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INVARIANT TORI FOR THE QUINTIC SCHRÖDINGER EQUATION WITH QUASI-PERIODIC FORCING ON THE TWO-DIMENSIONAL TORUS UNDER PERIODIC BOUNDARY CONDITIONS

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Abstract. This paper focuses on the quintic Schrödinger equation with quasi-periodic forcing on the two dimensional torus under periodic boundary conditions. By utilizing the measure estimation of infinitely many small divisors, for most values of the frequency vectors, the Hamiltonian of the linear part of the equation can be reduced to an autonomous system by a symplectic change of coordinates. By some transformations of coordinates, the Hamiltonian of the equation can be transformed into an angle-dependent block-diagonal normal form, which can be achieved by choosing the appropriate tangential sites. By an abstract KAM theorem, it is proved that the existence of a class of invariant tori implies the existence of a class of small-amplitude quasi-periodic solutions to the equation.

Keywords. Hamiltonian; KAM theory; Periodic boundary conditions; Quintic Schrödinger equations; Quasi-periodic forcing.

1. Introduction

It is known that KAM theory is an effective approach to solve the existence problem of finite-dimensional tori for nonlinear Hamiltonian partial differential equations (PDEs). The KAM theory is the use of a combination of Birkhoff normal form and KAM iterative technique. This method was originally proposed by Wayne [23], Kuksin [16], and Pöschel [19]. Later, it has been well developed in one dimensional Hamiltonian PDEs. We refer to [1, 12, 17, 18, 24] and the references therein.

When the spatial dimension exceeds one, the eigenvalues will be repeated, which leads to an infinite number of resonance problems. This make it more difficult to prove the second Melnikov non-degeneracy conditions. In this aspect, Bourgain [4, 5] made a breakthrough. instead of the KAM theory, he avoided the second Melnikov non-degeneracy conditions by

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solving angle dependent homological equations. This is known as the Craig-Wayne-Bourgain method, which combines the Lyapunov-Schmidt reduction technique and the Nash-Moser iterative scheme to solve the small divisor problem. This method was originally proposed by Craig-Wayne [7]. For further development, the readers may refer to the literatures [2, 3, 6, 22]. Although the CWB method only needs to satisfy the first Melnikov non-degeneracy conditions, it cannot give more properties of quasi-periodic solutions, such as their Whitney smooth dependence on parameters, zero Lyapunov exponents, and linear stability as KAM theory does.

In [13, 14], Geng and You constructed small-amplitude quasi-periodic solutions for the higher dimensional nonlinear beam equations and nonlocal Schrödinger equations by using the methods from an infinite dimensional KAM theory. In this aspect, Eliasson-Kuksin[9] constructed quasi-periodic solutions for more interesting higher dimensional Schrödinger equation by using the block diagonal normal form structure and the Töplitz-Lipschitz property of perturbation to overcome the difficulty caused by infinite resonances. However, all works mentioned above require artificial parameters, so they cannot be applied directly to the completely resonant Schrödinger equation without external parameters.

In [10], the authors constructed an infinite dimensional KAM theory for completely resonant NLS on $\mathbb{T}^2$ of the form $\mathrm{i}u_t-\Delta u+|u|^2u=0, \quad x\in\mathbb{T}^2, t\in\mathbb{R}$ with periodic boundary conditions. They simplified the proof in [9] by adding momentum conservation conditions and taking an appropriate admissible set of tangential sites. Procesi and Procesi [20] proved the existence of linearly stable and unstable quasi-periodic solutions via a KAM algorithm for the completely resonant cubic Schrödinger equations on $\mathbb{T}^d$ of the form $\mathrm{i}u_t-\Delta u=\kappa|u|^2u+\partial_{\bar{u}}G(|u|^2)$, $x\in\mathbb{T}^d:=\mathbb{R}^d/(2\pi\mathbb{Z})^d$, $t\in\mathbb{R}$. In [21], they discussed the following more general completely resonant nonlinear Schrödinger equations on $\mathbb{T}^d$ $\mathrm{i}u_t-\Delta u=k|u|^{2q}u+\partial_{\bar{u}}G(|u|^2)$, where $d\geq 1$ and $q\geq 1$. The above equation admitted linear stability of quasi-periodic solutions. It is worth mentioning that the quintic Schrödinger equation has attracted more and more attention in recent years. For the one-dimensional case, we refer to [12, 18]. In the higher dimensional space, Haus and Procesi [15] considered the quintic completed resonant Schrödinger equation on the $\mathbb{T}^2$

$$\mathrm{i}u_t-\Delta u+|u|^4u=0, \quad x\in\mathbb{T}^2, \quad t\in\mathbb{R}, \tag{1.1}$$

and exhibited orbits whose Sobolev norms grow with time. If the nonlinear term of the Schrödinger equation changes from cubic to quintic, its normal form will also change correspondingly. So, the KAM method [10], which cannot be applied directly to equation (1.1), needs to be adjusted accordingly. In [26], the authors, based on the idea in [10], constructed the quasi-periodic solutions of the equation (1.1) with small amplitude under periodic boundary conditions. This extension is nontrivial and substantial development. Coincidentally, Geng and Xue also discussed the existence of quasi-periodic solutions of equation (1.1) in [11]. Due to the different normal forms selected, the specific proof presented in [11] and [26] are quite different.

In this paper, we will focus on the existence of quasi-periodic solutions for the quintic Schrödinger equation with quasi-periodic forcing on the $\mathbb{T}^2$

$$\mathrm{i}u_t-\Delta u+\varepsilon\phi(t)(u+|u|^4u)=0, \quad x\in\mathbb{T}^2, \quad t\in\mathbb{R} \tag{1.2}$$

with periodic boundary conditions

$$u(t,x_1,x_2)=u(t,x_1+2\pi,x_2)=u(t,x_1,x_2+2\pi) \tag{1.3}$$

where ε is a small positive parameter, and $\phi(t)$ is a real analytic quasi-periodic function in t with frequency vector $\omega = (\omega_1, \omega_2 \dots, \omega_{b^*}) \in [\rho, 2\rho]^{b^*}$ for some constant $\rho > 0$. The method used to study the existence of quasi-periodic solutions of the above equation is based on the KAM theory in Zhang-Si [26]. The main steps are as follows. First, the linear part of the Hamiltonian system corresponding to (1.2) is converted to constant coefficients by using the reducibility theory in [25]. For, the conclusions on the reducibility, we refer to [1, 8]. Then, by taking an appropriate admissible set of tangential sites and symplectic transformation, the Hamiltonian system is transformed into some partial Birkhoff normal form of order six for using the KAM theory in [26]. Finally, the existence of the quasi-periodic solutions of the equation is proved by using the KAM theory in [26].

When the nonlinear term of Schrödinger equation (1.2) changes from $\varepsilon\phi(t)|u|^2u$ to $\varepsilon\phi(t)|u|^4u$, the admissible set of tangential sites and the normal form of the corresponding Hamiltonian system will change essentially. Therefore, we need to solve the following problems.

- The essence of the reducibility theory is to transform the linear part of the Hamiltonian system from non-autonomous systems to autonomous systems by infinitely many symplectic transformations, but these symplectic transformations act on the nonlinear part at the same time. Although this paper uses the reducibility theory in [25], it is still not trivial to prove that the nonlinear part $\varepsilon\widetilde{R}^6$ still satisfies the Töplitz-Lipschitz property after these symplectic transformations.
- When the degree of nonlinear terms changes, different admissible sets of tangential sites $\mathscr{R}$ are needed to be selected in order to minimize the number of items dependent on angular variables contained in the normal form. Therefore, in the process of transforming the Hamiltonian system into the normal form, it is necessary to select completely different symplectic transformations and solve completely different "small divisors" problems.
- When the nonlinear term changes from the third term to the fifth term, its corresponding tangential frequency $\widehat{\omega}$ changes from a first polynomial of parameter ξ to a second polynomial. Therefore, the first derivative of tangential frequency $\frac{\partial\widehat{\omega}}{\partial\xi}$ no longer satisfies the non-degenerate property. It is necessary to consider the second derivative $\frac{\partial^2\widehat{\omega}}{\partial\xi^2}$ and find the full rank matrix in it, so as to prove that the tangential frequency $\widehat{\omega}$ satisfies the corresponding non-degenerate property.
- Because the normal form of the quintic Schrödinger equation depends more complicated on the angular variables, the form of the normal frequency $\widehat{\Omega}$ becomes more complicated. Therefore, more skills are needed to prove that the normal frequency $\widehat{\Omega}$ satisfies the Melnikov's non-degeneracy.
- In order to ensure the simplicity of the normal form, the symplectic transformation selected X_F^1 is greatly changed by making the nonlinear term $\widetilde{R}^6$ more complex after the symplectic transformation. So it is challenging to prove that $\{\widetilde{R}^6, F\}$ still satisfies the Töplitz-Lipschitz property.

The following definition, which was inspired by Zhang-Si [26], quantifies the conditions that the tangential sites satisfy.

Definition 1.1. A finite set $\mathscr{R} = \{i_1^* = (x_1, y_1), \cdots, i_b^* = (x_b, y_b)\} \subset \mathbb{Z}^2$ is admissible if

(1) any three of them are not vertices of a rectangle;

(2) for any $\{i,j,d,l,n,m\}\subset\mathbb{Z}^2$, if they satisfy $i-j+d-l+n-m=0$ and $|i|^2-|j|^2+|d|^2-|l|^2+|n|^2-|m|^2=0$ or $i-j+d+l-n-m=0$ and $|i|^2-|j|^2+|d|^2+|l|^2-|n|^2-|m|^2=0$, then the intersection of $\{i,j,d,l,n,m\}$ and $\mathscr{R}$ contains at most four elements, i.e., $\sharp\big\{\{i,j,d,l,n,m\}\bigcap\mathscr{R}\big\}\le 4$;

(3) for any $n\in\mathbb{Z}^2\setminus\mathscr{R}$, there exists at most one triplet $\{i,j,m\}$ with $i,j\in\mathscr{R},m\in\mathbb{Z}^2\setminus\mathscr{R}$ such that $i-j+n-m=0$ and $|i|^2-|j|^2+|n|^2-|m|^2=0$. If such triplet exists, we say that n,m are resonant in the first type and denote all such n by $\mathscr{L}_1$;

(4) for any $n\in\mathbb{Z}^2\setminus\mathscr{R}$, there exists at most one triplet $\{i,j,m\}$ with $i,j\in\mathscr{R},m\in\mathbb{Z}^2\setminus\mathscr{R}$ such that $i+j-n-m=0$ and $|i|^2+|j|^2-|n|^2-|m|^2=0$. If such triplet exists, we say that n,m are resonant in the second type and denote all such n by $\mathscr{L}_2$.

(5) for any $n\in\mathbb{Z}^2\setminus\mathscr{R}$, there exists at most one quintuple $\{i,j,d,l,m\}$ with $i,j,d,l\in\mathscr{R},m\in\mathbb{Z}^2\setminus\mathscr{R}$ such that $i-j+d-l+n-m=0$ and $|i|^2-|j|^2+|d|^2-|l|^2+|n|^2-|m|^2=0$. If such quintuple exists, we say that n,m are resonant in the third type and denote all such n by $\mathscr{L}_3$;

(6) for any $n\in\mathbb{Z}^2\setminus\mathscr{R}$, there exists at most one quintuple $\{i,j,d,l,m\}$ with $i,j,d,l\in\mathscr{R},m\in\mathbb{Z}^2\setminus\mathscr{R}$ such that $i-j+d+l-n-m=0$ and $|i|^2-|j|^2+|d|^2+|l|^2-|n|^2-|m|^2=0$. If such quintuple exists, we say that n,m are resonant in the fourth type and denote all such n by $\mathscr{L}_4$;

(7) any $n\in\mathbb{Z}^2\setminus\mathscr{R}$ is not resonant of any two of the above four classes. Geometrically, any two of the above defined graphs cannot share vertex in $\mathbb{Z}^2\setminus\mathscr{R}$.

In [26], the authors gave a concrete example of the above admissible set and rigorously proved that the set satisfies all the conditions.

The main conclusion of this paper is as follows.

Theorem 1.2. (Main Theorem) *Let ρ and $\phi(t)$ be as above, and let $[\phi]\neq 0$, where $[\phi]$ denotes the time average of ϕ. Then, for each admissible set $\mathscr{R}=\{i_1^*=(x_1,y_1),\cdots,i_n^*=(x_b,y_b)\}\subset\mathbb{Z}^2$ with $b\ge 2$, and for $0<\gamma<1, 0<\rho<1$ and $\gamma'>0$ be small enough, there is $\varepsilon^*(\rho,\gamma,\gamma')>0$ such that, for arbitrary $0<\varepsilon<\varepsilon^*$, there is $S\subset[\rho,2\rho]^{b^*}$ with* meas $S>(1-\gamma)\rho^{b^*}$ *and there is $\Sigma_{\gamma'}\subset\Sigma:=S\times[\varepsilon^{\frac{3}{2}},2\varepsilon^{\frac{3}{2}}]^b$ with* meas$(\Sigma\setminus\Sigma_{\gamma'})=O(\sqrt[8]{\gamma'})$ *and an absolute constant C so that, for arbitrary $(\omega,\tilde{\xi}_{i_1^*},\dots,\tilde{\xi}_{i_b^*})\in\Sigma_{\gamma'}$, the Schrödinger equation (1.2)+(1.3) has a solution in the following $u(t,x)=\sum_{j\in\mathscr{R}}\big(1-f_j(\omega t,\omega,\varepsilon)\big)\sqrt{\frac{\tilde{\xi}_j}{4\pi^2}}(e^{\mathrm{i}\tilde{\omega}_j t}e^{\mathrm{i}<j,x>})+O(|\tilde{\xi}|^{3/2})+O(\varepsilon^{\rho})$, where $f_j(\vartheta,\omega,\varepsilon)=\varepsilon^{\rho}f_j^*(\vartheta,\omega,\varepsilon)$ is of period 2π in each component of ϑ and for $j\in\mathscr{R},\vartheta\in\Theta(\sigma_0/2),\omega\in S$, $|f_j^*(\vartheta,\omega,\varepsilon)|\le C$. In addition, the solution $u(t,x)$ is quasi-periodic in terms of t with the frequency vector $\tilde{\omega}=(\omega,(\tilde{\omega}_j)_{j\in\mathscr{R}})$, and $\tilde{\omega}_j=\varepsilon^{-4}|j|^2+O(|\tilde{\xi}|^2)+O(\varepsilon^{\rho})$.*

Remark 1.3. According to Definition 1.1, we suppose that $\mathscr{R}$ is an admissible set. Then $\{i,n\}\bigcap\{j,m\}=\emptyset$ holds for any $n\in\mathscr{L}_1$, $\{i,j\}\bigcap\{n,m\}=\emptyset$ holds for any $n\in\mathscr{L}_2$, $\{i,d,n\}\bigcap\{j,l,m\}=\emptyset$ holds for any $n\in\mathscr{L}_3$, and $\{i,d,l\}\bigcap\{j,n,m\}=\emptyset$ holds for any $n\in\mathscr{L}_4$.

2. The Hamiltonian Setting and Partial Birkhoff Normal Form of Schrödinger Equation

In this section, we first convert the linear part of the corresponding Hamiltonian system of the Schrödinger equation (1.2)+(1.3) from a non-autonomous system to an autonomous system by using the reducibility theory in [25]. Then, we convert the Hamiltonian system into a sixth-order Birkhoff normal form by using the symmetric transformation, which is applicable to the KAM theory in [26]. We first introduce some notations.

Let us define $\mathscr{R}=\{i_1^*,\cdots,i_b^*\}$ and $\mathbb{Z}_1^2=\mathbb{Z}^2\setminus\{i_1^*,\cdots,i_b^*\}$, where $\{i_1^*,\cdots,i_b^*\}$ is b vectors in $\mathbb{Z}^2$. Let l^a be the space of all complex vectors $w=(\dots,w_j,\dots)_{j\in\mathbb{Z}_1^2}$, where the norm is defined as follows $\|w\|_a=\sum_{j\in\mathbb{Z}_1^2}|w_j|e^{a|j|}<\infty$, $a>0$. Denote $\theta=(\theta_j)_{j\in\mathscr{R}}, I=(I_j)_{j\in\mathscr{R}}$, $w=(w_j)_{j\in\mathbb{Z}_1^2}$, and $\xi=(\xi_j)_{j\in\mathscr{R}}$, and introduce the phase space as follows $\mathscr{P}^a=\widehat{\mathbb{T}}^b\times\mathbb{C}^b\times l^a\times l^a\ni(\theta,I,w,\bar{w})$, where $\widehat{\mathbb{T}}^b$ is complex neighborhood of the usual (b)-torus $\mathbb{T}^b$. Denote

$$D_a(r,s):=\{(\theta,I,w,\bar{w})\in\mathscr{P}^a:|\mathrm{Im}\theta|<r,|I|<s^2,\|w\|_\rho<s,\|\bar{w}\|_\rho<s\}.$$

Let us define the weighted phase norms

$$|W^*|_a=|\theta|+\frac{1}{s^2}|I|+\frac{1}{s}\|w\|_a+\frac{1}{s}\|\bar{w}\|_a$$

for $W^*=(\theta,I,w,\bar{w})\in\mathscr{P}^a$. Let $\mathscr{O}$ be the parameter set. Denote $\alpha\equiv(\dots,\alpha_j,\dots)_{j\in\mathbb{Z}_1^2}$, $\beta\equiv(\dots,\beta_j,\dots)_{j\in\mathbb{Z}_1^2}$, α_j, and $\beta_j\in\mathbb{N}$ with finitely many nonzero components of positive integers. Suppose that $G(\theta,I,w,\bar{w})=\sum_{\alpha,\beta}G_{\alpha\beta}(\theta,I)w^\alpha\bar{w}^\beta$, where $w^\alpha\bar{w}^\beta$ is $\prod_j w_j^{\alpha_j}\bar{w}_j^{\beta_j}$, $G_{\alpha\beta}=\sum_{k,h}G_{kh\alpha\beta}I^he^{\mathrm{i}<k,\theta>}$ are C_W^8 functions with respect to parameter ξ. The norm of G is defined as follows

$$\|G\|_{D_a(r,s),\mathscr{O}}\equiv\sup_{\|w\|_a<s,\|\bar{w}\|_a<s}\sum_{\alpha,\beta}\|G_{\alpha\beta}\||w^\alpha||\bar{w}^\beta|,$$

where $G_{\alpha\beta}=\sum_{k\in\mathbb{Z}^b,h\in\mathbb{N}^b}G_{kh\alpha\beta}(\xi)I^he^{\mathrm{i}<k,\theta>}$, $|G_{\alpha\beta}\|\equiv\sum_{k,h}|G_{kh\alpha\beta}|_{\mathscr{O}}s^{2|h|}e^{|k|r}$, and $|G_{kh\alpha\beta}|_{\mathscr{O}}\equiv\sup_{\xi\in\mathscr{O}}\sum_{0\le\hat{i}\le8}|\partial_\xi^{\hat{i}}G_{kh\alpha\beta}|$. Denote X_G by the Hamiltonian vector field corresponding to G with respect to the symplectic structure $d\theta\wedge dI+\mathrm{i}dz\wedge d\bar{z}$, which is $X_G=(\partial_IG,-\partial_\theta G,\mathrm{i}\nabla_{\bar{z}}G,-\mathrm{i}\nabla_zG)$. We define its norm by

$$\begin{aligned}\|X_G\|_{D_a(r,s),\mathscr{O}}\ \equiv\ &\|G_I\|_{D_a(r,s),\mathscr{O}}+\frac{1}{s^2}\|G_\theta\|_{D_a(r,s),\mathscr{O}}\\&+\frac{1}{s}(\sum_{j\in\mathbb{Z}_1^2}\|G_{z_j}\|_{D_a(r,s),\mathscr{O}}e^{|j|a}+\sum_{j\in\mathbb{Z}_1^2}\|G_{\bar{z}_j}\|_{D_a(r,s),\mathscr{O}}e^{|j|a}).\end{aligned}$$

Suppose that the vector function $\breve{G}:D_a(r,s)\times\mathscr{O}\to\mathbb{C}^{\breve{m}}(\breve{m}<\infty)$ is C_W^8 functions with respect to parameter ξ, and define its norm as follows $\|\breve{G}\|_{D_a(r,s),\mathscr{O}}=\sum_{\hat{i}=1}^{\breve{m}}\|\breve{G}_{\hat{i}}\|_{D_a(r,s),\mathscr{O}}$. Suppose that the vector function $\hat{G}:D_a(r,s)\times\mathscr{O}\to l^{\bar{a}}$ is C_W^8 functions with respect to parameter ξ and define its norm as follows $\|\hat{G}\|_{\bar{a},D_a(r,s),\mathscr{O}}=\|(\|\hat{G}_{\hat{i}}\|_{D_a(r,s),\mathscr{O}})_{\hat{i}}\|_{\bar{a}}$. Suppose that $B(\eta;\xi)$ us an operator from $D_a(r,s)$ to $D_{\bar{a}}(\bar{r},\bar{s})$ for $(\eta;\xi)\in D_a(r,s)\times\mathscr{O}$, and define the operator norm as follows

$$|A(\eta;\xi)|_{\bar{a},a,D_a(r,s),\mathscr{O}}=\sup_{(\eta;\xi)\in D_a(r,s)\times\mathscr{O}}\sup_{W\neq0}\frac{|A(\eta;\xi)W^*|_{\bar{a}}}{|W^*|_a},$$

$$|A(\eta;\xi)|^*_{\bar{a},a,D_a(r,s),\mathscr{O}}=\sum_{0\le\hat{i}\le8}|\partial_\xi^{\hat{i}}A|_{\bar{a},a,D_a(r,s),\mathscr{O}}.$$

In the following, we rewrite the Schrödinger equation (1.2)

$$\mathrm{i}u_t=\Delta u-\varepsilon\phi(t)(u+|u|^4u),\quad x\in\mathbb{T}^2,\quad t\in\mathbb{R}.\tag{2.1}$$

Equation (2.1) is a Hamiltonian system $u_t = \mathrm{i}\frac{\partial H}{\partial \bar{u}}$ with hamiltonian function $H =< -\Delta u, u > +\varepsilon\phi(t)\int_{\mathbb{T}^2}(|u|^2 + \frac{1}{3}|u|^6)dx$, where $< \cdot,\cdot >$ denotes the scalar product in L^2. The eigenvalues and eigenfunctions of the operator $-\Delta$ with the periodic boundary conditions are $\lambda_j^0 = |j|^2 = {j_1}^2 + {j_2}^2$ and $\phi_j(x) = \sqrt{\frac{1}{4\pi^2}}e^{\mathrm{i}<j,x>}, \quad j = (j_1, j_2) \in \mathbb{Z}^2$. We introduce coordinates $q = (\cdots, q_j, \cdots)_{j\in\mathbb{Z}^2}$ and its complex conjugate $\bar{q} = (\cdots, \bar{q}_j, \cdots)_{j\in\mathbb{Z}^2}$ via the relations $u(t,x) = \sum_{j\in\mathbb{Z}^2} q_j(t)\phi_j(x)$, where $q, \bar{q} \in l^a$. The Hamiltonian system can be rewritten as

$$\dot{q}_j = \mathrm{i}\frac{\partial H}{\partial \bar{q}_j} \quad j \in \mathbb{Z}^2 \tag{2.2}$$

with $H = \Lambda + R$, where

$$\Lambda = \sum_{j\in\mathbb{Z}^2} |j|^2|q_j|^2 + \varepsilon\phi(t)|q_j|^2, \quad R = \frac{\varepsilon}{48\pi^4}\phi(t)\sum_{i-j+d-l+n-m=0} q_i\bar{q}_j q_d\bar{q}_l q_n\bar{q}_m.$$

Denoting by $\varphi(\vartheta)$ the shell of $\phi(t)$, we introduce action-angle variables $(J, \vartheta) \in \mathbb{R}^m \times \mathbb{T}^m$. Then (2.2) can be written as follows $\dot{\vartheta} = \omega$, $\dot{J} = -\frac{\partial H}{\partial \vartheta}$, and $\dot{q}_j = \mathrm{i}\frac{\partial H}{\partial \bar{q}_j}$, $j \in \mathbb{Z}^2$ with the corresponding Hamiltonian $H =< \omega, J > +\sum_{j\in\mathbb{Z}^2}(|j|^2 + \varepsilon\varphi(\vartheta))|q_j|^2 + \varepsilon R^6(x)$, where

$$R^6(x) = \frac{1}{48\pi^4}\sum_{i-j+d-l+n-m=0} R^6_{ijdlnm}(\vartheta)q_i\bar{q}_j q_d\bar{q}_l q_n\bar{q}_m$$

with

$$R^6_{ijdlnm}(\vartheta) = \begin{cases} \varphi(\vartheta), & i-j+d-l+n-m = 0, \\ 0, & i-j+d-l+n-m \neq 0. \end{cases} \tag{2.3}$$

Let

$$H = \bar{H} + \varepsilon R^6, \tag{2.4}$$

where

$$\bar{H} =< \omega, J > + \sum_{j\in\mathbb{Z}^2}(|j|^2 + \varepsilon\varphi(\vartheta))|q_j|^2 \tag{2.5}$$

and

$$R^6 = R^6(x) = \frac{1}{48\pi^4}\sum_{i-j+d-l+n-m=0} R^6_{ijdlnm}(\vartheta)q_i\bar{q}_j q_d\bar{q}_l q_n\bar{q}_m. \tag{2.6}$$

Next, we use the reducibility theory in [25] to transform the non-autonomous Hamiltonian function (2.5) into an autonomous Hamiltonian function. This process will involve an infinite number of small divisor problems. In order to solve these small divisor problems, the following parameter sets need to be introduced. For $0 \neq k \in \mathbb{Z}^{b^*}$, denote

$$\mathscr{I}_k^1 = \left\{\omega \in [\rho, 2\rho]^{b^*} : |< k, \omega >| \leq \frac{\rho}{C_*|k|^{b^*+1}}\right\}, \quad S^1 = \bigcup_{0\neq k\in\mathbb{Z}^{b^*}} \mathscr{I}_k^1.$$

By [25], we have, for arbitrary fixed $0 < \gamma < 1$, meas$S^1 \leq \frac{1}{2}\gamma\rho^{b^*}$. Letting $\underline{S} = [\rho, 2\rho]^{b^*}\backslash S^1$, we have meas$\underline{S} \geq (1 - \frac{\gamma}{2})\rho^{b^*}$. By the reducibility theory [25, Lemma 2.1], for given $\sigma_0 > 0$, $0 < \gamma < 1$, and $0 < \rho < 1$, there exists a $\varepsilon^*(\gamma) > 0$ such that, for any $0 < \varepsilon < \varepsilon^*(\gamma)$ and $\omega \in \underline{S}$, there is a symplectic transformation Σ_∞^0 defined on $D_a(\frac{\sigma_0}{2}, s) \times \underline{S}$ changing the Hamiltonian (2.5) into $H_0 := \bar{H} \circ \Sigma_\infty^0 =< \omega, J > +\sum_{j\in\mathbb{Z}^2} \lambda_j|q_j|^2$, where $\lambda_j = |j|^2 + \varepsilon[\phi] + \varepsilon^{(1+\rho)}\lambda_j^*$, $|\lambda_j^*|_{\underline{S}} \leq$

$C, \quad j\in\mathbb{Z}^2$, and for given $j\in\mathbb{Z}^2, \tilde{j}\in\mathbb{Z}^2\setminus\{0\}$, the limits $\lim_{\tilde{t}\to\infty}\partial_\omega^{\hat{s}}\lambda^*_{j+\tilde{j}\tilde{t}}$ exist for $\hat{s}=0,1,\cdots,8$ and

$$|\lambda^*_{j+\tilde{j}\tilde{t}}-\lim_{\tilde{t}\to\infty}\lambda^*_{j+\tilde{j}\tilde{t}}|_{\underline{S}}\leq\frac{1}{\tilde{t}}. \tag{2.7}$$

Moreover, there exists a constant $C>0$ such that $|\Sigma^0_\infty-id|^*_{\bar{a},a,D_a(\frac{\sigma_0}{2},s),\underline{S}}\leq C\varepsilon^\rho$, where id is identity mapping. The transformation Σ^0_∞ is linear and diagonal. Thus, there exists a function $\tilde{f}^*_{j,\infty}(\vartheta;\omega)$ such that $q_j\circ\Sigma^0_\infty=q_j+\varepsilon^\rho\tilde{f}^*_{j,\infty}(\vartheta;\omega)q_j$, where $|\tilde{f}^*_{j,\infty}(\vartheta;\omega)|_{\Theta(\sigma_0/2)\times\underline{S}}\leq C$, and, for any fixed $j\in\mathbb{Z}^2,\tilde{j}\in\mathbb{Z}^2\setminus\{0\}$, we have that the limits $\lim_{\tilde{t}\to\infty}\partial_\omega^{\hat{s}}\tilde{f}^*_{j+\tilde{j}\tilde{t},\infty}(\vartheta;\omega)$ exist for $\hat{s}=0,1,\cdots,8$ and

$$|\tilde{f}^*_{j+\tilde{j}\tilde{t},\infty}-\lim_{\tilde{t}\to\infty}\tilde{f}^*_{j+\tilde{j}\tilde{t},\infty}|_{\Theta(\sigma_0/2)\times\underline{S}}\leq\frac{1}{\tilde{t}}. \tag{2.8}$$

Thereby, the Hamiltonian (2.6) is transformed into

$$\tilde{R}^6=R^6\circ\Sigma^0_\infty=\frac{1}{48\pi^4}\sum_{i-j+d-l+n-m=0}\tilde{R}^6_{ijdlnm}(\vartheta;\omega)q_i\bar{q}_jq_d\bar{q}_lq_n\bar{q}_m, \tag{2.9}$$

where

$$\tilde{R}^6_{ijdlnm}(\vartheta;\omega)=R^6_{ijdlnm}(\vartheta)\left(1+\varepsilon^\rho R^{6*}_{ijdlnm}(\vartheta;\omega)\right), \tag{2.10}$$

with $|R^{6*}_{ijdlnm}(\vartheta;\omega)|_{\Theta(\sigma_0/2)\times\underline{S}}\leq C$. By (2.8), for given $i,j,d,l,n,m\in\mathbb{Z}^2$ and $\tilde{i},\tilde{j},\tilde{d},\tilde{l},\tilde{n},\tilde{m}\in\mathbb{Z}^2\setminus\{0\}$, $\hat{s}=0,1,\cdots,8$, we have $\lim\limits_{\tilde{t}\to\infty}\partial_\omega^{\hat{s}}\tilde{R}^6_{i+\tilde{i}\tilde{t},jdlnm}$, $\lim\limits_{\tilde{t}\to\infty}\partial_\omega^{\hat{s}}\tilde{R}^6_{i+\tilde{i}\tilde{t},j+\tilde{j}\tilde{t},dlnm}$, $\lim\limits_{\tilde{t}\to\infty}\partial_\omega^{\hat{s}}\tilde{R}^6_{i+\tilde{i}\tilde{t},j+\tilde{j}\tilde{t},d+\tilde{d}\tilde{t},lnm}$, $\lim\limits_{\tilde{t}\to\infty}\partial_\omega^{\hat{s}}\tilde{R}^6_{i+\tilde{i}\tilde{t},j+\tilde{j}\tilde{t},d+\tilde{d}\tilde{t},l+\tilde{l}\tilde{t},nm}$, $\lim\limits_{\tilde{t}\to\infty}\partial_\omega^{\hat{s}}\tilde{R}^6_{i+\tilde{i}\tilde{t},j+\tilde{j}\tilde{t},d+\tilde{d}\tilde{t},l+\tilde{l}\tilde{t},n+\tilde{n}\tilde{t},m}$, $\lim\limits_{\tilde{t}\to\infty}\partial_\omega^{\hat{s}}\tilde{R}^6_{i+\tilde{i}\tilde{t},j+\tilde{j}\tilde{t},d+\tilde{d}\tilde{t},l+\tilde{l}\tilde{t},n+\tilde{n}\tilde{t},m+\tilde{m}\tilde{t}}$ exist and

$$|\tilde{R}^6_{i+\tilde{i}\tilde{t},jdlnm}-\lim_{\tilde{t}\to\infty}\tilde{R}^6_{i+\tilde{i}\tilde{t},jdlnm}|_{\Theta(\sigma_0/2)\times\underline{S}}\leq\frac{1}{\tilde{t}}, \tag{2.11}$$

$$|\tilde{R}^6_{i+\tilde{i}\tilde{t},j+\tilde{j}\tilde{t},dlnm}-\lim_{\tilde{t}\to\infty}\tilde{R}^6_{i+\tilde{i}\tilde{t},j+\tilde{j}\tilde{t},dlnm}|_{\Theta(\sigma_0/2)\times\underline{S}}\leq\frac{1}{\tilde{t}}, \tag{2.12}$$

$$|\tilde{R}^6_{i+\tilde{i}\tilde{t},j+\tilde{j}\tilde{t},d+\tilde{d}\tilde{t},lnm}-\lim_{\tilde{t}\to\infty}\tilde{R}^6_{i+\tilde{i}\tilde{t},j+\tilde{j}\tilde{t},d+\tilde{d}\tilde{t},lnm}|_{\Theta(\sigma_0/2)\times\underline{S}}\leq\frac{1}{\tilde{t}}, \tag{2.13}$$

$$|\tilde{R}^6_{i+\tilde{i}\tilde{t},j+\tilde{j}\tilde{t},d+\tilde{d}\tilde{t},l+\tilde{l}\tilde{t},nm}-\lim_{\tilde{t}\to\infty}\tilde{R}^6_{i+\tilde{i}\tilde{t},j+\tilde{j}\tilde{t},d+\tilde{d}\tilde{t},l+\tilde{l}\tilde{t},nm}|_{\Theta(\sigma_0/2)\times\underline{S}}\leq\frac{1}{\tilde{t}}, \tag{2.14}$$

$$|\tilde{R}^6_{i+\tilde{i}\tilde{t},j+\tilde{j}\tilde{t},d+\tilde{d}\tilde{t},l+\tilde{l}\tilde{t},n+\tilde{n}\tilde{t},m}-\lim_{\tilde{t}\to\infty}\tilde{R}^6_{i+\tilde{i}\tilde{t},j+\tilde{j}\tilde{t},d+\tilde{d}\tilde{t},l+\tilde{l}\tilde{t},n+\tilde{n}\tilde{t},m}|_{\Theta(\sigma_0/2)\times\underline{S}}\leq\frac{1}{\tilde{t}}, \tag{2.15}$$

and

$$|\tilde{R}^6_{i+\tilde{i}\tilde{t},j+\tilde{j}\tilde{t},d+\tilde{d}\tilde{t},l+\tilde{l}\tilde{t},n+\tilde{n}\tilde{t},m+\tilde{m}\tilde{t}}-\lim_{\tilde{t}\to\infty}\tilde{R}^6_{i+\tilde{i}\tilde{t},j+\tilde{j}\tilde{t},d+\tilde{d}\tilde{t},l+\tilde{l}\tilde{t},n+\tilde{n}\tilde{t},m+\tilde{m}\tilde{t}}|_{\Theta(\sigma_0/2)\times\underline{S}}\leq\frac{1}{\tilde{t}}. \tag{2.16}$$

This means Hamiltonian (2.4) is converted by the transformation Σ^0_∞ into

$$H=H_0+\varepsilon\tilde{R}^6. \tag{2.17}$$

From [25], we give the following Lemma.

Lemma 2.1. *For $a\geq 0$, the gradients $\tilde{R}^6_q, \tilde{R}^6_{\bar{q}}$ are real analytic as maps from some neighborhood of origin in $l^a\times l^a$ into l^a with $\|\tilde{R}^6_q\|_a = O(\|q\|_a^5)$, $\|\tilde{R}^6_{\bar{q}}\|_a = O(\|q\|_a^5)$ uniformly for $(\vartheta,\omega)\in\Theta(\sigma_0/2)\times\underline{S}$.*

Proof. From (2.9), for $(\vartheta,\omega)\in\Theta(\sigma_0/2)\times\underline{S}$, we have

$$\left|\frac{\partial\tilde{R}^6}{\partial q_i}\right| \leq \frac{1}{48\pi^4}\sum_{j-d+l-n+m=i}|\tilde{R}^6_{ijdlnm}(\vartheta,\omega)||\bar{q}_j q_d\bar{q}_l q_n\bar{q}_m|.$$

According to (2.3) and (2.10), we have

$$\begin{aligned}
\|\tilde{R}^6_q\|_a &= \sum_{i\in\mathbb{Z}^2}|\tilde{R}^6_{q_i}|e^{|i|a} \leq \frac{1}{48\pi^4}\sum_{i\in\mathbb{Z}^2}\sum_{j-d+l-n+m=i} C|\bar{q}_j q_d\bar{q}_l q_n\bar{q}_m|e^{|j-d+l-n+m|a}\\
&\leq C\sum_{j\in\mathbb{Z}^2}|\bar{q}_j|e^{|j|a}\sum_{d\in\mathbb{Z}^2}|q_d|e^{|d|a}\sum_{l\in\mathbb{Z}^2}|\bar{q}_l|e^{|l|a}\sum_{n\in\mathbb{Z}^2}|q_n|e^{|n|a}\sum_{m\in\mathbb{Z}^2}|\bar{q}_m|e^{|m|a}\\
&= C\|\bar{q}\|_a^3\|q\|_a^2 = C\|q\|_a^5.
\end{aligned}$$

Similarly, we have $\|\tilde{R}^6_{\bar{q}}\|_a\leq C\|q\|_a^5$. □

Suppose that $\mathscr{R}$ is an admissible set. Denote $\mathbb{Z}^2_1 = \mathbb{Z}^2\setminus\mathscr{R}$. In order to solve the small divisor problem during the transformation from the Hamiltonian (2.17) to some six-order Birkhoff normal form, the following lemma is given, which will be proved in the Appendix.

Lemma 2.2. *Assume that $i,j,d,l,n,m\in\mathbb{Z}^2$ such that $i-j+d-l+n-m=0, |i|^2-|j|^2+|d|^2-|l|^2+|n|^2-|m|^2\neq 0$ and $\#(\mathscr{R}\cap\{i,j,d,l,n,m\})\geq 4$ or $i-j+d-l+n-m=0, |i|^2-|j|^2+|d|^2-|l|^2+|n|^2-|m|^2=0$, $\#(\mathscr{R}\cap\{i,j,d,l,n,m\})=4$ and $k\neq 0$. Denote $\rho>0, 0<\gamma<1$, and C_* large enough, and ε small enough, there exists a subset $S\subset\underline{S}\subset[\rho,2\rho]^{b^*}$ with $\operatorname{meas}S>(1-\gamma)\rho^{b^*}$, Then, for arbitrary $\omega\in S$,*

$$\left|\lambda_i-\lambda_j+\lambda_d-\lambda_l+\lambda_n-\lambda_m+<k,\omega>\right|\geq\frac{\rho}{C_*(|k|+\delta(|k|))^{b^*+2}},\quad\forall k\in\mathbb{Z}^{b^*},\tag{2.18}$$

with $\delta(x)=1$ as $x=0$ and $\delta(x)=0$ as $x\neq 0$.

Proposition 2.3. *Let $\mathscr{R}$ be an admissible set. There is a symplectic change of coordinates X^1_F that converts the Hamiltonian (2.17) into*

$$H^* = H\circ X^1_F = N^*+\mathscr{A}^*+\mathscr{B}^*+\overline{\mathscr{B}}^*+\mathscr{C}^*+\mathscr{D}^*+\overline{\mathscr{D}}^*+P^*\tag{2.19}$$

with

$$N^* = <\varepsilon^{-4}\omega, J> + \sum_{i\in\mathscr{R}}\breve{\omega}_i(\tilde{\xi})\tilde{I}_i + \sum_{n\in\mathbb{Z}^2_1}\widehat{\Omega}_n(\tilde{\xi})|z_n|^2,$$

$$\breve{\omega}_i(\tilde{\xi}) = \varepsilon^{-4}\lambda_i + \frac{[\tilde{R}^6_{iiiiii}]}{16\pi^4}\tilde{\xi}_i^2 + \sum_{j\in\mathscr{R},j\neq i}\left[\frac{3[\tilde{R}^6_{iiiijj}]}{16\pi^4}(\tilde{\xi}_j^2+2\tilde{\xi}_i\tilde{\xi}_j) + \sum_{d\in\mathscr{R},d\neq j,d\neq i}\frac{3[\tilde{R}^6_{iijjdd}]}{8\pi^4}\tilde{\xi}_j\tilde{\xi}_d\right],$$

$$\widehat{\Omega}_n(\tilde{\xi}) = \varepsilon^{-4}\lambda_n + \sum_{i\in\mathscr{R}}\left[\frac{3[\tilde{R}^6_{iiiinn}]}{16\pi^4}\tilde{\xi}_i^2 + \sum_{j\in\mathscr{R},j\neq i}\frac{3[\tilde{R}^6_{iijjnn}]}{8\pi^4}\tilde{\xi}_i\tilde{\xi}_j\right],$$

$$\mathscr{A}^* = \sum_{n\in\mathscr{L}_1}\sum_{d\in\mathscr{R}}\frac{3[\tilde{R}^6_{ijddnm}]}{4\pi^4}\sqrt{\tilde{\xi}_i\tilde{\xi}_j}\tilde{\xi}_d e^{\mathrm{i}(\tilde{\theta}_i-\tilde{\theta}_j)}z_n\bar{z}_m,$$

$$\mathscr{B}^* = \sum_{n\in\mathscr{L}_2}\sum_{d\in\mathscr{R}} \frac{3[\tilde{R}^6_{nimjdd}]}{4\pi^4}\sqrt{\tilde{\xi}_i\tilde{\xi}_j}\tilde{\xi}_d e^{\mathrm{i}(-\tilde{\theta}_i-\tilde{\theta}_j)} z_n z_m,$$

$$\overline{\mathscr{B}}^* = \sum_{n\in\mathscr{L}_2}\sum_{d\in\mathscr{R}} \frac{3[\tilde{R}^6_{injmdd}]}{4\pi^4}\sqrt{\tilde{\xi}_i\tilde{\xi}_j}\tilde{\xi}_d e^{\mathrm{i}(\tilde{\theta}_i+\tilde{\theta}_j)} \bar{z}_n\bar{z}_m,$$

$$\mathscr{C}^* = \sum_{n\in\mathscr{L}_3} \frac{3[\tilde{R}^6_{ijdlnm}]}{4\pi^4}\sqrt{\tilde{\xi}_i\tilde{\xi}_j\tilde{\xi}_d\tilde{\xi}_l} e^{\mathrm{i}(\tilde{\theta}_i-\tilde{\theta}_j+\tilde{\theta}_d-\tilde{\theta}_l)} z_n\bar{z}_m,$$

$$\mathscr{D}^* = \sum_{n\in\mathscr{L}_4} \frac{3[\tilde{R}^6_{jindml}]}{4\pi^4}\sqrt{\tilde{\xi}_i\tilde{\xi}_j\tilde{\xi}_d\tilde{\xi}_l} e^{\mathrm{i}(-\tilde{\theta}_i+\tilde{\theta}_j-\tilde{\theta}_d-\tilde{\theta}_l)} z_n z_m,$$

$$\overline{\mathscr{D}}^* = \sum_{n\in\mathscr{L}_4} \frac{3[\tilde{R}^6_{ijdnlm}]}{4\pi^4}\sqrt{\tilde{\xi}_i\tilde{\xi}_j\tilde{\xi}_d\tilde{\xi}_l} e^{\mathrm{i}(\tilde{\theta}_i-\tilde{\theta}_j+\tilde{\theta}_d+\tilde{\theta}_l)} \bar{z}_n\bar{z}_m,$$

$$\begin{aligned}P^* = O(&\varepsilon^2|\tilde{I}|^2+\varepsilon^2|\tilde{I}|\|z\|_a^2+\varepsilon^{\frac{7}{4}}|\tilde{\xi}|^{\frac{3}{2}}|\tilde{I}|^{\frac{1}{2}}\|z\|_a^2+\varepsilon^{\frac{7}{4}}|\tilde{\xi}|^{\frac{3}{2}}\|z\|_a^3\\ &+\varepsilon^{\frac{7}{2}}|\tilde{\xi}|\|z\|_a^4+\varepsilon^{\frac{9}{4}}|\tilde{\xi}|^{\frac{9}{2}}\|z\|_a+\varepsilon^4|\tilde{\xi}|^4\|z\|_a^2+\varepsilon^{\frac{23}{4}}|\tilde{\xi}|^{\frac{7}{2}}\|z\|_a^3), \end{aligned} \tag{2.20}$$

where $[\tilde{R}^6_{ijdlnm}]$ denotes the time average of $\tilde{R}^6_{ijdlnm}$, coinciding with the space average.

Proof. Denote $\tilde{R}^6_{ijdlnm}(\vartheta,\omega)=\sum_{k\in\mathbb{Z}^{b^*}}\tilde{R}^6_{ijdlnm,k}(\omega)e^{\mathrm{i}<k,\vartheta>}$. Let

$$F = \tfrac{\varepsilon}{48\pi^4}\sum_{i\in\mathscr{R}}\sum_{k\neq 0}\tfrac{\tilde{R}^6_{iiiiii,k}}{\mathrm{i}<k,\omega>}e^{\mathrm{i}<k,\vartheta>}|q_i|^6+\tfrac{9\varepsilon}{48\pi^4}\sum_{i,j\in\mathscr{R},i\neq j}\sum_{k\neq 0}\tfrac{\tilde{R}^6_{iiiijj,k}}{\mathrm{i}<k,\omega>}e^{\mathrm{i}<k,\vartheta>}|q_i|^4|q_j|^2$$

$$+\tfrac{3\varepsilon}{4\pi^4}\sum_{i,j,d\in\mathscr{R},i\neq j,i\neq d,j\neq d}\sum_{k\neq 0}\tfrac{\tilde{R}^6_{iijjdd,k}}{\mathrm{i}<k,\omega>}e^{\mathrm{i}<k,\vartheta>}|q_i|^2|q_j|^2|q_d|^2$$

$$+\tfrac{9\varepsilon}{48\pi^4}\sum_{i\in\mathscr{R},n\in\mathbb{Z}_1^2}\sum_{k\neq 0}\tfrac{\tilde{R}^6_{iiiinn,k}}{\mathrm{i}<k,\omega>}e^{\mathrm{i}<k,\vartheta>}|q_i|^4|q_n|^2+\tfrac{3\varepsilon}{4\pi^4}\sum_{i,j\in\mathscr{R},n\in\mathbb{Z}_1^2,i\neq j}\sum_{k\neq 0}\tfrac{\tilde{R}^6_{iijjnn,k}}{\mathrm{i}<k,\omega>}e^{\mathrm{i}<k,\vartheta>}|q_i|^2|q_j|^2|q_n|^2$$

$$+\tfrac{3\varepsilon}{4\pi^4}\sum_{n\in\mathscr{L}_1}\sum_{d\in\mathscr{R}}\sum_{k\neq 0}\tfrac{\tilde{R}^6_{ijddnm,k}}{\mathrm{i}(\lambda_i-\lambda_j+\lambda_n-\lambda_m+<k,\omega>)}e^{\mathrm{i}<k,\vartheta>}q_i\bar{q}_j|q_d|^2q_n\bar{q}_m$$

$$+\tfrac{3\varepsilon}{4\pi^4}\sum_{n\in\mathscr{L}_2}\sum_{d\in\mathscr{R}}\sum_{k\neq 0}\tfrac{\tilde{R}^6_{nimjdd,k}}{\mathrm{i}(-\lambda_i-\lambda_j+\lambda_n+\lambda_m+<k,\omega>)}e^{\mathrm{i}<k,\vartheta>}\bar{q}_i\bar{q}_j|q_d|^2q_nq_m$$

$$+\tfrac{3\varepsilon}{4\pi^4}\sum_{n\in\mathscr{L}_2}\sum_{d\in\mathscr{R}}\sum_{k\neq 0}\tfrac{\tilde{R}^6_{injmdd,k}}{\mathrm{i}(\lambda_i+\lambda_j-\lambda_n-\lambda_m+<k,\omega>)}e^{\mathrm{i}<k,\vartheta>}q_iq_j|q_d|^2\bar{q}_n\bar{q}_m$$

$$+\tfrac{3\varepsilon}{4\pi^4}\sum_{n\in\mathscr{L}_3}\sum_{k\neq 0}\tfrac{\tilde{R}^6_{ijdlnm,k}}{\mathrm{i}(\lambda_i-\lambda_j+\lambda_d-\lambda_l+\lambda_n-\lambda_m+<k,\omega>)}e^{\mathrm{i}<k,\vartheta>}q_i\bar{q}_jq_d\bar{q}_lq_n\bar{q}_m$$

$$+\tfrac{3\varepsilon}{4\pi^4}\sum_{n\in\mathscr{L}_4}\sum_{k\neq 0}\tfrac{\tilde{R}^6_{jindml,k}}{\mathrm{i}(-\lambda_i+\lambda_j-\lambda_d-\lambda_l+\lambda_n+\lambda_m+<k,\omega>)}e^{\mathrm{i}<k,\vartheta>}\bar{q}_iq_j\bar{q}_d\bar{q}_lq_nq_m$$

$$
\begin{aligned}
&+\frac{3\varepsilon}{4\pi^4}\sum_{n\in\mathscr{L}_4}\sum_{k\neq 0}\frac{\tilde{R}^6_{ijdnlm,k}}{\mathrm{i}(\lambda_i-\lambda_j+\lambda_d+\lambda_l-\lambda_n-\lambda_m+<k,\omega>)}e^{\mathrm{i}<k,\vartheta>}q_i\bar{q}_jq_dq_l\bar{q}_n\bar{q}_m\\
&+\frac{\varepsilon}{48\pi^4}\sum_{\substack{i-j+d-l+n-m=0\\ |i|^2-|j|^2+|d|^2-|l|^2+|n|^2-|m|^2\neq 0\\ \#(\mathscr{R}\cap\{i,j,d,l,n,m\})\geq 4}}\sum_{k\in\mathbb{Z}^{b^*}}\frac{\tilde{R}^6_{ijdlnm,k}e^{\mathrm{i}<k,\vartheta>}q_i\bar{q}_jq_d\bar{q}_lq_n\bar{q}_m}{\mathrm{i}(\lambda_i-\lambda_j+\lambda_d-\lambda_l+\lambda_n-\lambda_m+<k,\omega>)}. \qquad (2.21)
\end{aligned}
$$

and X_F^1 denote the time-1 map of the vector field of the Hamiltonian F. Denote

$$
q_j=\begin{cases} q_j, & j\in\mathscr{R},\\ z_j, & j\in\mathbb{Z}_1^2.\end{cases}
$$

Then the transformation of coordinates X_F^1 transforms H into

$$
\begin{aligned}
\widehat{H}&=H\circ X_F^1\\
&=H_0+\varepsilon\tilde{R}^6+\{H_0,F\}+\varepsilon\{\tilde{R}^6,F\}+\int_0^1(1-t)\{\{H,F\},F\}\circ X_F^t dt\\
&=<\omega,J>+\sum_{i\in\mathscr{R}}\lambda_i|q_i|^2+\sum_{n\in\mathbb{Z}_1^2}\lambda_n|w_n|^2+\sum_{i\in\mathscr{R}}\frac{\varepsilon[\tilde{R}^6_{iiiiii}]}{48\pi^4}|q_i|^6\\
&+\sum_{i,j\in\mathscr{R},i\neq j}\frac{9\varepsilon[\tilde{R}^6_{iiiijj}]}{48\pi^4}|q_i|^4|q_j|^2+\sum_{i,j,d\in\mathscr{R},i\neq j,i\neq d,j\neq d}\frac{3\varepsilon[\tilde{R}^6_{iijjdd}]}{4\pi^4}|q_i|^2|q_j|^2|q_d|^2\\
&+\sum_{i\in\mathscr{R},n\in\mathbb{Z}_1^2}\frac{9\varepsilon[\tilde{R}^6_{iiiinn}]}{48\pi^4}|q_i|^4|z_n|^2+\sum_{i,j\in\mathscr{R},n\in\mathbb{Z}_1^2,i\neq j}\frac{3\varepsilon[\tilde{R}^6_{iijjnn}]}{4\pi^4}|q_i|^2|q_j|^2|z_n|^2\\
&+\sum_{n\in\mathscr{L}_1}\sum_{d\in\mathscr{R}}\frac{3\varepsilon[\tilde{R}^6_{ijddnm}]}{4\pi^4}q_i\bar{q}_j|q_d|^2z_n\bar{z}_m+\sum_{n\in\mathscr{L}_2}\sum_{d\in\mathscr{R}}\frac{3\varepsilon[\tilde{R}^6_{nimjdd}]}{4\pi^4}\bar{q}_i\bar{q}_j|q_d|^2z_nz_m\\
&+\sum_{n\in\mathscr{L}_2}\sum_{d\in\mathscr{R}}\frac{3\varepsilon[\tilde{R}^6_{injmdd}]}{4\pi^4}q_iq_j|q_d|^2\bar{z}_n\bar{z}_m+\sum_{n\in\mathscr{L}_3}\frac{3\varepsilon[\tilde{R}^6_{ijdlnm}]}{4\pi^4}q_i\bar{q}_jq_d\bar{q}_lz_n\bar{z}_m\\
&+\sum_{n\in\mathscr{L}_4}\frac{3\varepsilon[\tilde{R}^6_{jindml}]}{4\pi^4}\bar{q}_iq_j\bar{q}_d\bar{q}_lz_nz_m+\sum_{n\in\mathscr{L}_4}\frac{3\varepsilon[\tilde{R}^6_{ijdnlm}]}{4\pi^4}q_i\bar{q}_jq_dq_l\bar{z}_n\bar{z}_m\\
&+O(\varepsilon|q|^3\|z\|_a^3+\varepsilon|q|^2\|z\|_a^4+\varepsilon|q|\|z\|_a^5+\varepsilon\|z\|_a^6+\varepsilon^2|q|^{10}\\
&+\varepsilon^2|q|^9\|z\|_a+\varepsilon^2|q|^8\|z\|_a^2+\varepsilon^2|q|^7\|z\|_a^3),
\end{aligned}
$$

where $\{\cdot,\cdot\}$ is the Poisson bracket with respect to the symplectic structure $d\vartheta\wedge dJ+\sum_{j\in\mathscr{R}}\mathrm{i}dq_j\wedge d\bar{q}_j+\sum_{n\in\mathbb{Z}_1^2}\mathrm{i}dw_n\wedge d\bar{w}_n$, (n,m) are resonant pairs, (i,j) and (i,j,d,l) are uniquely determined by (n,m). Let us introduce the action-angle variable by

$$
q_j=\sqrt{\tilde{I}_j+\tilde{\xi}_j}e^{\mathrm{i}\tilde{\theta}_j},\qquad \bar{q}_j=\sqrt{\tilde{I}_j+\tilde{\xi}_j}e^{-\mathrm{i}\tilde{\theta}_j},\quad j\in\mathscr{R}. \qquad (2.22)
$$

The symplectic change (2.22) transform the Hamiltonian $\widehat{H}$ into

$$
\begin{aligned}
\widehat{H} =& < \omega, J > + \sum_{i\in\mathscr{R}} \Bigg\{ \lambda_i + \frac{\varepsilon[\tilde{R}^6_{iiiiii}]\tilde{\xi}_i^2}{16\pi^4} \\
&+ \sum_{\substack{j\in\mathscr{R}\\ j\neq i}} \left[\frac{3\varepsilon[\tilde{R}^6_{iiiijj}](\tilde{\xi}_j^2+2\tilde{\xi}_i\tilde{\xi}_j)}{16\pi^4} + \sum_{\substack{d\in\mathscr{R}\\ d\neq j\\ d\neq i}} \frac{3\varepsilon[\tilde{R}^6_{iijjdd}]\tilde{\xi}_j\tilde{\xi}_d}{8\pi^4} \right] \Bigg\} \tilde{I}_i \\
&+ \sum_{n\in\mathbb{Z}_1^2} \left\{ \lambda_n + \sum_{i\in\mathscr{R}} \left[\frac{3\varepsilon[\tilde{R}^6_{iiiinn}]}{16\pi^4}\tilde{\xi}_i^2 + \sum_{j\in\mathscr{R}, j\neq i} \frac{3\varepsilon[\tilde{R}^6_{iijjnn}]}{8\pi^4}\tilde{\xi}_i\tilde{\xi}_j \right] \right\} |z_n|^2 \\
&+ \sum_{n\in\mathscr{L}_1}\sum_{d\in\mathscr{R}} \frac{3\varepsilon[\tilde{R}^6_{ijddnm}]}{4\pi^4}\sqrt{\tilde{\xi}_i\tilde{\xi}_j}\tilde{\xi}_d e^{\mathrm{i}(\tilde{\theta}_i-\tilde{\theta}_j)} z_n\bar{z}_m \\
&+ \sum_{n\in\mathscr{L}_2}\sum_{d\in\mathscr{R}} \frac{3\varepsilon[\tilde{R}^6_{nimjdd}]}{4\pi^4}\sqrt{\tilde{\xi}_i\tilde{\xi}_j}\tilde{\xi}_d e^{\mathrm{i}(-\tilde{\theta}_i-\tilde{\theta}_j)} z_n z_m \\
&+ \sum_{n\in\mathscr{L}_2}\sum_{d\in\mathscr{R}} \frac{3\varepsilon[\tilde{R}^6_{injmdd}]}{4\pi^4}\sqrt{\tilde{\xi}_i\tilde{\xi}_j}\tilde{\xi}_d e^{\mathrm{i}(\tilde{\theta}_i+\tilde{\theta}_j)} \bar{z}_n\bar{z}_m \\
&+ \sum_{n\in\mathscr{L}_3} \frac{3\varepsilon[\tilde{R}^6_{ijdlnm}]}{4\pi^4}\sqrt{\tilde{\xi}_i\tilde{\xi}_j\tilde{\xi}_d\tilde{\xi}_l} e^{\mathrm{i}(\tilde{\theta}_i-\tilde{\theta}_j+\tilde{\theta}_d-\tilde{\theta}_l)} z_n\bar{z}_m \\
&+ \sum_{n\in\mathscr{L}_4} \frac{3\varepsilon[\tilde{R}^6_{jindml}]}{4\pi^4}\sqrt{\tilde{\xi}_i\tilde{\xi}_j\tilde{\xi}_d\tilde{\xi}_l} e^{\mathrm{i}(-\tilde{\theta}_i+\tilde{\theta}_j-\tilde{\theta}_d-\tilde{\theta}_l)} z_n z_m \\
&+ \sum_{n\in\mathscr{L}_4} \frac{3\varepsilon[\tilde{R}^6_{ijdnlm}]}{4\pi^4}\sqrt{\tilde{\xi}_i\tilde{\xi}_j\tilde{\xi}_d\tilde{\xi}_l} e^{\mathrm{i}(\tilde{\theta}_i-\tilde{\theta}_j+\tilde{\theta}_d+\tilde{\theta}_l)} \bar{z}_n\bar{z}_m \\
&+ O(\varepsilon|\tilde{I}|^2 + \varepsilon|\tilde{I}|\|z\|_a^2 + \varepsilon|\tilde{\xi}|^{\frac{3}{2}}|\tilde{I}|^{\frac{1}{2}}\|z\|_a^2 + \varepsilon|\tilde{\xi}|^{\frac{3}{2}}\|z\|_a^3 + \varepsilon|\tilde{\xi}|\|z\|_a^4 \\
&\quad + \varepsilon^2|\tilde{\xi}|^{\frac{9}{2}}\|z\|_a + \varepsilon^2|\tilde{\xi}|^4\|z\|_a^2 + \varepsilon^2|\tilde{\xi}|^{\frac{7}{2}}\|z\|_a^3).
\end{aligned}
$$

By the scaling in time

$$
\tilde{\xi} \to \varepsilon^{\frac{3}{2}}\tilde{\xi}, \quad \tilde{I} \to \varepsilon^5\tilde{I}, \quad J \to \varepsilon^5 J, \quad \tilde{\theta} \to \tilde{\theta}, \quad \vartheta \to \vartheta, \quad z \to \varepsilon^{\frac{5}{2}}z, \quad \bar{z} \to \varepsilon^{\frac{5}{2}}\bar{z},
$$

we have the rescaled Hamiltonian $H^* = \varepsilon^{-9}\widehat{H}(\varepsilon^{\frac{3}{2}}\tilde{\xi}, \varepsilon^5\tilde{I}, \varepsilon^5 J, \vartheta, \tilde{\theta}, \varepsilon^{\frac{5}{2}}z, \varepsilon^{\frac{5}{2}}\bar{z})$. Then H^* satisfies (2.19) and (2.20), where $\xi \in [\varepsilon^{\frac{3}{2}}, 2\varepsilon^{\frac{3}{2}}]^b$. □

Remark 2.4. We also deal with the small divisor $\lambda_i - \lambda_j + \lambda_n - \lambda_m + < k, \omega >$ in Lemma 2.2, which is actually a special case of $d = l$ in (2.18).

3. An Infinite-Dimensional KAM Theorem for Partial Differential Equations

In this section, we need to state a KAM theorem, which was proved by Zhang-Si [26] for the main result (Theorem 1.2). Let us denote $H_0 = N + \mathscr{A} + \mathscr{B} + \overline{\mathscr{B}} + \mathscr{C} + \mathscr{D} + \overline{\mathscr{D}}$, where

$$N = \sum_{j\in\mathscr{R}} \widehat{\omega}_j(\xi) y_j + \sum_{n\in\mathbb{Z}_1^2} \widehat{\Omega}_n(\xi) z_n \bar{z}_n,$$

$$\mathscr{A} = \sum_{n\in\mathscr{L}_1} a_n(\xi) e^{\mathrm{i}(x_i - x_j)} z_n \bar{z}_m,$$

$$\mathscr{B} = \sum_{n\in\mathscr{L}_2} a_n(\xi) e^{-\mathrm{i}(x_i + x_j)} z_n z_m,$$

$$\overline{\mathscr{B}} = \sum_{n\in\mathscr{L}_2} \bar{a}_n(\xi) e^{\mathrm{i}(x_i + x_j)} \bar{z}_n \bar{z}_m,$$

$$\mathscr{C} = \sum_{n\in\mathscr{L}_3} a_n(\xi) e^{\mathrm{i}(x_i - x_j + x_d - x_l)} z_n \bar{z}_m,$$

$$\mathscr{D} = \sum_{n\in\mathscr{L}_4} a_n(\xi) e^{\mathrm{i}(-x_i + x_j - x_d - x_l)} z_n z_m,$$

and

$$\overline{\mathscr{D}} = \sum_{n\in\mathscr{L}_4} \bar{a}_n(\xi) e^{\mathrm{i}(x_i - x_j + x_d + x_l)} \bar{z}_n \bar{z}_m,$$

where (x,y) are b-dimensional angle-action coordinates, $(z,\bar{z})$ are infinite-dimensional coordinates, and the symplectic structure is defined by $\sum_{i\in\mathscr{R}} dx_i \wedge dy_i + \mathrm{i}\sum_{n\in\mathbb{Z}_1^2} dz_n \wedge d\bar{z}_n$. The tangential frequencies $\widehat{\omega} = (\widehat{\omega}_i)_{i\in\mathscr{R}}$ and the normal frequencies $\widehat{\Omega} = (\widehat{\Omega}_n)_{n\in\mathbb{Z}_1^2}$ depend on the b parameters $\xi \in \Pi \subset \mathbb{R}^b$, where Π is a closed bounded set with positive Lebesgue measure. For each ξ, there exists an invariant b-torus $\mathscr{T}_0^b = \mathbb{T}^b \times \{0,0,0\}$ with frequencies $\widehat{\omega}(\xi)$. The aim is to prove the persistence of a large part of this family of rotational tori under small perturbations $H = H_0 + P$ of H_0. To accomplish this, we make the following assumptions.

Assumption A1. (Non-degeneracy): $\forall \xi \in \Pi$,

$$\begin{cases} \operatorname{rank}\left\{ \dfrac{\partial \omega_{i_1^*}}{\partial \xi}, \cdots, \dfrac{\partial \omega_{i_b^*}}{\partial \xi} \right\} = \kappa, \\ \operatorname{rank}\left\{ \dfrac{\partial^{\beta^*} \omega}{\partial \xi^{\beta^*}} | \forall \beta^*, 1 \le |\beta^*| \le \min\{b - \kappa + 1, 5\} \right\} = b, \end{cases}$$

where κ is a given integer with $1 \le \kappa \le b$, $\frac{\partial \omega_{i_1^*}}{\partial \xi}, \cdots, \frac{\partial \omega_{i_b^*}}{\partial \xi}$ are vectors of all $1-$order partial derivatives in ξ, and, for a fixed β^*, $\frac{\partial^{\beta^*}\omega}{\partial \xi^{\beta^*}} = \left(\frac{\partial^{\beta^*} \omega_{i_1^*}}{\partial \xi^{\beta^*}}, \cdots, \frac{\partial^{\beta^*} \omega_{i_b^*}}{\partial \xi^{\beta^*}} \right)$. Moreover, $\omega(\xi)$ belongs to $C_W^8(\mathscr{O})$.

Assumption A2. (Asymptotics of normal frequencies): $\widehat{\Omega}_n = \varepsilon^{-\varsigma}|n|^2 + \widetilde{\Omega}_n, \quad \varsigma \ge 0$, where $\widetilde{\Omega}_n$'s are C_W^8 functions of ξ with C_W^8-norm bounded by some small positive constant L.

Assumption A3. (Melnikov's non-degeneracy): Denote $\mathscr{M}_n = \Omega_n$ for $n \in \mathbb{Z}_1^2 \setminus (\mathscr{L}_1 \cup \mathscr{L}_2 \cup \mathscr{L}_3 \cup \mathscr{L}_4)$, and denote

$$\mathscr{M}_n = \begin{pmatrix} \widehat{\Omega}_n + \widehat{\omega}_i & a_n \\ a_m & \widehat{\Omega}_m + \widehat{\omega}_j \end{pmatrix}, \quad n \in \mathscr{L}_1$$

$$\mathscr{M}_n = \begin{pmatrix} \widehat{\Omega}_n - \widehat{\omega}_i & -a_n \\ \bar{a}_m & -\widehat{\Omega}_m + \widehat{\omega}_j \end{pmatrix}, \quad n \in \mathscr{L}_2$$

$$\mathscr{M}_n = \begin{pmatrix} \widehat{\Omega}_n + \widehat{\omega}_i + \widehat{\omega}_d & a_n \\ a_m & \widehat{\Omega}_m + \widehat{\omega}_j + \widehat{\omega}_l \end{pmatrix}, \quad n \in \mathscr{L}_3,$$

and

$$\mathscr{M}_n = \begin{pmatrix} \widehat{\Omega}_n + \widehat{\omega}_j & -a_n \\ \bar{a}_m & -\widehat{\Omega}_m + \widehat{\omega}_i + \widehat{\omega}_d + \widehat{\omega}_l \end{pmatrix}, \quad n \in \mathscr{L}_4$$

where (n,m) are resonant pairs, (i,j) and (i,j,d,l) are uniquely determined by (n,m). We suppose that $\widehat{\omega}(\xi), \mathscr{M}_n(\xi) \in C_W^8(\Pi)$ and there is $\gamma', \tau > 0$ (here I is identity matrix)

$$|< k, \widehat{\omega} >| \geq \frac{\gamma'}{|k|^\tau}, \quad k \neq 0,$$

$$|\det(< k, \widehat{\omega} > I \pm \mathscr{M}_n)| \geq \frac{\gamma'}{|k|^\tau},$$

and

$$|\det(< k, \widehat{\omega} > I \pm \mathscr{M}_n \otimes I \pm I \otimes \mathscr{M}_{n'})| \geq \frac{\gamma'}{|k|^\tau}.$$

Assumption A4. (Regularity): $\mathscr{A} + \mathscr{B} + \bar{\mathscr{B}} + \mathscr{C} + \mathscr{D} + \bar{\mathscr{D}} + P$ is real analytic in $x, y, z, \bar{z}$ and C_W^8 Whitney smooth in ξ. In addition,

$$\|X_{\mathscr{A}}\|_{D_a(r,s),\Pi} + \|X_{\mathscr{B}}\|_{D_a(r,s),\Pi} + \|X_{\mathscr{C}}\|_{D_a(r,s),\Pi} + \|X_{\mathscr{D}}\|_{D_a(r,s),\Pi} < 1,$$

and

$$\|X_P\|_{D_a(r,s),\Pi} < \varepsilon.$$

Assumption A5. (Special Form): $\mathscr{A} + \mathscr{B} + \bar{\mathscr{B}} + \mathscr{C} + \mathscr{D} + \bar{\mathscr{D}} + P$ has a special form in the following

$$\begin{aligned} \mathscr{Y} &= \{\mathscr{A} + \mathscr{B} + \overline{\mathscr{B}} + \mathscr{C} + \mathscr{D} + \overline{\mathscr{D}} + P : \mathscr{A} + \mathscr{B} + \overline{\mathscr{B}} + \mathscr{C} + \mathscr{D} + \overline{\mathscr{D}} + P \\ &= \sum_{k \in \mathbb{Z}^b, \varpi \in \mathbb{N}^b, \alpha, \beta} (\mathscr{A} + \mathscr{B} + \overline{\mathscr{B}} + \mathscr{C} + \mathscr{D} + \overline{\mathscr{D}} + P)_{k\varpi\alpha\beta}(\xi) I^{\varpi} e^{\mathrm{i}<k,\theta>} z^\alpha \bar{z}^\beta \}, \end{aligned}$$

where k, α, β have the following relations $\sum_{\hat{s}=1}^{b} k_{\hat{s}} i_{\hat{s}}^* + \sum_{n \in \mathbb{Z}_1^2} (\alpha_n - \beta_n) n = 0$.

Assumption A6. (Töplitz-Lipschitz property): For any fixed $n, m \in \mathbb{Z}^2$, $\tilde{c} \in \mathbb{Z}^2 \setminus \{0\}$, the limits

$$\lim_{\tilde{t} \to \infty} \frac{\partial^2 (\mathscr{B} + \mathscr{D} + P)}{\partial z_{n+\tilde{c}\tilde{t}} \partial z_{m-\tilde{c}\tilde{t}}}, \lim_{\tilde{t} \to \infty} \frac{\partial^2 (\sum_{j \in \mathbb{Z}_1^2} \widetilde{\Omega}_j z_j \bar{z}_j + \mathscr{A} + \mathscr{C} + P)}{\partial z_{n+\tilde{c}\tilde{t}} \partial \bar{z}_{m+\tilde{c}\tilde{t}}}, \lim_{\tilde{t} \to \infty} \frac{\partial^2 (\bar{\mathscr{B}} + \bar{\mathscr{D}} + P)}{\partial \bar{z}_{n+\tilde{c}\tilde{t}} \partial \bar{z}_{m-\tilde{c}\tilde{t}}}$$

exist. Moreover, there exists $K>0$ such that, when $\tilde{t}>K, N+\mathscr{A}+\mathscr{B}+\bar{\mathscr{B}}+\mathscr{C}+\mathscr{D}+\bar{\mathscr{D}}+P$ satisfies

$$\left\|\frac{\partial^2(\mathscr{B}+\mathscr{D}+P)}{\partial z_{n+\tilde{c}\tilde{t}}\partial z_{m-\tilde{c}\tilde{t}}}-\lim_{\tilde{t}\to\infty}\frac{\partial^2(\mathscr{B}+\mathscr{D}+P)}{\partial z_{n+\tilde{c}\tilde{t}}\partial z_{m-\tilde{c}\tilde{t}}}\right\|_{D_a(r,s),\Pi}\leq\frac{\varepsilon}{\tilde{t}}e^{-|n+m|a},$$

$$\left\|\frac{\partial^2(\sum_{j\in\mathbb{Z}_1^2}\widetilde{\Omega}_j z_j\bar{z}_j+\mathscr{A}+\mathscr{C}+P)}{\partial z_{n+\tilde{c}\tilde{t}}\partial\bar{z}_{m+\tilde{c}\tilde{t}}}-\lim_{\tilde{t}\to\infty}\frac{\partial^2(\sum_{j\in\mathbb{Z}_1^2}\widetilde{\Omega}_j z_j\bar{z}_j+\mathscr{A}+\mathscr{C}+P)}{\partial z_{n+\tilde{c}\tilde{t}}\partial\bar{z}_{m+\tilde{c}\tilde{t}}}\right\|_{D_a(r,s),\Pi}\leq\frac{\varepsilon}{\tilde{t}}e^{-|n-m|a},$$

$$\left\|\frac{\partial^2(\bar{\mathscr{B}}+\bar{\mathscr{D}}+P)}{\partial\bar{z}_{n+\tilde{c}\tilde{t}}\partial\bar{z}_{m-\tilde{c}\tilde{t}}}-\lim_{\tilde{t}\to\infty}\frac{\partial^2(\bar{\mathscr{B}}+\bar{\mathscr{D}}+P)}{\partial\bar{z}_{n+\tilde{c}\tilde{t}}\partial\bar{z}_{m-\tilde{c}\tilde{t}}}\right\|_{D_a(r,s),\Pi}\leq\frac{\varepsilon}{\tilde{t}}e^{-|n+m|a}.$$

We can now state the basic KAM theorem, which is attributed to Zhang-Si [26].

Theorem 3.1. ([26] Theorem 2.1) *Suppose that the Hamiltonian $H=H_0+P=N+\mathscr{A}+\mathscr{B}+\overline{\mathscr{B}}+\mathscr{C}+\mathscr{D}+\overline{\mathscr{D}}+P$ satisfies* (A1) − (A6). *Let $\gamma'>0$ be small enough. Then there exists a positive constant $\varepsilon=\varepsilon(b,L,K,\tau,\gamma',r,s,a)$, so that if $\|X_P\|_{D_a(r,s),\Pi}<\varepsilon$, the following conclusions hold: there is a Cantor subset $\Pi_{\gamma'}\subset\Pi$ with* $\text{meas}(\Pi\setminus\Pi_{\gamma'})=O(\sqrt[8]{\gamma'})$ *and two changes (analytic in x and C^8_W in ξ) $\Psi:\mathbb{T}^b\times\Pi_{\gamma'}\to D_a(r,s)$, $\tilde{\omega}:\Pi_{\gamma'}\to\mathbb{R}^b$, where Ψ is $\frac{\varepsilon}{(\gamma')^8}$-close to the trivial embedding $\Psi_0:\mathbb{T}^b\times\Pi\to\mathbb{T}^b\times\{0,0,0\}$ and $\tilde{\omega}$ is ε-close to the unperturbed frequency $\widehat{\omega}$ so that, for any $\xi\in\Pi_{\gamma'}$ and $x\in\mathbb{T}^b$, the curve $t\to\Psi(x+\widehat{\omega}(\xi)t,\xi)$ is a quasi-periodic solution of the Hamiltonian equations governed by H.*

In order to use the above theorem to prove our problem, we need to deal with the parameter ω. For any given parameter $\omega^*\in S$, for $\omega\in\bar{\bar{\bar{S}}}:=\{\omega\in S\mid|\omega-\omega^*|\leq\varepsilon_*\}$, a new parameter $\bar{\omega}$ can be introduced by

$$\omega_{\hat{i}}=\omega^*_{\hat{i}}+\bar{\omega}^2_{\hat{i}},\quad\bar{\omega}=(\bar{\omega}_1,\cdots,\bar{\omega}_{b^*})^T\in[0,\sqrt{\varepsilon_*}]^{b^*}.\tag{3.1}$$

Then Hamiltonian (2.19) is changed into

$$H\;=\;<\widehat{\omega}(\xi),\hat{y}>+\sum_{n\in\mathbb{Z}_1^2}\widehat{\Omega}_n(\xi)z_n\bar{z}_n+\mathscr{A}+\mathscr{B}+\overline{\mathscr{B}}+\mathscr{C}+\mathscr{D}+\overline{\mathscr{D}}+P,\tag{3.2}$$

where $\widehat{\omega}(\xi)=(\varepsilon^{-4}\omega)\oplus\breve{\omega},\xi=\bar{\omega}\oplus\tilde{\xi},\hat{x}=\vartheta+\theta,\hat{y}=J\oplus I,\breve{\omega}_i(\xi)=\breve{\omega}_i(\tilde{\xi}),\widehat{\Omega}_n(\xi)=\widehat{\Omega}_n(\tilde{\xi})$ and $\breve{\omega}_i(\tilde{\xi}),\widehat{\Omega}_n(\tilde{\xi})$ are defined by (2.20), and $\mathscr{A},\mathscr{B},\overline{\mathscr{B}},\mathscr{C},\mathscr{D},\overline{\mathscr{D}}$ are the corresponding transformations of $\mathscr{A}^*,\mathscr{B}^*,\overline{\mathscr{B}}^*,\mathscr{C}^*,\mathscr{D}^*,\overline{\mathscr{D}}^*$ in the new coordinates. Denote $\Pi=[0,\sqrt{\varepsilon_*}]^{b^*}\times[\varepsilon^{\frac{3}{2}},2\varepsilon^{\frac{3}{2}}]^b$ and $\breve{\omega}(\xi)=\varepsilon^{-4}\tilde{\alpha}+A\tilde{\xi}$, where $\tilde{\alpha}=(\ldots,\lambda_i,\ldots)^T_{i\in\mathscr{R}}, A=(\ldots,A^*_i,\ldots)^T_{i\in\mathscr{R}}$. with

$$A^*_i=\frac{[\tilde{R}^6_{iiiiii}]}{16\pi^4}\tilde{\xi}^2_i+\sum_{\substack{j\in\mathscr{R}\\ j\neq i}}\left[\frac{3[\tilde{R}^6_{iiiijj}]}{16\pi^4}(\tilde{\xi}^2_j+2\tilde{\xi}_i\tilde{\xi}_j)+\sum_{\substack{d\in\mathscr{R}\\ d\neq i\\ d\neq j}}\frac{3[\tilde{R}^6_{iijjdd}]}{8\pi^4}\tilde{\xi}_j\tilde{\xi}_d\right].$$

4. Proof of the Main Theorem

In this section, we verify that Hamiltonian (3.2) satisfies the assumptions $(A1)-(A6)$.
$(A1)$: According to (3.1), for $\hat{i}=1,\cdots,b^*$, we have

$$\frac{\partial^2\omega_{\hat{i}}(\xi)}{\partial\bar{\omega}_{\hat{i}}\partial\bar{\omega}_{\hat{j}}}=\begin{cases}2, & \hat{j}=\hat{i},\\ 0, & 1\le\hat{j}\le b^*,\hat{j}\neq\hat{i},\end{cases}\tag{4.1}$$

and

$$\frac{\partial^2\omega_{\hat{i}}(\xi)}{\partial\bar{\omega}_{\hat{i}}\partial\tilde{\xi}_{i^*_{\hat{j}}}}=0,\quad 1\le\hat{j}\le b.\tag{4.2}$$

According to (2.20), for $\hat{l}=1,\cdots,b$, we have

$$\frac{\partial^2\breve{\omega}_{i^*_{\hat{l}}}(\xi)}{\partial\tilde{\xi}^2_{i^*_{\hat{l}}}}=\frac{1}{8\pi^4}[\tilde{R}^6_{i^*_{\hat{l}}i^*_{\hat{l}}i^*_{\hat{l}}i^*_{\hat{l}}i^*_{\hat{l}}i^*_{\hat{l}}}],\tag{4.3}$$

and

$$\frac{\partial^2\breve{\omega}_{i^*_{\hat{l}}}(\xi)}{\partial\tilde{\xi}_{i^*_{\hat{j}}}\partial\tilde{\xi}_{i^*_{\hat{l}}}}=\frac{\partial^2\breve{\omega}_{i^*_{\hat{l}}}(\xi)}{\partial\tilde{\xi}_{i^*_{\hat{l}}}\partial\tilde{\xi}_{i^*_{\hat{j}}}}=\frac{3}{8\pi^4}[\tilde{R}^6_{i^*_{\hat{l}}i^*_{\hat{l}}i^*_{\hat{l}}i^*_{\hat{l}}i^*_{\hat{j}}i^*_{\hat{j}}}],\quad 1\le\hat{j}\le b,\hat{j}\neq\hat{l}.\tag{4.4}$$

Denote

$$\mathscr{K}=\begin{pmatrix}
\varepsilon^{-4}\cdot\frac{\partial^2\omega_1(\xi)}{\partial\bar{\omega}_1^2} & \cdots & \varepsilon^{-4}\cdot\frac{\partial^2\omega_{b^*}(\xi)}{\partial\bar{\omega}_{b^*}\partial\bar{\omega}_1} & \frac{\partial^2\breve{\omega}_{i^*_1}(\xi)}{\partial\tilde{\xi}_{i^*_1}\partial\bar{\omega}_1} & \cdots & \frac{\partial^2\breve{\omega}_{i^*_b}(\xi)}{\partial\tilde{\xi}_{i^*_b}\partial\bar{\omega}_1}\\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\
\varepsilon^{-4}\cdot\frac{\partial^2\omega_1(\xi)}{\partial\bar{\omega}_1\partial\bar{\omega}_{b^*}} & \cdots & \varepsilon^{-4}\frac{\partial^2\omega_{b^*}(\xi)}{\partial\bar{\omega}_{b^*}^2} & \frac{\partial^2\breve{\omega}_{i^*_1}(\xi)}{\partial\tilde{\xi}_{i^*_1}\partial\bar{\omega}_{b^*}} & \cdots & \frac{\partial^2\breve{\omega}_{i^*_b}(\xi)}{\partial\tilde{\xi}_{i^*_b}\partial\bar{\omega}_{b^*}}\\
\varepsilon^{-4}\cdot\frac{\partial^2\omega_1(\xi)}{\partial\bar{\omega}_1\partial\tilde{\xi}_{i^*_1}} & \cdots & \varepsilon^{-4}\cdot\frac{\partial^2\omega_{b^*}(\xi)}{\partial\bar{\omega}_{b^*}\partial\tilde{\xi}_{i^*_1}} & \frac{\partial^2\breve{\omega}_{i^*_1}(\xi)}{\partial\tilde{\xi}^2_{i^*_1}} & \cdots & \frac{\partial^2\breve{\omega}_{i^*_b}(\xi)}{\partial\tilde{\xi}_{i^*_b}\partial\tilde{\xi}_{i^*_1}}\\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\
\varepsilon^{-4}\cdot\frac{\partial^2\omega_1(\xi)}{\partial\bar{\omega}_1\partial\tilde{\xi}_{i^*_b}} & \cdots & \varepsilon^{-4}\cdot\frac{\partial^2\omega_{b^*}(\xi)}{\partial\bar{\omega}_{b^*}\partial\tilde{\xi}_{i^*_b}} & \frac{\partial^2\breve{\omega}_{i^*_1}(\xi)}{\partial\tilde{\xi}_{i^*_1}\partial\tilde{\xi}_{i^*_b}} & \cdots & \frac{\partial^2\breve{\omega}_{i^*_b}(\xi)}{\partial\tilde{\xi}^2_{i^*_b}}
\end{pmatrix},$$

where $\xi\in\Pi$. Then $\mathscr{K}$ is the submatrix of matrix $\{\frac{\partial^2\widehat{\omega}}{\partial\xi^2}\}$. Now let us represent $\mathscr{K}$ as a partitioned matrix and denote

$$\mathscr{K}_{11}=\varepsilon^{-4}\cdot\begin{pmatrix}\frac{\partial^2\omega_1(\xi)}{\partial\bar{\omega}_1^2} & \cdots & \frac{\partial^2\omega_{b^*}(\xi)}{\partial\bar{\omega}_{b^*}}\\ \cdots & \cdots & \cdots\\ \frac{\partial^2\omega_1(\xi)}{\partial\bar{\omega}_1\partial\bar{\omega}_{b^*}} & \cdots & \frac{\partial^2\omega_{b^*}(\xi)}{\partial\bar{\omega}_{b^*}^2}\end{pmatrix},\quad \mathscr{K}_{12}=\begin{pmatrix}\frac{\partial^2\breve{\omega}_{i^*_1}(\xi)}{\partial\tilde{\xi}_{i^*_1}\partial\bar{\omega}_1} & \cdots & \frac{\partial^2\breve{\omega}_{i^*_b}(\xi)}{\partial\tilde{\xi}_{i^*_b}\partial\bar{\omega}_1}\\ \cdots & \cdots & \cdots\\ \frac{\partial^2\breve{\omega}_{i^*_1}(\xi)}{\partial\tilde{\xi}_{i^*_1}\partial\bar{\omega}_{b^*}} & \cdots & \frac{\partial^2\breve{\omega}_{i^*_b}(\xi)}{\partial\tilde{\xi}_{i^*_b}\partial\bar{\omega}_{b^*}}\end{pmatrix},$$

$$\mathscr{K}_{21}=\varepsilon^{-4}\cdot\begin{pmatrix}\frac{\partial^2\omega_1(\xi)}{\partial\bar{\omega}_1\partial\tilde{\xi}_{i^*_1}} & \cdots & \frac{\partial^2\omega_{b^*}(\xi)}{\partial\bar{\omega}_{b^*}\partial\tilde{\xi}_{i^*_1}}\\ \cdots & \cdots & \cdots\\ \frac{\partial^2\omega_1(\xi)}{\partial\bar{\omega}_1\partial\tilde{\xi}_{i^*_b}} & \cdots & \frac{\partial^2\omega_{b^*}(\xi)}{\partial\bar{\omega}_{b^*}\partial\tilde{\xi}_{i^*_b}}\end{pmatrix},\quad \mathscr{K}_{22}=\begin{pmatrix}\frac{\partial^2\breve{\omega}_{i^*_1}(\xi)}{\partial\tilde{\xi}^2_{i^*_1}} & \cdots & \frac{\partial^2\breve{\omega}_{i^*_b}(\xi)}{\partial\tilde{\xi}_{i^*_b}\partial\tilde{\xi}_{i^*_1}}\\ \cdots & \cdots & \cdots\\ \frac{\partial^2\breve{\omega}_{i^*_1}(\xi)}{\partial\tilde{\xi}_{i^*_1}\partial\tilde{\xi}_{i^*_b}} & \cdots & \frac{\partial^2\breve{\omega}_{i^*_b}(\xi)}{\partial\tilde{\xi}^2_{i^*_b}}\end{pmatrix}.$$

By (4.1), (4.2), (4.3), and (4.4), we have

$$\mathscr{K}_{21}=0,\qquad \lim_{\varepsilon\to 0}\mathscr{K}_{11}=2\varepsilon^{-4}\cdot \mathrm{I}_{b^*},$$
$$\lim_{\varepsilon\to 0}\mathscr{K}_{22}=\frac{[\phi]}{8\pi^4}\begin{pmatrix}1 & 3 & \cdots & 3\\ 3 & 1 & \cdots & 3\\ \cdots & \cdots & \cdots\cdots & \\ 3 & 3 & \cdots & 1\end{pmatrix}_{b\times b}:=[\phi]\widehat{\mathscr{K}_{22}},$$

where I_{b^*} denotes the unit $b^*\times b^*$-matrix. It is obvious that $\det(\widehat{\mathscr{K}_{22}})=\frac{1}{(8\pi^4)^b}[1+3(b-1)](-2)^{b-1}\neq 0$. So $\det(\mathscr{K})\neq 0$ when $0<\varepsilon\ll 1$, i.e., $\mathrm{rank}(\mathscr{K})=b^*+b$. Thus assumption (A1) is verified.

(A2) : Taking $\varsigma=4$, we have the proof immediately.

(A3) : From (3.2), $\mathscr{M}_n$ is defined as follows,

$$\mathscr{M}_n=\widehat{\Omega}_n\quad n\in\mathbb{Z}_1^2\setminus(\mathscr{L}_1\cup\mathscr{L}_2\cup\mathscr{L}_3\cup\mathscr{L}_4),$$

$$\mathscr{M}_n=\begin{pmatrix}\widehat{\Omega}_n+\breve{\omega}_i & \sum\limits_{d^*\in\mathscr{R}}\frac{3[\tilde{R}^6_{ijddnm}]}{4\pi^4}\sqrt{\xi_i\xi_j\xi_d}\\ \sum\limits_{d^*\in\mathscr{R}}\frac{3[\tilde{R}^6_{ijddmn}]}{4\pi^4}\sqrt{\xi_i\xi_j\xi_d} & \widehat{\Omega}_m+\breve{\omega}_j\end{pmatrix},n\in\mathscr{L}_1,$$

$$\mathscr{M}_n=\begin{pmatrix}\widehat{\Omega}_n-\breve{\omega}_i & -\sum\limits_{d^*\in\mathscr{R}}\frac{3[\tilde{R}^6_{nimjdd}]}{4\pi^4}\sqrt{\xi_i\xi_j\xi_d}\\ \sum\limits_{d^*\in\mathscr{R}}\frac{3[\tilde{R}^6_{injmdd}]}{4\pi^4}\sqrt{\xi_i\xi_j\xi_d} & -\widehat{\Omega}_m+\breve{\omega}_j\end{pmatrix},n\in\mathscr{L}_2,$$

$$\mathscr{M}_n=\begin{pmatrix}\widehat{\Omega}_n+\breve{\omega}_i+\breve{\omega}_d & \frac{3[\tilde{R}^6_{ijdlnm}]}{4\pi^4}\sqrt{\xi_i\xi_j\xi_d\xi_l}\\ \frac{3[\tilde{R}^6_{ijdlmn}]}{4\pi^4}\sqrt{\xi_i\xi_j\xi_d\xi_l} & \widehat{\Omega}_m+\breve{\omega}_j+\breve{\omega}_l\end{pmatrix},n\in\mathscr{L}_3,$$

$$\mathscr{M}_n=\begin{pmatrix}\widehat{\Omega}_n+\breve{\omega}_j & -\frac{3[\tilde{R}^6_{jindml}]}{4\pi^4}\sqrt{\xi_i\xi_j\xi_d\xi_l}\\ \frac{3[\tilde{R}^6_{ijdnlm}]}{4\pi^4}\sqrt{\xi_i\xi_j\xi_d\xi_l} & -\widehat{\Omega}_m+\breve{\omega}_i+\breve{\omega}_d+\breve{\omega}_l\end{pmatrix},n\in\mathscr{L}_4,$$

where $\mathscr{M}_n$ could be the transpose of itself, (n,m) are resonant pairs, (i,j) and (i,j,d,l) are uniquely determined by (n,m). We only prove (A3) for $\det[<k,\widehat{\omega}(\xi)>I\pm\mathscr{M}_n\otimes I\pm I\otimes\mathscr{M}_{n'}]$, which is complicated. For arbitrary $k\in\mathbb{Z}^{b^*+b}$, we define $k=(k_1,k_2),k_1\in\mathbb{Z}^{b^*},k_2\in\mathbb{Z}^b$, and denote

$$\begin{aligned}\mathscr{V}(\xi)&=<k,\widehat{\omega}(\xi)>I\pm\mathscr{M}_n\otimes I\pm I\otimes\mathscr{M}_{n'}\\ &=(\varepsilon^{-4}<k_1,\omega>+\varepsilon^{-4}<k_2,\tilde{\alpha}>+<k_2,A>)I\pm\mathscr{M}_n\otimes I\pm I\otimes\mathscr{M}_{n'}.\end{aligned}$$

By discussing the following two cases, we prove $|\mathscr{V}(\xi)|\geq\frac{\gamma}{|k|^\tau},(k\neq 0)$.

Case 1. Suppose that $k_1\neq 0$. In view of

$$\frac{\partial\big((\varepsilon^{-4}<k_2,\tilde{\alpha}>+<k_2,A>)I\pm\mathscr{M}_n\otimes I\pm I\otimes\mathscr{M}_{n'}\big)}{\partial\bar{\omega}}=O(\varepsilon^{-3+\rho})\cdot\frac{\partial\omega}{\partial\bar{\omega}},$$

and

$$\varepsilon^{-4}\cdot\frac{\partial<k_1,\omega>}{\partial\bar{\omega}}+O(\varepsilon^{-3+\rho})\cdot\frac{\partial\omega}{\partial\bar{\omega}}=\varepsilon^{-4}(k_1+O(\varepsilon^{1+\rho}))\cdot\frac{\partial\omega}{\partial\bar{\omega}}\neq 0,\quad 0<\varepsilon\ll 1$$

all the eigenvalues of $\mathscr{V}(\xi)$ are not identically zero.

Case 2. If $k_1=0$, then

$$\begin{aligned}\mathscr{V}(\xi) &= (\varepsilon^{-4}<k_1,\omega>+\varepsilon^{-4}<k_2,\tilde{\alpha}>+<k_2,A>)I\pm\mathscr{M}_n\otimes I\pm I\otimes\mathscr{M}_{n'}\\ &= (\varepsilon^{-4}<k_2,\tilde{\alpha}>+<k_2,A>)I\pm\mathscr{M}_n\otimes I\pm I\otimes\mathscr{M}_{n'},\end{aligned}$$

We assert that none of the eigenvalues of the matrix $\mathscr{V}(\xi)$ are zero. We only prove the case that $n,n'\in\mathscr{L}_4$. Denote

$$\mathscr{M}_n=\varepsilon^{-4}\mathscr{M}_{n,1}+\mathscr{M}_{n,2},\quad\forall n\in\mathscr{L}_2,$$

where

$$\mathscr{M}_{n,1}=\begin{pmatrix}\lambda_n+\lambda_j & 0\\ 0 & -\lambda_m+\lambda_i+\lambda_d+\lambda_l\end{pmatrix},\quad\mathscr{M}_{n,2}=\begin{pmatrix}\mathscr{M}_{n,2}^{11} & \mathscr{M}_{n,2}^{12}\\ \mathscr{M}_{n,2}^{21} & \mathscr{M}_{n,2}^{22}\end{pmatrix}$$

with

$$\begin{aligned}\mathscr{M}_{n,2}^{11}=&\sum_{i^*\in\mathscr{R}}\left[\frac{3[\tilde{R}^6_{i^*i^*i^*i^*nn}]}{16\pi^4}\tilde{\xi}^2_{i^*}+\sum_{\substack{j^*\in\mathscr{R}\\ j^*\neq i^*}}\frac{3[\tilde{R}^6_{i^*i^*j^*j^*nn}]}{8\pi^4}\tilde{\xi}_{i^*}\tilde{\xi}_{j^*}\right]+\frac{[\tilde{R}^6_{jjjjjj}]}{16\pi^4}\tilde{\xi}^2_j\\ &+\sum_{\substack{j^*\in\mathscr{R}\\ j^*\neq j}}\left[\frac{3[\tilde{R}^6_{jjjjj^*j^*}]}{16\pi^4}(\tilde{\xi}^2_{j^*}+2\tilde{\xi}_j\tilde{\xi}_{j^*})+\sum_{\substack{d^*\in\mathscr{R}\\ d^*\neq j^*\\ d^*\neq j}}\frac{3[\tilde{R}^6_{jjj^*j^*d^*d^*}]}{8\pi^4}\tilde{\xi}_{j^*}\tilde{\xi}_{d^*}\right]\end{aligned}$$

$$\mathscr{M}_{n,2}^{12}=-\frac{3}{4\pi^4}[\tilde{R}^6_{jindml}]\sqrt{\tilde{\xi}_i\tilde{\xi}_j\tilde{\xi}_d\tilde{\xi}_l},\qquad\mathscr{M}_{n,2}^{21}=\frac{3}{4\pi^4}[\tilde{R}^6_{ijdnlm}]\sqrt{\tilde{\xi}_i\tilde{\xi}_j\tilde{\xi}_d\tilde{\xi}_l},$$

$$\mathscr{M}_{n,2}^{22} = -\sum_{i^* \in \mathscr{R}} \left[\frac{3[\tilde{R}^6_{i^*i^*i^*i^*mm}]}{16\pi^4} \xi_{i^*}^2 + \sum_{\substack{j^* \in \mathscr{R} \\ j^* \neq i^*}} \frac{3[\tilde{R}^6_{i^*i^*j^*j^*mm}]}{8\pi^4} \xi_{i^*} \xi_{j^*} \right]$$

$$+ \frac{[\tilde{R}^6_{dddddd}]}{16\pi^4} \xi_d^2 + \frac{[\tilde{R}^6_{iiiiii}]}{16\pi^4} \xi_i^2 + \frac{[\tilde{R}^6_{llllll}]}{16\pi^4} \xi_l^2$$

$$+ \sum_{\substack{j^* \in \mathscr{R} \\ j^* \neq i}} \left[\frac{3[\tilde{R}^6_{iiiij^*j^*}]}{16\pi^4} (\xi_{j^*}^2 + 2\xi_i \xi_{j^*}) + \sum_{\substack{d^* \in \mathscr{R} \\ d^* \neq j^* \\ d^* \neq i}} \frac{3[\tilde{R}^6_{iij^*j^*d^*d^*}]}{8\pi^4} \xi_{j^*} \xi_{d^*} \right]$$

$$+ \sum_{\substack{j^* \in \mathscr{R} \\ j^* \neq d}} \left[\frac{3[\tilde{R}^6_{ddddj^*j^*}]}{16\pi^4} (\xi_{j^*}^2 + 2\xi_d \xi_{j^*}) + \sum_{\substack{d^* \in \mathscr{R} \\ d^* \neq j^* \\ d^* \neq d}} \frac{3[\tilde{R}^6_{ddj^*j^*d^*d^*}]}{8\pi^4} \xi_{j^*} \xi_{d^*} \right]$$

$$+ \sum_{\substack{j^* \in \mathscr{R} \\ j^* \neq l}} \left[\frac{3[\tilde{R}^6_{llllj^*j^*}]}{16\pi^4} (\xi_{j^*}^2 + 2\xi_l \xi_{j^*}) + \sum_{\substack{d^* \in \mathscr{R} \\ d^* \neq j^* \\ d^* \neq l}} \frac{3[\tilde{R}^6_{llj^*j^*d^*d^*}]}{8\pi^4} \xi_{j^*} \xi_{d^*} \right].$$

So, it turns out $\mathscr{V}(\xi) = \varepsilon^{-4}(< k_2, \tilde{\alpha} > I \pm \mathscr{M}_{n,1} \otimes I \pm I \otimes \mathscr{M}_{n',1}) + (< k_2, A > I \pm \mathscr{M}_{n,2} \otimes I \pm I \otimes \mathscr{M}_{n',2})$. According to $|n|^2 + |j|^2 = |i|^2 + |d|^2 + |l|^2 - |m|^2$ and (2.3),(2.10), we have

$$\lim_{\varepsilon \to 0} \mathscr{M}_{n,1} = \begin{pmatrix} |n|^2 + |j|^2 & 0 \\ 0 & |i|^2 + |d|^2 + |l|^2 - |m|^2 \end{pmatrix} := \widehat{\mathscr{M}_{n,1}},$$

$$\lim_{\varepsilon \to 0} \mathscr{M}_{n,2} = [\phi] \cdot \begin{pmatrix} \widehat{\mathscr{M}}_{n,2}^{11} & \widehat{\mathscr{M}}_{n,2}^{12} \\ \widehat{\mathscr{M}}_{n,2}^{21} & \widehat{\mathscr{M}}_{n,2}^{22} \end{pmatrix} := \widetilde{\mathscr{M}_{n,2}} := [\phi] \widehat{\mathscr{M}_{n,2}},$$

where

$$\widehat{\mathscr{M}}_{n,2}^{11} = \frac{1}{4\pi^4} \xi_j^2 - \frac{3}{8\pi^4} \sum_{d^* \in \mathscr{R}} \xi_j \xi_{d^*} + < \xi, 2\widehat{B}\xi >,$$

$$\widehat{\mathscr{M}}_{n,2}^{12} = -\frac{3}{4\pi^4} \sqrt{\xi_i \xi_j \xi_d \xi_l}, \quad \widehat{\mathscr{M}}_{n,2}^{21} = \frac{3}{4\pi^4} \sqrt{\xi_i \xi_j \xi_d \xi_l},$$

$$\widehat{\mathscr{M}}_{n,2}^{22} = \frac{1}{4\pi^4} (\xi_i^2 + \xi_d^2 + \xi_l^2) - \frac{3}{8\pi^4} \sum_{d^* \in \mathscr{R}} (\xi_i + \xi_d + \xi_l) \xi_{d^*} + < \xi, 2\widehat{B}\xi >$$

with $\tilde{\xi} = (\tilde{\xi}_{i_1^*}, \tilde{\xi}_{i_2^*}, \cdots, \tilde{\xi}_{i_b^*})$ and

$$\widehat{B} = \frac{3}{16\pi^4} \begin{pmatrix} 1 & 2 & \cdots & 2 \\ 2 & 1 & \cdots & 2 \\ \cdots & \cdots & \cdots & \cdots \\ 2 & 2 & \cdots & 1 \end{pmatrix}_{b\times b}. \tag{4.5}$$

Thus,

$$\begin{aligned}
\lim_{\varepsilon\to 0} \mathscr{V}(\xi) &= \varepsilon^{-4}(< k_2, \widehat{\alpha} > I \pm \widehat{\mathscr{M}_{n,1}} \otimes I \pm I \otimes \widehat{\mathscr{M}_{n',1}}) \\
&\quad + [\phi] \cdot (< k_2, \widehat{A} > I \pm \widehat{\mathscr{M}_{n,2}} \otimes I \pm I \otimes \widehat{\mathscr{M}_{n',2}}) \\
&= \varepsilon^{-4} \left(< k_2, \hat{\alpha} > \pm(|n|^2 + |j|^2) \pm (|i|^2 + |d|^2 + |l|^2 - |m|^2)\right) \\
&\quad + [\phi] \left(< k_2, \widehat{A} > + < \tilde{\xi}, (\pm 2\widehat{B} \pm 2\widehat{B})\tilde{\xi} >\right) I \\
&\quad \pm [\phi] \cdot \begin{pmatrix} \frac{\tilde{\xi}_j^2}{4\pi^4} - \frac{3\sum_{d^*\in\mathscr{R}} \tilde{\xi}_j \tilde{\xi}_{d^*}}{8\pi^4} & -\frac{3}{4\pi^4}\sqrt{\tilde{\xi}_i\tilde{\xi}_j\tilde{\xi}_d\tilde{\xi}_l} \\ \frac{3}{4\pi^4}\sqrt{\tilde{\xi}_i\tilde{\xi}_j\tilde{\xi}_d\tilde{\xi}_l} & \frac{(\tilde{\xi}_i^2 + \tilde{\xi}_d^2 + \tilde{\xi}_l^2)}{4\pi^4} - \frac{3\sum_{d^*\in\mathscr{R}} (\tilde{\xi}_i + \tilde{\xi}_d + \tilde{\xi}_l)\tilde{\xi}_{d^*}}{8\pi^4} \end{pmatrix} \otimes I \\
&\quad \pm [\phi] \cdot I \otimes \begin{pmatrix} \frac{\tilde{\xi}_{j'}^2}{4\pi^4} - \frac{3\sum_{d^*\in\mathscr{R}} \tilde{\xi}_{j'} \tilde{\xi}_{d^*}}{8\pi^4} & -\frac{3}{4\pi^4}\sqrt{\tilde{\xi}_{i'}\tilde{\xi}_{j'}\tilde{\xi}_{d'}\tilde{\xi}_{l'}} \\ \frac{3}{4\pi^4}\sqrt{\tilde{\xi}_{i'}\tilde{\xi}_{j'}\tilde{\xi}_{d'}\tilde{\xi}_{l'}} & \frac{(\tilde{\xi}_{i'}^2 + \tilde{\xi}_{d'}^2 + \tilde{\xi}_{l'}^2)}{4\pi^4} - \frac{3\sum_{d^*\in\mathscr{R}} (\tilde{\xi}_{i'} + \tilde{\xi}_{d'} + \tilde{\xi}_{l'})\tilde{\xi}_{d^*}}{8\pi^4} \end{pmatrix} := \widehat{\mathscr{V}}(\xi)
\end{aligned}$$

where $\hat{\alpha} = (|i_1^*|^2, |i_2^*|^2, \ldots, |i_n^*|^2)$ and $\widehat{A} = \lim_{\varepsilon\to 0} A = \left(\ldots, \widehat{A}_i^*, \ldots\right)^T_{i^*\in\mathscr{R}}$, with

$$\widehat{A}_i^* = \frac{[\phi]}{16\pi^4}\tilde{\xi}_{i^*}^2 + \sum_{\substack{j^*\in\mathscr{R} \\ j^*\neq i^*}} \left[\frac{3[\phi]}{16\pi^4}(\tilde{\xi}_{j^*}^2 + 2\tilde{\xi}_{i^*}\tilde{\xi}_{j^*}) + \sum_{\substack{d^*\in\mathscr{R} \\ d^*\neq i^* \\ d^*\neq j^*}} \frac{3[\phi]}{8\pi^4}\tilde{\xi}_{j^*}\tilde{\xi}_{d^*} \right].$$

The eigenvalues of $\widehat{\mathscr{V}}(\xi)$ are

$$\begin{aligned}
&\varepsilon^{-4} < k_2, \hat{\alpha} > \pm \varepsilon^{-4}(|n|^2 + |j|^2 - |i|^2) \pm \varepsilon^{-4}(|n'|^2 + |j'|^2 - |i'|^2) \\
&+ [\phi] \left(< k_2, \widehat{A} > + < \tilde{\xi}, (\pm 2\widehat{B} \pm 2\widehat{B})\tilde{\xi} >\right) \\
&\pm \frac{[\phi]}{16\pi^4} \Bigg\{ 2(\tilde{\xi}_i^2 + \tilde{\xi}_j^2 + \tilde{\xi}_d^2 + \tilde{\xi}_l^2) - 3(\sum_{d^*\in\mathscr{R}} \tilde{\xi}_{d^*})(\tilde{\xi}_i + \tilde{\xi}_j + \tilde{\xi}_d + \tilde{\xi}_l) \\
&\qquad \pm \sqrt{f_{\mathscr{L}_4,\mathscr{L}_4}(\tilde{\xi}_i, \tilde{\xi}_j, \tilde{\xi}_d, \tilde{\xi}_l)} \Bigg\}
\end{aligned}$$

$$\pm\frac{[\phi]}{16\pi^4}\Bigg\{2(\tilde{\xi}_{i'}^2+\tilde{\xi}_{j'}^2+\tilde{\xi}_{d'}^2+\tilde{\xi}_{l'}^2)-3(\sum_{d^*\in\mathscr{R}}\tilde{\xi}_{d^*})(\tilde{\xi}_{i'}+\tilde{\xi}_{j'}+\tilde{\xi}_{d'}+\tilde{\xi}_{l'})$$
$$\pm\sqrt{f_{\mathscr{L}_2,\mathscr{L}_2}(\tilde{\xi}_{i'},\tilde{\xi}_{j'},\tilde{\xi}_{d'},\tilde{\xi}_{l'})}\Bigg\},$$

where

$$f_{\mathscr{L}_2,\mathscr{L}_2}(\tilde{\xi}_i,\tilde{\xi}_j,\tilde{\xi}_d,\tilde{\xi}_l) = \left[2(\tilde{\xi}_i^2-\tilde{\xi}_j^2+\tilde{\xi}_d^2+\tilde{\xi}_l^2)-3(\sum_{d^*\in\mathscr{R}}\tilde{\xi}_{d^*})(\tilde{\xi}_i-\tilde{\xi}_j+\tilde{\xi}_d+\tilde{\xi}_l)\right]^2 -144\tilde{\xi}_i\tilde{\xi}_j\tilde{\xi}_d\tilde{\xi}_l.$$

Now, we have to show that none of the eigenvalues are equal to zero. The result was proved in [26]. Thus, we gave that none of the eigenvalues of $\mathscr{V}(\xi)$ are equal to zero as $0<\varepsilon\ll 1$. Moreover, when $n\in\mathscr{L}_1, n'\in\mathscr{L}_1$ or $n\in\mathscr{L}_1, n'\in\mathscr{L}_2$ or $n\in\mathscr{L}_1, n'\in\mathscr{L}_3$ and so on, the situations are similar. So, we omit them here. Thus we have none of the eigenvalues of $\mathscr{V}(\xi)$ are equal to zero for $k\neq 0$. By Lemma 3.1 in [10], we have that $\det(\mathscr{V}(\xi))$ is a polynomial function in the components of ξ with order at most eight. Thus $\left|\partial_\xi^8(\det(\mathscr{V}(\xi)))\right|\geq\frac{1}{2}|k|\neq 0$. By excluding some parameter set with measure $O(\sqrt[8]{\gamma'})$, we have $|\det(\mathscr{V}(\xi))|\geq\frac{\gamma'}{|k|^\tau},\quad k\neq 0$. (A3) is verified.

(A4) : Similar to [25], according to Lemma 2.1, it is easy to know that (A4) holds.

(A5) : Similar to [25], we have the proof immediately.

(A6) : We just need to verify that the perturbation term P satisfies (A6). From (2.21), we have that, for $\tilde{t}$ sufficient large and $\forall\tilde{c}\in\mathbb{Z}^2\setminus\{0\}$, all terms containing $z_{n+\tilde{c}\tilde{t}}\bar{z}_{m+\tilde{c}\tilde{t}}$ in the function F are

$$\begin{aligned}
\mathscr{F}_1 &= \frac{3\varepsilon}{4\pi^4}\sum_{n+\tilde{c}\tilde{t}\in\mathscr{L}_1}\sum_{d\in\mathscr{R}}\sum_{k\neq 0}\frac{\tilde{R}^6_{ijdd,n+\tilde{c}\tilde{t},m+\tilde{c}\tilde{t},k}e^{\mathrm{i}<k,\vartheta>}q_i\bar{q}_j|q_d|^2 z_{n+\tilde{c}\tilde{t}}\bar{z}_{m+\tilde{c}\tilde{t}}}{\mathrm{i}(<k,\omega>+\lambda_i-\lambda_j+\lambda_{n+\tilde{c}\tilde{t}}-\lambda_{m+\tilde{c}\tilde{t}})}\\
&= \frac{3\varepsilon}{4\pi^4}\sum_{n+\tilde{c}\tilde{t}\in\mathscr{L}_1}\sum_{d\in\mathscr{R}}\sum_{k\neq 0}\frac{\tilde{R}^6_{ijdd,n+\tilde{c}\tilde{t},m+\tilde{c}\tilde{t},k}e^{\mathrm{i}<k,\vartheta>}q_i\bar{q}_j|q_d|^2 z_{n+\tilde{c}\tilde{t}}\bar{z}_{m+\tilde{c}\tilde{t}}}{\mathrm{i}\left(<k,\omega>+\varepsilon^{1+\rho}(\lambda_i^*-\lambda_j^*+\lambda_{n+\tilde{c}\tilde{t}}^*-\lambda_{m+\tilde{c}\tilde{t}}^*)\right)},\\
\mathscr{F}_2 &= \frac{3\varepsilon}{4\pi^4}\sum_{n+\tilde{c}\tilde{t}\in\mathscr{L}_3}\sum_{k\neq 0}\frac{\tilde{R}^6_{ijdl,n+\tilde{c}\tilde{t},m+\tilde{c}\tilde{t},k}e^{\mathrm{i}<k,\vartheta>}q_i\bar{q}_j q_d\bar{q}_l z_{n+\tilde{c}\tilde{t}}\bar{z}_{m+\tilde{c}\tilde{t}}}{\mathrm{i}(<k,\omega>+\lambda_i-\lambda_j+\lambda_d-\lambda_l+\lambda_{n+\tilde{c}\tilde{t}}-\lambda_{m+\tilde{c}\tilde{t}})}\\
&= \frac{3\varepsilon}{4\pi^4}\sum_{n+\tilde{c}\tilde{t}\in\mathscr{L}_3}\sum_{k\neq 0}\frac{\tilde{R}^6_{ijdl,n+\tilde{c}\tilde{t},m+\tilde{c}\tilde{t},k}e^{\mathrm{i}<k,\vartheta>}q_i\bar{q}_j q_d\bar{q}_l z_{n+\tilde{c}\tilde{t}}\bar{z}_{m+\tilde{c}\tilde{t}}}{\mathrm{i}\left(<k,\omega>+\varepsilon^{1+\rho}(\lambda_i^*-\lambda_j^*+\lambda_d^*-\lambda_l^*+\lambda_{n+\tilde{c}\tilde{t}}^*-\lambda_{m+\tilde{c}\tilde{t}}^*)\right)}
\end{aligned}$$

and

$$\mathscr{F}_3 = \frac{\varepsilon}{48\pi^4}\cdot \sum_{\substack{i-j+d-l+n-m=0\\ |i|^2-|j|^2+|d|^2-|l|^2+|n+\tilde{c}\tilde{t}|^2-|m+\tilde{c}\tilde{t}|^2\neq 0\\ \#(\mathscr{R}\cap\{i,j,d,l,n+\tilde{c}\tilde{t},m+\tilde{c}\tilde{t}\})\geq 4}} \sum_{k\in\mathbb{Z}^{b^*}} \left(\frac{\tilde{R}^6_{ijdl,n+\tilde{c}\tilde{t},m+\tilde{c}\tilde{t},k} e^{\mathrm{i}<k,\vartheta>} q_i\bar{q}_j q_d\bar{q}_l z_{n+\tilde{c}\tilde{t}}\bar{z}_{m+\tilde{c}\tilde{t}}}{\mathrm{i}(<k,\omega>+\lambda_i-\lambda_j+\lambda_d-\lambda_l+\lambda_{n+\tilde{c}\tilde{t}}-\lambda_{m+\tilde{c}\tilde{t}})}\right)$$

$$= \frac{\varepsilon}{48\pi^4}\cdot \sum_{i,j,d,l,n,m,k} \frac{\tilde{R}^6_{ijdl,n+\tilde{c}\tilde{t},m+\tilde{c}\tilde{t},k} e^{\mathrm{i}<k,\vartheta>} q_i\bar{q}_j q_d\bar{q}_l z_{n+\tilde{c}\tilde{t}}\bar{z}_{m+\tilde{c}\tilde{t}}}{\mathscr{F}_3^*}$$

where

$$\mathscr{F}_3^* = \mathrm{i}(<k,\omega>+|i|^2-|j|^2+|d|^2-|l|^2+|n|^2-|m|^2 \\ +2\tilde{t}<n-m,\tilde{c}>+\varepsilon^{(1+\rho)}(\lambda_i^*-\lambda_j^*+\lambda_d^*-\lambda_l^*+\lambda^*_{n+\tilde{c}\tilde{t}}-\lambda^*_{m+\tilde{c}\tilde{t}})).$$

If $<n-m,\tilde{c}>\neq 0$, then

$$\left\|\frac{\partial^2\mathscr{F}_3}{\partial z_{n+\tilde{c}\tilde{t}}\partial\bar{z}_{m+\tilde{c}\tilde{t}}}-0\right\|_{D_a(r,s),\Pi} \leq \frac{\varepsilon}{\tilde{t}}e^{-|n-m|a};$$

If $<n-m,\tilde{c}>=0$, we show that $\mathscr{F}_3$ still satisfies (A6). From Cauchy's estimation and (2.7),(2.11)–(2.16), we know that, for any fixed $i,j,d,l,n,m\in\mathbb{Z}^2$ and $\tilde{c}\in\mathbb{Z}^2\setminus\{0\}$, the limits $\lim_{\tilde{t}\to\infty}\partial^{\hat{s}}_\omega \tilde{R}^6_{ijdl,n+\tilde{c}\tilde{t},m+\tilde{c}\tilde{t},k}$, and $\lim_{\tilde{t}\to\infty}\partial^{\hat{s}}_\omega(\lambda^*_{n+\tilde{c}\tilde{t}}-\lambda^*_{m+\tilde{c}\tilde{t}})$ exist for $\hat{s}=0,1,\cdots,8$ and

$$\|\tilde{R}^6_{ijdl,n+\tilde{c}\tilde{t},m+\tilde{c}\tilde{t},k}-\lim_{\tilde{t}\to\infty}\tilde{R}^6_{ijdl,n+\tilde{c}\tilde{t},m+\tilde{c}\tilde{t},k}\|^*_S \leq \frac{1}{\tilde{t}}e^{-|k|\sigma_0/2}, \tag{4.6}$$

$$\|\lambda^*_{n+\tilde{c}\tilde{t}}-\lambda^*_{m+\tilde{c}\tilde{t}}-\lim_{\tilde{t}\to\infty}(\lambda^*_{n+\tilde{c}\tilde{t}}-\lambda^*_{n+\tilde{c}\tilde{t}})\|^*_S \leq \frac{1}{\tilde{t}}. \tag{4.7}$$

Thus the limits $\lim_{\tilde{t}\to\infty}\frac{\partial^2\mathscr{F}_3}{\partial z_{d+\tilde{c}\tilde{t}}\partial\bar{z}_{l+\tilde{c}\tilde{t}}}$ exists and

$$\lim_{\tilde{t}\to\infty}\frac{\partial^2\mathscr{F}_3}{\partial z_{n+\tilde{c}\tilde{t}}\partial\bar{z}_{m+\tilde{c}\tilde{t}}} = \frac{\varepsilon}{48\pi^4}\sum_{k\neq 0}\frac{\lim_{\tilde{t}\to\infty}\tilde{R}^6_{ijdl,n+\tilde{c}\tilde{t},m+\tilde{c}\tilde{t},k}e^{\mathrm{i}<k,\vartheta>}q_i\bar{q}_jq_d\bar{q}_l}{\mathrm{i}\Lambda_{i,j,d,l}} \\ := \frac{\varepsilon}{48\pi^4}\sum_{k\neq 0}\lim_{\tilde{t}\to\infty}\breve{R}^6_{ijdl,n+\tilde{c}\tilde{t},m+\tilde{c}\tilde{t},k}e^{\mathrm{i}<k,\vartheta>}q_i\bar{q}_jq_d\bar{q}_l,$$

where

$$\Lambda_{i,j,d,l} = |i|^2-|j|^2+|d|^2-|l|^2+|n|^2-|m|^2+ \\ \varepsilon^{1+\rho}(\lambda_i^*-\lambda_j^*+\lambda_d^*-\lambda_l^*+\lim_{\tilde{t}\to\infty}(\lambda^*_{n+\tilde{c}\tilde{t}}-\lambda^*_{m+\tilde{c}\tilde{t}}))+<k,\omega>.$$

According to

$$\mathrm{i}\big[|i|^2-|j|^2+|d|^2-|l|^2+|n|^2-|m|^2+\varepsilon^{1+\rho}(\lambda_i^*-\lambda_j^*+\lambda_d^*-\lambda_l^* \\ +\lambda^*_{n+\tilde{c}\tilde{t}}-\lambda^*_{m+\tilde{c}\tilde{t}})+<k,\omega>\big]\cdot\left[\breve{R}^6_{ijdl,n+\tilde{c}\tilde{t},m+\tilde{c}\tilde{t},k}-\lim_{\tilde{t}\to\infty}\breve{R}^6_{ijdl,n+\tilde{c}\tilde{t},m+\tilde{c}\tilde{t},k}\right]$$

$$= \tilde{R}^6_{ijdl,n+\tilde{c}\tilde{t},m+\tilde{c}\tilde{t},k}-\lim_{\tilde{t}\to\infty}\tilde{R}^6_{ijdl,n+\tilde{c}\tilde{t},m+\tilde{c}\tilde{t},k}-\frac{\varepsilon^{1+\rho}\left[\lambda^*_{n+\tilde{c}\tilde{t}}-\lambda^*_{m+\tilde{c}\tilde{t}}-\lim_{\tilde{t}\to\infty}(\lambda^*_{n+\tilde{c}\tilde{t}}-\lambda^*_{m+\tilde{c}\tilde{t}})\right]\cdot\lim_{\tilde{t}\to\infty}\tilde{R}^6_{ijdl,n+\tilde{c}\tilde{t},m+\tilde{c}\tilde{t},k}}{\Lambda_{i,j,d,l}}$$

$$= \tilde{R}^6_{ijdl,n+\tilde{c}\tilde{t},m+\tilde{c}\tilde{t},k} - \lim_{\tilde{t}\to\infty} \breve{R}^6_{ijdl,n+\tilde{c}\tilde{t},m+\tilde{c}\tilde{t},k}$$

$$-\mathrm{i}\varepsilon^{1+\rho}\left[\lambda^*_{n+\tilde{c}\tilde{t}} - \lambda^*_{m+\tilde{c}\tilde{t}} - \lim_{\tilde{t}\to\infty}(\lambda^*_{n+\tilde{c}\tilde{t}} - \lambda^*_{m+\tilde{c}\tilde{t}})\right]\cdot \lim_{\tilde{t}\to\infty} \breve{R}^6_{ijdl,n+\tilde{c}\tilde{t},m+\tilde{c}\tilde{t},k}$$

and

$$\left\|\mathrm{i}\left[\lambda^*_{n+\tilde{c}\tilde{t}} - \lambda^*_{m+\tilde{c}\tilde{t}} - \lim_{\tilde{t}\to\infty}(\lambda^*_{n+\tilde{c}\tilde{t}} - \lambda^*_{m+\tilde{c}\tilde{t}})\right]\cdot \lim_{\tilde{t}\to\infty} \breve{R}^6_{ijdl,n+\tilde{c}\tilde{t},m+\tilde{c}\tilde{t},k}\right\|^*_S$$
$$\le \tfrac{1}{\tilde{t}}\cdot\left(\frac{C_*|k|^{b^*+2}}{\rho}\right)^8 \cdot C_1 e^{-|k|\sigma_0/2},$$

we have $\left\|\widetilde{G}_{ij,d+\tilde{c}\tilde{t},l+\tilde{c}\tilde{t},k} - \lim_{\tilde{t}\to\infty}\widetilde{G}_{ij,d+\tilde{c}\tilde{t},l+\tilde{c}\tilde{t},k}\right\|^*_S \le \frac{1}{\tilde{t}}\cdot\left(\frac{C_*|k|^{b^*+2}}{\rho}\right)^8\cdot C_2 e^{-|k|\sigma_0/2}$. Thus, for any $(\theta,\omega)\in\Theta(\sigma_0/3)\times S$ and by $i-j+d-l+n-m=0$,

$$\left\|\frac{\partial^2\mathscr{F}_3}{\partial z_{n+\tilde{c}\tilde{t}}\partial\bar{z}_{m+\tilde{c}\tilde{t}}} - \lim_{\tilde{t}\to\infty}\frac{\partial^2\mathscr{F}_3}{\partial z_{n+\tilde{c}\tilde{t}}\partial\bar{z}_{m+\tilde{c}\tilde{t}}}\right\|_{D_a(r,s),\Pi}$$

$$\le \sum_{k\neq 0}\frac{\varepsilon}{\tilde{t}}\cdot\left(\frac{C_*|k|^{b^*+2}}{\rho}\right)^{16}\cdot C_3 e^{-|k|\sigma_0/6}\cdot\|q\|_a^4\cdot e^{-|i-j+d-l|a} \le \frac{\varepsilon}{\tilde{t}}e^{-|n-m|a}.$$

Similarly, we have

$$\left\|\frac{\partial^2\mathscr{F}_1}{\partial z_{n+\tilde{c}\tilde{t}}\partial\bar{z}_{m+\tilde{c}\tilde{t}}} - \lim_{\tilde{t}\to\infty}\frac{\partial^2\mathscr{F}_1}{\partial z_{n+\tilde{c}\tilde{t}}\partial\bar{z}_{m+\tilde{c}\tilde{t}}}\right\|_{D_a(r,s),\Pi} \le \frac{\varepsilon}{\tilde{t}}e^{-|n-m|a}$$

and

$$\left\|\frac{\partial^2\mathscr{F}_2}{\partial z_{n+\tilde{c}\tilde{t}}\partial\bar{z}_{m+\tilde{c}\tilde{t}}} - \lim_{\tilde{t}\to\infty}\frac{\partial^2\mathscr{F}_2}{\partial z_{n+\tilde{c}\tilde{t}}\partial\bar{z}_{m+\tilde{c}\tilde{t}}}\right\|_{D_a(r,s),\Pi} \le \frac{\varepsilon}{\tilde{t}}e^{-|n-m|a}.$$

For $\hat{s}=0,1,2,3,4$, the limit $\lim_{\tilde{t}\to\infty}\partial^{\hat{s}}_\omega\frac{\partial^2 F}{\partial z_{n+\tilde{c}\tilde{t}}\partial\bar{z}_{m+\tilde{c}\tilde{t}}}$ exist, and

$$\left\|\frac{\partial^2 F}{\partial z_{n+\tilde{c}\tilde{t}}\partial\bar{z}_{m+\tilde{c}\tilde{t}}} - \lim_{\tilde{t}\to\infty}\frac{\partial^2 F}{\partial z_{n+\tilde{c}\tilde{t}}\partial\bar{z}_{m+\tilde{c}\tilde{t}}}\right\|_{D_a(r,s),\Pi} \le \frac{\varepsilon}{\tilde{t}}e^{-|n-m|a}.$$

Similarly,

$$\left\|\frac{\partial^2 F}{\partial z_{n+\tilde{c}\tilde{t}}\partial z_{m-\tilde{c}\tilde{t}}} - \lim_{\tilde{t}\to\infty}\frac{\partial^2 F}{\partial z_{n+\tilde{c}\tilde{t}}\partial z_{m-\tilde{c}\tilde{t}}}\right\|_{D_a(r,s),\Pi} \le \frac{\varepsilon}{\tilde{t}}e^{-|n+m|a},$$

and

$$\left\|\frac{\partial^2 F}{\partial \bar{z}_{n+\tilde{c}\tilde{t}}\partial\bar{z}_{m-\tilde{c}\tilde{t}}} - \lim_{\tilde{t}\to\infty}\frac{\partial^2 F}{\partial \bar{z}_{n+\tilde{c}\tilde{t}}\partial\bar{z}_{m-\tilde{c}\tilde{t}}}\right\|_{D_a(r,s),\Pi} \le \frac{\varepsilon}{\tilde{t}}e^{-|n+m|a}.$$

According to (2.9), we have

$$\varepsilon\tilde{R}^6 = \frac{\varepsilon}{48\pi^4}\sum_{i-j+d-l+n-m=0}\tilde{R}^6_{ijdlnm}(\vartheta;\omega)q_i\bar{q}_jq_d\bar{q}_lq_n\bar{q}_m.$$

Therefore, similar to the above discussion, we see that

$$\left\|\frac{\partial^2\varepsilon\tilde{R}^6}{\partial z_{n+\tilde{c}\tilde{t}}\partial\bar{z}_{m+\tilde{c}\tilde{t}}} - \lim_{\tilde{t}\to\infty}\frac{\partial^2\varepsilon\tilde{R}^6}{\partial z_{n+\tilde{c}\tilde{t}}\partial\bar{z}_{m+\tilde{c}\tilde{t}}}\right\|_{D_a(r,s),\Pi} \le \frac{\varepsilon}{\tilde{t}}e^{-|n-m|a},$$

$$\left\|\frac{\partial^2\varepsilon\tilde{R}^6}{\partial z_{n+\tilde{c}\tilde{t}}\partial z_{m-\tilde{c}\tilde{t}}} - \lim_{\tilde{t}\to\infty}\frac{\partial^2\varepsilon\tilde{R}^6}{\partial z_{n+\tilde{c}\tilde{t}}\partial z_{m-\tilde{c}\tilde{t}}}\right\|_{D_a(r,s),\Pi} \le \frac{\varepsilon}{\tilde{t}}e^{-|n+m|a},$$

and

$$\left\| \frac{\partial^2 \varepsilon \tilde{R}^6}{\partial \bar{z}_{n+\tilde{c}\tilde{t}} \partial \bar{z}_{m-\tilde{c}\tilde{t}}} - \lim_{\tilde{t}\to\infty} \frac{\partial^2 \varepsilon \tilde{R}^6}{\partial \bar{z}_{n+\tilde{c}\tilde{t}} \partial \bar{z}_{m-\tilde{c}\tilde{t}}} \right\|_{D_a(r,s),\Pi} \leq \frac{\varepsilon}{\tilde{t}} e^{-|n+m|a}.$$

So, both $\varepsilon \tilde{R}^6$ and F satisfy the Töplitz-Lipschitz property. According to Lemma 4.4 in [10], the Poisson bracket $\{\varepsilon \tilde{R}^6, F\}$ preserves Töplitz-Lipschitz property. Thus H satisfies (A6). From Theorem 3.1([26] Theorem 2.1), we can prove that Theorem 1.2 holds.

5. Appendix

Proof of Lemma 2.2. Case 1. Suppose that $i-j+d-l+n-m=0, |i|^2-|j|^2+|d|^2-|l|^2+|n|^2-|m|^2 \neq 0$ and $\#(\mathscr{R}\cap\{i,j,d,l,n,m\}) \geq 4$. Then $\left||i|^2-|j|^2+|d|^2-|l|^2+|n|^2-|m|^2\right| \geq 1$. Denote $h(\varepsilon)=\lambda_i-\lambda_j+\lambda_d-\lambda_l+\lambda_n-\lambda_m$. From $\lambda_{d^*}=|d^*|^2+\varepsilon[\phi]+\varepsilon^{(1+\rho)}\lambda^*_{d^*} \quad (d^*\in\mathbb{Z}^2)$, we have $h(\varepsilon)=|i|^2-|j|^2+|d|^2-|l|^2+|n|^2-|m|^2+\varepsilon^{(1+\rho)}(\lambda^*_i-\lambda^*_j+\lambda^*_d-\lambda^*_l+\lambda^*_n-\lambda^*_m)$.

Case 1.1. If $k=0$, then $|h(\varepsilon)+<k,\omega>|=|h(\varepsilon)|\geq 1-C\varepsilon^{(1+\rho)}\geq \frac{\rho}{C_*}$ when ε is small enough and C_* is large enough.

Case 1.2. Suppose that $k\neq 0$. Denote

$$\mathscr{I}^2_{ijdlnm,k}=\left\{\omega\in\underline{S}\subset[\rho,2\rho]^{b^*}: |h(\varepsilon)+<k,\omega>|<\frac{\rho}{C_*|k|^{b^*+2}}\right\},$$

and

$$S^2=\bigcup_{0\neq k\in\mathbb{Z}^{b^*}}\bigcup_{i,j,d,l,n,m}\mathscr{I}^2_{ijdlnm,k}.$$

Case 1.2.1. For $\#(\mathscr{R}\cap\{i,j,d,l,n,m\})=6$, we denote

$$\mathscr{I}^{2,1}_{ijdlnm,k}=\left\{\omega\in\underline{S}\subset[\rho,2\rho]^{b^*}: |h(\varepsilon)+<k,\omega>|<\frac{\rho}{C_*|k|^{b^*+1}}\right\},$$

$$S^{2,1}=\bigcup_{0\neq k\in\mathbb{Z}^{b^*}}\bigcup_{i\in\mathscr{R},j\in\mathscr{R},d\in\mathscr{R},l\in\mathscr{R},n\in\mathscr{R},m\in\mathscr{R}}\mathscr{I}^{2,1}_{ijdlnm,k}.$$

It follows that $\text{meas}\mathscr{I}^{2,1}_{ijdlnm,k}\leq\frac{2\rho^{b^*}}{C_*|k|^{b^*+2}}$. Letting $|k|_\infty=\max\{|k_1|,|k_2|,\ldots,|k_{b^*}|\}$, according to

$$\sum_{|k|_\infty=p}1\leq 2b^*(2p+1)^{b^*-1},\quad |k|_\infty\leq|k|\leq b^*|k|_\infty,$$

we have

$$\begin{aligned}\text{meas}S^{2,1} &= \text{meas}\bigcup_{0\neq k\in\mathbb{Z}^{b^*}}\bigcup_{i\in\mathscr{R},j\in\mathscr{R},d\in\mathscr{R},l\in\mathscr{R},n\in\mathscr{R},m\in\mathscr{R}}\mathscr{I}^{2,1}_{ijdlnm,k}\\ &\leq \sum_{0\neq k\in\mathbb{Z}^{b^*}} b^6\frac{2\rho^{b^*}}{C_*|k|^{b^*+2}}\leq\frac{C''_1}{C_*}\rho^{b^*}\sum_{0\neq k\in\mathbb{Z}^{b^*}}\frac{1}{|k|^{b^*+2}}\\ &\leq \frac{C'_1}{C_*}\rho^{b^*}\sum_{p=1}^{\infty}(2p+1)^{b^*-1}p^{-(b^*+2)}\leq\frac{C_1}{C_*}\rho^{b^*},\end{aligned}$$

where the constant C_1 depends on b,b^*. Thus $\text{meas}S^{2,1}\leq\frac{\gamma}{6}\rho^{b^*}$ when C_* is large enough.

Case 1.2.2. $\#(\mathscr{R}\cap\{i,j,d,l,n,m\})=5$. Without loss of generality, if $i,j,d,l,n\in\mathscr{R}$, $m\in\mathbb{Z}_1^2$ is true, we know that $m=i-j+d-l+n$ is at most b^5 different values. Denoting

$$\mathscr{I}^{2,2}_{ijdlnm,k}=\left\{\omega\in\underline{S}\subset[\rho,2\rho]^{b^*}:|h(\varepsilon)+<k,\omega>|<\frac{\rho}{C_*|k|^{b^*+1}}\right\},$$

$$S^{2,2}=\bigcup_{0\neq k\in\mathbb{Z}^{b^*}}\ \bigcup_{i\in\mathscr{R},j\in\mathscr{R},d\in\mathscr{R},l\in\mathscr{R},n\in\mathscr{R},m=i-j+d-l+n}\mathscr{I}^{2,2}_{ijdlnm,k},$$

we have meas$\mathscr{I}^{2,2}_{ijdlnm,k}\leq\frac{2\rho^{b^*}}{C_*|k|^{b^*+2}}$, and

$$\begin{aligned}\text{meas}S^{2,2}&=\text{meas}\bigcup_{0\neq k\in\mathbb{Z}^{b^*}}\ \bigcup_{i\in\mathscr{R},j\in\mathscr{R},d\in\mathscr{R},l\in\mathscr{R},n\in\mathscr{R},l=i-j+d-l+m}\mathscr{I}^{2,2}_{ijdlnm,k}\\&\leq\sum_{0\neq k\in\mathbb{Z}^{b^*}}b^{10}\frac{2\rho^{b^*}}{C_*|k|^{b^*+2}}\leq\frac{C_2}{C_*}\rho^{b^*}\end{aligned}$$

where the constant C_2 depends on b,b^*. Thus meas$S^{2,2}\leq\frac{\gamma}{6}\rho^{b^*}$ when C_* is large enough.

Case 1.2.3. $\#(\mathscr{R}\cap\{i,j,d,l,n,m\})=4$. Without loss of generality, if $i,j,d,l\in\mathscr{R},n,m\in\mathbb{Z}_1^2$ is true, then $m=i-j+d-l+n$ and

$$\begin{aligned}h(\varepsilon)&=|i|^2-|j|^2+|d|^2-|l|^2+|n|^2-|i-j+d-l+n|^2\\&\quad+\varepsilon^{(1+\rho)}(\lambda_i^*-\lambda_j^*+\lambda_d^*-\lambda_l^*+\lambda_n^*-\lambda_m^*)\\&=-2<i-j,d-j>+2<-i+j-d+l,n-l>\\&\quad+\varepsilon^{(1+\rho)}(\lambda_i^*-\lambda_j^*+\lambda_d^*-\lambda_l^*+\lambda_n^*-\lambda_m^*)\\&=W+\varepsilon^{(1+\rho)}(\lambda_i^*-\lambda_j^*+\lambda_d^*-\lambda_l^*+\lambda_n^*-\lambda_m^*)\end{aligned}$$

where $W=-2<i-j,d-j>+2<-i+j-d+l,n-l>\in\mathbb{Z}$. Denote

$$\mathscr{I}^{2,3}_{ijdlnm,k}=\left\{\omega\in\underline{S}\subset[\rho,2\rho]^{b^*}:|h(\varepsilon)+<k,\omega>|<\frac{\rho}{C_*|k|^{b^*+2}}\right\},$$

$$S^{2,3}=\bigcup_{0\neq k\in\mathbb{Z}^{b^*}}\ \bigcup_{i\in\mathscr{R},j\in\mathscr{R},d\in\mathscr{R},l\in\mathscr{R},n\in\mathbb{Z}_1^2,m=i-j+d-l+n}\mathscr{I}^{2,3}_{ijdlnm,k}.$$

For any fixed i,j,d,l,W, denote

$$v^*_{ijdlW}=\left\{n\in\mathbb{Z}_1^2:-2<i-j,d-j>+2<-i+j-d+l,n-l>=W\right\}$$

$$\lambda^*_{ijdlW,1}=\sup_{n\in v^*_{ijdlW}}\left\{\lambda_n^*-\lambda^*_{i-j+d-l+n}\right\},\quad\lambda^*_{ijdlW,2}=\inf_{n\in v^*_{ijdlW}}\left\{\lambda_n^*-\lambda^*_{i-j+d-l+n}\right\}$$

$$\begin{aligned}&\mathscr{I}^{2,3,1}_{ijdlW,k}=\Big\{\omega\in\underline{S}\subset[\rho,2\rho]^{b^*}:\\&|<k,\omega>+W+\varepsilon^{(1+\rho)}(\lambda_i^*-\lambda_j^*+\lambda_d^*-\lambda_l^*+\lambda^*_{ijdlW,1})|<\frac{\rho}{C_*|k|^{b^*+2}}\Big\},\end{aligned}$$

$$\begin{aligned}&\mathscr{I}^{2,3,2}_{ijdlW,k}=\Big\{\omega\in\underline{S}\subset[\rho,2\rho]^{b^*}:\\&|<k,\omega>+W+\varepsilon^{(1+\rho)}(\lambda_i^*-\lambda_j^*+\lambda_d^*-\lambda_l^*+\lambda^*_{ijdlW,2})|<\frac{\rho}{C_*|k|^{b^*+2}}\Big\}.\end{aligned}$$

Then,

$$\text{meas}\mathscr{I}^{2,3,1}_{ijdlW,k} \le \frac{2\rho^{b^*}}{C_*|k|^{b^*+3}}, \quad \text{meas}\mathscr{I}^{2,3,2}_{ijdlW,k} \le \frac{2\rho^{b^*}}{C_*|k|^{b^*+3}}.$$

It is obvious that, for $m = i-j+d-l+n, n \in v^*_{ijdlW}$,

$$\mathscr{I}^{2,3}_{ijdlnm,k} \subset \mathscr{I}^{2,3,1}_{ijdlW,k} \bigcup \mathscr{I}^{2,3,2}_{ijdlW,k}.$$

It is trivial that the sets $\mathscr{I}^{2,3,1}_{ijdlW,k}$ and $\mathscr{I}^{2,3,2}_{ijdlW,k}$ are empty when $|W| > |k|\rho+3$. Thus,

$$\begin{aligned} S^{2,3} = & \bigcup_{0\neq k\in\mathbb{Z}^{b^*}} \bigcup_{i\in\mathscr{R},j\in\mathscr{I},d\in\mathscr{R},l\in\mathscr{R}} \bigcup_{n\in\mathbb{Z}^2_1} \bigcup_{m=i-j+d-l+n} \mathscr{I}^{2,3}_{ijdlnm,k} \\ & \subset \bigcup_{0\neq k\in\mathbb{Z}^{m}} \bigcup_{i\in\mathscr{R},j\in\mathscr{I},d\in\mathscr{R},l\in\mathscr{R}} \bigcup_{W\in\mathbb{Z}} (\mathscr{I}^{2,3,1}_{ijdlW,k} \bigcup \mathscr{I}^{2,3,2}_{ijdlW,k}). \end{aligned}$$

It follows that

$$\begin{aligned} \text{meas}S^{2,3} & \le \text{meas} \bigcup_{0\neq k\in\mathbb{Z}^{b^*}} \bigcup_{i\in\mathscr{R},j\in\mathscr{I},d\in\mathscr{R},l\in\mathscr{R}} \bigcup_{W\in\mathbb{Z}} (\mathscr{I}^{2,3,1}_{ijdlW,k} \bigcup \mathscr{I}^{2,3,2}_{ijdlW,k}) \\ & = \text{meas} \bigcup_{0\neq k\in\mathbb{Z}^{b^*}} \bigcup_{i\in\mathscr{R},j\in\mathscr{I},d\in\mathscr{R},l\in\mathscr{R}} \bigcup_{1\le|W|\le|k|\rho+3} (\mathscr{I}^{2,3,1}_{ijdlW,k} \bigcup \mathscr{I}^{2,3,2}_{ijdlW,k}) \\ & \le \sum_{0\neq k\in\mathbb{Z}^{b^*}} 4b^4(|k|\rho+3)\frac{2\rho^{b^*}}{C_*|k|^{b^*+3}} \le \frac{C_3}{C_*}\rho^{b^*} \end{aligned}$$

where the constant C_3 depends on b, b^*. Thus, $\text{meas}\Omega^{2,3} \le \frac{\gamma}{6}\rho^{b^*}$ when C_* is large enough.

Case 2. Supposing that $i-j+d-l+n-m=0, |i|^2-|j|^2+|d|^2-|l|^2+|n|^2-|m|^2=0$, $\#(\mathscr{R}\cap\{i,j,d,l,n,m\}) = 4$ and $k\neq 0$. In view of $\omega\in\underline{S}$, we have

$$\begin{aligned} & \left|\lambda_i-\lambda_j+\lambda_d-\lambda_l+\lambda_n-\lambda_m+<k,\omega>\right| = \left|O(\varepsilon^{(1+\rho)})+<k,\omega>\right| \\ & \ge |<k,\omega>|-C\varepsilon^{(1+\rho)} \ge \frac{\rho}{|k|^{b^*+1}} - C\varepsilon^{(1+\rho)} > \frac{\rho}{C_*|k|^{b^*+2}} \end{aligned}$$

when ε is small enough and C_* is large enough. It is obvious that $S^2 \subset \left(S^{2,1}\cup S^{2,2}\cup S^{2,3}\right)$, and then $\text{meas}S^2 < \gamma$. Letting $S = \underline{S}\setminus S^2$, we see that it satisfies as required and $\text{meas}S \ge (1-\frac{\gamma}{2})\rho^{b^*}$. We prove that Lemma 2.2 holds.

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