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ON CONVERGENCE ANALYSIS OF THE INERTIAL MANN ITERATIVE PROCESS FOR WEAK CONTRACTION SEMIGROUPS

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Abstract. This paper aims to prove some convergence results of common fixed points for weak contraction semigroups. The inertial Mann iterative process was investigated in uniformly convex Banach spaces. Applications on common fixed points of Zamfirescu semigroups are also considered. Finally, examples are provided to illustrate our results.

Keywords. Fixed point; Inertial Mann iteration; Strong convergence; Weak contraction mapping; semigroup; Weak convergence.

1. Introduction

In mathematics, we can use fixed point results to solve many problems. For instance, the zero equation $f(x) = 0$, where $f : D \subseteq \mathbb{R} \to \mathbb{R}$ is a given mapping, can be solved by reformulation this equation to an equivalence fixed point problem $g(x) = x$, where g is an appropriate mapping, and then fixed point results are applicable. The Picard iterative method is often used to solve a fixed point problem for a self mapping g. This iterative method is as follows:

$$x_{n+1} = g(x_n), \ \ n = 0,1,2,\ldots, \tag{1.1}$$

with some initial guess x_0 in the domain of g. However, the Picard iterative method (1.1) does not converge to the fixed point of g in some cases. The other fixed point iterative method is needed to solve this situation. Nowadays, fixed point iterative methods have been applied in various fields such as biology, chemistry, economics, engineering, game theory, and physics, and the research in this field has become a popular topic.

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Recall that C is a nonempty convex subset of a normed space X and $T : C \to C$ is a given mapping. In 1953, the following Mann iterative method was introduced by Mann [17]:

$$\begin{cases} x_0 \in C, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n, \end{cases}$$

for all $n \in \mathbb{N} \cup \{0\}$, where $\{\alpha_n\} \subset [0,1]$ is a given sequence. The Mann iterative method is one of the main tools in the fixed point theory. Furthermore, there exists extensive literatures on the convergence of Mann iteration for different classes of operators considered on various spaces; see, e.g., [6, 7, 20, 23] and the references therein.

The minimization problem is one of the most important problems in optimization theory and nonlinear analysis. The most popular two methods for solving the minimization problem are the steepest descent method and the conjugate gradient method. Nowadays, many researchers presented different methods to solve the minimization problem. In 1964, Polyak [21] first introduced the inertial extrapolation method $\{x_n\}$ in a Hilbert space H and used it to solve the following scalar objective minimization problem over H:

$$\min_{x \in X} \varphi(x), \tag{1.2}$$

where $\varphi : H \to \mathbb{R}$ is differentiable. This method is defined by the following iteration:

$$\begin{cases} x_0, x_1 \in H, \\ x_{n+1} = x_n + \beta(x_n - x_{n-1}) - \alpha \nabla \varphi(x_n), \end{cases}$$

for all $n \in \mathbb{N}$, where $\alpha > 0$ and $\beta > 0$. Since then, inertial extrapolation has been an important technique to speed up the convergence rate. Several kinds of iterative methods have been constructed and improved by using the inertial technique to solve fixed point problems and various nonlinear problems, such as variational inequality problems, equilibrium problems, split variational inclusion problems, split equilibrium problems; see, e.g. [1, 4, 5, 10–12, 15] and the references therein. In general, the convergence rate of the Mann iterative method is slow. In 2008, Mainge [16] studied the problem of speeding up the Mann iterative method by combining inertial extrapolation and the Mann iterative method to the new iteration, which is called the classical inertial Mann iterative method, and it is defined by

$$\begin{cases} x_0, x_1 \in H, \\ y_n = x_n + \beta_n(x_n - x_{n-1}), \\ x_{n+1} = (1-\lambda_n)y_n + \lambda_n T y_n, \end{cases} \tag{1.3}$$

for all $n \in \mathbb{N}$, where $\{\lambda_n\}$, $\{\beta_n\} \subset (0,1)$ and T is a self-mapping on a Hilbert space H. Also, Mainge proved that $\{x_n\}$ converges weakly to a fixed point of T under the following conditions:

- (M_1) $\beta_n \in [0,b)$, where $b \in [0,1)$;
- (M_2) $\sum_{n=1}^{\infty} \beta_n \|x_n - x_{n-1}\|^2 < \infty$;
- (M_3) $\inf_{n \geq 1} \lambda_n > 0$ and $\sup_{n \geq 1} \lambda_n < 1$.

On the other hand, Berinde [3] introduced the concept of a new contractive condition as follows:

Definition 1.1. Let (X,d) be a metric space. A mapping $T : X \to X$ is called a weak contraction mapping in the sense of Berinde if there are constants $\delta \in [0,1)$ and $L \geq 0$ such that

$$d(Tx,Ty) \leq \delta d(x,y) + Ld(y,Tx), \tag{1.4}$$

for all $x,y \in X$.

Also, Berinde [3] proved the following two fixed point theorems:

Theorem 1.2 ([3]). *Let (X,d) be a complete metric space and $T : X \to X$ be a weak contraction mapping in the sense of Berinde, i.e., a mapping satisfying (1.4) with $\delta \in [0,1)$ and some $L \geq 0$. Then T has a fixed point. Moreover, for each $x_0 \in X$, the Picard iteration $\{x_n\}$, which is defined by*

$$x_n = Tx_{n-1}, \tag{1.5}$$

for all $n \in \mathbb{N}$, converges to some fixed point of T.

Theorem 1.3 ([3]). *Let (X,d) be a complete metric space and $T : X \to X$ be a weak contraction mapping in the sense of Berinde for which there exist constant $\theta \in [0,1)$ and $L_1 \geq 0$ such that*

$$d(Tx,Ty) \leq \theta d(x,y) + L_1 d(x,Tx), \tag{1.6}$$

for all $x,y \in X$. Then T has a unique fixed point and for any $x_0 \in X$, and the Picard iteration $\{x_n\}$ given by (1.5) converges to the unique fixed point of T.

A weak contractive condition (1.4) is one of the well-known generalized contractive conditions. It is obvious that each of the contractive conditions of Banach [2], Kannan [13], Chatterjea [8], and Zamfirescu [25] implies the condition (1.4) and also satisfies the uniqueness condition (1.6). Moreover, any quasi-contraction due to Ciric [9] with the contractive condition $k \in [0, \frac{1}{2})$ is a weak contraction mapping that satisfies the condition (1.6).

Recently, the concept of a semigroup of weak contraction mappings in the sense of Berinde is introduced in [14]. Surprisingly, the approximation of a common fixed point for a semigroup of weak contraction mappings in the sense of Berinde has not been extensively studied via the inertial Mann iterative method. In this paper, motivated by the above fact, we prove the convergence result of a common fixed point for a weak contraction semigroup in uniformly convex Banach spaces using the inertial Mann iteration method. Furthermore, we justify our main results through a numerical example.

2. Preliminaries

In the sequel, $\mathbb{N}$, $\mathbb{R}^+$, $\mathbb{R}_+$, and $\mathbb{R}$ denote the set of positive integers, the set of positive real numbers, the set of nonnegative real numbers, and the set of real numbers, respectively. Throughout this paper, let X be a uniformly convex Banach space, and let G be an unbounded subset of $[0,\infty)$ such that, for all $s,t \in G$,

$$s+t \in G \text{ and if } s > t, \text{ then } s-t \in G, \tag{2.1}$$

e.g., $G = [0,\infty)$, $G = \mathbb{N}$, or $G = \mathbb{N} \cup \{0\}$.

Definition 2.1. Let X be a uniformly convex Banach space. Let G be an unbounded subset of $[0,\infty)$ satisfying the condition (2.1) and $\tau = \{T_t : X \to X \mid t \in G\}$ be a family of self-mappings. A point $z \in X$ is said to be a common fixed point of τ if $T_t z = z$ for all $T_t \in \tau$. The set of all common fixed point of τ is denoted by $Fix(\tau)$, that is, $Fix(\tau) = \{z \in X \mid T_t z = z \text{ for all } T_t \in \tau\}$.

Definition 2.2. Let X be a Banach space and $T : X \to X$ be a self-mapping on X. A mapping T is said to be demiclosed at $y \in X$, if, for each sequence $\{x_n\}$ in X, $\{x_n\}$ converges weakly at $x \in X$ and $\{Tx_n\}$ converges strongly at y, then $Tx = y$.

Definition 2.3. ([19]) A Banach space X is said to satisfy the Opial's condition if for each sequence $\{x_n\}$ in X weakly convergent to $x \in X$, then

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - x_0\|$$

holds for all $x_0 \in X$ with $x_0 \neq x$.

The following lemmas are needed to prove our main results.

Lemma 2.4 ([18])**.** *Let $\{a_n\},\{b_n\}$, and $\{c_n\}$ be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leq (1+b_n)a_n + c_n,\ n \geq 1.$$

If $\sum_{n=1}^{\infty} b_n < \infty$ and $\sum_{n=1}^{\infty} c_n < \infty$, then the following assertions hold:

(1) $\lim_{n\to\infty} a_n$ *exists;*
(2) $\lim_{n\to\infty} a_n = 0$ *whenever* $\liminf_{n\to\infty} a_n = 0$.

Lemma 2.5 ([24])**.** *Let $\{a_n\} \subset \mathbb{R}^+,\{b_n\} \subset \mathbb{R}$, and $\{c_n\} \subset (0,1)$ be sequences satisfying the inequality*

$$a_{n+1} \leq (1-c_n)a_n + b_n,\ n \geq 1.$$

If (i) $\sum_{n=1}^{\infty} c_n = \infty$ and (ii) $\limsup_{n\to\infty} \frac{b_n}{c_n} \leq 0$ or $\sum_{n=1}^{\infty} |b_n| < \infty$, then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.6 ([22])**.** *Let X be a uniformly convex Banach space and $0 < \alpha_n < 1$ for all $n \in \mathbb{N}$. Suppose that $\{x_n\}$ and $\{y_n\}$ are two sequences in X. If $\limsup_{n\to\infty} \|x_n\| \leq r$, $\limsup_{n\to\infty} \|y_n\| \leq r$ and $\lim_{n\to\infty} \|\alpha_n x_n + (1-\alpha_n)y_n\| = r$ for some $r \geq 0$, then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

3. Weak Contraction Semigroups

In this section, we recall the definition of a semigroup of weak contraction mappings in the sense of Berinde, which was presented in [14].

Definition 3.1 ([14])**.** Let X be a uniformly convex Banach space and G be an unbounded subset of $[0,\infty)$ satisfying the condition (2.1). Then the family $\tau = \{T_t : X \to X \mid t \in G\}$ is said to be a weak contraction semigroup on X if the following conditions are satisfied:

(W_1) $T_{s+t}x = T_sT_tx$ for all $s,t \in G$ and $x \in X$;
(W_2) for all $x \in X$, the mapping $G \ni t \mapsto T_tx \in X$ is continuous;
(W_3) for each $t \in G$, $T_t : X \to X$ is a weak contraction mapping on X, i.e., there are constants $k_t \in [0,1)$ and $L_t \geq 0$ such that for all $x,y \in X$,

$$\|T_tx - T_ty\| \leq k_t\|x-y\| + L_t\|y - T_tx\|. \tag{3.1}$$

In the case that $k_t = k \in [0,1)$ and $L_t = L \geq 0$ for all $t \in G$ in (3.1), the family $\tau = \{T_t : X \to X \mid t \in G\}$ is said to be a (k,L)-weak contraction semigroup on X. Obviously, any (k,L)-weak contraction semigroup is a weak contraction semigroup.

Remark 3.2. By the homogeneity property of the norm, condition (3.1) implies, for all $x, y \in X$,

$$\|T_t x - T_t y\| \leq k_t \|x - y\| + L_t \|x - T_t y\|. \tag{3.2}$$

We now give the following example of a weak contraction semigroup.

Example 3.3. Let $X = \mathbb{R}$ (the set of real numbers) equipped with the usual norm and $\tau = \{T_t : X \to X \mid t \in \mathbb{N}\}$ such that, for each $t \in \mathbb{N}$, a mapping $T_t : X \to X$ is defined by

$$T_t x = \begin{cases} xa^{-2t} & \text{if } x \in [0, 0.2], \\ 0.2xa^{-2t} & \text{if } x \in (0.2, 1], \\ 0 & \text{if } x \in A, \end{cases} \tag{3.3}$$

where $a \in \mathbb{N} - \{1\}$ and $A = (-\infty, 0) \bigcup (1, \infty)$.

Now, we verify that τ is a weak contraction semigroup on X, i.e., the family $\tau = \{T_t : X \to X \mid t \in \mathbb{N}\}$ satisfies all conditions in Definition 3.1. In the first step, we show that the family τ satisfies (W_1), i.e., $T_{s+t}x = T_s T_t x$ for all $s, t \in \mathbb{N}$ and $x \in X$. For each $x \in X$ and $s, t \in \mathbb{N}$, there are three cases as follows:

- Case I: If $x \in [0, 0.2]$, then

$$T_{s+t}x \;=\; xa^{-2(s+t)} \;=\; a^{-2s}(xa^{-2t}) \;=\; a^{-2s}(T_t x) \;=\; T_s T_t x.$$

- Case II: If $x \in (0.2, 1]$, then

$$T_{s+t}x \;=\; 0.2xa^{-2(s+t)}$$

and so

$$T_s T_t x \;=\; T_s\left(0.2xa^{-2t}\right) \;=\; \left(0.2xa^{-2t}\right)a^{-2s} \;=\; 0.2xa^{-2(s+t)}.$$

- Case III: If $x \in A$, then

$$T_{s+t}x \;=\; 0 \;\text{ and }\; T_s T_t x \;=\; T_s(0) \;=\; 0.$$

Thus, the family τ satisfies (W_1). It is easy to see that the family τ satisfies (W_2), i.e., for all $x \in X$, the mapping $\mathbb{N} \ni t \mapsto T_t x \in X$ is continuous.

Finally, we verify that, for each $t \in \mathbb{N}$, T_t is a weak contraction mapping satisfying (3.1) with $k_t = 0.75$ and $L_t = 1$. Obviously, $\|T_t x - T_t y\| = 0$ for each $x, y \in X$ with $x = y$, so (3.1) holds. Assume that $x, y \in X$ with $x \neq y$. Consider the following cases.

- Case I: If $x, y \in A$, then $\|T_t x - T_t y\| = 0$ and so (3.1) is satisfied.
- Case II: If $x, y \in [0, 0.2]$, then

$$\begin{aligned} \|T_t x - T_t y\| &= |xa^{-2t} - ya^{-2t}| \\ &\leq \frac{|x-y|}{4} \\ &\leq 0.25|x-y| + |y - T_t x| \\ &\leq k_t\|x-y\| + L_t\|y - T_t x\|. \end{aligned}$$

- Case III: If $x, y \in (0.2, 1]$, then

$$\begin{aligned} \|T_t x - T_t y\| &= |0.2xa^{-2t} - 0.2ya^{-2t}| \\ &\leq \frac{0.2|x-y|}{4} \\ &\leq 0.05|x-y| + |y - T_t x| \\ &\leq k_t\|x-y\| + L_t\|y - T_t x\|. \end{aligned}$$

- Case IV: If $x \in [0, 0.2]$ and $y \in (0.2, 1]$, then

$$\begin{aligned} \|T_t x - T_t y\| &= |xa^{-2t} - 0.2ya^{-2t}| \\ &= |xa^{-2t} - 0.2xa^{-2t} + 0.2xa^{-2t} - 0.2ya^{-2t}| \\ &\leq |0.2xa^{-2t} - 0.2ya^{-2t}| + |0.8xa^{-2t}| \\ &\leq 0.05|x-y| + |y - xa^{-2t}| \\ &\leq k_t\|x-y\| + L_t\|y - T_t x\|. \end{aligned}$$

- Case V: If $x \in (0.2, 1]$ and $y \in [0, 0.2]$, then

$$\begin{aligned} & |0.2xa^{-2t} - ya^{-2t}| \\ = \;& |0.2xa^{-2t} - 0.2ya^{-2t} + 0.2ya^{-2t} - ya^{-2t}| \\ \leq \;& |0.2xa^{-2t} - 0.2ya^{-2t}| + |0.8ya^{-2t}| \\ \leq \;& 0.05|x-y| + 0.4|x - ya^{-2t}| \\ \leq \;& 0.05|x-y| + 0.4|x-y| + 0.4|y - 0.2xa^{-2t}| + 0.4|0.2xa^{-2t} - ya^{-2t}| \\ \leq \;& 0.45|x-y| + 0.4|y - 0.2xa^{-2t}| + 0.4|0.2xa^{-2t} - ya^{-2t}|, \end{aligned}$$

which implies that

$$(1 - 0.4)|0.2xa^{-2t} - ya^{-2t}| \leq 0.45|x-y| + 0.4|y - 0.2xa^{-2t}|.$$

Hence,

$$\begin{aligned} \|T_t x - T_t y\| &= |0.2xa^{-2t} - ya^{-2t}| \\ &\leq \frac{0.45}{0.6}|x-y| + \frac{0.4}{0.6}|y - 0.2xa^{-2t}| \\ &= 0.75|x-y| + \frac{2}{3}|y - 0.2xa^{-2t}| \\ &\leq k_t\|x-y\| + L_t\|y - T_t x\|. \end{aligned}$$

- Case VI: If $x \in A$ and $y \in [0, 0.2]$, then

$$\begin{aligned} \|T_t x - T_t y\| &= |0 - ya^{-2t}| \\ &\leq |y - 0| \\ &= \|y - T_t x\| \\ &\leq k_t\|x-y\| + L_t\|y - T_t x\|. \end{aligned}$$

- Case VII: If $x \in [0, 0.2]$ and $y \in A$, then

$$\begin{aligned} |xa^{-2t} - 0| &\leq 0.25|x| \\ &\leq 0.25|x-y| + 0.25|y - xa^{-2t}| + 0.25|xa^{-2t}|, \end{aligned}$$

which yields

$$(1-0.25)|xa^{-2t}| \leq 0.25|x-y|+0.25|y-xa^{-2t}|.$$

Hence, we obtain

$$\begin{aligned}\|T_tx-T_ty\| &= |xa^{-2t}| \\ &\leq \frac{0.25}{0.75}|x-y|+\frac{0.25}{0.75}|y-xa^{-2t}| \\ &= \frac{1}{3}|x-y|+\frac{1}{3}|y-xa^{-2t}| \\ &\leq k_t\|x-y\|+L_t\|y-T_tx\|.\end{aligned}$$

- Case VIII: If $x \in A$ and $y \in (0.2, 1]$, then

$$\begin{aligned}\|T_tx-T_ty\| &= |0-0.2ya^{-2t}| \\ &\leq |y-0| \\ &= \|y-T_tx\| \\ &\leq k_t\|x-y\|+L_t\|y-T_tx\|.\end{aligned}$$

- Case IX: If $x \in (0.2, 1]$ and $y \in A$, then

$$\begin{aligned}\|T_tx-T_ty\| &= |0.2xa^{-2t}-0| \\ &\leq 0.05|x| \\ &\leq 0.05|x-y|+0.05|y-0.2xa^{-2t}|+0.05|0.2xa^{-2t}|.\end{aligned}$$

This implies that

$$(1-0.05)|0.2xa^{-2t}| \leq 0.05|x-y|+0.05|y-0.2xa^{-2t}|.$$

Thus, we obtain

$$\begin{aligned}\|T_tx-T_ty\| &= |0.2xa^{-2t}| \\ &\leq \frac{0.05}{0.95}|x-y|+\frac{0.05}{0.95}|y-0.2xa^{-2t}| \\ &\leq k_t\|x-y\|+L_t\|y-T_tx\|.\end{aligned}$$

For all possible cases, it has been proven that, for all $t \in \mathbb{N}$, T_t is a weak contraction mapping on X satisfying (3.1) with $k_t = 0.75$ and $L_t = 1$. Therefore, the family $\tau = \{T_t : t \in \mathbb{N}\}$ is a weak contraction semigroup on X.

4. Convergence Theorems for Weak Contraction Semigroups

In this section, we obtain weak and strong convergence theorems for weak contraction semigroups using the inertial Mann iterative method in uniformly convex Banach spaces.

Let X be a uniformly convex Banach space. Let G be an unbounded subset of $[0, \infty)$ satisfying the condition (2.1) and $\tau = \{T_t : X \to X \mid t \in G\}$ be a weak contraction semigroup on X. Define an inertial Mann iterative process $\{x_n\}$ in X as follows:

$$\begin{cases} x_0, x_1 \in X, \\ y_n = x_n + \beta_n (x_n - x_{n-1}), \\ x_{n+1} = (1-\lambda_n) y_n + \lambda_n T_{t_n} y_n, \end{cases} \tag{IM_n}$$

for all $n \in \mathbb{N}$, where $\{t_n\} \subset G$, $\{\lambda_n\} \subset [0,1]$, and $\{\beta_n\} \subset [0,b]$ with $b \in [0,1)$. For each step with x_{n-1} and x_n given, we choose β_n such that $0 \leq \beta_n \leq \tilde{\beta}_n$, where

$$\tilde{\beta}_n = \begin{cases} \min\left\{b, \frac{\omega_n}{\|x_n - x_{n-1}\|}\right\} & \text{if } x_n \neq x_{n-1}, \\ b & \text{otherwise,} \end{cases} \tag{4.1}$$

where $\{\omega_n\} \subset [0,\infty)$.

Now, we present the following condition, which is needed to prove the results in this section.

Definition 4.1. Let X be a uniformly convex Banach space. Let G be an unbounded subset of $[0,\infty)$ satisfying the condition (2.1) and $\tau = \{T_t : X \to X \mid t \in G\}$ be a weak contraction semigroup on X. A family τ is said to satisfy the condition $(\mathbb{W})$ if there exist two constants $\tilde{k} \in [0,1)$ and $\tilde{L} \geq 0$ such that

$$\|T_t x - T_t y\| \leq \tilde{k}\|x-y\| + \tilde{L}\|x - T_t x\|, \tag{4.2}$$

for all $x, y \in X$ and for all $t \in G$.

In order to obtain our main results, we need the following lemmas.

Lemma 4.2. *Let X be a uniformly convex Banach space. Let G be an unbounded subset of $[0,\infty)$ satisfying the condition (2.1) and $\tau = \{T_t : X \to X \mid t \in G\}$ be a weak contraction semigroup on X. If τ satisfies the condition $(\mathbb{W})$ with $\tilde{k} \in [0,1)$ and $\tilde{L} \geq 0$, then the following condition holds*

$$\|T_s x - T_t x\| \leq (1+2\tilde{L}) \max\{\|x - T_s x\|, \|x - T_{t-s} x\|, \|x - T_{s-t} x\|\},$$

for all $x \in X$ and for all $s, t \in G$.

Proof. Let $x \in X$. It is easy to see that, for each given $s,t \in G$ with $s = t$, $\|T_s x - T_t x\| = 0$. For each given $s,t \in G$ with $s > t$, we obtain

$$\begin{aligned} \|T_s x - T_t x\| &= \|T_t T_{s-t} x - T_t x\| \\ &\leq \tilde{k}\|T_{s-t}x - x\| + \tilde{L}\|T_{s-t}x - T_s x\| \\ &\leq \tilde{k}\|T_{s-t}x - x\| + \tilde{L}\|T_{s-t}x - x\| + \tilde{L}\|x - T_s x\| \\ &= (\tilde{k}+\tilde{L})\|T_{s-t}x - x\| + \tilde{L}\|x - T_s x\| \\ &\leq (1+\tilde{L})\|T_{s-t}x - x\| + \tilde{L}\|x - T_s x\| \\ &\leq (1+2\tilde{L}) \max\{\|x - T_s x\|, \|x - T_{t-s} x\|, \|x - T_{s-t} x\|\}. \end{aligned}$$

If $s,t \in G$ such that $s < t$, we have

$$\begin{aligned} \|T_s x - T_t x\| &= \|T_s x - T_s T_{t-s} x\| \\ &\leq \tilde{k}\|x - T_{t-s}x\| + \tilde{L}\|T_s x - x\| \\ &\leq \|x - T_{t-s}x\| + \tilde{L}\|T_s x - x\| \\ &\leq (1+2\tilde{L}) \max\{\|x - T_s x\|, \|x - T_{t-s} x\|, \|x - T_{s-t} x\|\}. \end{aligned}$$

Therefore, we can conclude that

$$\|T_s x - T_t x\| \le (1+2\tilde{L}) \max\{\|x - T_s x\|, \|x - T_{t-s}x\|, \|x - T_{s-t}x\|\},$$

for all $x \in X$ and for all $s, t \in G$. □

Lemma 4.3. *Let X be a uniformly convex Banach space. Let G be an unbounded subset of $[0,\infty)$ satisfying the condition (2.1) and $\tau = \{T_t : X \to X \mid t \in G\}$ be a weak contraction semigroup on X such that $Fix(\tau) \neq \emptyset$ and τ satisfies condition $(\mathbb{W})$. Suppose that $\{x_n\}$ is a sequence defined by the iterative scheme (IM_n). Assume that $\{\omega_n\} \subset [0,\infty)$ in (4.1) satisfies $\sum_{n=1}^{\infty} \omega_n < \infty$. Then the following conclusions hold:*

- (i) *τ has a unique common fixed point;*
- (ii) *$\lim_{n\to\infty} \|x_n - z\|$ exists for all $z \in Fix(\tau)$;*
- (iii) *$\{x_n\}$ is bounded;*
- (iv) *for each $\{t_n\} \subset G$, $\lim_{n\to\infty} \|y_n - T_{t_n} y_n\| = 0$. Moreover, $\lim_{n\to\infty} \|x_n - T_t x_n\| = 0$ for all $t \in G$.*

Proof. (i) Since $Fix(\tau) \neq \emptyset$, there exists $z_0 \in X$ such that $T_t z_0 = z_0$ for all $t \in G$. First, we show that τ has a unique common fixed point. Suppose that z_1 and z_2 are common fixed points of τ, that is, $z_1 = T_t z_1$, and $z_2 = T_t z_2$ for all $t \in G$. Then, for each $t \in G$, we obtain from condition (4.2) that

$$\begin{aligned} \|z_1 - z_2\| &= \|T_t z_1 - T_t z_2\| \\ &\le \tilde{k}\|z_1 - z_2\| + \tilde{L}\|z_1 - T_t z_1\| \\ &\le \tilde{k}\|z_1 - z_2\|, \end{aligned}$$

so $\|z_1 - z_2\| = 0$, that is, $z_1 = z_2$. Therefore, τ has a unique common fixed point.

(ii) Suppose that $z \in Fix(\tau)$. For each $n \in \mathbb{N}$, we obtain

$$\begin{aligned} \|y_n - z\| &= \|x_n + \beta_n(x_n - x_{n-1}) - z\| \\ &\le \|x_n - z\| + \beta_n\|x_n - x_{n-1}\|. \end{aligned} \tag{4.3}$$

Then, we have

$$\begin{aligned} \|x_{n+1} - z\| &= \|(1-\lambda_n)y_n + \lambda_n T_{t_n} y_n - z\| \\ &\le (1-\lambda_n)\|y_n - z\| + \lambda_n\|T_{t_n} y_n - z\| \\ &\le (1-\lambda_n)\|y_n - z\| + \lambda_n(\tilde{k}\|y_n - z\| + \tilde{L}\|T_{t_n} z - z\|) \\ &= (1-\lambda_n)\|y_n - z\| + \lambda_n \tilde{k}\|y_n - z\| \\ &= (1-(1-\tilde{k})\lambda_n)\|y_n - z\| \\ &\le (1-(1-\tilde{k})\lambda_n)(\|x_n - z\| + \beta_n\|x_n - x_{n-1}\|) \\ &\le \|x_n - z\| + \beta_n\|x_n - x_{n-1}\| \end{aligned} \tag{4.4}$$

for all $n \in \mathbb{N}$. Since $\{\omega_n\} \subset [0,\infty)$ in (4.1) satisfies $\sum_{n=1}^{\infty} \omega_n < \infty$, we have

$$\sum_{n=1}^{\infty} \beta_n \|x_n - x_{n-1}\| < \infty. \tag{4.5}$$

From (4.4), (4.5), and Lemma 2.4, we obtain that $\lim_{n\to\infty} \|x_n - z\|$ exists, and then (ii) holds. This also implies that conclusion (iii) holds, that is, $\{x_n\}$ is bounded.

(iv) Assume that $z \in Fix(\tau)$. From conclusion (ii), there is $r \geq 0$ such that

$$\lim_{n\to\infty} \|x_n - z\| = r. \tag{4.6}$$

Then, by the definition in (IM_n), we have

$$r = \lim_{n\to\infty} \|x_{n+1} - z\| = \lim_{n\to\infty} \|(1-\lambda_n)(y_n - z) + \lambda_n(T_{t_n}y_n - z)\|. \tag{4.7}$$

From (4.3), (4.6), and the assumption, we obtain

$$\limsup_{n\to\infty} \|y_n - z\| \leq \limsup_{n\to\infty} \|x_n - z\| + \limsup_{n\to\infty} \beta_n \|x_n - x_{n-1}\| \leq r. \tag{4.8}$$

Since $z \in Fix(\tau)$ and τ satisfies condition ($\mathbb{W}$),

$$\begin{aligned} \|T_{t_n}y_n - z\| &= \|T_{t_n}y_n - T_{t_n}z\| \\ &\leq \tilde{k}\|y_n - z\| + \tilde{L}\|T_{t_n}z - z\| \\ &\leq \|y_n - z\|, \end{aligned}$$

which implies that

$$\limsup_{n\to\infty} \|T_{t_n}y_n - z\| \leq r. \tag{4.9}$$

From (4.7), (4.8), (4.9), and Lemma 2.6, we obtain

$$\lim_{n\to\infty} \|y_n - T_{t_n}y_n\| = 0.$$

Next, we verify that, for all $t \in G$, $\lim_{n\to\infty} \|x_n - T_t x_n\| = 0$. For each $t \in G$, by applying Lemma 4.2, we have

$$\begin{aligned} & \|x_n - T_t x_n\| \\ \leq\ & \|x_n - y_n\| + \|y_n - T_{t_n}y_n\| + \|T_{t_n}y_n - T_t y_n\| + \|T_t y_n - T_t x_n\| \\ \leq\ & \|x_n - y_n\| + \|y_n - T_{t_n}y_n\| + \|T_{t_n}y_n - T_t y_n\| + \tilde{k}\|x_n - y_n\| + \tilde{L}\|T_t y_n - y_n\| \\ \leq\ & \|x_n - y_n\| + \|y_n - T_{t_n}y_n\| + \|T_{t_n}y_n - T_t y_n\| + \tilde{k}\|x_n - y_n\| + \tilde{L}(\|y_n - T_{t_n}y_n\| + \|T_{t_n}y_n - T_t y_n) \\ =\ & (1+\tilde{k})\|x_n - y_n\| + (1+\tilde{L})\|y_n - T_{t_n}y_n\| + (1+\tilde{L})\|T_{t_n}y_n - T_t y_n\| \\ \leq\ & (1+\tilde{k})\beta_n\|x_n - x_{n-1}\| + (1+\tilde{L})\|y_n - T_{t_n}y_n\| + (1+\tilde{L})\|T_{t_n}y_n - T_t y_n\| \\ \leq\ & (1+\tilde{k})\beta_n\|x_n - x_{n-1}\| + (1+\tilde{L})M_n + (1+\tilde{L})(1+2\tilde{L})M_n \\ =\ & (1+\tilde{k})\beta_n\|x_n - x_{n-1}\| + (1+\tilde{L})(2+2\tilde{L})M_n \\ =\ & (1+\tilde{k})\beta_n\|x_n - x_{n-1}\| + 2(1+\tilde{L})^2 M_n, \end{aligned}$$

where $M_n := \max\{\|y_n - T_{t_n}y_n\|, \|y_n - T_{t-t_n}y_n\|, \|y_n - T_{t_n-t}y_n\|\}$. Taking the limit as $n \to \infty$ in the above inequality, we obtain $\lim_{n\to\infty} \|x_n - T_t x_n\| = 0$ for all $t \in G$. □

Lemma 4.4. *Let X be a uniformly convex Banach space. Let G be an unbounded subset of $[0,\infty)$ satisfying condition (2.1) and $\tau = \{T_t : X \to X \mid t \in G\}$ be a weak contraction semigroup on X such that τ satisfies condition ($\mathbb{W}$). Suppose that $\{x_n\}$ is a sequence defined by the iterative scheme (IM_n). Assume that X satisfies the Opial's condition. Then $I - T_t$ is demiclosed at zero for all $t \in G$, that is, for each $t \in G$, if $\{x_n\}$ converges weakly to $z \in X$ and $\lim_{n\to\infty} \|x_n - T_t x_n\| = 0$, then $T_t z = z$.*

Proof. Let $t \in G$. Suppose that $\{x_n\}$ converges weakly to $z \in X$ and $\lim_{n\to\infty} \|x_n - T_t x_n\| = 0$. Since X satisfies the Opial's condition,

$$\limsup_{n\to\infty} \|x_n - z\| < \limsup_{n\to\infty} \|x_n - y\|$$

holds for all $y \in X$, with $y \neq z$. Assume that $T_t z \neq z$. Then, by the condition ($\mathbb{W}$), we have

$$\begin{aligned} \|T_t z - x_n\| &\leq \|T_t z - T_t x_n\| + \|T_t x_n - x_n\| \\ &\leq \tilde{k}\|z - x_n\| + \tilde{L}\|T_t x_n - x_n\| + \|T_t x_n - x_n\| \\ &\leq \|z - x_n\| + (1 + \tilde{L})\|T_t x_n - x_n\|. \end{aligned} \tag{4.10}$$

Taking the limit superior as $n \to \infty$ in (4.10), we obtain $\limsup_{n\to\infty} \|T_t z - x_n\| \leq \limsup_{n\to\infty} \|z - x_n\|$, which is a contradiction. Hence, $T_t z = z$, which implies that $I - T_t$ is demiclosed at zero. This completes the proof. □

The following theorem is a weak convergence theorem for weak contraction semigroups using the inertial Mann iterative method in uniformly convex Banach spaces.

Theorem 4.5. *Let X be a uniformly convex Banach space. Let G be an unbounded subset of $[0,\infty)$ satisfying condition (2.1) and $\tau = \{T_t : X \to X \mid t \in G\}$ be a weak contraction semigroup on X such that $Fix(\tau) \neq \emptyset$ and τ satisfies condition ($\mathbb{W}$). Suppose that $\{x_n\}$ is a sequence defined by the iterative scheme (IM_n). Assume that X satisfies the Opial's condition and $\sum_{n=1}^{\infty} \omega_n < \infty$, where $\{\omega_n\} \subset [0,\infty)$ is a given sequence in (4.1). Then $\{x_n\}$ converges weakly to the unique common fixed point of τ.*

Proof. Let $z \in Fix(\tau)$. By using Lemma 4.3, we have that $Fix(\tau) = \{z\}$ and $\{x_n\}$ is bounded. Since X is uniformly convex, there exist a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\{x_{n_j}\}$ converges weakly to y for some $y \in X$.

Now we show that $\{x_n\}$ has a unique weak sub-sequential limit in $Fix(\tau)$. Let y_1 and y_2 be weak limits of the subsequence $\{x_{n_j}\}$ and $\{x_{n_k}\}$ of $\{x_n\}$, respectively. From the conclusion (iv) in Lemma 4.3, we have $\lim_{n\to\infty} \|x_n - T_t x_n\| = 0$ for all $t \in G$. It follows from Lemma 4.4 that $T_t y_1 = y_1$ and $T_t y_2 = y_2$ for all $t \in G$, which implies that $y_1, y_2 \in Fix(\tau)$. Since $Fix(\tau)$ is a singleton set, we have $y_1 = y_2 = z$. Therefore, $\{x_n\}$ converges weakly to the unique common fixed point of τ. □

In the sequel, we obtain a strong convergence theorem for the inertial Mann iterative process in uniformly convex Banach spaces. First, we establish the following useful lemma.

Lemma 4.6. *Let X be a uniformly convex Banach space. Let G be an unbounded subset of $[0,\infty)$ satisfying condition (2.1) and $\tau = \{T_t : X \to X \mid t \in G\}$ be a weak contraction semigroup on X such that $Fix(\tau) \neq \emptyset$ and τ satisfies condition ($\mathbb{W}$). Suppose that $\{x_n\}$ is a sequence defined by the iterative scheme (IM_n) with $\{\omega_n\} \subset [0,\infty)$ in (4.1) satisfies $\sum_{n=1}^{\infty} \omega_n < \infty$. If $\liminf_{n\to\infty} d(x_n, Fix(\tau)) = 0$ or $\limsup_{n\to\infty} d(x_n, Fix(\tau)) = 0$, then $\lim_{n\to\infty} d(x_n, Fix(\tau)) = 0$, where $d(x_n, Fix(\tau)) = \inf\{\|x_n - z\| | z \in Fix(\tau)\}$.*

Proof. Suppose that $\liminf_{n\to\infty} d(x_n, Fix(\tau)) = 0$ and $z \in Fix(\tau)$. Now, we show that $\lim_{n\to\infty} d(x_n, Fix(\tau))$ exists. Following the proof to see (4.4) in Lemma 4.3, for each $n \in \mathbb{N}$, we have

$$\|x_{n+1} - z\| \leq \|x_n - z\| + \beta_n \|x_n - x_{n-1}\|.$$

By taking the infimum all over $z \in Fix(\tau)$ on both sides, we have

$$d(x_{n+1}, Fix(\tau)) \leq d(x_n, Fix(\tau)) + \beta_n \|x_n - x_{n-1}\|. \tag{4.11}$$

Again, following the proof to see (4.5) in Lemma 4.3 and using Lemma 2.4, we obtain that $\lim_{n\to\infty} d(x_n, Fix(\tau))$ exists. This implies that $\lim_{n\to\infty} d(x_n, Fix(\tau)) = 0$. Similarly, we can show that if $\limsup_{n\to\infty} d(x_n, Fix(\tau)) = 0$, then $\lim_{n\to\infty} d(x_n, Fix(\tau)) = 0$. □

Theorem 4.7. *Let X be a uniformly convex Banach space. Let G be an unbounded subset of $[0,\infty)$ satisfying condition (2.1) and $\tau = \{T_t : X \to X \mid t \in G\}$ be a weak contraction semigroup on X such that $Fix(\tau) \neq \emptyset$ and τ satisfies condition $(\mathbb{W})$. Suppose that $\{x_n\}$ is a sequence defined by the iterative scheme (IM_n) with $\{\omega_n\} \subset [0,\infty)$ in (4.1) satisfying $\sum_{n=1}^{\infty} \omega_n < \infty$. Then the sequence $\{x_n\}$ converges strongly to the unique common fixed point of τ if and only if $\liminf_{n\to\infty} d(x_n, Fix(\tau)) = 0$ or $\limsup_{n\to\infty} d(x_n, Fix(\tau)) = 0$, where $d(x_n, Fix(\tau)) = \inf\{\|x_n - z\| : z \in Fix(\tau)\}$.*

Proof. Let $z \in Fix(\tau)$. Suppose that the sequence $\{x_n\}$ converges strongly to $z \in Fix(\tau)$. For a given $\varepsilon > 0$, there is $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$, $\|x_n - z\| < \varepsilon$. Taking the infimum all over $z \in Fix(\tau)$ in the above inequality, we obtain, for all $n \geq n_0$, $d(x_n, Fix(\tau)) < \varepsilon$. It follows that $\lim_{n\to\infty} d(x_n, Fix(\tau)) = 0$, which implies

$$\liminf_{n\to\infty} d(x_n, Fix(\tau)) = 0 \text{ and } \limsup_{n\to\infty} d(x_n, Fix(\tau)) = 0.$$

Conversely, we suppose that $\liminf_{n\to\infty} d(x_n, Fix(\tau)) = 0$ or $\limsup_{n\to\infty} d(x_n, Fix(\tau)) = 0$. Using Lemma 4.6, we see that $\lim_{n\to\infty} d(x_n, Fix(\tau)) = 0$. Now, we verify that $\{x_n\}$ is a Cauchy sequence in X. Let $\varepsilon > 0$. Then there is $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$, $d(x_n, Fix(\tau)) < \frac{\varepsilon}{8}$. It implies that, for all $n \geq n_0$, $\inf\{\|x_n - z\| : z \in Fix(\tau)\} < \frac{\varepsilon}{8}$, so there exists $z \in Fix(\tau)$ such that

$$\|x_n - z\| < \frac{\varepsilon}{4}. \tag{4.12}$$

Let $s_n = \beta_n \|x_n - x_{n-1}\|$ for all $n \in \mathbb{N}$. From the fact that $\sum_{n=1}^{\infty} s_n < \infty$, for a given $\varepsilon > 0$, there exists $n_1 \in \mathbb{N}$ such that, for all $k \geq n \geq n_1$,

$$\sum_{i=n}^{k} s_i < \frac{\varepsilon}{2}. \tag{4.13}$$

For each $m, n \in \mathbb{N}$, by (4.4), we have

$$\begin{aligned} \|x_{n+m} - z\| &\leq \|x_{n+m-1} - z\| + s_{n+m-1} \\ &\leq \|x_{n+m-2} - z\| + s_{n+m-2} + s_{n+m-1} \\ &\vdots \\ &\leq \|x_n - z\| + \sum_{i=n}^{n+m-1} s_i. \end{aligned} \tag{4.14}$$

For each m, $n \in \mathbb{N}$, it can be obtained that when $n \geq N = \max\{n_0, n_1\}$,

$$\begin{aligned}\|x_{n+m} - x_n\| &\leq \|x_{n+m} - z\| + \|x_n - z\| \\ &\leq 2\|x_n - z\| + \sum_{i=n}^{n+m-1} s_i \\ &< 2\frac{\varepsilon}{4} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

Hence, $\{x_n\}$ is a Cauchy sequence in X. By the completeness of X, the sequence $\{x_n\}$ converges strongly to a point $z_0 \in X$. Next, we show that z_0 is a common fixed point of τ. For each $n \in \mathbb{N}$ and $t \in G$, from condition $(\mathbb{W})$, we have

$$\begin{aligned}\|T_t z_0 - z_0\| &\leq \|T_t z_0 - T_t x_n\| + \|T_t x_n - x_n\| + \|x_n - z_0\| \\ &\leq \tilde{k}\|z_0 - x_n\| + \tilde{L}\|x_n - T_t x_n\| + \|T_t x_n - x_n\| + \|x_n - z_0\| \\ &= (1+\tilde{k})\|z_0 - x_n\| + (1+\tilde{L})\|x_n - T_t x_n\|. \end{aligned} \tag{4.15}$$

Taking the limit as $n \to \infty$ in (4.15) and using the conclusion (iv) in Lemma 4.3, we have $T_t z_0 = z_0$ for all $t \in G$, that is, $z_0 \in Fix(\tau)$. Therefore, $\{x_n\}$ converges strongly to the unique common fixed point of τ. □

Theorem 4.8. *Let X be a uniformly convex Banach space. Let G be an unbounded subset of $[0,\infty)$ satisfying condition (2.1) and $\tau = \{T_t : X \to X \mid t \in G\}$ be a weak contraction semigroup on X such that $Fix(\tau) \neq \emptyset$ and τ satisfies condition $(\mathbb{W})$. Suppose that $\{x_n\}$ is a sequence defined by the iterative scheme (IM_n). Assume that the following conditions hold:*

- (i) *$\{\lambda_n\} \subset [0,1]$ in (IM_n) satisfies* $\sum_{n=1}^{\infty} \lambda_n = \infty$;
- (ii) *$\{\omega_n\} \subset [0,\infty)$ in (4.1) satisfies* $\sum_{n=1}^{\infty} \omega_n < \infty$.

Then $\{x_n\}$ converges strongly to the unique common fixed point of τ.

Proof. Suppose that $z \in Fix(\tau)$. From the conclusion (i) in Lemma 4.3, we have $Fix(\tau) = \{z\}$. For each $n \in \mathbb{N}$, by (4.3), we have

$$\|y_n - z\| \leq \|x_n - z\| + \beta_n\|x_n - x_{n-1}\|,$$

and then

$$\begin{aligned}\|x_{n+1} - z\| &\leq (1-\lambda_n)\|y_n - z\| + \lambda_n\|T_{t_n} y_n - z\| \\ &\leq (1-\lambda_n)\|y_n - z\| + \lambda_n(\tilde{k}\|y_n - z\| + \tilde{L}\|T_{t_n} z - z\|) \\ &= (1-\lambda_n)\|y_n - z\| + \lambda_n\tilde{k}\|y_n - z\| \\ &= (1-(1-\tilde{k})\lambda_n)\|y_n - z\| \\ &\leq (1-(1-\tilde{k})\lambda_n)(\|x_n - z\| + \beta_n\|x_n - x_{n-1}\|) \\ &\leq (1-(1-\tilde{k})\lambda_n)\|x_n - z\| + \beta_n\|x_n - x_{n-1}\|. \end{aligned} \tag{4.16}$$

By assumptions (i)-(ii), (4.16), and Lemma 2.5, we obtain $\lim_{n\to\infty}\|x_n - z\| = 0$. It follows that $\{x_n\}$ converges strongly to the unique common fixed point of τ. □

In 1972, Zamfirescu [25] obtained a very interesting contraction mapping by combining the contractive conditions of Banach [2], Kannan [13], and Chatterjea [8], namely, a Zamfirescu contraction mapping. Now, we recall the concept of a Zamfirescu contraction mapping as follows:

Definition 4.9. Let $(X, \|\cdot\|)$ be a normed space. A mapping $T : X \to X$ is called a Zamfirescu contraction mapping if there are constants $\alpha \in [0,1)$ and $\beta, \gamma \in [0, \frac{1}{2})$ such that, for each $x, y \in X$, one of the following conditions hold:

(z_1) $\|Tx - Ty\| \leq \alpha\|x - y\|$;
(z_2) $\|Tx - Ty\| \leq \beta[\|x - Tx\| + \|y - Ty\|]$;
(z_3) $\|Tx - Ty\| \leq \gamma[\|x - Ty\| + \|y - Tx\|]$.

Proposition 4.10. *Let X be a normed space. If $T : X \to X$ is a Zamfirescu contraction mapping, then T is a weak contraction mapping. Moreover, any Zamfirescu contraction mapping also satisfies (1.6).*

Following the idea of a weak contraction semigroup in Definition 3.1, we introduce the notation of a Zamfirescu contraction semigroup as follows.

Definition 4.11. Let X be a uniformly convex Banach space and G be an unbounded subset of $[0, \infty)$ satisfying condition (2.1). Then the family $\tau = \{T_t : X \to X \mid t \in G\}$ is said to be a Zamfirescu contraction semigroup on X if the following conditions are satisfied:

(Z_1) $T_{s+t}x = T_sT_tx$ for all $s,t \in G$ and $x \in X$;
(Z_2) for all $x \in X$, the mapping $G \ni t \mapsto T_tx \in X$ is continuous;
(Z_3) for each $t \in G$, $T_t : X \to X$ be a Zamfirescu contraction mapping on X, i.e., there are constants $\alpha_t \in [0,1)$ and $\beta_t, \gamma_t \in [0, \frac{1}{2})$ such that for each $x, y \in X$, one of the following condition holds:
  $(\bar{z}_1)$ $\|T_tx - T_ty\| \leq \alpha_t\|x - y\|$;
  $(\bar{z}_2)$ $\|T_tx - T_ty\| \leq \beta_t[\|x - T_tx\| + \|y - T_ty\|]$;
  $(\bar{z}_3)$ $\|T_tx - T_ty\| \leq \gamma_t[\|x - T_ty\| + \|y - T_x\|]$.

It follows from Proposition 4.10 that one can obtain the following corollary.

Corollary 4.12. *Let X be a uniformly convex Banach space. Let G be an unbounded subset of $[0, \infty)$ satisfying condition (2.1) and $\tau = \{T_t : X \to X \mid t \in G\}$ be a Zamfirescu contraction semigroup on X such that $Fix(\tau) \neq \emptyset$ and τ satisfies condition $(\mathbb{W})$. Suppose that $\{x_n\}$ is a sequence defined by the iterative scheme (IM_n). Assume that the following conditions hold:*

(i) $\{\lambda_n\} \subset [0,1]$ in (IM_n) satisfies $\sum_{n=1}^{\infty} \lambda_n = \infty$;

(ii) $\{\omega_n\} \subset [0, \infty)$ in (4.1) satisfies $\sum_{n=1}^{\infty} \omega_n < \infty$.

Then the sequence $\{x_n\}$ converges strongly to the unique common fixed point of τ.

5. Numerical Examples

The aim of this section is to present a numerical example of Theorem 4.8.

Example 5.1. Let $X=\mathbb{R}$ be equipped with the usual norm and let $\tau=\{T_t : X\to X \mid t\in\mathbb{N}\}$ be defined by

$$T_t x = \begin{cases} xa^{-2t} & \text{if } x\in[0,0.2],\\ 0.2xa^{-2t} & \text{if } x\in(0.2,1],\\ 0 & \text{if } x\in A,\end{cases} \tag{5.1}$$

for all $t\in\mathbb{N}$, where $a\in\mathbb{N}-\{1\}$ and $A=(-\infty,0)\bigcup(1,\infty)$.

Obviously, $z=0$ is a unique common fixed point of τ, that is, $T_t(0)=0$ for all $t\in\mathbb{N}$. From Example 3.3, the family τ is a weak contraction semigroup on X. Now, we verify that τ satisfies the condition $(\mathbb{W})$ with $\tilde{k}=0.25$ and $\tilde{L}=1$, i.e., for all $x,y\in X$ and for all $t\in\mathbb{N}$, we have

$$\|T_tx-T_ty\|\le 0.25\|x-y\|+\|x-T_tx\|. \tag{5.2}$$

First, let $\mathfrak{C}=\{[0,0.2],(0.2,1],A\}$ and suppose that $B\in\mathfrak{C}$. For any $x,y\in B$, following the proof of Example 3.3, we obtain

$$\|T_tx-T_ty\|\le 0.25|x-y|\le \tilde{k}\|x-y\|+\tilde{L}\|x-T_tx\|,$$

which implies that (5.2) holds. Next, we divide the proof as follows.

- Case I: If $x\in(0.2,1]$ and $y\in[0,0.2]$, then

$$\begin{aligned}\|T_tx-T_ty\| &= |0.2xa^{-2t}-ya^{-2t}|\\ &= |0.2xa^{-2t}-0.2ya^{-2t}+0.2ya^{-2t}-ya^{-2t}|\\ &\le |0.2xa^{-2t}-0.2ya^{-2t}|+|0.8ya^{-2t}|\\ &\le 0.05|x-y|+|x-0.2xa^{-2t}|\\ &\le \tilde{k}\|x-y\|+\tilde{L}\|x-T_tx\|.\end{aligned}$$

- Case II: If $x\in[0,0.2]$ and $y\in(0.2,1]$, then

$$\begin{aligned}\|T_tx-T_ty\| &= |xa^{-2t}-0.2ya^{-2t}|\\ &= |0.2xa^{-2t}-0.2ya^{-2t}-0.2xa^{-2t}+xa^{-2t}|\\ &\le |0.2xa^{-2t}-0.2ya^{-2t}|+|0.8xa^{-2t}|\\ &\le 0.05|x-y|+|x-xa^{-2t}|\\ &\le \tilde{k}\|x-y\|+\tilde{L}\|x-T_tx\|.\end{aligned}$$

- Case III: If $x\in A$ and $y\in[0,0.2]$, then

$$\begin{aligned}\|T_tx-T_ty\| &= |0-ya^{-2t}|\\ &\le |xa^{-2t}-ya^{-2t}|+|xa^{-2t}|\\ &\le \frac{|x-y|}{4}+\frac{|x|}{4}\\ &= 0.25|x-y|+0.25|x-T_tx|\\ &\le \tilde{k}\|x-y\|+\tilde{L}\|x-T_tx\|.\end{aligned}$$

- Case IV: If $x \in [0, 0.2]$ and $y \in A$, then

$$\begin{aligned} \|T_t x - T_t y\| &= |xa^{-2t} - 0| \\ &\leq 0.25|x| \\ &\leq (1 - a^{-2t})|x| \\ &= |x - xa^{-2t}| \\ &\leq \tilde{k}\|x - y\| + \tilde{L}\|x - T_t x\|. \end{aligned}$$

- Case V: If $x \in A$ and $y \in (0.2, 1]$, then

$$\begin{aligned} \|T_t x - T_t y\| &= |0 - 0.2ya^{-2t}| \\ &\leq |0.2xa^{-2t} - 0.2ya^{-2t}| + |0.2xa^{-2t}| \\ &\leq \frac{0.2|x - y|}{4} + \frac{0.2|x|}{4} \\ &= 0.05|x - y| + 0.05|x - T_t x| \\ &\leq \tilde{k}\|x - y\| + \tilde{L}\|x - T_t x\|. \end{aligned}$$

- Case VI: If $x \in (0.2, 1]$ and $y \in A$, then

$$\begin{aligned} \|T_t x - T_t y\| &= |0.2xa^{-2t} - 0| \\ &\leq 0.05|x| \\ &\leq (1 - 0.2a^{-2t})|x| \\ &= |x - 0.2xa^{-2t}| \\ &\leq \tilde{k}\|x - y\| + \tilde{L}\|x - T_t x\|. \end{aligned}$$

Thus, for all possible cases, we can conclude that τ satisfies the condition ($\mathbb{W}$).

By applying Theorem 4.8, we obtain, for each x_0, $x_1 \in X$ and $\{t_n\} \subset \mathbb{N}$, the inertial Mann iteration $\{x_n\}$ defined in (IM_n), hwere $\{\lambda_n\} \subset [0, 1]$ satisfies $\sum_{n=1}^{\infty} \lambda_n = \infty$ and $\{\beta_n\} \subset [0, b]$ is chosen such that $0 \leq \beta_n \leq \tilde{\beta}_n$ with

$$\tilde{\beta}_n = \begin{cases} \min\left\{b, \frac{\omega_n}{\|x_n - x_{n-1}\|}\right\} & \text{if } x_n \neq x_{n-1}, \\ b & \text{otherwise}, \end{cases}$$

where $b \in [0, 1)$ and $\{\omega_n\} \subset [0, \infty)$ satisfies $\sum_{n=1}^{\infty} \omega_n < \infty$, converges strongly to 0.

Finally, there are three sets of experiments which are performed to see the convergence behavior of the inertial Mann iteration $\{x_n\}$. In the first set of experiments, we choose $a = 2$, $b = 0.5$, $\lambda_n = (n+1)^{-\frac{1}{2}}$, $\omega_n = (n+1)^{-2}$, $t_n = 2n + 5$, and test different three initial numbers, that is, $(x_0, x_1) = (2, 6)$, $(x_0, x_1) = (-5, -1)$ and $(x_0, x_1) = (-3, 7)$. Table 1 shows the value of $\{x_n\}$ with different initial numbers and Figure 1(a) illustrates the convergence behavior of $\{x_n\}$.

A second set of experiments, there are three sequences $\lambda_n = (n+1)^{-\frac{1}{2}}$, $\lambda_n = (n+2)^{-\frac{1}{3}}$, and $\lambda_n = (n+3)^{-\frac{1}{4}}$ for testing. Each sequence $\{\lambda_n\} \subset [0, 1]$ is tested on the inertial Mann iteration $\{x_n\}_{n=0}^{\infty}$ with $a = 4$, $b = 0.5$, $(x_0, x_1) = (3, -1)$, $\omega_n = (n+1)^{-2}$, and $t_n = 2n + 5$. The results of this experiment are shown in Table 2 and Figures 1(b).

In the final set of experiments, for fixed $a = 5$, $b = 0.5$, $(x_0, x_1) = (2, -0.5)$, $\lambda_n = (n+1)^{-\frac{1}{2}}$ and $t_n = n$, we have experimented with different choices of the sequence $\{\omega_n\} \subset [0, \infty)$

satisfying $\sum_{n=1}^{\infty} \omega_n < \infty$, as illustrated in Table 3 and Figure 1(c). From all experiments, it can be seen that the inertial Mann iteration $\{x_n\}$ converges to 0.

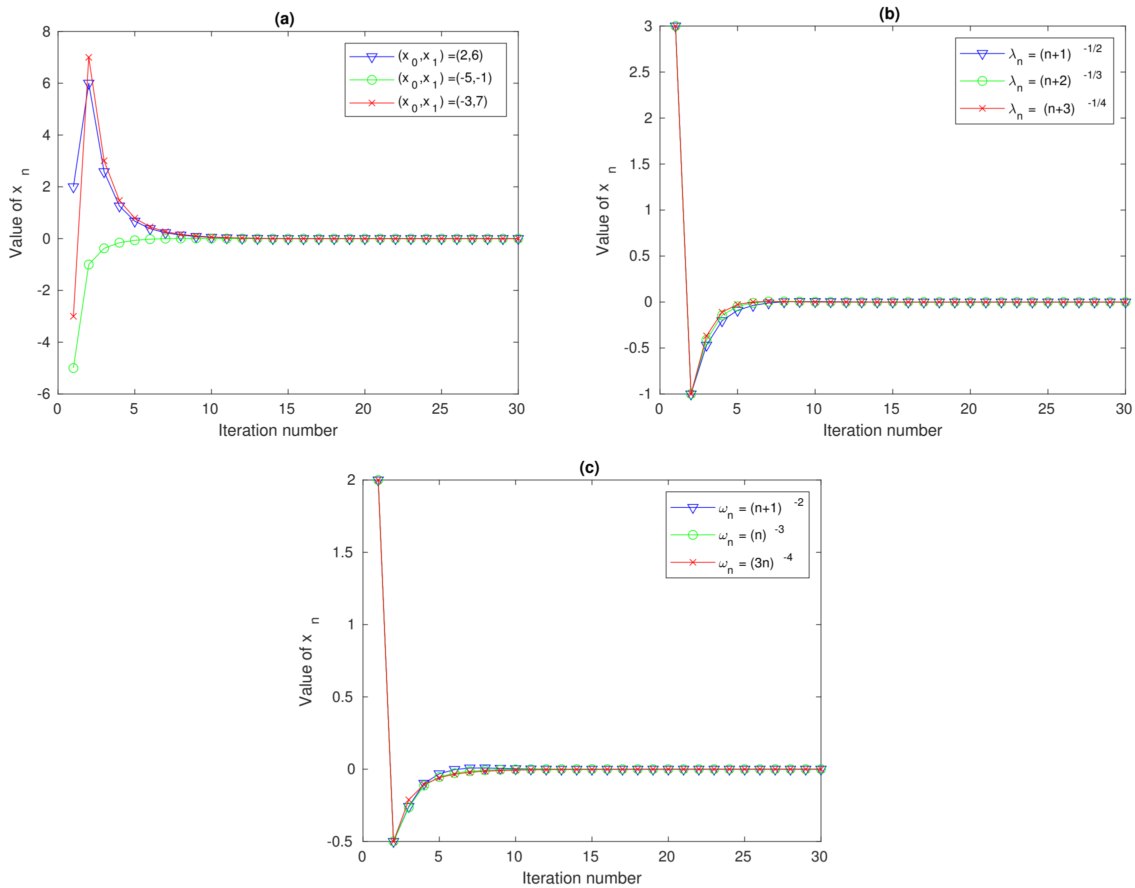

FIGURE 1. The convergence behavior of the inertial Mann iteration

TABLE 1. The values of $\{x_n\}$ with different initial numbers (x_0, x_1)

	The value of $\{x_n\}$		
	$(x_0,x_1) = (2,6)$	$(x_0,x_1) = (-5,-1)$	$(x_0,x_1) = (-3,7)$
x_0	2.0000	-5.0000	-3.0000
x_1	6.0000	-1.0000	7.0000
x_2	2.5829	-0.3757	3.0055
x_3	1.2602	-0.1566	1.4715
x_4	0.6745	-0.0645	0.7913
$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_{13}	0.0019	-0.0001	0.0047
$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_{18}	-0.0006	-0.0000	-0.0008
$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_{22}	-0.0000	-0.0000	-0.0000

TABLE 2. The values of $\{x_n\}$ with different choices of $\{\lambda_n\} \subset [0,1]$

	The value of $\{x_n\}$		
	$\lambda_n = (n+1)^{-\frac{1}{2}}$	$\lambda_n = (n+2)^{-\frac{1}{3}}$	$\lambda_n = (n+3)^{-\frac{1}{4}}$
x_0	3.0000	3.0000	3.0000
x_1	-1.0000	-1.0000	-1.0000
x_2	-0.4696	-0.4112	-0.3681
x_3	-0.2036	-0.1448	-0.1103
x_4	-0.0904	-0.0471	-0.0271
$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_{13}	0.0001	-0.0002	-0.0000
$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_{15}	-0.0002	-0.0000	-0.0000
$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_{19}	-0.0000	-0.0000	-0.0000

TABLE 3. The values of $\{x_n\}$ with different choices of $\{\omega_n\} \subset [0,\infty)$

	The value of $\{x_n\}$		
	$\omega_n = (n+1)^{-2}$	$\omega_n = n^{-3}$	$\omega_n = (3n)^{-4}$
x_0	2.0000	2.0000	2.0000
x_1	-0.5000	-0.5000	-0.5000
x_2	-0.2583	-0.2642	-0.2117
x_3	-0.0979	-0.1136	-0.1057
x_4	-0.0320	-0.0541	-0.0584
x_5	-0.0025	-0.0273	-0.0346
$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_{17}	-0.0001	0.0000	-0.0005
x_{18}	-0.0000	0.0000	-0.0004
$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_{29}	-0.0000	0.0000	-0.0000

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