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A GENERAL ITERATIVE ALGORITHM FOR SPLIT VARIATIONAL INCLUSION PROBLEMS AND FIXED POINT PROBLEMS OF A PSEUDOCONTRACTIVE MAPPING

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Abstract. In this paper, we introduce a general iterative algorithm based on the hybrid steepest descent method for finding a common element of the solution set of split variational inclusion problems and the fixed point set of a continuous pseudocontractive mapping. We establish strong convergence of the proposed iterative algorithm in a Hilbert space. We also find the minimum-norm element in the common set of two sets.

Keywords. Continuous pseudocontractive mapping; Fixed point problem; Maximal monotone operator; Minimum-norm point; Split variational inclusion problem.

1. Introduction

Let H_1 and H_2 be two real Hilbert spaces. Let D and Q be nonempty, closed, and convex subsets of H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator. Then the split feasibility problem (SFP) is to find a point $z \in H_1$ such that $z \in D \cap A^{-1}Q$. In 1994, the SFP was first introduced by Censor and Elfving [8], in finite-dimensional Hilbert spaces, for modeling inverse problems which arise from phase retrievals and in medical image reconstruction. Since then, the SFP has received much attention due to its wide applications in signal processing, image reconstruction, with particular progress in intensity-modulated radiation therapy(IMRT), approximation theory, control theory, biomedical engineering, communications, and geophysics; see, e.g., [2, 3, 4, 7] and the references therein.

In 2011, based on the split variational inequality problem (SVP) introduced by Censor et al. [9], Moudafi [16] proposed the following split monotone variational inclusion problem (SMVIP):

$$\text{find a point } x^* \in H_1 \text{ such that } 0 \in f_1(x^*) + B_1(x^*), \tag{1.1}$$

and

$$y^* = Ax^* \in H_2 \text{ solves } 0 \in f_2(y^*) + B_2(y^*), \tag{1.2}$$

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where $B_1 : H_1 \to 2^{H_1}$ and $B_2 : H_2 \to 2^{H_2}$ are multi-valued maximal monotone mappings, $A : H_1 \to H_2$ is a bounded linear operator, and $f_1 : H_1 \to H_1$ and $f_2 : H_2 \to H_2$ are two given single-valued operators. The SMVIP (1.1)-(1.2) includes, as special cases, several split problems, such as the split zero problem (SZP), the SVP, the SFP, and the split common fixed point problem (SCFPP) [3, 4, 5, 13, 16], which have already been studied and used in practice as a model in the IMRT treatment planning (see [7, 8]) and in many inverse problems arising for phase retrieval and other real-world problem; for instance, in computerized tomography, in sensor networks and date computation (see [3, 10]).

If $f_1 \equiv 0$ and $f_2 \equiv 0$, then SMVIP (1.1)-(1.2) reduces to the following split variational inclusion problem (SVIP):

$$\text{find a point } x^* \in H_1 \text{ such that } 0 \in B_1(x^*), \tag{1.3}$$

and

$$y^* = Ax^* \in H_2 \text{ solves } 0 \in B_2(y^*). \tag{1.4}$$

As we know, (1.3) is the variational inclusion problem, and we denote its solution set by SOLVIP(B_1). The SVIP (1.3)-(1.4) consists of a pair of variational inclusion problems, which need to be solved so that the image $y^* = Ax^*$ under a given bounded linear operator A, of the solution x^* of SVIP(1.3) in H_1 is the solution of another SVIP (1.4) in another space H_2. We denote the solution set of SVIP(1.4) by SOLVIP(B_2). The solution set of SVIP (1.3)-(1.4) is denote by $\Gamma = \{x^* \in H_1 : x^* \in \text{SOLVIP}(B_1) \text{ and } Ax^* \in \text{SOLVIP}(B_2)\}$.

A fixed point problem (FPP) is to find a fixed point z of a nonlinear mapping S with the property:

$$z \in C,\ Sz = z, \tag{1.5}$$

where C is a nonempty, closed, and convex subset of a Hilbert space H. We denote the fixed point set of S by $Fix(S)$.

Many authors considered the SVIP (1.3)-(1.4). In 2012, Byrne et al. [5] introduced the following iterative algorithm for the SVIP (1.3)-(1.4), which ensured the weak and strong convergence: for given $x_0 \in H_1$, compute iterative sequence $\{x_n\}$ generated by

$$x_{n+1} = J_\lambda^{B_1}(x_n + \eta A^*(J_\lambda^{B_2} - I)Ax_n),$$

where $J_\lambda^{B_i} = (I + \lambda B_i)^{-1}$ is the resolvent of B_i for $i = 1, 2$, and $\lambda > 0$, A^* is the adjoint of A, $L = \|AA^*\|$ and $\eta \in (0, \frac{2}{L})$.

In 2013, in order to study the SVIP (1.3)-(1.4) coupled with the FPP (1.5) of a nonexpansive mapping S, Kazmi and Rizvi [14] proposed the following iterative algorithm based on the work of Byrne et al. [5]:

$$\begin{cases} u_n = J_\lambda^{B_1}(x_n + \eta A^*(J_\lambda^{B_2} - I)Ax_n), \\ x_{n+1} = \alpha_n f x_n + (1 - \alpha_n) S u_n, \quad n \geq 0, \end{cases} \tag{1.6}$$

where $f : H_1 \to H_1$ is a contractive mapping and $\alpha_n \in (0, 1)$, and established the strong convergence of the sequence $\{x_n\}$ generated by (1.6) to the common element of the solution set Γ of SVIP (1.3)-(1.5) and the fixed point set $Fix(S)$ of S.

In 2015, combining the method (1.6) of Kazmi and Rizvi [14] and the method of Marino and Xu [15], Sitthithakerngkiet et al. [17] presented the following general iterative algorithm:

$$\begin{cases} u_n = J_\lambda^{B_1}(x_n + \eta A^*(J_\lambda^{B_2} - I)Ax_n), \\ x_{n+1} = \alpha_n \xi f x_n + (I - \alpha_n D)Su_n, \quad n \geq 0, \end{cases} \tag{1.7}$$

where $D : H_1 \to H_1$ is a strongly positive bounded linear operator and $\xi > 0$, and showed that the sequence $\{x_n\}$ generated by (1.7) converges strongly to a point of $\Gamma \cap Fix(S)$, which is the unique solution of a certain variational inequality related to D.

In 2018, in order to investigate the SVIP (1.3)-(1.4) coupled with the FPP (1.5) for a strictly pseudocontractive mapping T, Yang and Yuan [22] considered the following iterative algorithm based on Yamada's hybrid steepest descent method [21]:

$$\begin{cases} u_n = J_{\lambda_n}^{B_1}(x_n + \eta A^*(J_{\lambda_n}^{B_2} - I)Ax_n), \\ x_{n+1} = T_{\beta_n} u_n - \mu \alpha_n G T_{\beta_n} u_n, \quad n \geq 0, \end{cases} \tag{1.8}$$

where $T_{\beta_n} = (1 - \beta_n)T + \beta_n I$ and $G : H_1 \to H_1$ is a κ-Lipschitzian and ρ-strongly monotone mapping with constants κ, $\rho > 0$, and showed strong convergence of the sequence $\{x_n\}$ generated by (1.8) to a point of $\Gamma \cap Fix(T)$, which is the unique solution of a certain variational inequality related to G.

In this paper, motivated by the works of Byrne et al. [5], Kazmi and Rizvi [14], Sitthithakerngkiet et al. [17], and Yang and Yuan [22], we introduce a new general iterative algorithm based on the hybrid steepest descent method for finding a common element of the solution set Γ of SVIP (1.3)-(1.4) and the fixed point set $Fix(T)$ of a continuous pseudocontractive mapping T. Then we establish strong convergence of the sequence generated by the proposed iterative algorithm to a point of $\Gamma \cap Fix(T)$, which is a solution of a certain variational inequality. As a direct consequence, we find the unique minimum-norm element of $\Gamma \cap Fix(T)$. The results of in this paper are the supplement, extension, and generalization of the previous known results in this area; see, e.g., [5, 14, 17, 22]) and the references therein.

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$, and let C be a nonempty, closed, and convex subset of H.

We recall that

(i) a mapping $V : C \to H$ is said to be *l-Lipschitzian* if there exists a constant $l \geq 0$ such that

$$\|Vx - Vy\| \leq l\|x - y\| \ \text{ for all } x,\ y \in C;$$

(ii) a mapping $G : C \to H$ is said to be ρ-*strongly monotone* if there exists a constant $\rho > 0$ such that

$$\langle Gx - Gy, x - y \rangle \geq \rho \|x - y\|^2 \ \text{ for all } x,\ y \in C;$$

(iii) a mapping $T : C \to H$ is said to be *pseudocontractive* if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|(I - T)x - (I - T)y\|^2 \ \text{ for all } x,\ y \in C;$$

(iv) a mapping $T : C \to H$ is said to be *k-strictly pseudocontractive* [6] if there exists a constant $k \in [0,1)$ such that

$$\|Tx - Ty\|^2 \le \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2 \ \text{ for all } x,\ y \in C;$$

(v) a mapping $T : C \to H$ is said to be *nonexpansive* if

$$\|Tx - Ty\| \le \|x - y\|, \ \text{ for all } x,\ y \in C$$

where I is the identity mapping.

Clearly, the class of pseudocontractive mappings includes the class of strictly pseudocontractive mappings and the class of nonexpansive mappings as subclasses. Moreover, this inclusion is strict (see [1, Example 5.7.1 and Example 5.7.2]).

Let B be a mapping of H into 2^H. The effective domain of B is denoted by $dom(B)$, that is, $dom(B) = \{x \in H : Bx \neq \emptyset\}$. A set-valued mapping B is said to be a *monotone operator* on H if $\langle x - y, u - v\rangle \ge 0$ for all $x,\ y \in dom(B)$, $u \in Bx$, and $v \in By$. A monotone operator B on H is said to be *maximal* if its graph is not properly contained in the graph of any other monotone operator on H. For a maximal monotone operator B on H and $\lambda > 0$, we may define a single-valued operator $J_\lambda^B = (I + \lambda B)^{-1} : H \to dom(B)$, which is called the *resolvent* of B. Let B be a maximal monotone operator on H and let $B^{-1}0 = \{x \in H : 0 \in Bx\}$. It is well-known that $B^{-1}0 = Fix(J_\lambda^B)$ for all $\lambda > 0$ is closed and convex and the resolvent J_λ^B is firmly nonexpansive, that is,

$$\|J_\lambda^B x - J_\lambda^B y\|^2 \le \langle x - y, J_\lambda^B x - J_\lambda^B y\rangle \ \text{ for all } x,\ y \in H, \tag{2.1}$$

and that the resolvent identity

$$J_\lambda^B x = J_\mu^B\left(\frac{\mu}{\lambda}x + \left(1 - \frac{\mu}{\lambda}\right)J_\lambda^B x\right) \tag{2.2}$$

holds for all $\lambda,\ \mu > 0$ and $x \in H$.

In a real Hilbert space H, the following hold:

$$\|x - y\|^2 = \|x\|^2 + \|y\|^2 - 2\langle x, y\rangle, \tag{2.3}$$

and

$$\|\alpha x + \beta y\|^2 = \alpha\|x\|^2 + \beta\|y\|^2 - \alpha\beta\|x - y\|^2 \le \alpha\|x\|^2 + \beta\|y\|^2, \tag{2.4}$$

for all $x,\ y \in H$ and $\alpha,\ \beta \in (0,1)$ with $\alpha + \beta = 1$.

It is also well known ([11]) that every nonexpansive mapping $T : H \to H$ satisfies, for all $(x, y) \in H \times H$, the inequality

$$\langle (x - Tx) - (y - Ty), Ty - Tx\rangle \le \frac{1}{2}\|(Tx - x) - (Ty - y)\|^2,$$

and hence, for all $(x, y) \in H \times Fix(T)$,

$$\langle x - Tx, y - Tx\rangle \le \frac{1}{2}\|Tx - x\|^2. \tag{2.5}$$

For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| = \inf\{\|x - y\| : y \in C\}.$$

P_C is called the *metric projection* of H onto C. It is well known ([19]) that P_C is nonexpansive and P_C is characterized by the property

$$u = P_C x \Longleftrightarrow \langle x-u, u-y\rangle \geq 0 \text{ for all } x \in H,\ y \in C. \tag{2.6}$$

A mapping $T : H \to H$ is said to be *averaged* if it can be written as the average of the identity I and a nonexpansive mapping, that is,

$$T = (1-\alpha)I + \alpha S, \tag{2.7}$$

where α is a number in $(0,1)$ and $S : H \to H$ is nonexpansive. More precisely, when (2.7) holds, we say that T is α-averaged. We note that averaged mappings are nonexpansive. Further firmly nonexpansive mappings (in particular, projections and resolvents of maximal monotone operators) are averaged, and the composite of finitely many averaged mappimgs is averaged (see [4, 16]).

We need the following lemmas for the proof of our main results.

Lemma 2.1 ([1]). *In a real Hilbert space H, the following inequality holds:*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle, \quad \forall x,\ y \in H.$$

Lemma 2.2 ([18]). *Let $\{x_n\}$ and $\{z_n\}$ be bounded sequences in a real Banach space E, and let $\{\gamma_n\}$ be a sequence in $[0,1]$ which satisfies $0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < 1$. Suppose that $x_{n+1} = \gamma_n x_n + (1-\gamma_n) z_n$ for all $n \geq 1$ and $\limsup_{n\to\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then $\lim_{n\to\infty} \|z_n - x_n\| = 0$.*

Lemma 2.3 ([20]). *Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying*

$$s_{n+1} \leq (1-\xi_n)s_n + \xi_n \delta_n, \quad \forall n \geq 1,$$

where $\{\xi\}$ and $\{\delta_n\}$ satisfy the following conditions:

(i) $\{\xi_n\} \subset [0,1]$ *and* $\sum_{n=1}^{\infty} \xi_n = \infty$;
(ii) $\limsup_{n\to\infty} \delta_n \leq 0$ *or* $\sum_{n=1}^{\infty} \xi_n |\delta_n| < \infty$.

Then $\lim_{n\to\infty} s_n = 0$.

The following lemma is the Lemma 2.4 of Zegeye [23].

Lemma 2.4 ([23]). *Let C be a closed and convex subset of a real Hilbert space H. Let $T : C \to H$ be a continuous pseudocontractive mapping. Then, for $r > 0$ and $x \in H$, there exists $z \in C$ such that $\langle y-z, Tz\rangle - \frac{1}{r}\langle y-z, (1+r)z - x\rangle \leq 0$, $\forall y \in C$. For $r > 0$ and $x \in H$, define $T_r : H \to C$ by*

$$T_r x = \left\{ z \in C : \langle y-z, Tz\rangle - \frac{1}{r}\langle y-z, (1+r)z - x\rangle \leq 0, \quad \forall y \in C \right\}.$$

Then the following hold:

(i) T_r *is single-valued*;
(ii) T_r *is firmly nonexpansive, that is,*

$$\|T_r x - T_r y\|^2 \leq \langle x-y, T_r x - T_r y\rangle, \quad \forall x,\ y \in H;$$

(iii) $Fix(T_r) = Fix(T)$;
(iv) $Fix(T)$ *is a closed convex subset of C.*

The following lemmas can be easily proven from see [21], and therefore, we omit their proof.

Lemma 2.5. *Let H be a real Hilbert space. Let $V : H \to H$ be an l-Lipschitzian mapping with a constant $l \geq 0$, and let $G : H \to H$ be a κ-Lipschitzian and ρ-strongly monotone mapping with constants $\kappa,\ \rho > 0$. Then, for $0 \leq \gamma l < \mu\rho$,*

$$\langle(\mu G - \gamma V)x - (\mu G - \gamma V)y, x - y\rangle \geq (\mu\rho - \gamma l)\|x - y\|^2 \text{ for all } x,\ y \in H.$$

That is, $\mu G - \gamma V$ is strongly monotone with constant $\mu\rho - \gamma l$.

Lemma 2.6. *Let H be a real Hilbert space H. Let $G : H \to H$ be a κ-Lipschitzian and ρ-strongly monotone mapping with constants $\kappa > 0$ and $\rho > 0$. Let $0 < \mu < \frac{2\rho}{\kappa^2}$ and $0 < t < \sigma \leq 1$. Then $\sigma I - t\mu G : H \to H$ is a contractive mapping with a constant $\sigma - t\tau$, where $\tau = 1 - \sqrt{1 - \mu(2\rho - \mu\kappa^2)}$.*

Lemma 2.7 ([12])**.** *Assume that T is nonexpansive self mapping of a closed convex subset of C of a Hilbert space H. If T has a fixed point, then $I - T$ is demiclosed, i.e., whenever $\{x_n\}$ is a sequence in C converging weakly to some $x \in C$ and the sequence $\{(I - T)x_n\}$ converges strongly to some y, it follows that $(I - T)x = y$, where I is the identity mapping H.*

We have the following lemma, which is a direct consequence of the definition of resolvent mapping.

Lemma 2.8 ([14])**.** *SVIP(1.3)-(1.4) is equivalent to find $x^* \in H_1$ such that $y^* = Ax^* \in H_2$*

$$x^* = J_\lambda^{B_1} x^* \textit{ and } y^* = J_\nu^{B_2} y^* \textit{ for some } \lambda,\ \nu > 0.$$

In the following, we write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x. $x_n \to x$ implies that $\{x_n\}$ converges strongly to x.

3. Iterative Algorithms

Throughout the rest of this paper, we always assume the following:

- H_1 and H_2 are real Hilbert spaces with inner product $\langle\cdot,\cdot\rangle$ and induced norm $\|\cdot\|$;
- $A : H_1 \to H_2$ is a bounded linear operator;
- $A^* : H_2 \to H_1$ is the adjoint of A;
- L is the spectral radius of the operator A^*A
- $B_1 : H_1 \to 2^{H_1}$ is a maximal monotone operator with $dom(B_1) \subset H_1$;
- $B_2 : H_2 \to 2^{H_2}$ is a maximal monotone operator with $dom(B_2) \subset H_2$
- $B_1^{-1}0$ is the set of zero points of B_1, that is, $B_1^{-1}0 = \{z \in H_1 : 0 \in B_1 z\}$;
- $B_2^{-1}0$ is the set of zero points of B_2, that is, $B_2^{-1}0 = \{z \in H_2 : 0 \in B_2 z\}$;
- $J_{\lambda_n}^{B_1} : H_1 \to \mathrm{dom}(B_1)$ is the resolvent of B_1 for $\lambda_n \in (0,\infty)$ and $\liminf_{n\to\infty} \lambda_n > 0$;
- $J_{\nu_n}^{B_2} : H_2 \to \mathrm{dom}(B_2)$ is the resolvent of B_2 for $\nu_n \in (0,\infty)$ and $\liminf_{n\to\infty} \nu_n > 0$;
- $G : H_1 \to H_1$ is a κ-Lipschitzian and ρ-strongly monotone mapping with constants $\kappa,\ \rho > 0$;
- $V : H_1 \to H_1$ is an l-Lipschitzian mapping with constant $l \in [0,\infty)$;
- μ and γ, which are two positive constants, satisfy $0 < \mu < \frac{2\rho}{\kappa^2}$ and $0 \leq \gamma l < \tau$, where $\tau = 1 - \sqrt{1 - \mu(2\rho - \mu\kappa^2)}$;
- $T : H_1 \to H_1$ is a continuous pseudocontractive mapping with $Fix(T) \neq \emptyset$;

- $T_{r_n} : H_1 \to H_1$ is a mapping defined by

$$T_{r_n}x = \left\{ z \in H_1 : \langle Tz, y - z \rangle - \frac{1}{r_n} \langle y - z, (1 + r_n)z - x \rangle \leq 0, \quad \text{for all} y \in H_1 \right\}$$

  for $x \in H_1$ and $r_n \in (0, \infty)$, and $\liminf_{n\to\infty} r_n > 0$;
- $\Gamma \neq \emptyset$ is the solution set of SVIP (1.3)-(1.4). That is, $\Gamma = \{x^* \in H_1 : x^* \in \text{SOLVIP}(B_1)$ and $Ax^* \in \text{SOLVIP}(B_2)\} \neq \emptyset$.
- $\Omega := \Gamma \cap Fix(T) \neq \emptyset$.

By Lemma 2.4, we note that T_{r_n} is nonexpansive, and $Fix(T_{r_n}) = Fix(T)$.

Now, we propose a new iterative algorithm which generates a sequence $\{x_n\}$ in an explicit way: for an arbitrarily chosen $x_0 \in C$,

$$\begin{cases} z_n = J_{\lambda_n}^{B_1}(x_n + \eta_n A^*(J_{\nu_n}^{B_2} - I)Ax_n), \\ x_{n+1} = \beta_n x_n + (1 - \beta_n)T_{r_n}(\alpha_n \gamma V x_n + (I - \alpha_n \mu G)z_n), \quad n \geq 0, \end{cases} \tag{3.1}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are two real sequences in $(0,1)$, $\{r_n\}$, $\{\lambda_n\}$, and $\{\nu_n\}$ are three positive real sequences, and $\{\eta_n\}$ is a real sequence in $(0, \frac{1}{L})$. We will establish strong convergence of this sequence to a common element of Ω.

Theorem 3.1. *Let the sequence $\{x_n\}$ be generated iteratively by the explicit algorithm (3.1). Let $\{\alpha_n\}$, $\{\beta_n\} \subset (0,1)$, $\{r_n\}, \{\lambda_n\}, \{\nu_n\} \subset (0,\infty)$ and $\{\eta_n\} \subset (0, \frac{1}{L})$ satisfy the following conditions:*

(C1) $\lim_{n\to\infty} \alpha_n = 0$;
(C2) $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(C3) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;
(C4) $0 < r \leq r_n < \infty$ *and* $\lim_{n\to\infty} |r_{n+1} - r_n| = 0$;
(C5) $0 < \lambda \leq \lambda_n < \infty$ *and* $\lim_{n\to\infty} |\lambda_{n+1} - \lambda_n| = 0$;
(C6) $0 < \nu \leq \nu_n < \infty$ *and* $\lim_{n\to\infty} |\nu_{n+1} - \mu_n| = 0$;
(C7) $0 < \eta \leq \eta_n < \frac{1}{L}$ *and* $\lim_{n\to\infty} |\eta_{n+1} - \eta_n| = 0$.

Then $\{x_n\}$ converges strongly to a point $q \in \Omega$, which is the unique solution of the variational inequality

$$\langle (\mu G - \gamma V)q, p - q \rangle \geq 0 \ \text{ for all } p \in \Omega. \tag{3.2}$$

Proof. First, let $Q = P_\Omega$, where $\Omega := \Gamma \cap Fix(T)$. Then, by the closedness and convexity of Γ and $Fix(T)$(Lemma 2.4 (iv)), P_Ω is well-defined. Also, it is easy to show that $Q(I - \mu G + \gamma V) : H_1 \to H_1$ is a contractive mapping with a constant $1 - (\tau - \gamma l)$. In fact, from Lemma 2.6, we have

$$\begin{aligned} \|Q(I - \mu G + \gamma V)x - Q(I - \mu G + \gamma V)y\| &\leq \|(I - \mu G + \gamma V)x - (I - \mu G + \gamma V)y\| \\ &\leq \|(I - \mu G)x - (I - \mu G)y\| + \gamma \|Vx - Vy\| \\ &\leq (1 - (\tau - \gamma l))\|x - y\| \end{aligned}$$

for any x, $y \in H_1$. So, $Q(I - \mu G + \gamma V)$ is a contractive mapping with a constant $1 - (\tau - \gamma l) < 1$. Thus, by Banach contraction principle, there exists a unique element $q \in H_1$ such that $q = P_\Omega(I - \mu G + \gamma V)q$. Equivalently, by (2.6), q is a solution of the variational inequality (3.2). We note that the uniqueness of a solution to variational inequality (3.2) is a consequence of the strong monotonicity of $\mu G-\gamma V$ (due to Lemma 2.5). Below, we will use $q\in\Omega$ to denote the unique solution to variational inequality (3.2).

From now on, we put $K_n=I+\eta_nA^*(J^{B_2}_{\nu_n}-I)A$, $u_n=x_n+\eta_nA^*(J^{B_2}_{\nu_n}-I)Ax_n=K_nx_n$, $z_n=J^{B_1}_{\lambda_n}u_n$, and $y_n=\alpha_n\gamma Vx_n+(I-\alpha_n\mu G)z_n$ for $n\geq 0$. Let $p\in\Omega$. Since

$$\begin{aligned}\|z_n-p\|^2&=\|J^{B_1}_{\lambda_n}u_n-J^{B_1}_{\lambda_n}p\|^2\\&=\|J^{B_1}_{\lambda_n}(x_n+\eta_nA^*(J^{B_2}_{\nu_n}-I)Ax_n-J^{B_1}_{\lambda_n}p\|^2\\&\leq\|x_n+\eta_nA^*(J^{B_2}_{\nu_n}-I)Ax_n-p\|^2\\&=\|x_n-p\|^2+\eta_n^2\|A^*(J^{B_2}_{\nu_n}-I)Ax_n\|^2+2\eta_n\langle x_n-p,A^*(J^{B_2}_{\nu_n}-I)Ax_n\rangle,\end{aligned}\tag{3.3}$$

we have

$$\begin{aligned}\|z_n-p\|^2\leq\|x_n-p\|^2&+\eta_n^2\langle(J^{B_2}_{\nu_n}-I)Ax_n,AA^*(J^{B_2}_{\nu_n}-I)Ax_n\rangle\\&+2\eta_n\langle x_n-p,A^*(J^{B_2}_{\nu_n}-I)Ax_n\rangle.\end{aligned}\tag{3.4}$$

Observe that

$$\eta_n^2\langle J^{B_2}_{\nu_n}-I)Ax_n,AA^*(J^{B_2}_{\nu_n}-I)Ax_n\rangle\leq L\eta_n^2\|(J^{B_2}_{\nu_n}-I)Ax_n\|^2.\tag{3.5}$$

Moreover, from (2.5), we obtain

$$\begin{aligned}2\eta_n\langle x_n-p,A^*(J^{B_2}_{\nu_n}-I)Ax_n\rangle&=2\eta_n\langle A(x_n-p),(J^{B_2}_{\nu_n}-I)Ax_n\rangle\\&=2\eta_n\langle A(x_n-p)+(J^{B_2}_{\nu_n}-I)Ax_n-(J^{B_2}_{\nu_n}-I)Ax_n,(J^{B_2}_{\nu_n}-I)Ax_n\rangle\\&=2\eta_n[\langle(J^{B_2}_{\nu_n}Ax_n-Ap,J^{B_2}_{\nu_n}-I)Ax_n\rangle-\|(J^{B_2}_{\nu_n}-I)Ax_n\|^2]\\&\leq2\eta_n[\frac{1}{2}\|(J^{B_2}_{\nu_n}-I)Ax_n\|^2-\|(J^{B_2}_{\nu_n}-I)Ax_n\|^2]\\&=-\eta_n\|(J^{B_2}_{\nu_n}-I)Ax_n\|^2.\end{aligned}\tag{3.6}$$

From (3.3), (3.4), (3.5), and (3.6), we have

$$\begin{aligned}\|z_n-p\|^2=\|J^{B_1}_{\lambda_n}u_n-J^{B_1}_{\lambda_n}p\|^2&\leq\|u_n-p\|^2\\&=\|x_n+\eta_nA^*(J^{B_2}_{\nu_n}-I)Ax_n-p\|^2\\&\leq\|x_n-p\|^2+\eta_n(L\eta_n-1)\|(J^{B_2}_{\nu_n}-I)Ax_n\|^2\\&\leq\|x_n-p\|^2\ \ (\text{by }\eta_n\in(0,\frac{1}{L})).\end{aligned}\tag{3.7}$$

Now, we divide the proof into the following steps.

Step 1. Show that $\{x_n\}$ is bounded.

In view of $p=J^{B_1}_{\lambda_n}p$, $p=K_np$, and $p=T_{r_n}p$, we obtain from (3.7) that

$$\begin{aligned}\|y_n-p\|&=\|\alpha_n\gamma Vx_n+(I-\alpha_n\mu G)z_n-p\|\\&\leq\alpha_n\|\gamma Vx_n-\gamma Vp\|+\alpha_n\|\gamma Vp-\mu Gp\|+(1-\alpha_n\tau)\|u_n-p\|\\&\leq\alpha_n\gamma l\|x_n-p\|+\alpha_n\|\gamma Vp-\mu Gp\|+(1-\alpha_n\tau)\|x_n-p\|\\&=(1-(\tau-\gamma l)\alpha_n)\|x_n-p\|+\alpha_n\|\gamma Vp-\mu Gp\|.\end{aligned}\tag{3.8}$$

Thus, since T_{r_n} is nonexpansive (by Lemma 2.4), we deduce from (3.7) and (3.8) that

$$\begin{aligned}
\|x_{n+1}-p\| &\le \beta_n\|x_n-p\|+(1-\beta_n)\|T_{r_n}y_n-p\| \\
&\le \beta_n\|x_n-p\|+(1-\beta_n)\|y_n-p\| \\
&\le \beta_n\|x_n-p\|+(1-\beta_n)[(1-(\tau-\gamma l)\alpha_n)\|x_n-p\|+\alpha_n\|\gamma Vp-\mu Gp\|] \\
&= (1-(1-\beta_n)(\tau-\gamma l)\alpha_n)\|x_n-p\|+(1-\beta_n)(\tau-\gamma l)\alpha_n\frac{\|\gamma Vp-\mu Gp\|}{\tau-\gamma l} \\
&\le \max\left\{\|x_n-p\|,\frac{\|\gamma Vp-\mu Gp\|}{\tau-\gamma l}\right\}.
\end{aligned}$$

Using an induction, we have

$$\|x_n-p\|\le \max\left\{\|x_0-p\|,\frac{\|\gamma Vp-\mu Gp\|}{\tau-\gamma l}\right\}.$$

Hence, $\{x_n\}$ is bounded. So, $\{y_n\}$, $\{u_n\}=\{K_nx_n\}$, $\{z_n\}=\{J_{\lambda_n}^{B_1}u_n\}$, $\{Gx_n\}$, $\{GJ_{\lambda_n}^{B_1}u_n\}$, $\{w_n\}$ $=\{T_{r_n}y_n\}$, and $\{Vx_n\}$ are bounded. It follows from (3.1) and condition (C1) that

$$\|y_n-z_n\|=\|y_n-J_{\lambda_n}^{B_1}u_n\|=\alpha_n\|\gamma Vx_n-\mu GJ_{\lambda_n}^{B_1}u_n\|\to 0 \ \text{ as } n\to\infty.$$

Step 2. Show that $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$.

For this purpose, we derive

$$\begin{aligned}
\|y_n-y_{n-1}\| &= \|\alpha_n\gamma Vx_n+(I-\alpha_n\mu G)J_{\lambda_n}^{B_1}u_n-(\alpha_{n-1}\gamma Vx_{n-1}+(I-\alpha_{n-1}\mu G)J_{\lambda_{n-1}}^{B_1}u_{n-1})\| \\
&\le \|(\alpha_n-\alpha_{n-1})\gamma Vx_{n-1}+\alpha_n(\gamma Vx_n-\gamma Vx_{n-1})\| \\
&\quad +\|(I-\alpha_n\mu G)J_{\lambda_n}^{B_1}u_n-(I-\alpha_n\mu G)J_{\lambda_{n-1}}^{B_1}u_{n-1}\| \\
&\quad +\|(I-\alpha_n\mu G)J_{\lambda_{n-1}}^{B_1}u_{n-1}-(I-\alpha_{n-1}\mu G)J_{\lambda_{n-1}}^{B_1}u_{n-1}\| \\
&\le |\alpha_n-\alpha_{n-1}|\|\gamma Vx_{n-1}\|+\alpha_n\gamma l\|x_n-x_{n-1}\| \\
&\quad +(1-\alpha_n\tau)\|J_{\lambda_n}^{B_1}u_n-J_{\lambda_{n-1}}^{B_1}u_{n-1}\|+|\alpha_n-\alpha_{n+1}|\|\mu GJ_{\lambda_{n-1}}^{B_1}u_{n-1}\| \\
&= |\alpha_n-\alpha_{n-1}|(\|\gamma Vx_{n-1}\|+\|\mu GJ_{\lambda_{n-1}}^{B_1}u_{n-1}\|) \\
&\quad +\alpha_n\gamma l\|x_n-x_{n-1}\|+(1-\alpha_n\tau)\|J_{\lambda_n}^{B_1}u_n-J_{\lambda_{n-1}}^{B_1}u_{n-1}\| \\
&\le |\alpha_n-\alpha_{n-1}|M_1+\alpha_n\gamma l\|x_n-x_{n-1}\|+(1-\alpha_n\tau)\|J_{\lambda_n}^{B_1}u_n-J_{\lambda_{n-1}}^{B_1}u_{n-1}\|,
\end{aligned} \tag{3.9}$$

where $M_1>0$ is an appropriate constant. From the resolvent identity (2.2) and condition (C5) $(0<\lambda\le\lambda_n$ for $n\ge 0)$, we have

$$\begin{aligned}
\|J_{\lambda_n}^{B_1}u_n-J_{\lambda_{n-1}}^{B_1}u_{n-1}\| &= \left\|J_{\lambda_{n-1}}^{B_1}\left(\frac{\lambda_{n-1}}{\lambda_n}u_n+\left(1-\frac{\lambda_{n-1}}{\lambda_n}\right)J_{\lambda_n}^{B_1}u_n\right)-J_{\lambda_{n-1}}^{B_1}u_{n-1}\right\| \\
&\le \left\|\frac{\lambda_{n-1}}{\lambda_n}(u_n-u_{n-1})+\left(1-\frac{\lambda_{n-1}}{\lambda_n}\right)(J_{\lambda_n}^{B_1}u_n-u_{n-1})\right\| \\
&\le \|u_n-u_{n-1}\|+\frac{|\lambda_n-\lambda_{n-1}|}{\lambda_n}\|J_{\lambda_n}^{B_1}u_n-u_n\| \\
&\le \|u_n-u_{n-1}\|+\frac{|\lambda_n-\lambda_{n-1}|}{\lambda}M_2,
\end{aligned} \tag{3.10}$$

where $M_2 > 0$ is an appropriate constant. Again, since $K_n = I + \eta_n A^*(J_{\nu_n}^{B_2} - I)A$ is nonexpansive as averaged ([16]), we calculate

$$\begin{aligned}
&\|u_n - u_{n-1}\| \\
&= \|(I + \eta_n A^*(J_{\nu_n}^{B_2} - I)A)x_n - (I + \eta_{n-1}A^*(J_{\nu_{n-1}}^{B_2} - I)A)x_{n-1}\| \\
&= \|K_n x_n - K_n x_{n-1}\| + \|K_n x_{n-1} - K_{n-1}x_{n-1}\| \\
&\le \|x_n - x_{n-1}\| + \|(x_{n-1} + \eta_n A^*(J_{\nu_n}^{B_2} - I)Ax_{n-1}) - (x_{n-1} + \eta_{n-1}A^*(J_{\nu_{n-1}}^{B_2} - I)Ax_{n-1})\| \\
&\le \|x_n - x_{n-1}\| + \|\eta_n A^*(J_{\nu_n}^{B_2} - I)Ax_{n-1} - \eta_{n-1}A^*(J_{\nu_n}^{B_2} - I)Ax_{n-1}\| \\
&\quad + \|\eta_{n-1}A^*(J_{\nu_n}^{B_2} - I)Ax_{n-1} - \eta_{n-1}A^*(J_{\nu_{n-1}}^{B_2} - I)Ax_{n-1}\| \\
&\le \|x_n - x_{n-1}\| + |\eta_n - \eta_{n-1}|\|A^*(J_{\nu_n}^{B_2} - I)Ax_{n-1}\| \\
&\quad + \eta_{n-1}\|A^*\|\|J_{\nu_n}^{B_2}(Ax_{n-1}) - J_{\nu_{n-1}}^{B_2}(Ax_{n-1})\| \\
&\le \|x_n - x_{n-1}\| + |\eta_n - \eta_{n-1}|M_3 + \frac{1}{L}\|A^*\|\|J_{\nu_n}^{B_2}(Ax_{n-1}) - J_{\nu_{n-1}}^{B_2}(Ax_{n-1})\|,
\end{aligned} \tag{3.11}$$

where $M_3 > 0$ is an appropriate constant. From the resolvent identity (2.2) and condition (C6) ($0 < \nu \le \nu_n$ for $n \ge 0$), we deduce

$$\begin{aligned}
&\|J_{\nu_n}^{B_2}(Ax_{n-1}) - J_{\nu_{n-1}}^{B_2}(Ax_{n-1})\| \\
&= \left\| J_{\nu_{n-1}}^{B_2}\left(\frac{\nu_{n-1}}{\nu_n}Ax_{n-1} + \left(1 - \frac{\nu_{n-1}}{\nu_n}\right)J_{\nu_n}^{B_2}(Ax_{n-1})\right) - J_{\nu_{n-1}}^{B_2}(Ax_{n-1})\right\| \\
&\le \left|1 - \frac{\nu_{n-1}}{\nu_n}\right| \|J_{\nu_n}^{B_2}(Ax_{n1}) - Ax_{n-1}\| \\
&\le \frac{|\nu_n - \nu_{n-1}|}{\nu_n}\|(J_{\nu_n}^{B_2} - I)Ax_{n-1}\| \\
&\le \frac{|\nu_n - \nu_{n-1}|}{\nu}M_4,
\end{aligned} \tag{3.12}$$

where $M_4 > 0$ is an appropriate constant. Substituting (3.12) into (3.11), we arrive at

$$\|u_n - u_{n-1}\| \le \|x_n - x_{n-1}\| + |\eta_n - \eta_{n-1}|M_3 + \frac{1}{L}\|A^*\||\nu_n - \nu_{n-1}|\frac{M_4}{\nu}. \tag{3.13}$$

From (3.9), (3.10) and (3.13), we drive

$$\begin{aligned}
\|y_n - y_{n-1}\| &\le |\alpha_n - \alpha_{n-1}|M_1 + \alpha_n\gamma l\|x_n - x_{n-1}\| + (1 - \alpha_n\tau)\|J_{\lambda_n}^{B_1}u_n - J_{\lambda_{n-1}}^{B_1}u_{n-1}\| \\
&\le |\alpha_n - \alpha_{n-1}|M_1 + \alpha_n\gamma l\|x_n - x_{n-1}\| \\
&\quad + (1 - \alpha_n\tau)\left(\|u_n - u_{n-1}\| + |\lambda_n - \lambda_{n-1}|\frac{M_2}{\lambda}\right) \\
&\le |\alpha_n - \alpha_{n-1}|M_1 + \alpha_n\gamma l\|x_n - x_{n-1}\| \\
&\quad + (1 - \alpha_n\tau)\Big[\|x_n - x_{n-1}\| + |\eta_n - \eta_{n-1}|M_3 \\
&\quad + \frac{1}{L}\|a^*\||\nu_n - \nu_{n-1}|\frac{M_4}{\nu} + |\lambda_n - \lambda_{n-1}|\frac{M_2}{\lambda}\Big].
\end{aligned} \tag{3.14}$$

On the other hand, let $w_n = T_{r_n} y_n$. It follows from $w_{r_{n-1}} = T_{r_{n-1}} y_{n-1}$ that

$$\langle y - w_n, Tw_n \rangle - \frac{1}{r_n} \langle y - w_n, (1 + r_n) w_n - y_n \rangle \le 0 \text{ for all } y \in H_1, \tag{3.15}$$

and

$$\langle y - w_{n-1}, Tw_{n-1} \rangle - \frac{1}{r_{n-1}} \langle y - w_{n-1}, (1 + r_{n-1}) w_{n-1} - y_{n-1} \rangle \le 0 \text{ for all } y \in H_1. \tag{3.16}$$

Putting $y := z_{n-1}$ in (3.15) and $y := z_n$ in (3.16), we obtain

$$\langle w_{n-1} - w_n, Tw_n \rangle - \frac{1}{r_n} \langle w_{n-1} - w_n, (1 + r_n) w_n - y_n \rangle \le 0, \tag{3.17}$$

and

$$\langle w_n - w_{n-1}, Tw_{n-1} \rangle - \frac{1}{r_{n-1}} \langle w_n - w_{n-1}, (1 + r_{n-1}) w_{n-1} - y_{n-1} \rangle \le 0. \tag{3.18}$$

Adding up (3.17) and (3.18), we obtain

$$\langle w_{n-1} - w_n, Tw_n - Tw_{n-1} \rangle - \left\langle w_{n-1} - w_n, \frac{(1 + r_n) w_n - y_n}{r_n} - \frac{(1 + r_{n-1}) w_{n-1} - y_{n-1}}{r_{n-1}} \right\rangle \le 0. \tag{3.19}$$

Using the fact that T is pseudocontractive, we conclude from (3.19) that

$$\left\langle w_{n-1} - w_n, \frac{w_n - y_n}{r_n} - \frac{w_{n-1} - y_{n-1}}{r_{n-1}} \right\rangle \ge 0,$$

and hence

$$\left\langle w_{n-1} - w_n, w_n - w_{n-1} + w_{n-1} - y_n - \frac{r_n}{r_{n-1}} (w_{n-1} - y_{n-1}) \right\rangle \ge 0. \tag{3.20}$$

From (3.20) and condition (C4) ($0 < r \le r_n$ for $n \ge 0$), we derive

$$\begin{aligned} \|w_n - w_{n-1}\|^2 &\le \left\langle w_{n-1} - w_n, y_{n-1} - y_n + (1 - \frac{r_n}{r_{n-1}})(w_{n-1} - y_{n-1}) \right\rangle \\ &\le \|w_{n-1} - w_n\| \left(\|y_{n-1} - y_n\| + \frac{|r_n - r_{n-1}|}{r} \|w_{n-1} - y_{n-1}\| \right). \end{aligned}$$

Thus,

$$\begin{aligned} \|w_n - w_{n-1}\| &\le \|y_{n-1} - y_n\| + \frac{|r_n - r_{n-1}|}{r} \|w_{n-1} - y_{n-1}\| \\ &\le \|y_{n-1} - y_n\| + \frac{|r_n - r_{n-1}|}{r} M_5, \end{aligned} \tag{3.21}$$

where $M_5 > 0$ is an appropriate constant. Substituting (3.14) into (3.21) yields

$$\begin{aligned} \|w_n - w_{n-1}\| &= \|T_{r_n} y_n - T_{r_{n-1}} y_{n-1}\| \\ &\le (1 - \alpha_n(\tau - \gamma l)) \|x_n - x_{n-1}\| + |\alpha - \alpha_{n-1}| M_1 + |\eta_n - \eta_{n-1}| M_3 \\ &\quad + \frac{1}{L} \|A^*\| |\nu_n - \nu_{n-1}| \frac{M_4}{\nu} + |\lambda_n - \lambda_{n-1}| \frac{M_2}{\lambda} + |r_n - r_{n-1}| \frac{M_5}{r}. \end{aligned} \tag{3.22}$$

In view of conditions (C1), (C4), (C5), (C6), and (C7), we find from (3.22) that

$$\limsup_{n \to \infty} (\|w_n - w_{n-1}\| - \|x_n - x_{n-1}\| \le 0.$$

Thus, by Lemma 2.2, we have

$$\lim_{n\to\infty}\|w_n - x_n\| = \lim_{n\to\infty}\|T_{r_n}y_n - x_n\| = 0. \tag{3.23}$$

Since $x_{n+1} - x_n = (1-\beta_n)(w_n - x_n)$, we conclude from (3.23) and condition (3) that

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0.$$

Step 3. Show that $\lim_{n\to\infty}\|u_n - z_n\| = \lim_{n\to\infty}\|u_n - J_{\lambda_n}^{B_1}u_n\| = 0$.

To this end, by (3.7), we see that

$$\begin{aligned} \|u_n - p\|^2 &= \|x_n + \eta_n A^*(J_{\nu_n}^{B_2} - I)Ax_n - p\|^2 \\ &\leq \|x_n - p\|^2 + \eta_n(L\eta_n - 1)\|(J_{\nu_t}^{B_2} - I)Ax_n\|^2 \\ &\leq \|x_n - p\|^2 \ \ (\text{by } \eta_n \in (0, \frac{1}{L})). \end{aligned} \tag{3.24}$$

Again, since $J_{\lambda_n}^{B_1}$ is firmly nonexpansive, it follows from (2.1) and (2.3) that

$$\begin{aligned} \|z_n - p\|^2 &\leq \langle J_{\lambda_n}^{B_1}u_n - J_{\lambda_n}^{B_1}p, u_n - p\rangle \\ &= \frac{1}{2}[\|u_n - p\|^2 + \|z_n - p\|^2 - \|u_n - z_n\|^2] \end{aligned}$$

and hence

$$\|z_n - p\|^2 \leq \|u_n - p\|^2 - \|u_n - z_n\|^2 \leq \ \|x_n - p\|^2 - \|u_n - z_n\|^2. \tag{3.25}$$

Thus, by (2.4), (3.1), and (3.25), we obtain

$$\begin{aligned} \|x_{n+1} - p\|^2 &\leq \beta_n\|x_n - p\|^2 + (1-\beta_n)\|T_{r_n}y_n - p\|^2 \\ &\leq \beta_n\|x_n - p\|^2 + (1-\beta_n)\|y_n - p\|^2 \\ &= \beta_n\|x_n - p\|^2 + (1-\beta_n)\|\alpha_n(\gamma Vx_n - \mu Gp) + (I - \alpha_n\mu G)J_{\lambda_n}^{B_1}u_n - (I - \alpha_n\mu G)p\|^2 \\ &\leq \beta_n\|x_n - p\|^2 + (1-\beta_n)(\alpha_n\|\gamma Vx_n - \mu Gp\| + (1-\alpha_n\tau)\|z_n - p\|)^2 \\ &\leq \beta_n\|x_n - p\|^2 + (1-\beta_n)(\alpha_n\|\gamma Vx_n - \mu Gp\| + \|z_n - p\|)^2 \\ &\leq \beta_n\|x_n - p\|^2 + (1-\beta_n)[\alpha_n M_6 + \|x_n - p\|^2 - \|u_n - z_n\|^2] \\ &\leq \|x_n - p\|^2 + (1-\beta_n)(\alpha_n M_6 - \|u_n - z_n\|^2), \end{aligned} \tag{3.26}$$

where $M_6 > 0$ is an appropriate constant, and so

$$\begin{aligned} \|u_n - z_n\|^2 \leq & \alpha_n M_6 + \frac{1}{1-\beta_n}(\|x_n - p\|^2 - \|x_{n+1} - p\|^2) \\ &\leq \alpha_n M_6 + \frac{1}{1-\beta_n}(\|x_n - p\| + \|x_{n-1} - p\|)\|x_n - x_{n+1}\| \\ &\leq \alpha_n M_6 + \frac{M_7}{1-\beta_n}\|x_n - x_{n+1}\|, \end{aligned} \tag{3.27}$$

where $M_7 > 0$ is an appropriate constant. Therefore, by conditions (C1) and (C3), and Step 2, we derive from (3.27) that $\|u_n - z_n\| \to 0$ as $n \to \infty$.

Step 4. Show that $\lim_{n\to\infty}\|z_n - x_n\| = \lim_{n\to\infty}\|J_{\lambda_n}^{B_1}u_n - x_n\| = 0$.

From (3.24) and (3.26), we derive

$$\begin{aligned}
\|x_{n+1}-p\|^2 &\leq \beta_n\|x_n-p\|^2+(1-\beta_n)[\alpha_n M_6+(1-\alpha_n\tau)\|z_n-p\|^2] \\
&\leq \beta_n\|x_n-p\|^2+(1-\beta_n)[\alpha_n M_6+\|u_n-p\|^2-\|u_n-z_n\|^2] \\
&\leq \beta_n\|x_n-p\|^2+(1-\beta_n)[\alpha_n M_6+\|u_n-p\|^2] \\
&\leq \|x_n-p\|^2+(1-\beta_n)[\alpha_n M_6+\|x_n-p\|^2+\eta_n(L\eta_n-1)\|(J^{B_2}_{\nu_n}-I)Ax_n\|^2] \\
&= \|x_n-p\|^2+(1-\beta_n)\alpha_n M_6+(1-\beta_n)\eta_n(L\eta_n-1)\|(J^{B_2}_{\nu_n}-I)Ax_n\|^2,
\end{aligned}$$

and hence

$$\begin{aligned}
\eta_n(1-L\eta_n)\|(J^{B_2}_{\nu_n}-I)Ax_n\|^2 &\leq \alpha_n M_6+\frac{1}{1-\beta_n}(\|x_n-p\|^2-\|x_{n+1}-p\|^2) \\
&\leq \alpha_n M_6+\frac{M_7}{1-\beta_n}\|x_n-x_{n+1}\|.
\end{aligned}$$

Since $(1-L\eta_n)>0$ and $0<\liminf_{n\to\infty}\eta_n$, we induce from conditions (C1) and (C3) and Step 2 that

$$\|(J^{B_2}_{\nu_n}-I)Ax_n\|\to 0 \text{ as } n\to\infty. \tag{3.28}$$

From (2.1), (2.3), (3.7), and $\eta_n\in(0,\frac{1}{L})$, we have

$$\begin{aligned}
\|z_n-p\|^2 &= \|J^{B_1}_{\lambda_n}(x_n+\eta_n A^*(J^{B_2}_{\nu_n}-I)Ax_n)-p\|^2 \\
&= \|J^{B_1}_{\lambda_n}(x_n+\eta_n A^*(J^{B_2}_{\nu_n}-I)Ax_n)-J^{B_1}_{\lambda_n}p\|^2 \\
&\leq \langle z_n-p, x_n+\eta_n A^*(J^{B_2}_{\nu_n}-I)Ax_n-p\rangle \\
&= \frac{1}{2}\{\|z_n-p\|^2+\|x_n+\eta_n A^*(J^{B_2}_{\nu_n}-I)Ax_n-p\|^2 \\
&\qquad -\|(z_n-p)-(x_n+\eta_n A^*(J^{B_2}_{\nu_n}-I)Ax_n-p)\|^2\} \\
&\leq \frac{1}{2}\{\|z_n-p\|^2+\|x_n-p\|^2+\eta_n(L\eta_n-1)\|(J^{B_2}_{\nu_n}-I)Ax_n\|^2 \\
&\qquad -\|(z_n-x_n)-\eta_n A^*(J^{B_2}_{\nu_n}-I)Ax_n\|^2\} \\
&\leq \frac{1}{2}\{\|z_n-p\|^2+\|x_n-p\|^2-[\|z_n-x_n\|^2 \\
&\qquad +\eta_n^2\|A^*(J^{B_2}_{\nu_n}-I)Ax_n\|^2-2\eta_n\langle z_n-x_n, A^*(J^{B_2}_{\nu_n}-I)Ax_n\rangle]\} \\
&\leq \frac{1}{2}\{\|z_n-p\|^2+\|x_n-p\|^2-\|z_n-x_n\|^2+2\eta_n\|A(z_n-x_n)\|\|(J^{B_2}_{\nu_n}-I)Ax_n\|\}.
\end{aligned} \tag{3.29}$$

Hence, by (3.29), we obtain

$$\|z_n-p\|^2\leq\|x_n-p\|^2-\|z_n-x_n\|^2+2\eta_n\|A(z_n-x_n)\|\|(J^{B_2}_{\nu_n}-I)Ax_n\|. \tag{3.30}$$

Again, from (2.4), (3.1) and (3.30), we derive that

$$\begin{aligned}
\|x_{n+1}-p\|^2 &\le \beta_n\|x_n-p\|^2+(1-\beta_n)[\alpha_n\|\gamma Vx_n-\mu Gp\|+(1-\alpha_n\tau)\|z_n-p\|]^2\\
&\le \beta_n\|x_n-p\|^2+(1-\beta_n)[\alpha_nM_6+\|z_n-p\|^2]\\
&\le \beta_n\|x_n-p\|^2\\
&\quad+(1-\beta_n)[\alpha_nM_6+(\|x_n-p\|^2-\|z_n-x_n\|^2+2\eta_n\|A(z_n-x_n)\|\|(J^{B_2}_{\nu_n}-I)Ax_n\|)]\\
&= \|x_n-p\|^2+(1-\beta_n)\alpha_nM_6-(1-\beta_n)\|z_n-x_n\|^2\\
&\quad+2(1-\beta_n)\eta_n\|A(z_n-x_n)\|\|(J^{B_2}_{\nu_n}-I)Ax_n\|
\end{aligned}$$

and hence

$$\begin{aligned}
\|z_n-x_n\|^2 \le&\alpha_nM_6+2\eta_n\|A(z_n-x_n)\|\|(J^{B_2}_{\nu_n}-I)Ax_n\|+\frac{1}{1-\beta_n}(\|x_n-p\|^2-\|x_{n+1}-p\|^2)\\
\le&\alpha_nM_6+2\eta_n\|A(z_n-x_n)\|\|(J^{B_2}_{\nu_n}-I)Ax_n\|\\
&+\frac{1}{1-\beta_n}(\|x_n-p\|+\|x_{n+1}-p\|)\|x_n-x_{n+1}\|\\
\le&\alpha_nM_6+2\eta_n\|A(z_n-x_n)\|\|(J^{B_2}_{\nu_n}-I)Ax_n\|+\frac{1}{1-\beta_n}\|x_n-x_{n+1}\|M_7.
\end{aligned} \tag{3.31}$$

From conditions (C1) and (C3), Step 2, (3.28), and (3.31), we obtain

$$\|z_n-x_n\|\to 0 \text{ as } n\to\infty.$$

Step 5. Show that $\lim_{n\to\infty}\|x_n-u_n\|=0$.

In fact, by Steps 3 and 4, we have

$$\|x_n-u_n\|\le\|x_n-z_n\|+\|z_n-u_n\|\to 0 \text{ as } n\to\infty.$$

Step 6. Show that $\lim_{n\to\infty}\|x_n-y_n\|=0$.

In fact, since

$$\begin{aligned}
\|x_n-y_n\| &= \|x_n-(\alpha_n\gamma Vx_n+(I-\alpha_n\mu G)z_n)\|\\
&\le \alpha_n\|\mu Gx_n-\gamma Vx_n\|+\|(I-\alpha_n\mu G)x_n-(I-\alpha_n\mu G)z_n\|\\
&\le \alpha_nM_8+(1-\alpha_n\tau)\|x_n-z_n\|\\
&\le \alpha_nM_8+\|x_n-z_n\|,
\end{aligned}$$

where $M_8>0$ is an appropriate constant, by condition (C1) and Step 4, we obtain

$$\|x_n-y_n\|\to 0 \text{ as } n\to\infty.$$

Step 7. Show that $\lim_{n\to\infty}\|y_n-T_{r_n}y_n\|=\lim_{n\to\infty}\|y_n-w_n\|=0$.

Indeed, from (3.23) and Step 6, we obtain

$$\|y_n-T_{r_n}y_n\|\le\ \|y_n-x_n\|+\|x_n-w_n\|\to 0 \text{ as } n\to\infty.$$

Step 8. Show that

$$\limsup_{n\to\infty}\langle(\gamma V-\mu G)q,y_n-q\rangle\le 0.$$

For this purpose, we can choose a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ such that

$$\lim_{i\to\infty}\langle(\gamma V-\mu G)q, y_{n_i}-q\rangle = \limsup_{n\to\infty}\langle(\gamma V-\mu G)q, y_n-q\rangle.$$

Since $\{y_{n_i}\}$ is bounded, there exists a subsequence $\{y_{n_{i_j}}\}$ of $\{y_{n_i}\}$ which converges weakly to some point z. Without loss of generality, we can assume that $y_{n_i} \rightharpoonup z$. First of all, we show $z \in \Omega$. To this end, we divide its proof into three steps.

(i) We prove that $z \in Fix(T)$. To show this, put $w_n = T_{r_n}y_n$. Then, by Lemma 2.4, we have

$$\langle y-w_n, Tw_n\rangle - \frac{1}{r_n}\langle y-w_n, (1+r_n)w_n-y_n\rangle \le 0 \ \text{ for all } y\in H_1. \tag{3.32}$$

Put $v_\varepsilon = \varepsilon v+(1-\varepsilon)z$ for $\varepsilon\in(0,1]$ and $v\in H_1$. Then $v_\varepsilon\in H_1$, and we derive from (3.32) and pseudocontractivity of T that

$$\begin{aligned}\langle w_n-v_\varepsilon, Tv_\varepsilon\rangle &\ge \langle w_n-v_\varepsilon, Tv_\varepsilon\rangle + \langle v_\varepsilon-w_n, Tw_n\rangle - \frac{1}{r_n}\langle v_\varepsilon-w_n, (1+r_n)w_n-y_n\rangle\\ &= -\langle v_\varepsilon-w_n, Tv_\varepsilon-Tw_n\rangle - \frac{1}{r_n}\langle v_\varepsilon-w_n, w_n-y_n\rangle - \langle v_\varepsilon-w_n, w_n\rangle\\ &\ge -\|v_\varepsilon-w_n\|^2 - \frac{1}{r_n}\langle v_\varepsilon-w_n, w_n-y_n\rangle - \langle v_\varepsilon-w_n, w_n\rangle\\ &= -\langle v_\varepsilon-w_n, v_\varepsilon\rangle - \langle v_\varepsilon-w_n, \frac{w_n-y_n}{r_n}\rangle.\end{aligned} \tag{3.33}$$

Since $\{y_n\}$ and $\{w_n\}$ have the same asymptotical behavior (due to Step 7), $\{w_{n_i}\}$ converges weakly to z as $i\to\infty$. Also, by Step 7, we have $\frac{\|w_n-y_n\|}{r_n} \le \frac{\|w_n-y_n\|}{r} \to 0$. So, replacing n by n_i and letting $i\to\infty$, we derive from (3.33) that

$$\langle z-v_\varepsilon, Tv_\varepsilon\rangle \ge \langle z-v_\varepsilon, v_\varepsilon\rangle$$

and

$$-\langle v-z, Tv_\varepsilon\rangle \ge -\langle v-z, v_\varepsilon\rangle \ \text{ for all } v\in H_1.$$

Letting $\varepsilon\to 0$ and using the fact that T is continuous, we obtain

$$-\langle v-z, Tz\rangle \ge -\langle v-z, z\rangle \ \text{ for all } v\in H_1. \tag{3.34}$$

Let $v = Tz$ in (3.34). Then, $z = Tz$, that is, $z\in Fix(T)$.

(ii) We prove that $z\in B_1^{-1}0$. To this end, let $z_n = J_{\lambda_n}^{B_1}u_n$. Then it follows that

$$u_n \in (I+\lambda_n B_1)z_n, \ \text{ that is, } \ \frac{u_n-z_n}{\lambda_n}\in B_1z_n.$$

Since B is monotone, we know that, for any $v\in B_1u$,

$$\langle z_n-u, \frac{u_n-z_n}{\lambda_n}-v\rangle \ge 0. \tag{3.35}$$

And, by Step 3, Step 5, and Step 6, we obtain that

$$\|y_n-z_n\| \le \|y_n-x_n\| + \|x_n-u_n\| + \|u_n-z_n\| \to 0 \text{ as } n\to\infty$$

and $\frac{\|u_n-z_n\|}{\lambda_n} \le \frac{\|u_n-z_n\|}{\lambda} \to 0$ as $n\to\infty$. Thus, $u_{n_i}\rightharpoonup z$ and $z_{n_i}\rightharpoonup z$ as $i\to\infty$. By replacing n by n_i in (3.35) and letting $i\to\infty$, we have $\langle z-u, -v\rangle \ge 0$. Since B_1 is maximal monotone, we get $0\in B_1z$, that is, $z\in SOLVIP(B_1)$.

(iii) We prove that $Az \in SOLVIP(B_2)$. In fact, since $\{x_n\}$ and $\{u_n\}$ have the same asymptotical behavior (due to Step 5), $\{Ax_{n_i}\}$ converges weakly to Az. Again, let $\hat{\nu} > 0$. Then, using the resolvent identity (2.2), condition (C6) ($0 < \nu \leq \nu_n$ for $n \geq 0$), and (3.28), we estimate

$$\begin{aligned}
\|J^{B_2}_{\nu_{n_i}}Ax_{n_i} - J^{B_2}_{\hat{\nu}}Ax_{n_i}\| &= \left\| J^{B_2}_{\hat{\nu}}\left(\frac{\hat{\nu}}{\nu_{n_i}}Ax_{n_i} + \left(1 - \frac{\hat{\nu}}{\nu_{n_i}}\right)J^{B_2}_{\nu_{n_i}}Ax_{n_i}\right) - J^{B_2}_{\hat{\nu}}Ax_{n_i}\right\| \\
&\leq \left|1 - \frac{\hat{\nu}}{\nu_{n_i}}\right| \|J^{B_2}_{\nu_{n_i}}Ax_{n_i} - Ax_{n_i}\| \\
&\leq \frac{|\nu_{n_i} - \hat{\nu}|}{\nu}\|(J^{B_2}_{\nu_{n_i}} - I)Ax_{n_i}\| \to 0 \text{ as } i \to \infty.
\end{aligned} \tag{3.36}$$

Hence, from (3.36), it follows that

$$\lim_{i\to\infty}\|(J^{B_2}_{\hat{\nu}} - I)Ax_{n_i}\| = \lim_{i\to\infty}\|(J^{B_2}_{\nu_{n_i}} - I)Ax_{n_i}\| = 0. \tag{3.37}$$

Since $J^{B_2}_{\hat{\nu}}$ is nonexpansive, we obtain from (3.37) and Lemma 2.7 that $Az = J^{B_2}_{\hat{\nu}}(Az)$, that is, $Az \in SOLVIP(B_2)$ This along with (i) and (ii) obtains $z \in \Omega$.

Thus,

$$\begin{aligned}
\limsup_{n\to\infty}\langle(\gamma V - \mu G)q, y_n - q\rangle &= \lim_{i\to\infty}\langle(\gamma V - \mu G)q, y_{n_i} - q\rangle \\
&= \langle(\gamma V - \mu G)q, z - q\rangle \leq 0.
\end{aligned}$$

Step 9. Show that $\lim_{n\to\infty}\|x_n - q\| = 0$.

Indeed, from Lemma 2.1, we derive

$$\begin{aligned}
\|y_n - q\|^2 &= \|\alpha_n\gamma Vx_n + (I - \alpha_n\mu G)z_n - q\|^2 \\
&= \|\alpha_n(\gamma Vx_n - \gamma Vq) + \alpha_n(\gamma Vq - \mu Gq) + (I - \alpha_n\mu G)z_n - (I - \alpha_n\mu G)q\|^2 \\
&\leq \|\alpha_n(\gamma Vx_n - \gamma Vq) + (I - \alpha_n\mu G)z_n - (I - \alpha_n\mu G)q\|^2 + 2\alpha_n\langle\gamma Vq - \mu Gq, y_n - q\rangle \\
&\leq (\alpha_n\gamma l\|x_n - q\| + (1 - \alpha_n\tau)\|x_n - q\|)^2 + 2\alpha_n\langle\gamma Vq - \mu Gq, y_n - q\rangle \\
&= ((1 - (\tau - \gamma l)\alpha_n)\|x_n - q\|)^2 + 2\alpha_n\langle\gamma Vq - \mu Gq, y_n - q\rangle.
\end{aligned} \tag{3.38}$$

Thus, by (2.3), (3.1), and (3.38), we obtain

$$\begin{aligned}
\|x_{n+1} - q\|^2 &\leq \beta_n\|x_n - q\|^2 + (1 - \beta_n)\|T_{r_n}y_n - q\|^2 \\
&\leq \beta_n\|x_n - q\|^2 + (1 - \beta_n)\|y_n - q\|^2 \\
&\leq \beta_n\|x_n - q\|^2 + (1 - \beta_n)((1 - (\tau - \gamma l)\alpha_n)\|x_n - q\|)^2 \\
&\quad + 2(1 - \beta_n)\alpha_n\langle\gamma Vq - \mu Gq, y_n - q\rangle \\
&\leq \beta_n\|x_n - q\|^2 + (1 - \beta_n)(1 - (\tau - \gamma l)\alpha_n)\|x_n - q\|^2 \\
&\quad + 2(1 - \beta_n)\alpha_n\langle\gamma Vq - \mu Gq, y_n - q\rangle \\
&= (1 - (1 - \beta_n)(\tau - \gamma l)\alpha_n)\|x_n - q\|^2 + 2(1 - \beta_n)(\tau - \gamma l)\alpha_n\frac{\langle\gamma Vq - \mu Gq, y_n - q\rangle}{\tau - \gamma l} \\
&= (1 - \xi_n)\|x_n - q\|^2 + \xi_n\delta_n,
\end{aligned}$$

where $\xi_n = (1-\beta_n)(\tau-\gamma l)\alpha_n$ and $\delta_n = \frac{2\langle \gamma V q - \mu G, y_n - q\rangle}{\tau - \gamma l}$. From conditions (C1), (C2), and (C3), it is easy to see from Step 8 that $\xi_n \to 0$, $\sum_{n=1}^{\infty} \xi_n = \infty$, and $\limsup_{n\to\infty} \delta_n \leq 0$. Hence, by Lemma 2.3, we conclude $\lim_{n\to\infty} \|x_n - q\| = 0$. This completes the proof. □

By taking $V \equiv 0$, $G \equiv I$, and $\mu = 1$ in Theorem 3.1, we obtain the following result.

Corollary 3.2. *Let the sequence $\{x_n\}$ be generated by*

$$\begin{cases} z_n = J_{r_n}^{B_1}(x_n + \eta_n A^*(J_{\nu_n}^{B_2} - I)Ax_n), \\ x_{n+1} = \beta_n x_n + (1-\beta_n)T_{r_n}((1-\alpha_n)z_n), \quad n \geq 0, \end{cases}$$

where $\{\alpha_n\}$, $\{\beta_n\} \subset (0,1)$, $\{r_n\}, \{\lambda_n\}, \{\nu_n\} \subset (0,\infty)$, and $\{\eta_n\} \subset (0, \frac{1}{L})$ satisfy the conditions (C1), (C2), (C3), (C4), (C5), (C6), *and* (C7) *in Theorem 3.1. Then $\{x_n\}$ converges strongly to a point $q \in \Omega$, which is the minimum-norm element of Ω.*

Proof. From (3.2) with $V \equiv 0$, $G \equiv I$, and $\mu = 1$, we derive $0 \leq \langle q, p-q\rangle$ for all $p \in \Omega$. This obviously implies that $\|q\|^2 \leq \langle p, q\rangle \leq \|p\|\|q\|$ for all $p \in \Omega$. It turns out that $\|q\| \leq \|p\|$ for all $p \in \Omega$. Therefore, q is the minimum-norm point of Ω. □

If, in Theorem 3.1, we take $T \equiv I$, identity mapping on H_1, then we obtain the following result.

Corollary 3.3. *Let the sequence $\{x_n\}$ be generated by*

$$\begin{cases} z_n = J_{\lambda_n}^{B_1}(x_n + \eta_n A^*(J_{\nu_n}^{B_2} - I)Ax_n), \\ x_{n+1} = \beta_n x_n + (1-\beta_n)(\alpha_n \gamma V x_n + (I - \alpha_n \mu G)z_n), \quad n \geq 0. \end{cases}$$

where $\{\alpha_n\}$, $\{\beta_n\} \subset (0,1)$, $\{\lambda_n\}, \{\nu_n\} \subset (0,\infty)$, and $\{\eta_n\} \subset (0, \frac{1}{L})$ satisfy the conditions (C1), (C2), (C3), (C5), (C6), *and* (C7) *in Theorem 3.1. Then $\{x_n\}$ converges strongly to a point $q \in \Gamma$, which is the unique solution of the following variational inequality: $\langle(\mu G - \gamma V)q, p-q\rangle \geq 0$ for all $p \in \Gamma$.*

By taking $V \equiv 0$, $G \equiv I$, and $\mu = 1$ in Corollary 3.3, we obtain the following result.

Corollary 3.4. *Let the sequence $\{x_n\}$ be generated by*

$$\begin{cases} z_n = J_{\lambda_n}^{B_1}(x_n + \eta_n A^*(J_{\nu_n}^{B_2} - I)Ax_n), \\ x_{n+1} = \beta_n x_n + (1-\beta_n)(1-\alpha_n)z_n, \quad n \geq 0. \end{cases}$$

Let $\{\alpha_n\}$, $\{\beta_n\} \subset (0,1)$, $\{r_n\}, \{\lambda_n\}, \{\nu_n\} \subset (0,\infty)$ and $\{\eta_n\} \subset (0, \frac{1}{L})$ satisfy the conditions (C1), (C2), (C3), (C5), (C6) *and* (C7) *in Theorem 3.1. Then $\{x_n\}$ converges strongly to a point $q \in \Gamma$, which is the minimum-norm element of Γ.*

Remark 3.5. 1) It is worth pointing out that our iterative algorithms are new different from those announced by several authors; see, e.g., [5, 14, 17, 22] and the references therein. In particular, we use the variable parameters r_n, λ_n, ν_n, and η_n in comparison with the corresponding iterative algorithms in [5, 14, 17, 22] and the references therein.

2) Our general iterative algorithm (3.1) is very different from iterative algorithms (1.6) of [14], (1.7) of [17] and (1.8) of [22] in Introduction 1 because the first iterative steps $u_n = J_{\lambda}^{B_1}(x_n + \eta A^*(J_{\lambda}^{B_2} - I)Ax_n)$ in [14, 17] and $u_n = J_{\lambda_n}^{B_1}(x_n + \eta A^*(J_{\lambda_n}^{B_2} - I)Ax_n)$ in [22] are

replaced by the first step $z_n = J_{\lambda_n}^{B_1}(x_n + \eta_n A^*(J_{\nu_n}^{B_2} - I)Ax_n)$ in our iterative algorithm (3.1), and the second iterative steps $x_{n+1} = \alpha_n f x_n + (1-\alpha_n)Su_n$ in [14], $x_{n+1} = \alpha_n \xi f x_n + (I - \alpha_n D)Su_n$ in [17] and $x_{n+1} = T_{\beta_n}u_n - \mu\alpha_n GT_{\beta_n}u_n$ in [22] are replaced by the second step $x_{n+1} = \beta_n x_n + (1-\beta_n)T_{r_n}(\alpha_n\gamma V x_n + (I - \alpha_n\mu G)z_n)$ in our iterative algorithm (3.1).

3) Theorem 3.1 supplements, improves, and develops the corresponding results in [14, 17, 22] in following aspects:
   (a) The nonexpansive mapping S in [14, 17] and the strictly pseudocontractive mapping T in [22] is extended to the case of the pseudocontractive mapping T.
   (b) A strongly positive bounded linear operator D in [17] is extended to the case of a κ-Lipschitzian and ρ-strongly monotone mapping G. (In fact, from the definitions, it follows that a strongly positive bounded linear operator D (i.e., there exists a constant $\overline{\gamma} > 0$ with the property: $\langle Dx, x\rangle \geq \overline{\gamma}\|x\|^2, \ \ x \in H_1$) is a $\|D\|$-Lipschitzian and $\overline{\gamma}$-strongly monotone mapping).
   (c) The contractive mapping f with a constant $\alpha \in (0,1)$ in [14, 17] is extended to the case of a Lipschizian mapping V with a constant $l \geq 0$.
   (d) The condition $\sum_{n=1}^{\infty}|\alpha_{n+1} - \alpha_n| < \infty$ on the control parameter $\{\alpha_n\}$ in [14, 17, 22] was dispensed.
4) Corollary 3.2 is a new result for finding a minimum norm point of $\Gamma \cap Fix(T)$.
5) Corollary 3.4 is also a new result for finding a minimum norm point of Γ.

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