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THREE SOLUTIONS FOR A NONLINEAR EQUATION INVOLVING p-TRIHARMONIC OPERATORS

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Abstract. The existence of at least three weak solutions for a nonlinear elliptic Navier boundary value problem involving the p-triharmonic operator is investigated. The main tools used for obtaining our results are two critical points theorems established in [B. Ricceri, A three critical points theorem revisited, Nonlinear Anal. 9 (2009), 3084-3089] and [G. Bonanno, S.A. Marano, On the structure of the critical set of non-differentiable functionals with a weak compactness condition, Appl. Anal. 89 (2010), 1-10].

Keywords. Nonlinear problems; Variational methods; p-triharmonic operators.

1. INTRODUCTION

The study of the differential equations with $p(x)$-growth conditions is an interesting and attractive topic and has been the objective of considerable attention in recent years; see, e.g., [1, 2, 3, 4] and the references therein. The interest in studying such problems was stimulated by their applications in elastic mechanics, fluid dynamics, the thermo-convective flows of non-Newtonian fluids and the image processing. For more information on modeling physical phenomena by these equations, we refer to [5, 6, 7].

The operator $\Delta^3_{p(x)}u := \operatorname{div}\left(\Delta(|\nabla\Delta u|^{p(x)-2}\nabla\Delta u)\right)$ is the $p(x)$-triharmonic operator, where $p(\cdot)\in C(\overline{\Omega})$, which is a natural generalization of the p-triharmonic operator (where $p>1$ is a constant).

The following nonlinear Navier boundary value problem involving the $p(x)$-Kirchhoff type triharmonic operator

$$\begin{cases} -M\left(\int_\Omega \frac{1}{p(x)}|\nabla\Delta u|^{p(x)}dx\right)\Delta^3_{p(x)}u = \lambda\zeta(x)|u|^{\alpha(x)-2}u - \lambda\xi(x)|u|^{\beta(x)-2}u, \qquad x\in\Omega,\\ u=\Delta u=\Delta^2 u=0, \qquad x\in\partial\Omega, \end{cases}$$

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where $\Omega \subset \mathbb{R}^N$ $(N > 3)$ is a bounded domain with smooth boundary, λ is a positive parameter, $p \in C^0(\overline{\Omega})$ with $1 < p(x) < \frac{N}{3}$ for any $x \in \overline{\Omega}$ and $\zeta, \xi, \alpha, \beta \in C^0(\overline{\Omega})$, was analysed by Rahal in [8].

Motivated by the results, in this paper, we establish the existence of three weak solutions for the following Navier boundary value problem involving the p-triharmonic operator:

$$\begin{cases} -\Delta_p^3 u = \lambda f(x,u) + \mu g(x,u), & x \in \Omega, \\ u = \Delta u = \Delta^2 u = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ $(N > 3)$ is a bounded domain with boundary of class C^1, λ is a positive parameter, μ is a non-negative parameter, $f, g \in C^0(\overline{\Omega} \times \mathbb{R})$ and $p > \max\{1, \frac{N}{3}\}$.

In recent years, many authors investigated the problem of finding solutions of problems with Navier boundary value condition, we refer the reader to [1, 2, 3, 4, 8, 9, 10, 11, 12, 13]. As we see, most of them include p or $p(x)$-biharmonic operators.

In [12], Yin and Liu studied the following $p(x)$-biharmonic elliptic problem with Navier boundary conditions:

$$\begin{cases} \Delta_{p(x)}^2 u = \lambda a(x) f(x,u) + \mu g(x,u), & x \in \Omega, \\ u = \Delta u = 0, & x \in \partial\Omega, \end{cases} \tag{1.2}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is a bounded domain with boundary of class C^1, λ, μ are non-negative parameters, $p(\cdot) \in C^0(\overline{\Omega})$ with $\max\{2, \frac{N}{2}\} < p^- := \inf_{x\in\overline{\Omega}} p(x) \leq p^+ := \sup_{x\in\overline{\Omega}} p(x)$. By the three critical points theorem obtained by Ricceri [14], they established the existence of three weak solutions to problem (1.2).

In [1], by using the critical point theory, the existence of infinitely many weak solutions for a class of Navier boundary value problem depending on two parameters and involving the $p(x)$-biharmonic operator

$$\begin{cases} \Delta_{p(x)}^2 u = \lambda f(x,u) + \mu g(x,u), & x \in \Omega, \\ u = \Delta u = 0, & x \in \partial\Omega, \end{cases}$$

where λ is a positive parameter, μ is a non-negative parameter, $f, g \in C^0(\overline{\Omega} \times \mathbb{R})$ and $p(\cdot) \in C^0(\overline{\Omega})$, was discussed.

In [9], Candito and Molica Bisci proved the existence of three weak solutions for the following elliptic Navier boundary problem

$$\begin{cases} \Delta(|\Delta u|^{p-2}\Delta u) = \lambda f(x,u), & x \in \Omega, \\ u = \Delta u = 0, & x \in \partial\Omega, \end{cases}$$

where λ is a positive parameter, p is a constant and f is a suitable continuous function defined on $\overline{\Omega} \times \mathbb{R}$.

To the best of our knowledge, there are just a few contributions to the study of the problems involving $p-$triharmonic operators. The paper is organised as follows. Section 2, is devoted to our abstract framework. Section 3, the last section, is dedicated to the main results.

2. Preliminaries

From now on, we assume that Ω is an open bounded subset of $\mathbb{R}^N$, $p > \max\{1, \frac{N}{3}\}$, while

$$X := W^{3,p}(\Omega) \cap W_0^{1,p}(\Omega)$$

endowed with the norm

$$\|u\| = \left(\int_\Omega |\nabla\Delta u|^p \, dx\right)^{\frac{1}{p}} \tag{2.1}$$

for $u \in X$.

Proposition 2.1 (see [15])**.** *If $\Omega \subset \mathbb{R}^N$ is a bounded domain, then the embedding $X \hookrightarrow C^0(\overline{\Omega})$ is compact whenever $p > \frac{N}{3}$.*

From Proposition 2.1, we see that there exists a positive constant k depending on p, N and Ω such that

$$\|u\|_\infty = \sup_{x\in\overline{\Omega}} |u(x)| \le k\|u\|, \quad \forall u \in X. \tag{2.2}$$

Corresponding to f and g, we introduce the functions $F, G : \Omega \times \mathbb{R} \to \mathbb{R}$, respectively, as follows

$$F(x,t) := \int_0^t f(x,\xi)\, d\xi, \qquad G(x,t) := \int_0^t g(x,\xi)\, d\xi$$

for all $x \in \Omega$ and $t \in \mathbb{R}$.

For every $u \in X$, let us define $\Phi, \Psi, J : X \to \mathbb{R}$ by putting

$$\Phi(u) := \frac{\|u\|^p}{p}, \qquad \Psi(u) = -\int_\Omega F(x,u(x))dx, \qquad J(u) = -\int_\Omega G(x,u(x))dx.$$

By standard arguments, we have that Φ is Gâteaux differentiable and sequentially weakly lower semicontinuous and its Gâteaux derivative is the functional $\Phi'(u) \in X^*$, given by

$$\Phi'(u)(v) = \int_\Omega |\nabla\Delta u|^{p-2}\nabla\Delta u \cdot \nabla\Delta v dx$$

for any $v \in X$. Furthermore, the differential $\Phi' : X \to X^*$ admits a continuous inverse (see [12, Lemma 3.1]). Similar arguments as those used in [16] imply that $\Psi, J \in C^1(X,\mathbb{R})$ with the derivatives given by

$$\Psi'(u)(v) = -\int_\Omega f(x,u(x))v(x)\, dx$$
$$J'(u)(v) = -\int_\Omega g(x,u(x))v(x)\, dx$$

for any $u, v \in X$. Moreover, thanks to the compact embedding $X \hookrightarrow C^0(\overline{\Omega})$, the operator $\Psi' : X \to X^*$ is compact.

Our analysis is based on the following theorems. These tools have been successfully applied to different problems in [2, 4, 9, 13, 17].

Theorem 2.2 (Ricceri, [14])**.** *Let X be a reflexive real Banach space, and $I \subseteq \mathbb{R}$ an interval. Let $\Phi : X \to \mathbb{R}$ be a sequentially weakly lower semicontinuous C^1 functional, bounded on each bounded subset of X, whose derivative admits a continuous inverse on X^*, and let $\Psi : X \to \mathbb{R}$ be a C^1 functional with compact derivative. Assume that*

$$\lim_{\|x\|\to+\infty} (\Phi(x) + \lambda\Psi(x)) = +\infty$$

for all $\lambda \in I$, and there exists $h \in \mathbb{R}$ such that

$$\sup_{\lambda\in I}\inf_{x\in X} (\Phi(x) + \lambda(\Psi(x) + h)) < \inf_{x\in X}\sup_{\lambda\in I} (\Phi(x) + \lambda(\Psi(x) + h)). \tag{2.3}$$

Then, there exist a non-empty open set interval $A \subseteq I$ and a positive real number q with the following property: for every $\lambda \in A$ and every C^1 functional $J : X \to \mathbb{R}$ with a compact derivative, there exists $\delta > 0$ such that, for each $\mu \in [0, \delta]$, the equation

$$\Phi'(u) + \lambda \Psi'(u) + \mu J'(u) = 0$$

has at least three solutions in X whose norms are less than q.

In the proof of our first main result, we also use the following result to verify the minimax inequality in theorem 2.2.

Theorem 2.3 (Bonanno, [18]). *Let X be a non-empty set and Φ, Ψ two real functions on X. Assume that $\Phi(x) \geq 0$ for every $x \in X$ and there exists $u_0 \in X$ such that $\Phi(u_0) = \Psi(u_0) = 0$. Further, assume that exist $u_1 \in X$, $r > 0$ such that*

(κ_1) $\Phi(u_1) > r$,
(κ_2) $\sup_{\Phi(x)<r}(-\Psi(x)) < r\frac{-\Psi(u_1)}{\Phi(u_1)}$.

Then, for every $\nu > 1$ and for every $h \in \mathbb{R}$ satisfying

$$\sup_{\Phi(x)<r}(-\Psi(x)) + \frac{r\frac{-\Psi(u_1)}{\Phi(u_1)} - \sup_{\Phi(x)<r}(-\Psi(x))}{\nu} < h < r\frac{-\Psi(u_1)}{\Phi(u_1)},$$

one has

$$\sup_{\lambda\in\mathbb{R}} \inf_{x\in X}(\Phi(x) + \lambda(\Psi(x) + h)) < \inf_{x\in X} \sup_{\lambda\in[0,\sigma]}(\Phi(x) + \lambda(\Psi(x) + h)),$$

where

$$\sigma = \frac{\nu r}{r\frac{-\Psi(u_1)}{\Phi(u_1)} - \sup_{\Phi(x)<r}(-\Psi(x))}.$$

Finally, we recall the following tool, obtained by Bonanno and Marano in [19], that we recall in a convenient form.

Theorem 2.4 ([19, Theorem 2.6]). *Let X be a reflexive real Banach space. Let $\Phi : X \to \mathbb{R}$ be a sequentially weakly lower semicontinuous, coercive and continuously Gâteaux differentiable functional whose Gâteaux derivative admits a continuous inverse on X^*, and let $\Psi : X \to \mathbb{R}$ be a sequentially weakly lower semicontinuous, continuously Gâteaux differentiable functional whose Gâteaux derivative is compact such that*

$$\Phi(0) = \Psi(0) = 0.$$

Assume that there exist $r > 0$ and $\bar{x} \in X$, with $r < \Phi(\bar{x})$ such that

(i) $\sup_{\Phi(x)\leq r}\Psi(x) < r\Psi(\bar{x})/\Phi(\bar{x})$,
(ii) *for each λ in*

$$\Lambda_r := \left(\frac{\Phi(\bar{x})}{\Psi(\bar{x})}, \frac{r}{\sup_{\Phi(x)\leq r}\Psi(x)}\right),$$

the functional $\Phi + \lambda\Psi$ is coercive.

Then, for each $\lambda \in \Lambda_r$, the functional $\Phi + \lambda\Psi$ has at least three distinct critical points in X.

3. Main results

In this section, we present our main results.

Let $\tau := \sup_{x\in\Omega} \operatorname{dist}(x,\partial\Omega)$. Simple calculations show that there is $x^0 \in \Omega$ such that $B(x^0,\tau) \subset \Omega$, where $B(x^0,\tau)$ denotes the ball in $\mathbb{R}^N$ with center x^0 and radius of τ. Also, put

$$\sigma_{p,N}(\tau) := \int_{\frac{\tau}{2}}^{\tau} |\tau^3(N+1) - 8\tau^2 s(N+2) + 15\tau s^2(N+3) - 8s^3(N+4)|^p s^{N-1} ds.$$

Finally, let us denote

$$K_{p,N}(\tau) := \frac{\tau^{6p}\Gamma(\frac{N}{2})}{2^{6p+1}(3k)^p \pi^{\frac{N}{2}} \sigma_{p,N}(\tau)},$$

where Γ denotes the Gamma function and k is defined by (2.2).

Theorem 3.1. *Let $f \in C^0(\overline{\Omega}\times\mathbb{R})$ and denote $F(x,\xi) := \int_0^{\xi} f(x,t)\,dt$ for all $(x,\xi) \in \overline{\Omega}\times\mathbb{R}$. Assume that there exist two positive constants c and d, with $d > c(K_{p,N}(\tau))^{\frac{1}{p}}$ such that*

- (f_1) *$F(x,\xi) \geq 0$ for each $(x,\xi) \in (\Omega \setminus B(x^0,\frac{\tau}{2})) \times [0,d]$;*
- (f_2) *$\operatorname{meas}(\Omega) \sup_{(x,\xi)\in\Omega\times[-c,c]} F(x,\xi) \leq \left(\frac{c}{d}\right)^p K_{p,N}(\tau) \int_{B(x^0,\frac{\tau}{2})} F(x,d)\,dx$, where $\operatorname{meas}(\Omega)$ is the Lebesgue measure of Ω;*
- (f_3) *there exist a function $\alpha \in L^1(\Omega)$ and a positive constant $\gamma < p$ such that*
$$F(x,\xi) \leq \alpha(x)(1+|\xi|^{\gamma}),$$
for almost every $x \in \Omega$ and for every $\xi \in \mathbb{R}$.

Then, there exist a number $\rho \in \mathbb{R}$ and an open interval $\Lambda \subseteq [0,+\infty)$ with the following property: for every $\lambda \in \Lambda$ and for every L^1–Carathéodory function $g : \Omega\times\mathbb{R} \to \mathbb{R}$ with

- (g_1) *$\sup_{\{|s|\leq\xi\}} |g(\cdot,s)| \in L^1(\Omega)$, for all $\xi > 0$,*

there exists $\delta > 0$ such that, for each $\mu \in [0,\delta]$, problem (1.1) has at least three weak solutions whose norms are less than ρ.

Proof. Our aim is to apply Theorem 2.2 with $X = W^{3,p}(\Omega) \cap W_0^{1,p}(\Omega)$ endowed with the norm introduced in (2.1). We take Φ, Ψ, J as in the previous section. Put

$$I_{\lambda,\mu}(u) := \Phi(u) + \lambda\Psi(u) + \mu J(u)$$

for $u \in X$ and $\lambda, \mu \in [0,+\infty)$. Note that the weak solutions of (1.1) are exactly the critical points of $I_{\lambda,\mu}$.

If we fix $u \in X$ with $\|u\| > 1$, by condition (f_3) and for each $\lambda > 0$, we have

$$\begin{aligned}\Phi(u) + \lambda\Psi(u) &\geq \frac{\|u\|^p}{p} - \lambda \int_{\Omega} \alpha(x)\left(1+|u|^{\gamma}\right)dx \\ &\geq \frac{\|u\|^p}{p} - \lambda\|\alpha\|_{L^1(\Omega)}(1+k^{\gamma}\|u\|^{\gamma}).\end{aligned}$$

So, the coercivity of functional $\Phi + \lambda\Psi$ is obtained, and therefore the first assumption of Theorem 2.2 holds true. Next, we put

$$w(x) := \begin{cases} 0 & \text{if } \ x \in \overline{\Omega}\setminus B(x^0,\tau), \\ 64\frac{L^3}{\tau^6}(\tau-L)^3 d & \text{if } \ x \in B(x^0,\tau)\setminus B(x^0,\frac{\tau}{2}), \\ d & \text{if } \ x \in B(x^0,\frac{\tau}{2}), \end{cases} \tag{3.1}$$

where

$$L = dist(x, x^0) = \sqrt{\sum_{i=1}^{N} (x_i - x_i^0)^2}.$$

Then, we deduce that

$$\frac{\partial w(x)}{\partial x_i} = \begin{cases} 0 & \text{if} \quad x \in \overline{\Omega} \setminus B(x^0, \tau) \cup B(x^0, \frac{\tau}{2}), \\ \frac{64d(x_i - x_i^0)}{\tau^6}(3L\tau^3 - 12\tau^2 L^2 + 15\tau L^3 - 6L^4) & \text{if} \quad x \in B(x^0, \tau) \setminus B(x^0, \frac{\tau}{2}), \end{cases}$$

$$\frac{\partial^2 w(x)}{\partial x_i^2} = \begin{cases} 0 & \text{if } x \in \overline{\Omega} \setminus B(x^0, \tau) \cup B(x^0, \frac{\tau}{2}), \\ \frac{64d}{\tau^6}(3L\tau^3 - 12\tau^2 L^2 + 15\tau L^3 - 6L^4) + & \\ \frac{64d(x_i - x_i^0)^2}{\tau^6}\left(\frac{3\tau^3}{L} - 24\tau^2 + 45\tau L - 24L^2\right) & \text{if } x \in B(x^0, \tau) \setminus B(x^0, \frac{\tau}{2}), \end{cases}$$

$$\sum_{i=1}^{N} \frac{\partial^2 w(x)}{\partial x_i^2} = \begin{cases} 0 & \text{if } x \in \overline{\Omega} \setminus B(x^0, \tau) \cup B(x^0, \frac{\tau}{2}), \\ \frac{64d}{\tau^6}(3L\tau^3(N+1) - 12\tau^2 L^2(N+2) + & \\ 15\tau L^3(N+3) - 6L^4(N+4)) & \text{if } x \in B(x^0, \tau) \setminus B(x^0, \frac{\tau}{2}), \end{cases}$$

$$\frac{\partial \Delta w(x)}{\partial x_i} = \begin{cases} 0 \qquad \text{if } x \in \overline{\Omega} \setminus B(x^0, \tau) \cup B(x^0, \frac{\tau}{2}), \\ \frac{64d(x_i - x_i^0)}{\tau^6}\left(\frac{3\tau^3(N+1)}{L} - 24\tau^2(N+2) + 45\tau L(N+3) - 24L^2(N+4)\right) \\ \text{if} \quad x \in B(x^0, \tau) \setminus B(x^0, \frac{\tau}{2}), \end{cases}$$

and

$$|\nabla \Delta w(x)| = \frac{64d}{\tau^6}|3\tau^3(N+1) - 24\tau^2 L(N+2) + 45\tau L^2(N+3) - 24L^3(N+4)|.$$

Now, if we let $r = \frac{1}{p}\left(\frac{c}{k}\right)^p$, then

$$\Phi(w) = \frac{\|w\|^p}{p} = \frac{2^{6p+1}(3d)^p \pi^{\frac{N}{2}}}{p\tau^{6p}\Gamma\left(\frac{N}{2}\right)}\sigma_{p,N}(\tau) > \frac{d^p}{pk^p K_{p,N}(\tau)} > \frac{1}{p}\left(\frac{c}{k}\right)^p = r. \tag{3.2}$$

Since, $0 \leq w(x) \leq d$, for each $x \in \Omega$, the condition (f_1) ensures that

$$\int_{\Omega \setminus B(x^0, \tau)} F(x, w(x))\, dx + \int_{B(x^0, \tau) \setminus B(x^0, \frac{\tau}{2})} F(x, w(x))\, dx \geq 0.$$

Hence, by condition (f_2) and (3.2), one has

$$\begin{aligned} \text{meas}(\Omega) \sup_{(x,\xi) \in \Omega \times [-c,c]} F(x, \xi) &\leq \left(\frac{c}{d}\right)^p K_{p,N}(\tau) \int_{B(x^0, \frac{\tau}{2})} F(x, d)\, dx \\ &\leq \left(\frac{c}{k\|w\|}\right)^p \int_{B(x^0, \frac{\tau}{2})} F(x, d)\, dx \\ &\leq \left(\frac{c}{k\|w\|}\right)^p \int_{\Omega} F(x, w(x))\, dx. \end{aligned} \tag{3.3}$$

Taking $\sup_{x\in\Omega}|u(x)|\le k\|u\|$ into account, we obtain that $|u(x)|\le c$ for all $u\in X$ with $\Phi(u)\le r$. So, by (3.2) and (3.3), one has

$$\begin{aligned}
\sup_{\Phi(u)\le r}(-\Psi(u)) &\le \int_\Omega \sup_{\Phi(u)\le r} F(x,u)\,dx\\
&\le \int_\Omega \sup_{|t|\le c} F(x,u)\,dx\\
&\le \operatorname{meas}(\Omega) \sup_{(x,\xi)\in\Omega\times[-c,c]} F(x,\xi)\\
&\le \left(\frac{c}{k\|w\|}\right)^p \int_\Omega F(x,w(x))\,dx = r\frac{-\Psi(w)}{\Phi(w)}.
\end{aligned}$$

Fixing any $\nu>1$, it is easy to see that

$$\sup_{u\in\Phi^{-1}(]-\infty,r[)}(-\Psi(u)) + \frac{r\frac{-\Psi(w)}{\Phi(w)} - \sup_{u\in\Phi^{-1}(]-\infty,r[)}(-\Psi(u))}{\nu} < r\frac{-\Psi(w)}{\Phi(w)}.$$

If h satisfies

$$\sup_{u\in\Phi^{-1}(]-\infty,r[)}(-\Psi(u)) + \frac{r\frac{-\Psi(w)}{\Phi(w)} - \sup_{u\in\Phi^{-1}(]-\infty,r[)}(-\Psi(u))}{\nu} < h < r\frac{-\Psi(w)}{\Phi(w)},$$

applying Theorem 2.3 with $u_0=0$ and $u_1=w$, we obtain

$$\sup_{\lambda\ge 0}\inf_{x\in X}(\Phi(x)+\lambda(\Psi(x)+h)) < \inf_{x\in X}\sup_{\lambda\ge 0}(\Phi(x)+\lambda(\Psi(x)+h)).$$

Therefore, the assumption (2.3) of Theorem 2.2 holds true. □

Let $h_1\in C(\Omega)$ be a positive function and $h_2\in C(\mathbb{R})$ be a function. Let $f(x,u)=h_1(x)h_2(u)$ for each $(x,u)\in\Omega\times\mathbb{R}$,

$$H(t)=\int_0^t h_2(\xi)d\xi$$

for all $t\in\mathbb{R}$ and $\alpha_1(x)=\frac{\alpha(x)}{h_1(x)}$. Then, applying Theorem 3.1, we have following result.

Theorem 3.2. *Assume that there exist two positive constants c and d, with $d>c(K_{p,N}(\tau))^{\frac{1}{p}}$ such that*

(f_1) $F(x,\xi)\ge 0$ *for each* $(x,\xi)\in(\Omega\setminus B(x^0,\frac{\tau}{2}))\times[0,d]$;

(f_2) $\operatorname{meas}(\Omega)\sup_{(x,\xi)\in\Omega\times[-c,c]}F(x,\xi)\le\left(\frac{c}{d}\right)^p K_{p,N}(\tau)\frac{H(d)}{H(c)}\int_{B(x^0,\frac{\tau}{2})}h_1(x)\,dx$;

(f_3) *there exist a negative function $\alpha\in L^1(\Omega)$ and a positive constant $\gamma<p$ such that*

$$F(x,\xi)\le\alpha_1(x)(1+|\xi|^\gamma),$$

for almost every $x\in\Omega$ and for every $\xi\in\mathbb{R}$.

Then, there exist a real positive $\rho\in\mathbb{R}$ and an open interval $\Lambda\subseteq[0,+\infty)$ with the following property: for every $\lambda\in\Lambda$ and for every L^1-Carathéodory function $g:\Omega\times\mathbb{R}\to\mathbb{R}$, thus satisfying

(g_1) $\sup_{\{|s|\le\xi\}}|g(\cdot,s)|\in L^1(\Omega)$, *for all $\xi>0$,* *there exists $\delta > 0$ such that, for each $\mu \in [0,\delta]$, the problem*

$$\begin{cases} -\Delta_p^3 u = \lambda h_1(x)h_2(u) + \mu g(x,u), & x \in \Omega, \\ u = \Delta u = \Delta^2 u = 0, & x \in \partial\Omega, \end{cases}$$

has at least three weak solutions whose norms are less than ρ.

A simple consequence of Theorem 3.2 is as follows.

Theorem 3.3. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Put $F(\xi) = \int_\Omega f(t)dt$ for $\xi \in \mathbb{R}$ and assume that there exist two positive constants c and d, with $d > c(K_{p,N}(\tau))^{\frac{1}{p}}$ such that*

(f_1) *$F(\xi) \geq 0$ for each $\xi \in [0,d]$;*

(f_2) $\operatorname{meas}(\Omega)\sup_{[-c,c]} F(\xi) \leq \left(\frac{c}{d}\right)^p K_{p,N}(\tau)\frac{\tau^N \pi^{\frac{N}{2}}}{2^{N-1}\Gamma(\frac{N}{2})}F(d)$;

(f_3) *there exist a negative constant α and a positive constant $\gamma < p$ such that*

$$F(\xi) \leq \alpha(1 + |\xi|^\gamma)$$

for every $\xi \in \mathbb{R}$.

Then, there exist a real positive $\rho \in \mathbb{R}$ and an open interval $\Lambda \subseteq [0,+\infty)$ with the following property: for every $\lambda \in \Lambda$ and for every L^1-Carathéodory function $g : \Omega \times \mathbb{R} \to \mathbb{R}$, thus satisfying

(g_1) $\sup_{\{|s|\leq\xi\}} |g(\cdot,s)| \in L^1(\Omega)$, *for all $\xi > 0$,*

there exists $\delta > 0$ such that, for each $\mu \in [0,\delta]$, the problem

$$\begin{cases} -\Delta_p^3 u = \lambda f(u) + \mu g(x,u), & x \in \Omega, \\ u = \Delta u = \Delta^2 u = 0, & x \in \partial\Omega, \end{cases}$$

has at least three weak solutions whose norms are less than ρ.

Our other main result reads as follows.

Theorem 3.4. *Let $f \in C^0(\overline{\Omega} \times \mathbb{R})$ and denote $F(x,\xi) := \int_0^\xi f(x,t)\,dt$ for all $(x,\xi) \in \overline{\Omega} \times \mathbb{R}$. Assume that there exist two positive constants d and c, with $d > c(K_{p,N}(\tau))^{\frac{1}{p}}$ such that*

(f_1) *$F(x,\xi) \geq 0$ for each $(x,\xi) \in (\Omega \setminus B(x^0,\frac{\tau}{2})) \times [0,d]$;*

(f_2)

$$\frac{\int_\Omega \max_{|t|\leq c} F(x,\xi)\,dx}{c^p} < K_{p,N}(\tau)\frac{\int_B (x^0,\frac{\tau}{2})F(x,d)\,dx}{d^p};$$

(f_3) *there exist a function $\alpha \in L^1(\Omega)$ and a positive constant $\gamma < p$ such that*

$$F(x,\xi) \leq \alpha(x)(1 + |\xi|^\gamma),$$

for almost every $x \in \Omega$ and for every $\xi \in \mathbb{R}$.

Then, for every λ in

$$\Lambda := \left(\frac{2^{6p+1}3^p \pi^{\frac{N}{2}} \sigma_{p,N}(\tau) d^p}{p\tau^{6p}\Gamma(N/2)\int_{B(x^0,\frac{\tau}{2})} F(x,d)\,dx}, \frac{c^p}{pk^p \int_\Omega \max_{|t|\leq c} F(x,t)\,dx} \right),$$

problem

$$\begin{cases} -\Delta_p^3 u = \lambda f(x,u), & x \in \Omega, \\ u = \Delta u = \Delta^2 u = 0, & x \in \partial\Omega, \end{cases}$$

has at least three weak solutions.

Proof. Let X, Φ, Ψ and w be as in the proof of Theorem 3.1. Using Theorem 2.4, we obtain the desired conclusion immediately. $\square$

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REFERENCES

[1] G.A. Afrouzi and S. Shokooh, Existence of infinitely many solutions for quasilinear problems with a $p(x)$-biharmonic operator, Electron. J. Differ. Equ. 317 (2015), 1-14.
[2] F. Cammaroto and L. Vilasi, Multiple solutions for a Kirchhoff-type problem involving the $p(x)$- biharmonic operator, Nonlinear Anal. 74 (2011), 1841-1852.
[3] B. Ge, Q.M. Zhou, Y.H. Wu, Eigenvalues of the $p(x)$-biharmonic operator with indefinite weight, Zeitschrift für angewandte Mathematik und Physik, 66 (2015), 1007-1021.
[4] L. Kong, Multiple solutions for fourth order elliptic problems with $p(x)$-biharmonic operators, Opusc. Math. 36 (2016), 253-264.
[5] E. Acerbi, G. Mingione, Regularity results for a class of functionals with non-standard growth, Arch. Ration. Mech. Anal. 156 (2001), 121–140.
[6] M. Růžička, Electrorheological Fluids: Modeling and Mathematical Theory, vol. 1748 of Lecture Notes in Mathematics, Springer, Berlin, Germany, 2000.
[7] W.M. Winslow, Induced fibration of suspensions, J. Appl. Phys. 20 (1949), 1137–1140.
[8] B. Rahal, Existence results of infinitely many solutions for $p(x)-$Kirchhoff type triharmonic operator with Navier boundary conditions, J. Math. Anal. Appl. 478 (2019), 1133-1146.
[9] P. Candito, G. Molica Bisci, Multiple solutions for a Navier boundary value problem involving the p- biharmonic operator, Discrete Contin. Dyn. Syst. Ser. S. 5 (2012), 741-751.
[10] P. Candito, L. Li, R. Livrea, Infinitely many solutions for a perturbed nonlinear Navier boundary value problem involving the p-biharmonic, Nonlinear Anal. 75 (2012), 6360-6369.
[11] S. Shokooh, G.A. Afrouzi, Existence results of infinitely many solutions for a class of $p(x)-$ biharmonic problems, Comput. Methods Differ. Equ. 5 (2017), 310-323.
[12] H. Yin, Y. Liu, Existence of three solutions for a Navier boundary value problem involving the $p(x)$-biharmonic, Bull. Korean. Math. Soc. 6 (2013), 1817-1826.
[13] H. Yin, M. Xu, Existence of three solutions for a Navier boundary value problem involving the $p(x)$-biharmonic operator, Ann. Polon. Math. 109 (2013), 47-54.
[14] B. Ricceri, A three critical points theorem revisited, Nonlinear Anal. 9 (2009), 3084-3089.
[15] P. Drábek, J. Milota, Methods of nonlinear analysis: Application to differential equations, Birkhäuser, Berlin, 2006.
[16] X.L. Fan, D. Zhao, On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$, J. Math. Anal. Appl. 263 (2001), 424-446.
[17] G. A. Afrouzi, V. Rădulescu, S. Shokooh, Multiple solutions of Neumann problems: an Orlicz–Sobolev Space Setting, Bull. Malays. Math. Sci. Soc. 40 (2017), 1591–1611.
[18] G. Bonanno, Some remarks on a three critical points theorem, Nonlinear Anal. 54 (2003), 651-665.
[19] G. Bonanno, S.A. Marano, On the structure of the critical set of non-differentiable functionals with a weak compactness condition, Appl. Anal. 89 (2010), 1-10.