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A GENERALIZED $\mathscr{F}$-CONTRACTION MAPPING FOR COUPLED FIXED POINT THEOREMS AND AN APPLICATION TO A TWO-PERSON GAME

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Abstract. To relate the existence of equilibria to the existence of fixed points is the main aim of this paper. Game theory is to study a situation under some conflict among certain agents (or players). Each one seeks for their own minimum loss in such situation. Moreover, game can be divided into two main broad categories characterized as cooperative or non-cooperative manner. In other words, cooperative game allows agents to collaborate, but non-cooperative does not. Due to the equivalence of the existence of a solution of a non-cooperative equilibrium and a couple fixed point, the existence problem of the non-cooperative equilibrium of two person games is clarified by applying some coupled fixed point theorems in partial metric spaces.

Keywords. Couple fixed point; Generalized $\mathscr{F}$-contraction; Partial metric space; Non-cooperative equilibrium.

1. Introduction

There are several interesting problems, which can be modeled in form of either linear or nonlinear equations. To solve these problems, the first thing should be done is to make sure that the equations have a solution. The mathematical analysis of this issue usually leads to the theory of fixed points. In mathematics, a point that is mapped to itself by a mapping is said to be a fixed point of the mapping. Let $F : X \to X$ be a mapping on a set X. A point $x_0 \in X$ is said to be a fixed point of F if and only if $F(x_0) = x_0$. For example, let f be defined on the set of the real line by $f(x) = x^2 - 4x + 3$. Then the roots of $f(x) = 0$ can be solved by rewriting it in the form $F(x) = x$, where $F(x)$ can be expressed in many forms. One is $F(x) = x^2 - 3x + 3$. It

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can be seen that 1 and 3 are fixed points of F. Moreover, 1 and 3 are the roots of the equation $f(x) = 0$. It is clear that if the mappings have a fixed point, then the corresponding equations absolutely have a solution.

These existence theorems of fixed points play an important role in many fields, such as, differential equations, integral equations, operational research, computer science and so on. A lot of real world problems can be solved via fixed point methods. There are many classical existence theorems of fixed points of single-valued and multi-valued nonlinear mappings in nonlinear functional analysis. For example, Brouwer [3] proved that a continuous mappings on a convex and compact set always have a fixed point. In 1922, a mapping on a complete metric space, which satisfies a contraction mapping had been proved that it has a unique fixed point by Banach [2] and this famous theorem is known as Banach contraction principle. The principle is a powerful tool in various fields of pure and applied matheamtics. In 1987, Guo and Lakshmikantham [7] extended the concept of fixed points to coupled fixed points. That is, the determined mapping is changed to be the mapping mapped from Cartesian product of the two same sets to itself. Let $F : X \times X \to X$ be a mapping on a set X. A point $(x_0, y_0) \in X \times X$ is said to be a coupled fixed point of F if $F(x_0, y_0) = x_0$ and $F(y_0, x_0) = y_0$. In 2004, Oltra and Valero [11] studied the Banach contraction principle in the framework of a partial metric space. In 2011, Ilić et al. [8] also extended the results in such a direction. A partial metric space introduced by Matthews [9] is slightly different from the classical metric space for some properties. After that, Piri et al. [13] introduced an interesting contraction, named the $\mathscr{F}$-contraction, to prove some fixed point theorems in 2014. Recently, Duc et al. [5] investigated (coupled) fixed points by using F-contractions in a partial metric space in 2015. Moreover, they considered an application of the results in the game theory.

In economics, game theory applies the knowledge from the fixed point theory to solve some problems in the area [4, 6, 10]. One of the most common ways of representing a game, especially, non-cooperative, is described under the requirement that there are at least 2 players such that the preference of players must be defined over the set of all possible outcomes of all the player. That is, each player may be affected not only about his own action but also about the actions taken by the other players. Consequently, a game [12] is defined to be any situation in which

(1) there are at least two players,

(2) each player has a number of possible strategies and the courses of action which the players may choose to follow,

(3) the strategies chosen by each player determine the outcome of the game which is a collection of numerical payoffs, one to each players.

Let us consider subsets X and Y. To determine a pair $(x, y) \in X \times Y$, called a bistrategy, we consider the problems only under the static framework for 2 players. With the limitation, game theory provides useful tools for many clarifying concepts.

For any loss function or cost functions, a player behaves so as to minimize his losses as far as possible. However, the direction of the inequality for utility functions and gain functions is inverted, that is, the player behaves to maximize his evaluation function.

We now suppose that player 1 and player 2 separately choose their strategies by using their loss functions f^1 and f^2, respectively, and define from $X \times Y$ to $\mathbb{R}$, which, in $\mathbb{R}$, the values mapped by f^1 and f^2 can be determined by the total order $\geq$.

Definition 1.1. [1] A pair $(\bar{x},\bar{y}) \in X \times Y$ is called a non-cooperative equilibrium (or Nash equilibrium) of the game if and only if

$$f^1(\bar{x},\bar{y}) = \min_{x\in X} f^1(x,\bar{y}) \text{ and } f^2(\bar{x},\bar{y}) = \min_{y\in Y} f^2(\bar{x},y).$$

The aim of this paper is to ensure the non-cooperative equilibrium of two-person game. To achieve the goal, we present an existence theorem of a coupled fixed point by using a generalized $\mathscr{F}$-contraction mapping in partial metric spaces. We also apply the fixed point results to the problems in such games.

2. Preliminaries

From now on, we always assume that X is a partial metric space. In this section, we gather the basic definitions and important tools, which are utilized in the main results.

Definition 2.1. [7] A pair $(x,y) \in X \times X$ is called a coupled fixed point of the mapping $T : X \times X \to X$ if

$$T(x,y) = x, \ T(y,x) = y.$$

Definition 2.2. [9] Let X be a nonempty set. The mapping $p : X \times X \to [0,\infty)$ is said to be a partial metric on X if, for any $x,y,z \in X$, the following conditions hold true:

(P1) $x = y$ if and only if $p(x,x) = p(y,y) = p(x,y)$;
(P2) $p(x,x) \le p(x,y)$;
(P3) $p(x,y) = p(y,x)$;
(P4) $p(x,z) \le p(x,y) + p(y,z) - p(y,y)$.

The pair (X,p) is then called a partial metric space (in short, PMS).

Next, we consider a class of functions which plays a crucial role in this work. Let $F : \mathbb{R}_0^+$ be a mapping satisfying the restrictions:

(F1) F is strictly increasing and continuous;
(F2) for each sequence $(a_n) \subseteq \mathbb{R}_0^+$, $\lim_{n\to\infty} a_n = 0$ if and only if $\lim_{n\to\infty} F(a_n) = -\infty$;
(F3) there exist $k \in (0,1)$ such that $\lim_{n\to\infty} a^k F(\alpha) = 0$.

We denote by $\mathscr{F}$ the family of all such functions that satisfy the conditions (F1)-(F3).

Definition 2.3. [14] Let (X,p) be a partial metric space. A sequence $(x_n) \subseteq X$ is called a $0-$*Cauchy* sequence if $\lim_{n,m\to\infty} p(x_n,x_m) = 0$. Moreover, we say that (X,p) is $0-$complete if every $0-$*Cauchy* sequence in X is a convergence sequence, with respect to the partial metric p to a point $x \in X$ such that $p(x,x) = 0$.

If (X,p) is complete, then it is $0-$complete, but the converse does not hold.

Example 2.4. [5] Let $X = [0,\infty)$ and define $p(x,y) = \max\{x,y\}$, for all $x,y \in X$. Then (X,p) is a complete partial metric space. It is clear that p is not a (usual) metric.

Example 2.5. [5] Let $X = [0,\infty) \cap \mathbb{Q}$, where $\mathbb{Q}$ is the set of rational numbers. Define

$$p(x,y) = \max\{x,y\},$$

for all $x,y \in X$. Then (X,p) is a 0-complete partial metric space, which is not complete.

In this paper, we introduce the following mapping.

Definition 2.6. Let $T : X \times X \to X$ be a mapping. The mapping T is said to be generalized $\mathscr{F}$-contraction if there exist $\alpha, \beta \in [0, \frac{1}{2})$ such that

$$\tau + F(p(T(x,y), T(u,v))) \leq F(\alpha p(x,u) + \beta p(y,v))$$

for all $x \leq u, y \geq v$, for some $F \in \mathscr{F}$ and $\tau > 0$.

Remark 2.7. If $\alpha = \beta = 1$, the generalized $\mathscr{F}$-contraction is reduced to the classical $\mathscr{F}$-contraction.

3. Existence of a Couple Fixed Point

In this part, we provide the results of existence of a couple fixed point theorem including a consequent corollary under some assumptions.

Theorem 3.1. *Let $(X, \leq)$ be a partially ordered set and suppose there exists a partial metric p on X such that (X, p) is a 0-complete partial metric space. Let $T : X \times X \to X$ be a continuous mapping with the mixed monotone property on X. Suppose also that T is a generalized $\mathscr{F}$-contraction. Moreover, there exist $x_0, y_0 \in X$ such that $x_0 \leq T(x_0, y_0), T(y_0, x_0) \leq y_0$. Then T has a coupled fixed point, that is, there exist $x, y \in X$ such that $x = T(x,y)$ and $y = T(y,x)$.*

Proof. Let $x_0, y_0 \in X$ be such that $x_0 \leq T(x_0, y_0)$ and $y_0 \geq T(x_0, y_0)$. Let $x_1 = T(x_0, y_0)$ and $y_1 = T(y_0, x_0)$. Then $x_0 \leq x_1$ and $y_0 \geq y_1$. Again, let $x_2 = T(x_1, y_1)$ and $y_2 = T(y_1, x_1)$. We obtain $x_1 \leq x_2$ and $y_1 \geq y_2$ because T satisfies the mixed monotone property. Keep going this process, it turns out that there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that $x_{n+1} = T(x_n, y_n)$, $y_{n+1} = T(y_n, x_n)$, and

$$x_0 \leq x_1 \leq x_2 \leq \cdots \leq x_n \leq x_{n+1} \cdots, \quad y_0 \geq y_1 \geq y_2 \geq \cdots \geq y_n \geq y_{n+1} \cdots.$$

Now, for each $n = 0, 1, 2, \cdots$, we have

$$\begin{aligned} \tau + F(p(x_n, x_{n+1})) &= \tau + F(p(T(x_{n-1}, y_{n-1}), T(x_n, y_n))) \\ &\leq F(\alpha p(x_{n-1}, x_n) + \beta p(y_{n-1}, y_n)) \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} \tau + F(p(y_n, y_{n+1})) &= \tau + F(p(T(y_{n-1}, x_{n-1}), T(y_n, x_n))) \\ &\leq F(\alpha p(y_{n-1}, y_n) + \beta p(x_{n-1}, x_n)). \end{aligned} \tag{3.2}$$

Since F is strictly increasing, we obtain from (3.1) and (3.2) that

$$p(x_n, x_{n+1}) \leq \alpha p(x_{n-1}, x_n) + \beta p(y_{n-1}, y_n)$$

and

$$p(y_n, y_{n+1}) \leq \alpha p(y_{n-1}, y_n) + \beta p(x_{n-1}, x_n).$$

Therefore, by letting $p_n = p(x_n, x_{n+1}) + p(y_n, y_{n+1})$, we have

$$p_n \leq (\alpha + \beta)(p(x_{n-1}, x_n) + p(y_{n-1}, y_n)) = (\alpha + \beta) p_{n-1}$$

for all $n \in \mathbb{N}$. Since $\alpha + \beta \leq 1$, we obtain $p_n \leq p_{n-1}$ for all $n \in \mathbb{N}$. Consequently,

$$\tau + F(p_n) \leq F(p_{n-1})$$

for all $n \in \mathbb{N}$. It follows that

$$F(p_n) \leq F(p_{n-1}) - \tau \leq \cdots \leq F(p_0) - n\tau \tag{3.3}$$

for all $n \in \mathbb{N}$. If we take limit as $n \to \infty$ in (3.3), we obtain $\lim_{n\to\infty} F(p_n) = -\infty$. From property (F2), we have that $\lim_{n\to\infty} p_n = 0$. Using property (F3), we can say that there exists $k \in (0,1)$ such that $\lim_{n\to\infty} p_n^k F(p_n) = 0$. Using inequation (3.3), we obtain

$$p_n^k F(p_n) - p_n^k F(p_0) \leq p_n^k (F(p_0) - n\tau) - p_n^k F(p_0) = -n\tau p_n^k \leq 0. \tag{3.4}$$

If we take limit as $n \to \infty$ in (3.4), then $\lim_{n\to\infty} n p_n^k = 0$. So, there exists $n_0 \in \mathbb{N}$ such that $n p_n^K \leq 1$ for all $n \geq n_0$. Hence, $p_n \leq \frac{1}{n^{\frac{1}{k}}}$, for all $n \geq n_0$. We consider $m,n \in \mathbb{N}$ such that, $m > n > n_0$,

$$\begin{aligned} p(x_m,x_n) + p(y_m,y_n) &\leq p(x_m,x_{m+1})p(y_m,y_{m+1}) + \cdots + p(x_{n-1},x_n) + p(y_{n-1},y_n) \\ &= p_{m-1} + p_{m-2} + \cdots + p_m \\ &= \sum_{i=n}^{m-1} p_i \leq \sum_{i=1}^{\infty} p_i \leq \sum_{i=1}^{\infty} \frac{1}{i^{\frac{1}{k}}}. \end{aligned} \tag{3.5}$$

Since the series $\sum_{i=1}^{\infty} \frac{1}{i^{\frac{1}{k}}}$ are convergence, $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in X. From the 0-completeness of (X,p), we can say that there exist $x,y \in X$ such that

$$\lim_{n\to\infty} x_n = x \quad \text{and} \quad \lim_{n\to\infty} y_n = y.$$

From property of partial metric, we obtain

$$p(T(x,y),x) \leq p(T(x,y),x_{n+1}) + p(x_{n+1},x)$$

and

$$p(T(x,y),x) - p(x_{n+1},x) \leq p(T(x,y),x_{n+1}). \tag{3.6}$$

In addition, from $\tau + F(p(T(x,y),T(u,v))) \leq F(\alpha p(x,u) + \beta p(y,v))$, we have

$$F(p(T(x,y),T(x_n,y_n))) < \tau + F(p(T(x,y),T(x_n,y))) \leq F(\alpha p(x,x_n) + \beta p(y,y_n)).$$

From property of (F1), we further obtain

$$p(T(x,y),T(x_n,y_n)) < \alpha p(x,x_n) + \beta p(y,y_n). \tag{3.7}$$

In view of (3.6) and (3.7), we obtain

$$p(T(x,y),x) - p(x_{n+1},x) < \alpha p(x,x_n) + \beta p(y,y_n).$$

Letting $n \to \infty$ yields that $p(T(x,y),x) \to 0$, which in turn implies that $T(x,y) = x$. Similarly, we have also $T(x,y) = y$. Then (x,y) is a coupled fixed point of T. This completes the proof. □

Theorem 3.2. *Let $(X,\leq)$ be a partially ordered set and suppose there exists a partial metric p on X such that (X,p) is a 0-complete partial metric space. Let $T : X \times X \to X$ be a continuous mapping with the mixed monotone property on X. Assume that T is a generalized $\mathscr{F}$-contraction. Moreover, there exist $x_0, y_0 \in X$ such that $x_0 \leq T(x_0,y_0), T(y_0,x_0) \leq y_0$. Furthermore, assume that X has the properties:*

(a) *for a non-decreasing sequence $\{x_n\}$ in X, if it converges to x, then $x_n \leq x$ for all n.*
(b) *for a non-increasing sequence $\{y_n\}$ in X, if it converges to x, then $y_n \geq y$ for all n.*

Then T has a coupled fixed point, that is, there exist $x,y \in X$ such that $x = T(x,y), y = T(y,x)$.

Proof. We claim that

$$\lim_{n\to\infty} p(x_{n+1}, F(u,v)) = 0,$$

where $x_{n+1} = T(x_n, y_n)$ and $x_n \leq u$ and $y_n \geq v$. Since

$$\begin{aligned} \tau + F(p(x_{n+1}, T(u,v)) &= \tau + F(p(T(x_n,y_n), T(u,v)) \\ &\leq F(\alpha p(x_n,u) + \beta p(y_n,v)), \end{aligned}$$

we obtain that

$$p(T(x_n,y_n), T(u,v)) \leq \alpha p(x_n,u) + \beta p(y_n,v).$$

We have to prove that

$$\lim_{n\to\infty} (\alpha p(x_n,u) + \beta p(y_n,v)) = 0.$$

This implies that

$$\lim_{n\to\infty} \alpha p(x_n,u) = 0$$

and

$$\lim_{n\to\infty} \beta p(y_n,v) = 0.$$

In view of the 0-complete partial metric space, there exist $u,v \in X$ such that

$$\begin{aligned} \lim_{n\to\infty} \alpha p(x_n,u) &= \alpha \lim_{n\to\infty} p(x_n,u) \\ &= \alpha p(u,u) \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} \lim_{n\to\infty} \beta p(y_n,v) &= \beta \lim_{n\to\infty} p(y_n,v) \\ &= \beta p(v,v) \\ &= 0. \end{aligned}$$

It remains to show that $u = T(u,v)$ and $v = T(v,u)$. Indeed, since $u \leq u$ and $v \geq v$, we have

$$\tau + F(p(T(u,v), T(u,v))) \leq F(\alpha p(u,u) + \beta p(v,v)) = F(0).$$

By the definition of F, we see that $\lim_{n\to\infty} F(0) = -\alpha$. Therefore,

$$\lim_{n\to\infty} p(x_{n+1}, F(u,v)) = \lim_{n\to\infty} p(T(x_n,y_n), T(u,v)) = 0.$$

□

Corollary 3.3. *Let $(X, \leq)$ be a partially ordered set and suppose there exists a partial metric p on X such that (X, p) is a 0-complete partial metric space. Let $T : X \times X \to X$ be a continuous mapping with the mixed monotone property on X. Assume that T is a generalized $\mathscr{F}$-contraction such that*

$$\tau + F(p(T(x,y), T(u,v))) \leq F(\frac{p(x,x_n) + p(y,y_n)}{2}).$$

Moreover, there exit $x_0, y_0 \in X$ such that $x_0 \leq T(x_0,y_0), T(y_0,x_0) \leq y_0$. Then T has a coupled fixed point, that is, there exist $x,y \in X$ such that $x = T(x,y)$ and $y = T(y,x)$. Also, assume that X has the properties;

(a) *T is continuous; or*
(b) *if a non-decreasing sequence $\{x_n\}$ in X converges to x, then $x_n \leq x$ for all n;*

(c) *if a non-increasing sequence $\{y_n\}$ in X converges to x, then $y_n \geq y$ for all n.*

Then T has a coupled fixed point, that is, there exist $x, y \in X$ such that $x = T(x,y), y = T(y,x)$.

4. Non-Cooperative Equilibrium Problem for Two Players

A two person game $\mathscr{G}$ in a normal form consists of the following data:

(1) topological space S_1 and S_2, called strategies for player 1 and player 2, respectively;
(2) a topological subspace $U \subseteq S_1 \times S_2$, which is a feasible set of all possible strategy pairs;
(3) a biloss operator, i.e., $f : U \to \mathbb{R}^2$ is defined by

$$f(s_1,s_2) = (f_1(s_1,s_2), f_2(s_1,s_2)),$$

where $f_i : U \to \mathbb{R}$ is the loss function of player i determined all strategies.

Recall that a pair (s_1', s_2') is said to be a non-cooperative equilibrium if

$$f_1(s_1', s_2') \leq f_1(s_1, s_2')$$

for all $s_1 \in S_1$ and

$$f_2(s_1', s_2') \leq f_2(s_1', s_2)$$

for all $s_2 \in S_2$.

Next, define mappings $C : S_2 \to S_1$ and $D : S_1 \to S_2$ such that the following equations hold:

$$f_1(C(s_2), s_2) = \min_{s_1 \in S_1} f_1(s_1, s_2)$$

and

$$f_2(s_1, D(s_1)) = \min_{s_2 \in S_2} f_2(s_1, s_2).$$

Such mappings C and D are called optimal decision rules. Then, any solution (s_1', s_2') satisfying the system

$$C(s_2') = s_1' \quad \text{and} \quad D(s_1') = s_2'$$

is a non-cooperative equilibrium. Let $T_0 : S_1 \times S_2 \to S_1 \times S_2$ be a mapping defined by

$$T_0(s_1, s_2) = (C(s_2), D(s_1)).$$

Observe that the existence of a solution to a non-cooperative equilibrium problem is equivalent to the existence of a coupled fixed point.

In this section, we also consider the same strategy sets for each player, that is, $S_1 = S_2 =: S$. Also, we determine $C(s) = D(s)$ for all $s \in S$. It implies that if $f_1(s_1,s_2) = f_2(s_1,s_2)$ for all $(s_1,s_2) \in S \times S$, then $C(s) = D(s)$ for all $s \in S$, but the converse is not true.

Define $T : S \times S \to \mathbb{R}$ by $T(x,y) = C(y)$ for all $x, y \in S$. Suppose that T has a coupled fixed point $(a,b) \in S \times S$. It follows that

$$a = T(a,b) = C(b) \quad \text{and} \quad b = T(b,a) = C(a).$$

That is, (a,b) is also a non-cooperative equilibrium of such setting.

Theorem 4.1. *Let $\mathscr{G}$ be a two-person game, and let S be a strategy set. Suppose that the optimal decision rule is a monotone continuous function C and there exist $\alpha,\beta \in [0,\frac{1}{2})$ such that*

$$\tau + F(p(C(x),C(y)) \leq F(\min\{\alpha,\beta\}p(x,y))$$

for all $x,y \in S$ with $x \leq y$ for some $F \in \mathscr{F}$ and $\tau > 0$. Moreover, there exit $x_0, y_0 \in \mathbb{R}_0^+$ such that $x_0 \leq C(y_0)$ and $y_0 \geq C(x_0)$. Then the two-person game $\mathscr{G}$ has a non-cooperative equilibrium.

Proof. Let $T : S \times S \to S$ be defined by

$$T(x,y) = C(y)$$

for all $x,y \in S$. Since C is continuous, then T is also continuous. Since C is monotone, we have that T has the mixed monotone property on X. For all $x,y,u,v \in \mathbb{R}_0^+$ with $x \leq u$, $y \geq v$ we have

$$p(T(x,y),T(u,v) = p(C(y),C(v)).$$

We also obtain that

$$\begin{aligned} \tau + F(p(T(x,y),T(u,v)) &= \tau + F(p(C(y),C(v))) \\ &\leq F(\min\{\alpha,\beta\}p(y,v)) \end{aligned}$$

for every $x \leq u$ and $y \geq v$. Since $\alpha p(x,u) + \beta p(y,v) \geq \min\{\alpha,\beta\}p(x,y)$, and F is increasing, we obtain that

$$\tau + F(p(T(x,y),T(u,v)) \leq F(\alpha p(x,u) + \beta p(y,v))$$

for every $x \leq u$ and $y \geq v$. By Theorem 3.1, we can conclude that T has a coupled fixed point. This implies that the two person game $\mathscr{G}$ has a non-cooperative equilibrium. □

Corollary 4.2. *Let $\mathscr{G}$ be a two-person game, and let S be a strategy set. Suppose that (S,d) is a metric space and the optimal decision rule is a monotone continuous function C, and there exist $\alpha,\beta \in [0,\frac{1}{2})$ such that*

$$\tau + F(p(C(x),C(y)) \leq F(\min\{\alpha,\beta\}p(x,y))$$

for all $x,y \in S$ with $x \leq y$ for some $F \in \mathscr{F}$ and $\tau > 0$. Moreover, there exit $x_0, y_0 \in \mathbb{R}_0^+$ such that $x_0 \leq C(y_0)$ and $y_0 \geq C(x_0)$. Then the two-person game $\mathscr{G}$ has a non-cooperative equilibrium.

Example 4.3. Consider $S = \mathbb{R}_0^+$ endowed with metric $d(x,y) = |x-y|$ for all $x,y \in S$. Let $\mathscr{G}$ be a two person game with biloss operator

$$\begin{aligned} f_1(s_1,s_2) &= 9s_1^2(1+s_2)e^{\tau} - 6s_1, \\ f_2(s_1,s_2) &= 9s_2^2(1+s_1)e^{\tau} - 6s_2, \end{aligned}$$

where $s_1,s_2 \in S$ and $\tau > 0$. Choose $\alpha = \frac{1}{3} \in [0,\frac{1}{2})$ and $\beta = \frac{2}{5} \in [0,\frac{1}{2})$. Then, we define

$$C(s_2) = s_1 = \tfrac{e^{-\tau}}{3(1+s_1)} \text{ and } D(s_1) = s_2 = \tfrac{e^{-\tau}}{3(1+s_2)}.$$

In this case, we consider that $C(s_2) = C(s) = D(s) = D(s_1)$ and C, D are continuous maps. Then,

$$d(C(s),D(s)) \leq \frac{1}{3}e^{-\tau}d(x,y) = \min\{\alpha,\beta\}e^{-\tau}d(x,y).$$

It implies that $\ln(d(C(x),D(y))) \leq \ln e^{-\tau} + \ln(\min\{\alpha,\beta\}d(x,y))$. Since we can set $F(x) = \ln(x) \in \mathscr{F}$, we consider $x_0 = 0$ and $y_0 = 1$. It follows that $C(x_0) = \frac{1}{3}e^{-\tau}$ and $C(y_0) = \frac{1}{6}e^{-\tau}$ which can be seen that $y_0 \geq C(x_0)$ and $x_0 \leq C(y_0)$. By Corollary 4.2, this two-person game has an equilibrium.

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REFERENCES

[1] J.-P. Aubin, Optima and equilibria, Springer-Verlag, Berlin, 1998.
[2] S. Banach, Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales, Fund. Math. 3 (1922) 133-181.
[3] L.E.J. Brouwer, Über Abbildung von Mannigfaltigkeiten, Math. Ann. 71 (1912) 598.
[4] A. Capătă, Existence of solutions of bilevel strong vector equilibrium problems and their applications, J. Nonlinear Var. Anal. 5 (2021) 371-389.
[5] T.T. Duc, A. Hobiny, E. Karapınar, A solution for the non-cooperative equilibrium problem of two person via fixed point theory, J. Inequal. Appl. 158 (2015) 18.
[6] C.-O. Ewald, Games, fixed points and Mathematical Economics, School of Economics and Finance University of St. Andrews, 2003.
[7] D.J. Guo, V. Lakshmikantham, Coupled fixed points of nonlinear operators with applications, Nonlinear Anal. 11 (1987) 623-632.
[8] D. Ilić, V. Pavlović, V. Rakočević, Some new extensions of Banach's contraction principle to partial metric space, Appl. Math. Lett. 24 (2011) 1326-1330.
[9] S.G. Matthews, Partial metric topology, Papers on general topology and applications (Flushing, NY, 1992), New York Acad. Sci., New York, 728 (1994) 183-197.
[10] T. Migot, M.-G. Cojocaru, Nonsmooth dynamics of generalized Nash games, J. Nonlinear Var. Anal. 4 (2020) 27-44.
[11] S. Oltra, O. Valero, Banach's fixed point theorem for partial metric spaces, Rend. Istit. Mat. Univ. Trieste, 36 (2004) 17-26.
[12] M. J. Osborne, A. Rubinstein, A course in game theory, MIT Press, Cambridge, 1994.
[13] H. Piri, P. Kumam, Some fixed point theorems concerning F-contraction in complete metric spaces, Fixed Point Theory Appl. 210 (2014) 11.
[14] S. Romaguera, A Kirk type characterization of completeness for partial metric spaces, Fixed Point Theory Appl. 2010 (2010) 493298.