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CORRIGENDUM TO "THE EXISTENCE OF SOLUTIONS FOR INTEGRAL BOUNDARY VALUE PROBLEMS WITH P-LAPLACIAN OPERATOR ON INFINITE INTERVAL" [JOURNAL OF NONLINEAR FUNCTIONAL ANALYSIS, VOL. 2021 (2021), ARTICLE ID 13]

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Abstract. A corrigendum to [F. Fenizri, R. Khaldi, A. Guezane-Lakoud, The existence of solutions for integral boundary value problems with p-Laplacian operators on infinite interval, J. Nonlinear Funct. Anal. 2021 (2021), Article ID 13] is given.

Keywords. Boundary value problems; Integral boundary condition; Existence of solution; p-Laplacian operator.

1. Statement

This corrigendum corrects the lemmas and theorems in their above titled paper. The authors would like to apologise for any inconvenience caused, and thank the editor and the anonymous reader, who pointed out the inaccuracies and gaps. In this corrigendum, all the symbols and labels used in this paper are the same in the original published paper. To avoid the repetition, we only give the preliminaries and main results.

2. Preliminaries

In this section, we introduce some necessary lemmas and preliminaries, which can be found in [1, 2, 3], to facilitate the analysis of the problem 1.1 (see the original paper).

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Definition 2.1. The fractional integral of order $\alpha > 0$ of a function $f : [0,+\infty) \to \mathbb{R}$ is given by

$$I_{0^+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\, ds,$$

provided that the right side is pointwise defined.

Definition 2.2. The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a function $f : [0,+\infty) \to \mathbb{R}$ is given by

$$D_{0^+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t \frac{f(s)}{(t-s)^{\alpha-n+1}} ds = \left(\frac{d}{dt}\right)^n I_{0^+}^{n-\alpha} f(t),$$

provided that the right side is pointwise defined, where $n = \lceil \alpha \rceil$, ($\lceil \alpha \rceil$ denotes the ceiling of the number).

Lemma 2.3. *We have the following relations:*
1) $D_{0^+}^{\alpha} t^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} t^{\beta-\alpha-1}$, $0 \le \alpha < \beta$.
2) $I_{0^+}^{\alpha} t^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)} t^{\beta+\alpha-1}$, $\alpha, \beta > 0$.
3) $D_{0^+}^{\alpha} t^{\alpha-j} = 0$, *for* $j = 1, 2, ..., \lceil \alpha \rceil$.
4) $D_{0^+}^{\alpha} I_{0^+}^{\alpha} u(t) = u(t)$, *for a.e.* t *and* $u \in L^1(0,+\infty)$, $\alpha > 0$.

The following Lemma is significant in the proof of the existence Theorem.

Lemma 2.4. *(Krasnoselskii) Let C be a non-empty closed convex subset of a Banach space X. Suppose that A and T map C into X such that:*
i) $Ax + Ty \in C$ *for all* $x, y \in C$,
ii) A is a contraction mapping,
iii)T is completely continuous.
Then, there exists an $x \in C$ *with* $Ax + Tx = x$.

Define the space

$$X = \Big\{ u : u \in C(\mathbb{R}^+, \mathbb{R}),\ D_{0^+}^{\alpha-1} u \in C(\mathbb{R}^+, \mathbb{R}), \sup_{t\in\mathbb{R}^+} \frac{|u(t)|}{1+t^{\alpha-1}} < +\infty,\ \sup_{t\in\mathbb{R}^+} \left| D_{0^+}^{\alpha-1} u(t) \right| < +\infty \Big\},$$

with the norm

$$\|u\| = \max \left\{ \sup_{t\in\mathbb{R}^+} \frac{|u(t)|}{1+t^{\alpha-1}},\ \sup_{t\in\mathbb{R}^+} \left| D_{0^+}^{\alpha-1} u(t) \right| \right\}.$$

The space $(X, \|.\|)$ is a Banach space as essentially shown in [4].

To prove the existence results, the following lemma will be useful.

Lemma 2.5 (p. 1081)**.** [4] *Let $Z \subseteq X$ be a bounded set. Then Z is relatively compact in X if the following conditions hold:*
i) for any $u \in Z$, $\frac{u(t)}{1+t^{\alpha-1}}$ *and* $D_{0^+}^{\alpha-1} u(t)$ *are equicontinuous on any compact interval of* $\mathbb{R}^+$.
ii) given $\varepsilon > 0$, *there exists a constant* $l = l(\varepsilon) > 0$ *such that*

$$\left| \frac{u(t_1)}{1+t_1^{\alpha-1}} - \frac{u(t_2)}{1+t_2^{\alpha-1}} \right| < \varepsilon \text{ and } \left| D_{0^+}^{\alpha-1} u(t_1) - D_{0^+}^{\alpha-1} u(t_2) \right| < \varepsilon,$$

for any $t_1, t_2 \geq l$, $u \in Z$.

Throughout this paper, we assume that the following conditions hold:

(C1) For all $\rho > 0$, *there exists* $M_\rho > 0$ *such that* $\left|f\left(t, \left(1+t^{\alpha-1}\right)x, y\right)\right| \leq M_\rho$, *for all* $t > 0$, $x, y \in [-\rho, \rho]$.

(C2) The function a is not identical zero on any closed subinterval of $[0, +\infty)$ *and*

$$I_{0^+}^{\delta} a \in L^{q-1}[0, +\infty).$$

3. Existence of solutions

In this section, we prove the existence result of solutions for the boundary value problem (1.1) (see the original paper). First, let us state some lemmas.

Lemma 3.1. *Assume* $e \in L^1(0, +\infty)$. *Then the solution of the integral equation*

$$u(t) = \int_0^{+\infty} k(t,s)e(s)ds + \int_0^{+\infty} G(t,s)u(s)ds,$$

in the space X, *is a solution of the following linear boundary value problem*

$$\begin{cases} D_{0^+}^{\alpha} u(t) + e(t) = 0, & 2 < \alpha \leq 3, \ a.e.\ t > 0, \\ u(0) = 0,\ D_{0^+}^{\alpha-1} u(+\infty) = \int_0^{+\infty} g(s)u(s)ds, & \\ D_{0^+}^{\alpha-2} u(0) = \int_0^{+\infty} h(s)u(s)ds, & \end{cases} \tag{3.1}$$

where

$$G(t,s) = \frac{g(s)t^{\alpha-1} + (\alpha-1)h(s)t^{\alpha-2}}{\Gamma(\alpha)}, \tag{3.2}$$

and

$$k(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} t^{\alpha-1} - (t-s)^{\alpha-1}, & 0 \leq s \leq t < +\infty, \\ t^{\alpha-1}, & 0 \leq t \leq s < +\infty. \end{cases} \tag{3.3}$$

Proof. Let $u \in X$ be a solution of the following integral equation

$$\begin{aligned} u(t) = &-\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} e(s)ds + \frac{t^{\alpha-1}}{\Gamma(\alpha)}\left[\int_0^{+\infty} e(s)ds + \int_0^{+\infty} g(s)u(s)ds\right] \\ &+ \frac{t^{\alpha-2}}{\Gamma(\alpha-1)} \int_0^{+\infty} h(s)u(s)ds \end{aligned} \tag{3.4}$$

By applying the operators $I_{0^+}^{3-\alpha}$to the two sides of (3.4), we see that the term $I_{0^+}^{3-\alpha} I_{0^+}^{\alpha} = I_{0^+}^{3}$ has the first and second derivative existing for all t, and a third derivative existing almost everywhere. So, using Lemma 2.3, we obtain

$$D_{0^+}^{\alpha-1} u(t) = -\int_0^t e(s)ds + \int_0^{+\infty} e(s)ds + \int_0^{+\infty} g(s)u(s)ds,$$

and

$$\begin{aligned} D_{0^+}^{\alpha-2} u(t) = &-\int_0^t (t-s)e(s)ds \\ &+ t\left[\int_0^{+\infty} e(s)ds + \int_0^{+\infty} g(s)u(s)ds\right] + \int_0^{+\infty} h(s)u(s)ds. \end{aligned}$$

Since $e\in L^{1}(0,+\infty)$, by differentiating the first relation above, we obtain

$$D_{0^+}^{\alpha}u(t)=-e(t),\ a.e.t>0$$

So, it is easy to see $u(0)=0$, $D_{0^+}^{\alpha-1}u(+\infty)=\int_0^{+\infty}g(s)u(s)ds$, and $D_{0^+}^{\alpha-2}u(0)=\int_0^{+\infty}h(s)u(s)ds$, thus we obtain (3.1) and the assertion of Lemma 3.1 follows. The proof is complete. □

Lemma 3.2. *The solution of the following integral equation*

$$\begin{aligned}u(t)=&\int_0^{+\infty}k(t,s)\phi_q\left(\frac{1}{\Gamma(\delta)}\int_0^s(s-r)^{\delta-1}a(r)f(r,u(r),D_{0^+}^{\alpha-1}u(r)dr\right)ds\\&+\int_0^{+\infty}G(t,s)u(s)ds,\end{aligned}\tag{3.5}$$

in the space X, is a solution of the boundary value problem (1.1) (see the original paper), where $k(t,s)$ and $G(t,s)$ are defined by (3.3) and (3.2), respectively.

Proof. Firstly, we prove that af, $\phi_q\left(I_{0^+}^{\delta}(af)\right)\in L^{1}(0,+\infty)$. Indeed, for $u\in X$ with $\rho=\|u\|$, by using (C1), we have

$$\int_0^{+\infty}a(t)f(t,u(t),D_{0^+}^{\alpha-1}u(t)dt\leq M_\rho\int_0^{+\infty}a(s)ds<+\infty,$$

and by conditions (C1) and (C2), we have

$$\begin{aligned}&\int_0^{+\infty}\left|\phi_q\left(\frac{1}{\Gamma(\delta)}\int_0^s(s-r)^{\delta-1}a(r)f(r,u(r),D_{0^+}^{\alpha-1}u(r)dr\right)\right|ds\\=&\int_0^{+\infty}\left|\frac{1}{\Gamma(\delta)}\int_0^s(s-r)^{\delta-1}a(r)f(r,u(r),D_{0^+}^{\alpha-1}u(r)dr\right|^{q-1}ds\\\leq&(M_\rho)^{q-1}\int_0^{+\infty}\left|I_{0^+}^{\delta}a(s)\right|^{q-1}ds<+\infty.\end{aligned}$$

Now, from Lemma 3.1, we have the solution $u\in X$ of (3.4) is a solution of the following boundary value problem

$$\begin{cases}D_{0^+}^{\alpha}u(t)&=-\phi_q\left(\frac{1}{\Gamma(\delta)}\int_0^t(t-s)^{\delta-1}a(s)f(s,u(s),D_{0^+}^{\alpha-1}u(s))ds\right),2<\alpha\leq3,\ a.e.\ t>0\\u(0)&=0,D_{0^+}^{\alpha-1}u(+\infty)=\int_0^{+\infty}g(s)u(s)ds,\\D_{0^+}^{\alpha-2}u(0)&=\int_0^{+\infty}h(s)u(s)ds.\end{cases}$$

Applying ϕ_p to the differential equation in the above boundary value problem, we have the boundary condition $I_{0^+}^{1-\delta}(\phi_p(D_{0^+}^{\alpha}u(0)))=0$ and then, we apply $D_{0^+}^{\delta}$ to the obtained equation to obtain that u is a solution of the boundary value problem (1.1). The proof is complete. □

Lemma 3.3. *The functions $k(t,s)$ and $G(t,s)$ are nonnegative and satisfy:*

1) $0\leq\frac{k(t,s)}{1+t^{\alpha-1}}\leq\frac{1}{\Gamma(\alpha)}$ for all $(t,s)\in(0,+\infty)\times[0,+\infty)$.

2) $0\leq\frac{G(t,s)}{1+t^{\alpha-1}}\leq\frac{1}{\Gamma(\alpha)}(g(s)+(\alpha-1)h(s))$ for all $(t,s)\in(0,+\infty)\times[0,+\infty)$.

Define the operators $A,T:X\to X$ by

$$Au(t)=\int_0^{+\infty}G(t,s)u(s)ds,$$

and

$$Tu(t)=\int_0^{+\infty}k(t,s)\phi_q\left(\frac{1}{\Gamma(\delta)}\int_0^s(s-r)^{\delta-1}a(r)f(r,u(r),D_{0^+}^{\alpha-1}u(r)dr\right)ds.$$

It follows that

$$D_{0^+}^{\alpha-1}Au(t)=\int_0^{+\infty}g(s)u(s)ds.$$

Since

$$\begin{aligned}Tu(t) &= \frac{t^{\alpha-1}}{\Gamma(\alpha)}\int_0^{+\infty}\phi_q\left(\frac{1}{\Gamma(\delta)}\int_0^s(s-r)^{\delta-1}a(r)f(r,u(r),D_{0^+}^{\alpha-1}u(r)dr\right)ds\\ &\quad-\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\phi_q\left(\frac{1}{\Gamma(\delta)}\int_0^s(s-r)^{\delta-1}a(r)f(r,u(r),D_{0^+}^{\alpha-1}u(r)dr\right)ds.\end{aligned}$$

we obtain

$$\begin{aligned}D_{0^+}^{\alpha-1}Tu(t) &= \int_0^{+\infty}\phi_q\left(\frac{1}{\Gamma(\delta)}\int_0^s(s-r)^{\delta-1}a(r)f(r,u(r),D_{0^+}^{\alpha-1}u(r)dr\right)ds\\ &\quad-\int_0^t\phi_q\left(\frac{1}{\Gamma(\delta)}\int_0^s(s-r)^{\delta-1}a(r)f(r,u(r),D_{0^+}^{\alpha-1}u(r)dr\right)ds\\ &= \int_t^{+\infty}\phi_q\left(\frac{1}{\Gamma(\delta)}\int_0^s(s-r)^{\delta-1}a(r)f(r,u(r),D_{0^+}^{\alpha-1}u(r)dr\right)ds.\end{aligned}$$

Theorem 3.4. *If conditions (C1) and (C2) hold, then T is completely continuous.*

Proof. First, we prove that T is a continuous operator. Let $u_n\to u$ as $n\to+\infty$ in X. We show that $\|Tu_n-Tu\|\to 0$ as $n\to+\infty$. Indeed, since

$$\left|\phi_q\left(\frac{1}{\Gamma(\delta)}\int_0^s(s-r)^{\delta-1}a(r)f(r,u_n(r),D_{0^+}^{\alpha-1}u_n(r)dr\right)\right|\le M_\rho^{q-1}\phi_q\left(I_{0^+}^{\delta}a(s)\right),$$

where $\|u_n\|\le\rho$, for all n, then, by the Lebesgue dominated convergence theorem, the continuity of f and condition (C2), we obtain

$$\begin{aligned}&\int_0^{+\infty}\Big|\phi_q\Big(\frac{1}{\Gamma(\delta)}\int_0^s(s-r)^{\delta-1}a(r)f(r,u_n(r),D_{0^+}^{\alpha-1}u_n(r)dr\Big)\\ &\quad-\phi_q\Big(\frac{1}{\Gamma(\delta)}\int_0^s(s-r)^{\delta-1}a(r)f(r,u(r),D_{0^+}^{\alpha-1}u(r)dr\Big)\Big|ds\to 0,\ \text{as } n\to+\infty.\end{aligned}\tag{3.6}$$

Now, we have

$$\begin{aligned}&\Big|\frac{Tu_n(t)-Tu(t)}{1+t^{\alpha-1}}\Big|\\ &=\Big|\int_0^{+\infty}\frac{k(t,s)}{1+t^{\alpha-1}}\phi_q(\frac{1}{\Gamma(\delta)}\int_0^s(s-r)^{\delta-1}a(r)f(r,u_n(r),D_{0^+}^{\alpha-1}u_n(r)dr)ds\\ &\quad-\int_0^{+\infty}\frac{k(t,s)}{1+t^{\alpha-1}}\phi_q(\frac{1}{\Gamma(\delta)}\int_0^s(s-r)^{\delta-1}a(r)f(r,u(r),D_{0^+}^{\alpha-1}u(r)dr)ds\Big|\\ &\le\frac{1}{\Gamma(\alpha)}\int_0^{+\infty}|\phi_q(\frac{1}{\Gamma(\delta)}\int_0^s(s-r)^{\delta-1}a(r)f(r,u_n(r),D_{0^+}^{\alpha-1}u_n(r)dr)\\ &\quad-\phi_q(\frac{1}{\Gamma(\delta)}\int_0^s(s-r)^{\delta-1}a(r)f(r,u(r),D_{0^+}^{\alpha-1}u(r)dr)|ds,\end{aligned}$$

and

$$\begin{aligned}
&\left|D_{0^+}^{\alpha-1}Tu_n(t)-D_{0^+}^{\alpha-1}Tu(t)\right|\\
&=\Big|\int_t^{+\infty}\phi_q\Big(\frac{1}{\Gamma(\delta)}\int_0^s(s-r)^{\delta-1}a(r)f(r,u_n(r),D_{0^+}^{\alpha-1}u_n(r)dr\Big)ds\\
&\quad-\int_t^{+\infty}\phi_q\Big(\frac{1}{\Gamma(\delta)}\int_0^s(s-r)^{\delta-1}a(r)f(r,u(r),D_{0^+}^{\alpha-1}u(r)dr\Big)ds\Big|\\
&\le\int_0^{+\infty}\Big|\phi_q\Big(\frac{1}{\Gamma(\delta)}\int_0^s(s-r)^{\delta-1}a(r)f(r,u_n(r),D_{0^+}^{\alpha-1}u_n(r)dr\Big)\\
&\quad-\phi_q\left(\frac{1}{\Gamma(\delta)}\int_0^s(s-r)^{\delta-1}a(r)f(r,u(r),D_{0^+}^{\alpha-1}u(r)dr\right)\Big|ds,
\end{aligned}$$

from which and (3.6), we obtain

$$\begin{aligned}
&\sup_{t\in\mathbb{R}^+}\Big|\frac{Tu_n(t)-Tu(t)}{1+t^{\alpha-1}}\Big|\\
&\le\frac{1}{\Gamma(\alpha)}\int_0^{+\infty}\Big|\phi_q\Big(\frac{1}{\Gamma(\delta)}\int_0^s(s-r)^{\delta-1}a(r)f(r,u_n(r),D_{0^+}^{\alpha-1}u_n(r)dr\Big)\\
&\quad-\phi_q\Big(\frac{1}{\Gamma(\delta)}\int_0^s(s-r)^{\delta-1}a(r)f(r,u(r),D_{0^+}^{\alpha-1}u(r)dr\Big)\Big|ds\to 0,\ \text{as } n\to+\infty.
\end{aligned}$$

and

$$\begin{aligned}
&\sup_{t\in\mathbb{R}^+}\left|D_{0^+}^{\alpha-1}Tu_n(t)-D_{0^+}^{\alpha-1}Tu(t)\right|\\
&\le\int_0^{+\infty}\Big|\phi_q\Big(\frac{1}{\Gamma(\delta)}\int_0^s(s-r)^{\delta-1}a(r)f(r,u_n(r),D_{0^+}^{\alpha-1}u_n(r)dr\Big)\\
&\quad-\phi_q\Big(\frac{1}{\Gamma(\delta)}\int_0^s(s-r)^{\delta-1}a(r)f(r,u(r),D_{0^+}^{\alpha-1}u(r)dr\Big)\Big|ds\to 0,\ \text{as } n\to+\infty.
\end{aligned}$$

So, $\|Tu_n-Tu\|\to 0$ as $n\to+\infty$. Thus T is continuous.

Next, we show that T is a compact operator. Let $B_m=\{u\in X,\|u\|\le m\}$ be a nonempty bounded closed subset of X. From condition (C1), we see that there exists $M_m>0$ such that

$$\sup_{t\in[0,+\infty)}\left\{\left|f(t,u(t),D_{0^+}^{\alpha-1}u(t))\right|,u\in B_m\right\}\le M_m.$$

For any $u\in B_m$, we find from condition (C2) that

$$\begin{aligned}
\left|\frac{Tu(t)}{1+t^{\alpha-1}}\right|&=\left|\int_0^{+\infty}\frac{k(t,s)}{1+t^{\alpha-1}}\phi_q\left(\frac{1}{\Gamma(\delta)}\int_0^s(s-r)^{\delta-1}a(r)f(r,u(r),D_{0^+}^{\alpha-1}u(r)dr\right)ds\right|\\
&\le\frac{M_m^{q-1}}{\Gamma(\alpha)}\int_0^{+\infty}\phi_q\left(\frac{1}{\Gamma(\delta)}\int_0^s(s-r)^{\delta-1}a(r)dr\right)ds<+\infty,
\end{aligned}\tag{3.7}$$

and

$$\begin{aligned}\left|D_{0^+}^{\alpha-1}Tu(t)\right| &= \left|\int_t^{+\infty}\phi_q\left(\frac{1}{\Gamma(\delta)}\int_0^s(s-r)^{\delta-1}a(r)f(r,u(r),D_{0^+}^{\alpha-1}u(r)dr\right)ds\right|\\ &\le \int_0^{+\infty}\left|\phi_q\left(\frac{1}{\Gamma(\delta)}\int_0^s(s-r)^{\delta-1}a(r)f(r,u(r),D_{0^+}^{\alpha-1}u(r)dr\right)\right|ds\\ &\le M_m^{q-1}\int_0^{+\infty}\phi_q\left(\frac{1}{\Gamma(\delta)}\int_0^s(s-r)^{\delta-1}a(r)dr\right)ds<+\infty.\end{aligned}$$

So, $T(B_m)$ is bounded. Moreover, for any $T_0>0$, let $l=[0,T_0]$ be a compact interval. For all $t_1,t_2\in l$ with $t_1<t_2$, we have

$$\begin{aligned}&\Big|\frac{Tu(t_1)}{1+t_1^{\alpha-1}}-\frac{Tu(t_2)}{1+t_2^{\alpha-1}}\Big|\\ &=\frac{1}{\Gamma(\alpha)}\Big|\Big(\frac{t_2^{\alpha-1}}{1+t_2^{\alpha-1}}-\frac{t_1^{\alpha-1}}{1+t_1^{\alpha-1}}\Big)\int_0^{+\infty}\phi_q\Big(\frac{1}{\Gamma(\delta)}\int_0^s(s-r)^{\delta-1}a(r)f(r,u(r),D_{0^+}^{\alpha-1}u(r)dr\Big)ds\\ &-\int_0^{t_1}\Big(\frac{(t_2-s)^{\alpha-1}}{1+t_2^{\alpha-1}}-\frac{(t_1-s)^{\alpha-1}}{1+t_1^{\alpha-1}}\Big)\phi_q\Big(\frac{1}{\Gamma(\delta)}\int_0^s(s-r)^{\delta-1}a(r)f(r,u(r),D_{0^+}^{\alpha-1}u(r)dr\Big)ds\\ &-\int_{t_1}^{t_2}\frac{(t_2-s)^{\alpha-1}}{1+t_2^{\alpha-1}}\phi_q\Big(\frac{1}{\Gamma(\delta)}\int_0^s(s-r)^{\delta-1}a(r)f(r,u(r),D_{0^+}^{\alpha-1}u(r)dr\Big)ds\Big|\to 0,\text{ as } t_1\to t_2,\end{aligned}$$

and

$$\begin{aligned}&\left|D_{0^+}^{\alpha-1}Tu(t_2)-D_{0^+}^{\alpha-1}Tu(t_1)\right|\\ =&\left|\int_{t_2}^{+\infty}\phi_q\left(\frac{1}{\Gamma(\delta)}\int_0^s(s-r)^{\delta-1}a(r)f(r,u(r),D_{0^+}^{\alpha-1}u(r)dr\right)ds\right.\\ &\left.-\int_{t_1}^{+\infty}\phi_q\left(\frac{1}{\Gamma(\delta)}\int_0^s(s-r)^{\delta-1}a(r)f(r,u(r),D_{0^+}^{\alpha-1}u(r)dr\right)ds\right|\\ \le& M_m^{q-1}\int_{t_1}^{t_2}\phi_q\left(\frac{1}{\Gamma(\delta)}\int_0^s(s-r)^{\delta-1}a(r)dr\right)ds\to 0\text{ as } t_1\to t_2.\end{aligned}$$

Therefore, $\frac{Tu(t)}{1+t^{\alpha-1}}$ and $D_{0^+}^{\alpha-1}Tu(t)$ are equicontinuous.

Next, we show that $T(B_m)$ is equiconvergent at $+\infty$. For any $u\in B_m$, we have

$$\begin{aligned}&\int_0^{+\infty}\phi_q\left(\frac{1}{\Gamma(\delta)}\int_0^s(s-r)^{\delta-1}a(r)\left|f(r,u(r),D_{0^+}^{\alpha-1}u(r)\right|dr\right)ds\\ \le& M_m^{q-1}\int_0^{+\infty}\phi_q\left(\frac{1}{\Gamma(\delta)}\int_0^s(s-r)^{\delta-1}a(r)dr\right)ds<+\infty.\end{aligned}$$

For a given $\varepsilon>0$, there exists a constant $L>0$ such that

$$\int_L^{+\infty}\phi_q\left(\frac{1}{\Gamma(\delta)}\int_0^s(s-r)^{\delta-1}(r)\left|f(r,u(r),D_{0^+}^{\alpha-1}u(r)\right|dr\right)ds<\varepsilon.$$

On the other hand, since $\lim_{t\to+\infty}\frac{t^{\alpha-1}}{1+t^{\alpha-1}}=1$, there exists a constant $T_1>L$ such that for any $t_1,\,t_2\geq T_1$

$$\left|\frac{t_1^{\alpha-1}}{1+t_1^{\alpha-1}}-\frac{t_2^{\alpha-1}}{1+t_2^{\alpha-1}}\right|<\varepsilon.$$

Since, for $0\leq s\leq L$, we have

$$\lim_{t\to+\infty}\frac{k(t,s)}{1+t^{\alpha-1}}=\lim_{t\to+\infty}\left(\frac{t^{\alpha-1}}{1+t^{\alpha-1}}-\frac{(t-s)^{\alpha-1}}{1+t^{\alpha-1}}\right)=0.$$

Then, for $0\leq s\leq L$, there exists a constant $T_2>L$ such that, for any $t_1,t_2\geq T_2$,

$$\left|\frac{k(t_1,s)}{1+t_1^{\alpha-1}}-\frac{k(t_2,s)}{1+t_2^{\alpha-1}}\right|<\varepsilon.$$

For any $t_1,t_2\geq T_3=\max\{T_1,T_2\}$, by Lemma 2.3, one has

$$\begin{aligned}
&\left|\frac{Tu(t_1)}{1+t_1^{\alpha-1}}-\frac{Tu(t_2)}{1+t_2^{\alpha-1}}\right|\\
&\leq\int_0^{+\infty}\left|\frac{k(t_1,s)}{1+t_1^{\alpha-1}}-\frac{k(t_2,s)}{1+t_2^{\alpha-1}}\right|\phi_q\left(\frac{1}{\Gamma(\delta)}\int_0^s(s-r)^{\delta-1}a(r)\left|f(r,u(r),D_{0^+}^{\alpha-1}u(r))\right|dr\right)ds\\
&=\int_0^{L}\left|\frac{k(t_1,s)}{1+t_1^{\alpha-1}}-\frac{k(t_2,s)}{1+t_2^{\alpha-1}}\right|\phi_q\left(\frac{1}{\Gamma(\delta)}\int_0^s(s-r)^{\delta-1}a(r)\left|f(r,u(r),D_{0^+}^{\alpha-1}u(r))\right|dr\right)ds\\
&+\int_L^{+\infty}\left|\frac{k(t_1,s)}{1+t_1^{\alpha-1}}-\frac{k(t_2,s)}{1+t_2^{\alpha-1}}\right|\phi_q\left(\frac{1}{\Gamma(\delta)}\int_0^s(s-r)^{\delta-1}a(r)\left|f(r,u(r),D_{0^+}^{\alpha-1}u(r))\right|dr\right)ds\\
&\leq\varepsilon\int_0^{+\infty}\phi_q\left(\frac{1}{\Gamma(\delta)}\int_0^s(s-r)^{\delta-1}a(r)\left|f(r,u(r),D_{0^+}^{\alpha-1}u(r))\right|dr\right)ds\\
&+\frac{2}{\Gamma(\alpha)}\int_L^{+\infty}\phi_q\left(\frac{1}{\Gamma(\delta)}\int_0^s(s-r)^{\delta-1}a(r)\left|f(r,u(r),D_{0^+}^{\alpha-1}u(r))\right|dr\right)ds\\
&\leq\varepsilon\int_0^{+\infty}\phi_q\left(\frac{1}{\Gamma(\delta)}\int_0^s(s-r)^{\delta-1}a(r)\left|f(r,u(r),D_{0^+}^{\alpha-1}u(r))\right|dr\right)ds+\frac{2}{\Gamma(\alpha)}\varepsilon\\
&\leq\left(M_m^{q-1}\int_0^{+\infty}\phi_q\left(\frac{1}{\Gamma(\delta)}\int_0^s(s-r)^{\delta-1}a(r)dr\right)ds+\frac{2}{\Gamma(\alpha)}\right)\varepsilon,
\end{aligned}$$

and

$$\begin{aligned}
&\left|D_{0^+}^{\alpha-1}Tu(t_1)-D_{0^+}^{\alpha-1}Tu(t_2)\right|\\
\leq\;&\int_{t_1}^{t_2}\phi_q\left(\frac{1}{\Gamma(\delta)}\int_0^s(s-r)^{\delta-1}a(r)\left|f(r,u(r),D_{0^+}^{\alpha-1}u(r))\right|dr\right)ds\\
\leq\;&\int_L^{+\infty}\phi_q\left(\frac{1}{\Gamma(\delta)}\int_0^s(s-r)^{\delta-1}(r)\left|f(r,u(r),D_{0^+}^{\alpha-1}u(r)\right|dr\right)ds<\varepsilon.
\end{aligned}$$

Consequently, by criterion of compactness, Lemma 2.5, $T\left(B_m\right)$ is relatively compact, and then T is a compact operator. Hence, T is completely continuous. The proof is completed. □

Theorem 3.5. *Assume conditions (C1) and (C2) hold and that*
(C3) $0 < 1 + (\alpha-1)\lambda_2 < \Gamma(\alpha)$, $0 < \lambda_1 < 1$ *and there exists a constant* $R > 0$ *such that*

$$M_R^{q-1}\int_0^{+\infty}\left|I_{0^+}^{\delta}a(s)\right|^{q-1}ds \le (1-\lambda_1)R,$$

where M_R *is given by condition (C1). Then the boundary value problem (1.1) (see the original paper) has at least one solution.*

Proof. Let $B_R = \{u \in X : \|u\| \le R\}$. Then, B_R is a nonempty bounded closed convex subset of X. We show that A is a contraction mapping. For any $u, v \in X$, we have

$$\begin{aligned}\left|\frac{Au(t)-Av(t)}{1+t^{\alpha-1}}\right| &\le \frac{1}{\Gamma(\alpha)}\int_0^{+\infty}(g(s)+(\alpha-1)h(s))\left(1+s^{\alpha-1}\right)\left|\frac{u(s)-v(s)}{1+s^{\alpha-1}}\right|ds\\ &\le \frac{\lambda_1+(\alpha-1)\lambda_2}{\Gamma(\alpha)}\sup_{s\in\mathbb{R}^+}\frac{|u(s)-v(s)|}{1+s^{\alpha-1}}\\ &\le \lambda\|u-v\|,\end{aligned} \tag{3.8}$$

and

$$\begin{aligned}\left|D_{0^+}^{\alpha-1}Au(t)-D_{0^+}^{\alpha-1}Av(t)\right| &\le \int_0^{+\infty}g(s)\left(1+s^{\alpha-1}\right)\left|\frac{u(s)-v(s)}{1+s^{\alpha-1}}\right|ds\\ &\le \lambda_1\sup_{s\in\mathbb{R}^+}\frac{|u(s)-v(s)|}{1+s^{\alpha-1}}\\ &\le \lambda\|u-v\|,\end{aligned}$$

where

$$\lambda = \max\left\{\frac{\lambda_1+(\alpha-1)\lambda_2}{\Gamma(\alpha)},\lambda_1\right\}.$$

It follows that

$$\|Au-Av\| \le \lambda\|u-v\|.$$

It is easy to derive by condition (C3) that $0 < \lambda < 1$. So, A is a contraction mapping.

Next, we prove that $Au + Tv \in B_R$ for any u, $v \in B_R$. In fact, from (3.7), (3.8), and condition (C3), one has

$$\begin{aligned}&\left|\frac{Au(t)+Tv(t)}{1+t^{\alpha-1}}\right|\\ &\le \left|\frac{Au(t)}{1+t^{\alpha-1}}\right|+\left|\frac{Tv(t)}{1+t^{\alpha-1}}\right|\\ &\le \frac{\lambda_1+(\alpha-1)\lambda_2}{\Gamma(\alpha)}\|u\|+\frac{M_R^{q-1}}{\Gamma(\alpha)}\int_0^{+\infty}\phi_q\left(\frac{1}{\Gamma(\delta)}\int_0^s(s-r)^{\delta-1}a(r)dr\right)ds\\ &\le \frac{\lambda_1+(\alpha-1)\lambda_2}{\Gamma(\alpha)}R+\frac{1-\lambda_1}{\Gamma(\alpha)}R = \frac{1+(\alpha-1)\lambda_2}{\Gamma(\alpha)}R < R,\end{aligned}$$

and

$$\begin{aligned}
&\left|D_{0^+}^{\alpha-1}Au(t)+D_{0^+}^{\alpha-1}Tv(t)\right| \\
&=\left|\int_0^{+\infty} g(s)u(s)ds+\int_t^{+\infty}\phi_q\left(\frac{1}{\Gamma(\delta)}\int_0^s (s-r)^{\delta-1}a(r)f(r,v(r),D_{0^+}^{\alpha-1}v(r))dr\right)ds\right| \\
&\le \int_0^{+\infty} g(s)\left(1+s^{\alpha-1}\right)\frac{|u(s)|}{1+s^{\alpha-1}}ds \\
&+M_R^{q-1}\int_0^{+\infty}\phi_q\left(\frac{1}{\Gamma(\delta)}\int_0^s (s-r)^{\delta-1}a(r)dr\right)ds \\
&\le \lambda_1 R+M_R^{q-1}\int_0^{+\infty}\left|I_{0^+}^{\delta}a(s)\right|^{q-1}ds \\
&\le \lambda_1 R+(1-\lambda_1)R=R.
\end{aligned}$$

Hence, $\|Au+Tv\|\le R$, which implies that $Au+Tv\in B_R$. Therefore, by Krasnoselskii fixed point theorem,the boundary value problem has at least one solution in X. The proof is complete. □

Example 3.6. Consider the following fractional boundary value problem

$$\begin{cases}
D_{0^+}^{\frac{1}{2}}(\phi_p(D_{0^+}^{\frac{5}{2}}u(t)))+e^{-t-|u(t)|}\left(D_{0^+}^{\frac{3}{2}}u(t)+\frac{1}{7}\right)=0, a.e.\ 0<t<+\infty, \\
u(0)=0, D_{0^+}^{\frac{3}{2}}u(+\infty)=\int_0^{+\infty}\frac{e^{-t}}{5\left(1+t^{\frac{3}{2}}\right)}u(s)ds, \\
D_{0^+}^{\frac{1}{2}}u(0)=\int_0^{+\infty}\frac{1}{15(1+t)^2\left(1+t^{\frac{3}{2}}\right)}u(s)ds,\ I_{0^+}^{\frac{1}{2}}(\phi_p(D_{0^+}^{\frac{3}{2}}u(0)))=0,
\end{cases}$$

where $p=\frac{4}{3}$, $\alpha=\frac{5}{2}$, $\delta=\frac{1}{2}$, and $a(t)=e^{-t}$. Let us check that conditions (Ci), $i=1,2,3$, are fulfilled.

(C1) If $x,y\in[-\rho,\rho]$, then

$$\left|f(t,\left(1+t^{\frac{3}{2}}\right)x,y)\right|=e^{-\left(1+t^{\frac{3}{2}}\right)|x|}\left|y+\frac{1}{7}\right|\le\left(\rho+\frac{1}{7}\right)=M_\rho.$$

(C2) Since $p=\frac{4}{3}$, then $q=4$. We next prove that $I_{0^+}^{\delta}a\in L^{q-1}[0,+\infty)$. Indeed,

$$\begin{aligned}
\int_0^{+\infty}\left|I_{0^+}^{\frac{1}{2}}a(s)\right|^3 ds &\le \int_0^1\left|I_{0^+}^{\frac{1}{2}}a(s)\right|^3 ds+\int_1^{+\infty}s^{\frac{3}{2}}\left|I_{0^+}^{\frac{1}{2}}a(s)\right|^3 ds \\
&\le \int_0^1\left(\int_0^s (s-r)^{-\frac{1}{2}}e^{-s}dr\right)^3 ds+\int_0^{+\infty}s^{\frac{3}{2}}\left|I_{0^+}^{\frac{1}{2}}a(s)\right|^3 ds \\
&\le \int_0^1\left(\int_0^s (s-r)^{-\frac{1}{2}}dr\right)^3 ds+\int_0^{+\infty}s^{\frac{3}{2}}\left|I_{0^+}^{\frac{1}{2}}a(s)\right|^3 ds \\
&= \frac{16}{5}+\int_0^{+\infty}s^{\frac{3}{2}}\left|I_{0^+}^{\frac{1}{2}}a(s)\right|^3 ds.
\end{aligned}$$

To estimate the integral in the right hand side of the above inequality, we use the following inequality (see [3, (5.46), p. 104]):$\int_0^{+\infty}s^{\delta\mu}\left|I_{0^+}^{\delta}a(s)\right|^{\mu}ds\le c^{\mu}\int_0^{+\infty}|a(s)|^{\mu}ds$, where $a\in$

$L_\mu[0,+\infty), 1<\mu<+\infty, \delta>0$, and $c=\frac{\Gamma(\frac{1}{\nu})}{\Gamma(\delta+\frac{1}{\nu})}, \frac{1}{\mu}+\frac{1}{\nu}=1$. Then,

$$\int_0^{+\infty}\left|I_{0^+}^{\frac{1}{2}}a(s)\right|^3 ds \leq \int_0^1\left(\int_0^s (s-r)^{-\frac{1}{2}}dr\right)^3 ds + c^3\int_0^{+\infty}|a(s)|^3 ds$$
$$=\frac{16}{5}+\left(\frac{\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{1}{2}+\frac{2}{3}\right)}\right)^3\int_0^{+\infty}e^{-3s}ds \simeq 4.2366<+\infty.$$

(C3) We have $\lambda_1=\int_0^{+\infty}g(s)\left(1+s^{\alpha-1}\right)ds=\int_0^{+\infty}\frac{e^{-s}}{5\left(1+s^{\frac{3}{2}}\right)}\left(1+s^{\frac{3}{2}}\right)ds=\frac{1}{5}$, so,$0<\lambda_1<1$,

and

$$\lambda_2=\int_0^{+\infty}h(s)\left(1+s^{\alpha-1}\right)ds=\int_0^{+\infty}\frac{1}{15(1+s)^2\left(1+s^{\frac{3}{2}}\right)}\left(1+s^{\frac{3}{2}}\right)ds=\frac{1}{15}.$$

Thus $0<1+(\alpha-1)\lambda_2=1.1<\Gamma(\alpha)=\frac{3}{4}\sqrt{\pi}\simeq 1.3293$. By taking $R=\frac{1}{8}$, we have

$$M_R^3\int_0^{+\infty}\left|I_{0^+}^{\frac{1}{2}}a(s)\right|^3 ds \leq \left(\frac{1}{8}+\frac{1}{7}\right)^3\times 4.2366 \simeq 8.1419\times 10^{-2}.$$

and $(1-\lambda_1)R=0.1$. Hence, $M_R^{q-1}\int_0^{+\infty}\left|I_{0^+}^{\delta}a(s)\right|^{q-1}ds<(1-\lambda_1)R$. By Theorem 3.5, we conclude that the problem has at least one solution in X.

4. Conclusion

We established the existence of solutions for a fractional differential equation with p-Laplacian operators on unbounded domain. To do this, we converted the problem posed into the sum of a contraction and a compact operator, and then we apply the Krasnoselskii fixed point theorem. To overcome the difficulties arising from the infinite interval, we used a criterion of compactness, Lemma 2.5.

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