suppose that

$$u_n\to u\in\left(H(E_Q(\Omega),\frac{1}{k_2})\right)^p,$$

and let $\alpha>0$ be such that $d(k_2|u|,E_Q(\Omega))<\alpha<1$ and $d(k_2|u|,E_Q(\Omega))<1-\alpha<1$. We have $\frac{k_2}{\alpha}|u|\in K_Q(\Omega)$ and $\frac{k_2}{1-\alpha}|u|\in K_Q(\Omega)$ (see Theorem 10.1 [29]). For $\lambda>4k_1$ with $\frac{4b(x)}{\lambda}\in K_R(\Omega,\rho)$, we get

$$\begin{aligned}&\int_\Omega R\left(\tfrac{|f(x,u_n)-f(x,u)|}{\lambda}\right)\rho(x)dx\\
&\quad\le\int_\Omega R\left(\tfrac{2b(x)+k_1R^{-1}\left(\rho^{-1}(x)Q(k_2|u_n|)\right)+k_1R^{-1}\left(\rho^{-1}(x)Q(k_2|u|)\right)}{\lambda}\right)\rho(x)dx\\
&\quad\le\tfrac{1}{2}\int_\Omega R\left(\tfrac{4b(x)}{\lambda}\right)\rho(x)dx+\tfrac{1-\alpha}{4}\int_\Omega Q\left(\tfrac{k_2}{1-\alpha}|u_n-u|\right)\rho(x)dx\\
&\qquad+\tfrac{\alpha}{4}\int_\Omega Q\left(\tfrac{k_2}{\alpha}|u|\right)dx+\tfrac{1}{4}\int_\Omega Q\left(k_2|u|\right)dx.\end{aligned}$$

Since $Q(k_2|u|)\le Q\left(\frac{k_2}{\alpha}|u|\right)$, the last inequality becomes

$$\begin{aligned}&\int_\Omega R\left(\tfrac{|f(x,u_n)-f(x,u)|}{\lambda}\right)\rho(x)dx\\
&\quad\le\frac{1}{2}\int_\Omega R\left(\frac{4b(x)}{\lambda}\right)\rho(x)dx+\int_\Omega Q\left(\frac{k_2}{1-\alpha}|u_n-u|\right)dx+\int_\Omega Q\left(k_2|u|\right)dx,\end{aligned}$$

which implies from the Vitali's theorem that $f(x,u_n)\to f(x,u)$ (mod) in $L_R(\Omega,\rho)$, for a subsequence, denoted again by u_n (which holds for the whole sequence).

(2) Let $u\in\left(B_{L_Q(\Omega)}(0,\frac{1}{k_2})\right)^p$ and let $\lambda\ge 2k$ such that $\int_\Omega R\left(\frac{2b(x)}{\lambda}\right)\rho(x)dx\le 1$. By the growth condition (2.8) and the convexity of R, we get

$$\int_\Omega R\left(\frac{|f(x,u)|}{\lambda}\right)\rho(x)dx\le\frac{1}{2}\int_\Omega R\left(\frac{2b(x)}{\lambda}\right)\rho(x)dx+\frac{1}{2}\int_\Omega Q\left(k_2|u(x)|\right)dx\le 1,$$

which implies (2).

(3) Suppose that $b(x)\in E_{R_1}(\Omega,\rho)$ with $R_1\prec\prec R$. As in (1), since $L_R(\Omega,\rho)\subset E_{R_1}(\Omega,\rho)$), we can show that $f(x,u)\in E_{R_1}(\Omega,\rho)$ for all $u\in\left(E_Q(\Omega),\frac{1}{k_2})\right)^p$. Suppose now that

$$u_n\to u\in\left(E_Q(\Omega),\frac{1}{k_2})\right)^p\ \text{ in }(L_Q(\Omega))^p,$$

We next show that $f(x,u_n) \to f(x,u)$ (mod) in $E_{R_1}(\Omega,\rho)$. Fixing $\varepsilon > 0$, we have as above

$$\begin{aligned}&\int_\Omega R_1\left(\frac{|f(x,u_n)-f(x,u)|}{\varepsilon}\right)\rho(x)dx \\ &\leq \frac{1}{4}\int_\Omega R_1\left(\frac{4b(x)}{\varepsilon}\right)\rho(x)dx + \frac{1}{4}\int_\Omega R_1\left(\frac{4k_1}{\varepsilon}R^{-1}\left(\rho^{-1}(x)Q(k_2|u_n|)\right)\right)\rho(x)dx \\ &\quad + \frac{1}{4}\int_\Omega R_1\left(\frac{4k_1}{\varepsilon}R^{-1}\left(\rho^{-1}(x)Q(k_2|u|)\right)\right)\rho(x)dx.\end{aligned}$$

Since $R_1 \prec\prec R$, we have that there exists $K^{'}$ such that $R_1(\frac{4k_1}{\varepsilon}t) \leq R(t) + K^{'}$ for all $t \geq 0$. Then, the last inequality can be written as the form

$$\begin{aligned}&\int_\Omega R_1\left(\frac{|f(x,u_n)-f(x,u)|}{\varepsilon}\right)\rho(x)dx \\ &\leq \frac{1}{4}\int_\Omega R_1\left(\frac{4b(x)}{\varepsilon}\right)\rho(x)dx + \frac{1}{4}\int_\Omega Q(k_2|u_n|)dx \\ &\quad + \frac{1}{4}\int_\Omega Q(k_2|u|)dx + \frac{K^{'}}{2}\int_\Omega \rho(x)dx.\end{aligned}$$

As in (1), by using the Vitali's theorem, we get

$$\int_\Omega R_1\left(\frac{|f(x,u_n)-f(x,u)|}{\varepsilon}\right)\rho(x)dx \to 0 \text{ as } n \to \infty,$$

for a subsequence (which holds for the whole sequence). Since ε is arbitrary, we conclude the desired conclusion immediately. □

3. Main results

Let Y be a closed subspace of $W^1L_M(\Omega,\rho)$ for $\sigma(\prod L_{M,\rho}, \prod E_{\overline{M},\rho})$ and let

$$Y_0 = Y \cap W^1L_M(\Omega,\rho),$$

such that Y is the closure of Y_0 for $\sigma(\prod L_{M,\rho}, \prod E_{\overline{M},\rho})$. In the next, we consider the complementary system (Y,Y_0,Z,Z_0) generated by Y, that is, Y_0^* can be identified to Z and Z_0^* can be identified to Y by the means $\langle.,.\rangle$. Let the mapping T (associated to the operator A) be defined from $D(T) = \{u \in Y, a_0(x,u,\nabla u) \in L_{\overline{M}}(\Omega), a_i(x,u,\nabla u) \in L_{\overline{M}}(\Omega,\rho)\}$ into Z by the formula

$$a(u,v) = \int_\Omega a_0(x,u,\nabla u)v(x)dx + \sum_{1\leq i\leq N}\int_\Omega a_i(x,u,\nabla u)\frac{\partial v(x)}{\partial x_i}\rho(x)dx \quad \forall v \in Y_0.$$

We consider the complementary system

$$(Y,Y_0,Z,Z_0) = (W_0^1L_M(\Omega,\rho), W_0^1E_M(\Omega,\rho), W^{-1}E_{\overline{M}}(\Omega,\rho), W^{-1}L_{\overline{M}}(\Omega,\rho)).$$

As in [1], we define the entropy solution of our problem.

Definition 3.1. An entropy solution of problem (1.1) is a measurable function if $T_k(u) \in W_0^1L_M(\Omega,\rho)$ for every $k > 0$ and

$$\int_\Omega a(x,u,\nabla u)\nabla T_k(u-\phi)\rho(x)dx + \int_\Omega g(x,u)T_k(u-\phi)dx \leq \int_\Omega fT_k(u-\phi)dx$$

for every $\phi \in W_0^1E_M(\Omega,\rho) \cap L^\infty(\Omega)$.

The following is the main result of this paper.

Theorem 3.2. *Let* (1.2) -(1.7) *and* (2.4)–(2.6) *hold. Let* $\rho(x)$ *be a weight function on* Ω *satisfing* (2.2). *Then, there exist an entropy solution, u, of problem* (1.1).

3.1. Main Lemma.

Lemma 3.3. *Let u be a mesurable function such that* $T_k(u)$ *belongs to* $W_0^1L_M(\Omega,\rho)$ *for every* $k>0$. *Then*

$$\int_\Omega a(x,u,\nabla\phi)\nabla T_k(u-\phi)\,dx \le \int_\Omega f\,T_k(u-\phi)dx, \tag{3.1}$$

is equivalent to

$$\int_\Omega a(x,u,\nabla u)\nabla T_k(u-\phi)\rho(x)\,dx + \int_\Omega g(x,u)T_k(u-\phi)dx = \int_\Omega f\,T_k(u-\phi)dx, \tag{3.2}$$

for every $\phi\in W_0^1L_M(\Omega,\rho)$, *and for every* $k>0$.

Proof. It is obvious that (3.2) can imply (3.1). Thus, it remains to prove that (3.1) implies (3.2). Let h and k be positive real numbers, and let $\lambda\in]-1,1[$ and $\Psi\in W_0^1L_M(\Omega,\rho)\cap L^\infty(\Omega)$. Choose $\phi=T_h(u-\lambda T_k(u-\Psi))\in W_0^1L_M(\Omega,\rho)\cap L^\infty(\Omega)$ as test a function in (3.1). Then

$$I_{hk}\le J_{hk}, \tag{3.3}$$

where

$$I_{hk} = \int_\Omega a(x,u,\nabla T_h(u-\lambda T_k(u-\Psi)))\nabla T_k(u-T_h(u-\lambda T_k(u-\Psi)))\,dx + \int_\Omega g(x,u)T_k\left(u-T_h\left(u-\lambda T_k(u-\psi)\right)\right)dx = I'_{hk}+I''_{hkk}$$

and

$$J_{hk} = \int_\Omega f\,T_k(u-T_h(u-\lambda T_k(u-\Psi)))\,dx.$$

Putting

$$A_{hk}=\{x\in\Omega, |u-T_h(u-\lambda T_k(u-\Psi))|\le k\},$$

and

$$B_{hk}=\{x\in\Omega, |u-\lambda T_k(u-\Psi)|\le h\}.$$

we obtain

$$\begin{aligned} I'_{hk} = &\int_{A_{kh}\cap B_{hk}} a(x,u,\nabla T_h(u-\lambda T_k(u-\Psi)))\nabla T_k(u-T_h(u-\lambda T_k(u-\Psi)))\,dx \\ &+ \int_{A_{kh}\cap B_{hk}^C} a(x,u,\nabla T_h(u-\lambda T_k(u-\Psi)))\nabla T_k(u-T_h(u-\lambda T_k(u-\Psi)))\,dx \\ &+ \int_{A_{kh}^C} a(x,u,\nabla T_h(u-\lambda T_k(u-\Psi)))\nabla T_k(u-T_h(u-\lambda T_k(u-\Psi)))\,dx. \end{aligned}$$

Since $\nabla T_k(u-T_h(u-\lambda T_k(u-\Psi)))$ is different to zero only on A_{kh}, we have

$$\int_{A_{kh}^C} a(x,u,\nabla T_h(u-\lambda T_k(u-\Psi)))\nabla T_k(u-T_h(u-\lambda T_k(u-\Psi)))\,dx = 0. \tag{3.4}$$

Moreover, if $x \in B_{hk}^C$, we have $\nabla T_h(u - \lambda T_k(u - \Psi)) = 0$. Using (1.4), we deduce that

$$\int_{A_{kh}\cap B_{hk}^C} a(x,u,\nabla T_h(u-\lambda T_k(u-\Psi)))\nabla T_k(u-T_h(u-\lambda T_k(u-\Psi)))\,dx$$
$$= \int_{A_{kh}\cap B_{hk}^C} a(x,u,0)\nabla T_k(u-T_h(u-\lambda T_k(u-\Psi)))\,dx = 0. \tag{3.5}$$

From (3.4) and (3.5), we obtain

$$I'_{hk} = \int_{A_{kh}\cap B_{hk}} a(x,u,\nabla T_h(u-\lambda T_k(u-\Psi)))\nabla T_k(u-T_h(u-\lambda T_k(u-\Psi)))\,dx.$$

Letting $h \to +\infty, |\lambda| \le 1$, we have

$$A_{kh} \to \{x, |\lambda||T_k(u-\Psi)| \le h\} = \Omega, \tag{3.6}$$

$$B_{hk} \to \Omega \quad \textit{which implies} \quad A_{kh}\cap B_{hk} \to \Omega. \tag{3.7}$$

Using the Lebesgue theorem, we conclude that

$$\lim_{h\to+\infty}\int_{A_{kh}\cap B_{hk}} a(x,u,\nabla T_h(u-\lambda T_k(u-\Psi)))\nabla T_k(u-T_h(u-\lambda T_k(u-\Psi)))\,dx$$
$$= \lambda\int_\Omega a(x,u,\nabla(u-\lambda T_k(u-\Psi)))\nabla T_k(u-\Psi)\,dx, \tag{3.8}$$

which implies that

$$\lim_{h\to+\infty} I'_{hk} = \lambda\int_\Omega a(x,u,\nabla(u-\lambda T_k(u-\Psi)))\nabla T_k(u-\Psi)\,dx.$$

Moreover, it is easy to see that

$$\lim_{h\to+\infty}\int_\Omega g(x,u)T_k(u-T_h(u-\lambda T_k(u-\psi)))\,dx = \lambda\int_\Omega g(x,u)T_k[u-\psi]dx,$$

which implies that

$$\lim_{h\to+\infty} I_{hk} = \lambda\int_\Omega a(x,u,\nabla(u-\lambda T_k(u-\Psi)))\nabla T_k(u-\Psi)\,dx + \lambda\int_\Omega g(x,u)T_k[u-\psi]dx. \tag{3.9}$$

On the other hand, we have

$$J_{hk} = \int_\Omega f\; T_k(u-T_h(u-\lambda T_k(u-\Psi)))\,dx.$$

Then

$$\lim_{h\to+\infty}\int_\Omega f\; T_k(u-T_h(u-\lambda T_k(u-\Psi)))\,dx = \lambda\int_\Omega fT_k(u-\Psi)dx, \tag{3.10}$$

i.e.,

$$\lim_{h\to+\infty} J_{hk} = \lambda\int_\Omega fT_k(u-\Psi)dx. \tag{3.11}$$

Using (3.9) and (3.11), and passing to the limit in (3.3), we obtain

$$\lambda\left(\int_\Omega a(x,u,\nabla(u-\lambda T_k(u-\Psi)))\nabla T_k(u-\Psi)\,dx + \int_\Omega g(x,u)T_k[u-\psi]dx\right)$$
$$\le \lambda\left(\int_\Omega fT_k(u-\Psi)dx\right)$$

for every $\Psi \in W_0^1 L_M(\Omega,\rho) \cap L^\infty(\Omega)$, and for every $k > 0$. Choose $\lambda > 0$. Dividing by λ, and letting λ tend to zero, we obtain

$$\int_\Omega a(x,u,\nabla u)\nabla T_k(u-\Psi)\,dx + \int_\Omega g(x,u)T_k[u-\psi]dx \le \int_\Omega f T_k(u-\Psi)dx. \tag{3.12}$$

For $\lambda < 0$, dividing by λ, and letting λ tend to zero, we obtain

$$\int_\Omega a(x,u,\nabla u)\nabla T_k(u-\Psi)\,dx + \int_\Omega g(x,u)T_k[u-\psi]dx \ge \int_\Omega f T_k(u-\Psi)dx. \tag{3.13}$$

Combining (3.12) and (3.13), we conclude

$$\int_\Omega a(x,u,\nabla u)\nabla T_k(u-\Psi)\,dx + \int_\Omega g(x,u)T_k[u-\psi]dx = \int_\Omega f T_k(u-\Psi)dx. \tag{3.14}$$

This completes the proof of Lemma 3.3.

3.2. **Proof of Theorem 3.2.**

3.2.1. *Approximate problem and a priori estimate.* For $n \in \mathbb{N}$, define $f_n := T_n(f)$. Let u_n be a solution in $W_0^1 L_\varphi(\Omega)$ to the problem

$$\begin{cases} -\operatorname{div}(a(x,u_n,\nabla u_n)) + g_n(x,u_n) = f_n & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.15}$$

where

$$g_n(x,s) = \frac{g(x,s)}{1+\frac{1}{n}|g(x,s)|},$$

which exists due to [30]. Choosing $T_k(u_n)$ as a test function in (3.15), we have

$$\int_\Omega a(x,u_n,\nabla u_n)\nabla T_k(u_n)\,dx + \int_\Omega g_n(x,u_n)\,T_k(u_n)\,dx = \int_\Omega f_n T_k(u_n)dx.$$

Note that $g_n(x,u_n)\,T_k(u_n) \ge 0$. Using $\nabla T_k(u_n) = \nabla u_n \chi_{\{|u_n|\le k\}}$ and assumption (1.4), we obtain

$$\int_\Omega a(x,u_n,\nabla u_n)\nabla T_k(u_n)\,dx \ge \int_\Omega M(\lambda_1|\nabla T_k(u_n)|)dx.$$

It follows that

$$\int_\Omega \rho(x)M(\lambda_1|\nabla T_k(u_n)|)dx \le k\|f\|_{L^1(\Omega)}, \tag{3.16}$$

and

$$\int_\Omega \rho(x)M(\lambda_1|\nabla T_k(u_n)|)dx \le C_1 k, \tag{3.17}$$

where C_1 is a constant, which is independent of n.

3.2.2. *Local convergence of u_n in measure.* Taking $\frac{1}{\lambda}|T_k(u_n)|$ in (3.15) and using (3.17), one has

$$\int_\Omega \rho(x)M(\lambda_1\frac{|\nabla T_k(u_n)|}{\lambda})dx \leq \int_\Omega \rho(x)M(\lambda_1|\nabla T_k(u_n)|)dx \leq C_1 k. \tag{3.18}$$

Then, we deduce by using (3.18) that

$$\begin{aligned} meas\{|u_n|>k\} &\leq \frac{1}{\inf\limits_k M(\frac{k}{\lambda})}\int_{\{|u_n|>k\}} M(\frac{|u_n(x)|}{\lambda})dx \\ &\leq \frac{1}{\inf\limits_k M(\frac{k}{\lambda})}\int_\Omega M(\frac{1}{\lambda}|T_k(u_n)|)dx \\ &\leq \frac{C_1 k}{\inf\limits_k M(\frac{k}{\lambda})} \quad \forall n, \quad \forall k\geq 0. \end{aligned} \tag{3.19}$$

For any $\beta>0$, we have

$$meas\{|u_n-u_m|>\beta\} \leq meas\{|u_n|>k\}+meas\{|u_m|>k\}+meas\{|T_k(u_n)-T_k(u_m)|>\beta\},$$

and then

$$meas\{|u_n-u_m|>\beta\} \leq \frac{2C_1 k}{\inf\limits_k M(\frac{k}{\lambda})}+meas\{|T_k(u_n)-T_k(u_m)|>\beta\}. \tag{3.20}$$

By using (3.17) and the Poincaré inequality in weighted Orlicz-Sobolev spaces, we deduce that $(T_k(u_n))$ is bounded in $W_0^1L_M(\Omega,\rho)$, and then there exists $\omega_k \in W_0^1L_M(\Omega,\rho)$ such that $T_k(u_n) \rightharpoonup \omega_k$ weakly in $W_0^1L_M(\Omega,\rho)$ for $\sigma(\Pi L_M,\Pi E_{\overline{M}})$; strongly in $E_{\overline{M}}(\Omega,\rho)$ and a.e. in Ω. Consequently, we can assume that $(T_k(u_n))_n$ is a Cauchy sequence in measure in Ω. Let $\varepsilon>0$. From (3.20) and the fact that $\frac{2C_1k}{\inf\limits_k M(\frac{k}{\lambda})}\to 0$ as $k\to+\infty$, we have that there exists some $k=k(\varepsilon)>0$ such that $meas\{|u_n-u_m|>\lambda\}<\varepsilon$, for all $n,m\geq h_0\,(k(\varepsilon),\lambda)$. This proves that u_n is a Cauchy sequence in measure. Thus, u_n converges almost everywhere to some measurable function u.

Finally, there exist a subsequence of $\{u_n\}_n$, still indexed by n, and a function $u\in W_0^1L_M(\Omega,\rho)$ such that

$$\begin{cases} u_n \rightharpoonup u \quad \text{weakly in } W_0^1L_M(\Omega,\rho) \text{ for } \sigma(\Pi L_{M,\rho},\Pi E_{\overline{M},\rho}) \\ u_n \longrightarrow u \quad \text{strongly in } E_M(\Omega,\rho) \text{ and a.e. in } \Omega. \end{cases} \tag{3.21}$$

3.2.3. *Equi-integrability of nonlinearities.* we need to prove that

$$g_n(x,u_n)\to g(x,u) \text{ strongly in } L^1(\Omega). \tag{3.22}$$

In particular, it is enough to prove the equi-integrable of $g_n(x,u_n)$ to this purpose. Taking $T_{l+1}(u_n)-T_l(u_n)$ as a test function in (3.15), we obtain

$$\int_\Omega a(x,u_n,\nabla u_n)\nabla(T_{l+1}(u_n)-T_l(u_n))dx+\int_\Omega g_n(x,u_n)(T_{l+1}(u_n)-T_l(u_n))\,dx$$
$$=\int_\Omega f\,(T_{l+1}(u_n)-T_l(u_n))\,dx$$

which implies that

$$\int_{\{l\leq|u_n|\leq l+1\}} a\left(x,u_n.\nabla u_n\right),\nabla u_n dx+\int_{\{|u_n|\geq l+1\}}\left|g_n\left(x,u_n\right)\right|dx$$
$$\leq c\int_{\{|u_n|\geq l\}}|f|dx.$$

From (1.4), we have

$$\int_{\{|u_n|\geq l+1\}}\left|g_n\left(x,u_n\right)\right|dx\leq c\int_{\{|u_n|\geq l\}}|f_n|\,dx.$$

Letting $\varepsilon>0$, we find that there exist $l(\varepsilon)\geq 1$ such that

$$\int_{\{|u_n|>l(\varepsilon)\}}\left|g_n\left(x,u_n\right)\right|dx\leq\frac{\varepsilon}{2}. \tag{3.23}$$

For any measurable subset $E\subset\Omega$, we have

$$\int_E\left|g_n\left(x,u_n\right)\right|dx\leq\int_{E\cap\{|u_n|\leq l(\varepsilon)\}}\left|g_n\left(x,u_n\right)\right|dx+\int_{E\cap\{|u_n|>l(\varepsilon)\}}\left|g_n\left(x,u_n\right)\right|dx$$
$$\leq\int_E\left|h_{l(\varepsilon)}(x)\right|dx+\int_{E\cap\{|u_n|>l(\varepsilon)\}}\left|g_n\left(x,u_n\right)\right|dx.$$

In view of (1.6), we have that there exist $\eta(\varepsilon)>0$ such that

$$\int_E\left|h_{l(\varepsilon)}(x)\right|dx\leq\frac{\varepsilon}{2}, \tag{3.24}$$

for all E such that meas $(E)<\eta(\varepsilon)$

Finally, by combining (3.23) and (3.24), one easily obtains $\int_E\left|g_n\left(x,u_n\right)\right|dx\leq\varepsilon$ for all E such that meas $(E)<\eta(\varepsilon)$.

3.2.4. *An intermediate Inequality.* In this step, we shall prove that, for $\phi\in W_0^1L_M(\Omega,\rho)\cap L^\infty(\Omega)$,

$$\int_\Omega a(x,u_n,\nabla\phi)\nabla T_k(u_n-\phi)\,dx+\int_\Omega g_n\left(x,u_n\right)T_k\left(u_n-\phi\right)dx\leq\int_\Omega f_nT_k(u_n-\phi)dx. \tag{3.25}$$

We choose now $T_k(u_n-\phi)$ as a test function in (3.15) with ϕ in $W_0^1L_M(\Omega,\rho)\cap L^\infty(\Omega)$. Then,

$$\int_\Omega a(x,u_n,\nabla u_n)\nabla T_k(u_n-\phi)\,dx+\int_\Omega g_n\left(x,u_n\right)T_k\left(u_n-\phi\right)dx=\int_\Omega f_nT_k(u_n-\phi)dx. \tag{3.26}$$

Adding and substracting the term $\int_\Omega a(x,u_n,\nabla\phi),\nabla T_k(u_n-\phi)\,dx$, i.e.,

$$\int_\Omega a(x,u_n,\nabla u_n)\nabla T_k(u_n-\phi)\,dx+\int_\Omega a(x,u_n,\nabla\phi)\nabla T_k(u_n-\phi)\,dx \tag{3.27}$$
$$-\int_\Omega a(x,u_n,\nabla\phi)\nabla T_k(u_n-\phi)\,dx+\int_\Omega g_n\left(x,u_n\right)T_k\left(u_n-\phi\right)dx=\int_\Omega f_nT_k(u_n-\phi)dx.$$

Thanks to (1.3) and the definition of truncation functions, we have

$$\int_\Omega(a(x,u_n,\nabla u_n)-a(x,u_n,\nabla\phi))\nabla T_k(u_n-\phi)\,dx\geq 0. \tag{3.28}$$

Combining (3.26) and (3.28), we obtain (3.25).

3.2.5. *Passing to the limit.* We shall prove that, for $\phi \in W_0^1 L_M(\Omega,\rho)\cap L^\infty(\Omega)$,
$$\int_\Omega a(x,u,\nabla\phi)\nabla T_k(u-\phi)\,dx+\int_\Omega g(x,u)\,T_k(u-\phi)\,dx\le\int_\Omega fT_k(u-\phi)dx.$$
First, we claim that
$$\int_\Omega a(x,u_n,\nabla\phi)\nabla T_k(u_n-\phi)\,dx\to\int_\Omega a(x,u,\nabla\phi)\nabla T_k(u-\phi)\,dx \text{ as } n\to+\infty.$$
Since $T_M(u_n)\rightharpoonup T_M(u)$ weakly in $W_0^1 L_M(\Omega,\rho)$, with $M=k+\|\phi\|_\infty$, then
$$T_k(u_n-\phi)\rightharpoonup T_k(u-\phi) \text{ in } W_0^1 L_M(\Omega,\rho), \tag{3.29}$$
which gives
$$\frac{\partial T_k}{\partial x_i}(u_n-\phi)\rightharpoonup\frac{\partial T_k}{\partial x_i}(u-\phi) \text{ weakly in } L_M(\Omega,\rho) \qquad \forall i=1,..,N. \tag{3.30}$$
Note that
$$a(x,T_M(u_n),\nabla\phi)\to a(x,T_M(u),\nabla\phi) \text{ strongly in } (L_{\overline{M}}(\Omega))^N.$$
Thanks to assumption (1.2), we obtain
$$|a_i(x,T_M(u_n),\nabla\phi)|\le|\phi_i(x)|+K_i\overline{P}^{-1}(\rho^{-1}(x)M(c_2|T_M(u_n)|))+K_i\overline{M}^{-1}M(c_1|\nabla\phi|),$$
where β and μ are positive constants. Since $T_M(u_n)\rightharpoonup T_M(u)$ weakly in $W_0^1 L_M(\Omega,\rho)$ and $W_0^1 L_M(\Omega,\rho)\hookrightarrow\hookrightarrow L_{\overline{M}}(\Omega,\rho)$, then $T_M(u_n)\rightharpoonup T_M(u)$ strongly in $L_M(\Omega,\rho)$ and a.e. in Ω. Hence,
$$|a(x,T_M(u_n),\nabla\phi)|\to|a(x,T_M(u),\nabla\phi)| \text{ a.e. in } \Omega.$$
and
$$|\phi_i(x)|+K_i\overline{P}^{-1}(\rho^{-1}(x)M(c_2|T_M(u_n)|))+K_i\overline{M}^{-1}M(c_1|\nabla\phi|)\to$$
$$|\phi_i(x)|+K_i\overline{P}^{-1}(\rho^{-1}(x)M(c_2|T_M(u)|))+K_i\overline{M}^{-1}M(c_1|\nabla\phi|),$$
a.e. in Ω. Then, By Vitali's theorem, we deduce that
$$a(x,T_M(u_n),\nabla\phi)\to a(x,T_M(u),\nabla\phi) \text{ strongly in } (L_{\overline{M}}(\Omega,\rho))^N, \text{ as } n\to\infty. \tag{3.31}$$
Combining (3.30) and (3.31), we obtain
$$\int_\Omega a(x,u_n,\nabla\phi)\nabla T_k(u_n-\phi)\,dx\to\int_\Omega a(x,u,\nabla\phi)\nabla T_k(u-\phi)\,dx \text{ as } n\to+\infty. \tag{3.32}$$
Second, we show that
$$\int_\Omega f_nT_k(u_n-\phi)dx\to\int_\Omega fT_k(u-\phi)dx. \tag{3.33}$$
We have $f_nT_k(u_n-\phi)\to fT_k(u-\phi)$ a.e. in Ω and $|fT_k(u_n-\phi)|\le k|f|$. By using Vitali's theorem, we obtain (3.33). Similarly, thanks to (3.22), we can show that
$$\int_\Omega g_n(x,u_n)\,T_k(u_n-\phi)\,dx\to\int_\Omega g(x,u)T_k(u-\phi)dx \text{ as } n\to\infty. \tag{3.34}$$
Thanks to (3.32), (3.33) and (3.34), $\forall\phi\in W_0^1 L_M(\Omega,\rho)\cap L^\infty(\Omega)$, we deduce
$$\int_\Omega a(x,u,\nabla\phi)\nabla T_k(u-\phi)\,dx+\int_\Omega g(x,u)T_k(u-\phi)dx\le\int_\Omega fT_k(u-\phi)dx.$$
In view of the main Lemma, we can deduce that u is an entropy solution of problem (1.1). This completes the proof of Theorem 3.2.

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