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FORCING STRONG CONVERGENCE OF A VISCOSITY ITERATION FOR NONEXPANSIVE AND COCOERCIVE OPERATORS IN A HILBERT SPACE

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Abstract. A viscosity iteration is introduced and considered for nonexpansive operators and the variational inequality with cocoercive operators. A common element theorem of strong convergence is established in the setting of Hilbert space without compact restrictions.

Keywords. Variational inequalities; Fixed point; Nearest point projection; Strong convergence.

1. Introduction-Preliminaries

In the real world, there are a lot of nonlinear phenomena, which can be modelled into variational inequalities and variational inclusions, such as, single processing, image recovery, machine learning; see, e.g., [1,11,14,20,21,23,31] and the references therein. Fixed point methods are powerful and popular for dealing various nonlinear operator equations and inequalities in abstract spaces, in particular, for variational inequalities and variational inclusions. Recently, various efficient fixed point methods have been introduced and investigated; see, e.g., [4,6,7,15,16] and the references therein. Let T be a nonlinear operator on a Hilbert space H, which is endowed with inner product $\langle\cdot,\cdot\rangle$ and induced norm $\|\cdot\|$. The fixed point set of T is presented by $Fix(T)$. Recall that T is said to be contractive iff there is a real number $a\in(0,1)$ such that

$$\|Tx-Ty\|\leq a\|x-y\|, \quad \forall x,y\in H.$$

Recall that T is said to be nonexpansive iff

$$\|Tx-Ty\|\leq\|x-y\|, \quad \forall x,y\in H.$$

Recall that T is said to be firmly nonexpansive iff

$$\|Tx-Ty\|^2\leq\langle x-y,Tx-Ty\rangle, \quad \forall x,y\in H.$$

It is clear the class of firmly nonexpansive mappings is a special class of nonexpansive mappings. One knows the projection operator (see below) is firmly nonexpansive. The class of

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nonexpansive operators is significant in various nonlinear equations and mathematical programming computation. It also has a wide real applications in applied and industrial fields. For various iterative methods, Mann iteration is popular for dealing with fixed points of nonexpansive operators. It reads

$$x_{n+1} = (1-\alpha_n)Tx_n + \alpha_n x_n,$$

where $\{\alpha_n\}$ is a real number sequence in the interval $(0,1)$. However, the Mann iteration is weakly convergent only in infinite dimensional spaces; see, e.g., [8] and the references therein. To force the strong convergence without possible compact assumptions, various regularized methods have been investigated in Hilbert spaces and Banach spaces recently; see, e.g., [10, 12, 13, 19, 24] and the references therein. One of the efficient regularized method is the Halpern iteration, which reads

$$x_{n+1} = (1-\alpha_n)Tx_n + \alpha_n x,$$

where $\{\alpha_n\}$ is a real number sequence in the interval $(0,1)$, and x is a fixed anchor. With some conditions on $\{\alpha_n\}$, it was proved that $\{x_n\}$ converges to x, which is a special fixed point of T, that is, the nearest point in $Fix(T)$ to x. Halpern [9] pointed out that the conditions (c1) $\alpha_n \to 0$ as $n \to \infty$ and (c2) $\sum_{n=1}^{\infty} \alpha_n = \infty$ are necessary if the Halpern iteration scheme converges in norm. In view of (c2), the Halpern iteration may not be a fast iteration. Recently, a number of researchers investigated the problem of removing (c2) with the aid of projections; see, e.g., [17, 25, 28] and the references therein. In 2000, Moudafi [18] further proposed the viscosity approximation iteration, which reads as follows

$$x_{n+1} = (1-\alpha_n)Tx_n + \alpha_n Sx_n,$$

where S is a contraction. This approximation method, which improves the property of the class of nonexpansive mappings, is popular from the viewpoint of variational inequalities. Indeed, the fixed point also solves a monotone variational inequality with S. Another popular regularized method is the hybrid projection method, which was considered by Nakajo and Takahashi [19] for fixed points of nonexpansive mappings first. Indeed, they studied the following algorithm

$$\begin{cases} x_0 \in C, \\ y_n = (1-\alpha_n)Tx_n + \alpha_n x_n, \\ Q_n = \{x \in C : \langle x_n - x, x_n - x_0 \rangle \le 0\}, \\ C_n = \{x \in C : \|x - y_n\| \le \|x - x_n\|\}, \\ x_{n+1} = Proj_{Q_n \cap C_n} x_0, \end{cases}$$

where C is a closed, convex, and nonempty subset of H, and $Proj_{Q_n \cap C_n}$ is the nearest point projection onto the intersection set. They obtained a strong convergence theorem for nonexpansive mappings in a real Hilbert spaces without compact assumption on T. For more general nonlinear mappings though the projection-based method, we refer to [2, 3, 5, 22, 30] and the references therein.

Let C be a convex and closed subset of a real Hilbert space H. From now on, $Proj_C$ is borrowed to denote the nearest projection onto subset C, i.e., $Proj_C(x) := \arg\min\{\|x-y\|, y \in C\}$. Let A be a nonlinear mapping on H. Recall that A is said to be

(1) *strongly monotone* iff there exists a positive constant ξ such that $\langle Ax - Ay, x - y \rangle \ge \xi \|x - y\|^2$, $\forall x, y \in H$.

(2) *monotone* iff $\langle Ax - Ay, x - y \rangle \ge 0$, $\forall x, y \in H$;

(3) *cocoercive* iff there exists a positive constant ξ such that $\langle Ax-Ay, x-y\rangle \geq \xi\|Ax-Ay\|^2$, $\forall x,y\in H$.

Let $B: H \rightrightarrows H$ be a mutil-valued nonlinear mapping. Next, we turn our attention to the class of mutil-valued mappings. B is said to be a monotone mapping if and only if for all $x,y\in H$, $f\in By$ and $e\in Bx \implies \langle e-f, x-x\rangle > 0$. The symbol $B^{-1}(0)$ is used to stand for the set of zero points of B. Mapping B is said to be a maximally monotone mapping iff the graph of B, $Graph(B)$, is not contained in the graph of any other monotone mapping properly. Let $J^B_\beta = (Id+\beta B)^{-1}$, where Id is the identity mapping, and β is a constant. This operator is called the resolvent of B. Its domain is denoted by $Dom(B)$ in this paper. It is clear $B^{-1}(0) = Fix(J^B_\beta)$.

Consider the following variational inclusion problem, which finds a point $x\in C$ such that $x\in (B+A)^{-1}(0)$, where B is a multi-valued maximally monotone mapping, and A is a ξ-cocoercive mapping. For the inclusion problem, splitting methods (FB, PR, and DR) are popular for zero points of the sum of the monotone mappings. Splitting methods were considered by many authors for image recovery, signal processing and machine learning. The FB-type splitting method means an iterative method for which each iteration involves only with the individual operators not the sum. In this paper, with the condition that the solution set is nonempty, we consider finding a $\theta\in C$ such that $\theta\in F(T)\cap(B+A)^{-1}(0)$, where T is a nonexpansive mapping with a nonempty fixed-point set, B is a multi-valued maximally monotone mapping, and A is a ξ-cocoercive mapping. We establish a strong convergence with the aid of hybrid projection and FB splitting in a Hilbert space. Our strong convergence theorem requires less restriction on parameter sequences and the operators.

To show our main findings, we also need the following necessary tools.

Nearest point projection operator $Proj_C$ has the following property:

$$\|Proj_C y - Proj_C x\|^2 \leq \langle y-x, Proj_C y - Proj_C(x)\rangle, \quad \forall x,y\in H.$$

Lemma 1.1. *[27] Let H be a Hilbert space, and let C be a convex, closed, and nonempty subset of H. Let T be a nonexpansive mapping on C. Then $Fix(T)$ is convex and closed.*

Remark 1.1. Let H be a Hilbert space, and let C be a convex, closed, and nonempty subset of H. Let $A: C\to H$ be a ξ-cocoercive mapping, and let $B: H \rightrightarrows H$ be a multi-valued maximally monotone operator. Then $Fix(J^B_\beta(Id-\beta A)) = (B+A)^{-1}(0)$, where β is some constant, and Id is the identity mapping. Besides, the resolvent is firmly nonexpansive. From Lemma 1.1, we have that $(B+A)^{-1}(0)$ is convex and closed.

Lemma 1.2. *[27] Let H be a Hilbert space, and let C be a convex, closed, and nonempty subset of H. Let T be a nonexpansive mapping on C. Then $Id-T$ is demiclosed (Let $\{x_n\}$ be a sequence weakly converging to x, and let $Tx_n - x_n \to \infty$ as $n\to\infty$. Then x is a fixed point of T).*

Lemma 1.3. *[29] Let $\{a_n\}$ be a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1-t_n)a_n + b_n + c_n, \quad \forall n\geq 0,$$

where $\{c_n\}$ is a sequence of nonnegative real numbers, $\{t_n\}\subset(0,1)$, and $\{b_n\}$ is a sequence of real numbers. Assume that

(a) $\sum_{n=0}^{\infty} c_n < \infty$;

(b) $\limsup_{n\to\infty} \frac{b_n}{t_n} \leq 0$, $\sum_{n=0}^{\infty} t_n = \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 1.4. *[26] Let H be a real Hilbert spaces, and let C be a convex, closed, and nonempty subset of space H. Let B be a monotone mapping of C into H and $N_C v$ the normal cone to C at $v \in C$, i.e.,*

$$N_C v = \{w \in H : \langle v-u, w\rangle \geq 0, \quad \forall u \in C\}$$

and define a mapping T on C by

$$Tv = \begin{cases} Bv + N_C v, & v \in C \\ \emptyset, & v \notin C. \end{cases}$$

Then T is maximal monotone and $0 \in Tv$ if and only if $\langle Bv, u-v\rangle \geq 0$ for all $u \in C$.

2. Main Results

Theorem 2.1. *Let H be a real Hilbert space. Let T be a nonexpansive mapping with a nonempty fixed point set on C, where C is a convex, closed, and nonempty subset of space H. Let $A : C \to H$ be a c-cocoercive mapping. Let $f : C \to C$ be a ξ-contractive mapping. Let $\{x_n\}$ be the iterative sequence generated in the following process: $x_0 \in C$ and*

$$\begin{cases} w_n = (1-\alpha_n)x_n + \alpha_n f(x_n), \\ z_n = P_C(Id - r_n A)w_n, \\ y_n = (1-\beta_n)Tz_n + \beta_n Err_n, \\ x_{n+1} = \gamma_n x_n + (1-\gamma_n)y_n, \quad \forall n \geq 0, \end{cases}$$

where Id is the identity on H, $\{Err_n\}$ is a bounded error sequence in C, $\{r_n\}$ is a positive real number sequence in $(0,2c)$, and $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are real number sequences in $(0,1)$. Assumed that $0 < r \leq r_n \leq r' < 2c$ for some $r, r' \in (0,2c)$; $\sum_{n=0}^{\infty} \alpha_n = \infty$ with $\lim_{n\to\infty} \alpha_n = 0$; $\sum_{n=0}^{\infty} \beta_n < \infty$; $0 < \gamma \leq \gamma_n \leq \gamma' < 1$ for some $\gamma, \gamma \in (0,1)$, $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, and $\sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty$, $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$. If $\Omega := F(T) \cap Sol(A,C)$ is not empty, then $\{x_n\}$ converges strongly to a point $\bar{x} \in \Omega$, and $\bar{x} = P_\Omega f(\bar{x})$.

Proof. One fixes a common solution x' in Ω. Hence,

$$\begin{aligned} \|x' - w_n\| &= \|(1-\alpha_n)(x'-x_n) + \alpha_n(x' - f(x')) + \alpha_n(f(x') - f(x_n))\| \\ &\leq (1-\alpha_n)\|x'-x_n\| + \alpha_n\|x' - f(x')\| + \alpha_n\|f(x_n) - f(x')\| \\ &\leq (1-\alpha_n(1-\xi))\|x'-x_n\| + \alpha_n\|x' - f(x')\|. \end{aligned} \tag{2.1}$$

Note that A is c-cocoercive, which yields that $Id - r_n A$ is nonexpansive for any n. Indeed,

$$\begin{aligned} \|(Id - r_n A)y - (Id - r_n A)x\|^2 &= \|y-x\|^2 - 2r_n\langle y-x, Ay - Ax\rangle + \|r_n Ay - r_n Ax\|^2 \\ &\leq (r_n - 2c)r_n\|Ay - Ax\|^2 + \|y-x\|^2, \end{aligned}$$

that is, $\|(Id - rA)y - (Id - rA)x\| \leq \|y-x\|$ for any $y, x \in C$. From (2.1), one has

$$\begin{aligned} \|x' - z_n\| &= \|P_C(x' - r_n Ax') - P_C(Id - r_n A)w_n\| \\ &\leq \|(x' - r_n Ax') - (Id - r_n A)w_n\| \\ &\leq \|x' - w_n\| \\ &\leq (1-\alpha_n(1-\xi))\|x'-x_n\| + \alpha_n\|x' - f(x')\|. \end{aligned} \tag{2.2}$$

It follows from (2.2) that

$$\begin{aligned}\|x'-y_n\| &\le (1-\beta_n)\|Tx'-Tz_n\|+\beta_n\|x'-Err_n\|\\ &\le (1-\beta_n)\|x'-z_n\|+\beta_n\|x'-Err_n\|\\ &\le [1-\alpha_n(1-\beta_n)(1-\xi)]\|x'-x_n\|+\alpha_n(1-\beta_n)\|x'-f(x')\|+\beta_n\|x'-Err_n\|.\end{aligned}$$

So,

$$\begin{aligned}\|x'-x_{n+1}\| &\le \lambda_n\|x'-x_n\|+(1-\lambda_n)\|x'-y_n\|\\ &\le \lambda_n\|x'-x_n\|+(1-\lambda_n)[1-\alpha_n(1-\beta_n)(1-\xi)]\|x'-x_n\|\\ &\quad+\alpha_n(1-\beta_n)(1-\lambda_n)\|x'-f(x')\|+\beta_n(1-\lambda_n)\|x'-Err_n\|.\end{aligned}$$

Note that $\sum_{n=1}^{\infty}\beta_n<\infty$ and $\{Err_n\}$ is bounded. By mathematical induction, one asserts a bounded sequence $\{x_n\}$. Hence, $\{y_n\}$, $\{z_n\}$, and $\{w_n\}$ are also bounded. Observe that

$$w_n-w_{n-1}=(1-\alpha_n)(x_n-x_{n-1})+\alpha_n(f(x_n)-f(x_{n-1}))+(\alpha_n-\alpha_{n-1})(f(x_{n-1})-x_{n-1}).$$

It yields

$$\begin{aligned}&\|w_n-w_{n-1}\|\\ &\le (1-\alpha_n)\|x_n-x_{n-1}\|+\alpha_n\|f(x_n)-f(x_{n-1})\|+|\alpha_n-\alpha_{n-1}|\|f(x_{n-1})-x_{n-1}\|\\ &\le (1-\alpha_n(1-\xi))\|x_n-x_{n-1}\|+|\alpha_n-\alpha_{n-1}|\|f(x_{n-1})-x_{n-1}\|.\end{aligned}\tag{2.3}$$

Observe that

$$\begin{aligned}&\|z_n-z_{n-1}\|\\ &\le \|(Id-r_nA)w_n-(Id-r_{n-1}A)w_{n-1}\|\\ &\le \|(Id-r_nA)w_n-(Id-r_nA)w_{n-1}\|+\|(Id-r_nA)w_{n-1}-(Id-r_{n-1}A)w_{n-1}\|\\ &\le \|w_{n-1}-w_n\|+|r_{n-1}-r_n|Aw_{n-1}\|.\end{aligned}\tag{2.4}$$

(2.3) and (2.4) send us to

$$\begin{aligned}&\|z_n-z_{n-1}\|\\ &\le (1-\alpha_n(1-\xi))\|x_n-x_{n-1}\|+|\alpha_n-\alpha_{n-1}|\|f(x_{n-1})-x_{n-1}\|+|r_{n-1}-r_n|\|Aw_{n-1}\|.\end{aligned}\tag{2.5}$$

On the other hand,

$$\begin{aligned}y_n-y_{n-1}&=(1-\beta_n)(Tz_n-Tz_{n-1})+\beta_n(Err_n-Err_{n-1})\\ &\quad+(\beta_n-\beta_{n-1})(Err_{n-1}-Tz_{n-1}).\end{aligned}$$

It results in

$$\begin{aligned}\|y_n-y_{n-1}\| &\le (1-\beta_n)\|Tz_n-Tz_{n-1}\|+\beta_n\|Err_n-Err_{n-1}\|\\ &\quad+|\beta_n-\beta_{n-1}|\|Err_{n-1}-Tz_{n-1}\|\\ &\le (1-\beta_n)\|z_n-z_{n-1}\|+\beta_n\|Err_n-Err_{n-1}\|\\ &\quad+|\beta_n-\beta_{n-1}|\|Err_{n-1}-Tz_{n-1}\|.\end{aligned}\tag{2.6}$$

Combing (2.5) with (2.6), one sees that

$$\begin{aligned}&\|y_n-y_{n-1}\|\\ &\le (1-\alpha_n(1-\xi))\|x_n-x_{n-1}\|+|\alpha_n-\alpha_{n-1}|\|f(x_{n-1})-x_{n-1}\|\\ &\quad+|r_{n-1}-r_n|\|Aw_{n-1}\|+\beta_n\|Err_n-Err_{n-1}\|+|\beta_n-\beta_{n-1}|\|Err_{n-1}-Tz_{n-1}\|.\end{aligned}\tag{2.7}$$

Observe that

$$x_{n+1}-x_n=\gamma_n(x_n-x_{n-1})+(1-\gamma_n)(y_n-y_{n-1})+(\gamma_n-\gamma_{n-1})(x_{n-1}-y_{n-1}).$$

It yields that

$$\|x_{n+1}-x_n\|\leq \gamma_n\|x_n-x_{n-1}\|+(1-\gamma_n)\|y_n-y_{n-1}\|+|\gamma_n-\gamma_{n-1}|\|x_{n-1}-y_{n-1}\|. \tag{2.8}$$

Combing (2.7) and (2.8), one finds

$$\begin{aligned}\|x_{n+1}-x_n\|\leq{}& [1-\alpha_n(1-\xi)(1-\gamma_n)]\|x_n-x_{n-1}\|\\ &+|\alpha_n-\alpha_{n-1}|\|f(x_{n-1})-x_{n-1}\|+|r_{n-1}-r_n|\|Aw_{n-1}\|+\beta_n\|Err_n-Err_{n-1}\|\\ &+|\beta_n-\beta_{n-1}|\|Err_{n-1}-Tz_{n-1}\|+|\gamma_n-\gamma_{n-1}|\|x_{n-1}-y_{n-1}\|.\end{aligned}$$

From the restrictions and conditions on the parameter control sequences, one asserts from Lemma 1.3 that $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$. Observe that

$$\|x'-w_n\|^2\leq(1-\alpha_n)\|x'-x_n\|^2+\alpha_n\|x'-f(x_n)\|^2.$$

It follows that

$$\begin{aligned}\|x'-z_n\|^2&\leq\|(Id-r_nA)x'-(Id-r_nA)w_n\|^2\\ &\leq\|x'-w_n\|^2-(2c-r_n)r_n\|Ax'-Aw_n\|^2\\ &\leq(1-\alpha_n)\|x'-x_n\|^2+\alpha_n\|x'-f(x_n)\|^2-(2c-r_n)r_n\|Ax'-Aw_n\|^2.\end{aligned}$$

Furthermore,

$$\begin{aligned}\|x'-y_n\|^2&\leq(1-\beta_n)\|Tx'-Tz_n\|^2+\beta_n\|x'-Err_n\|^2\\ &\leq(1-\beta_n)\|x'-z_n\|^2+\beta_n\|x'-Err_n\|^2\\ &\leq(1-\alpha_n)(1-\beta_n)\|x'-x_n\|^2+\alpha_n(1-\beta_n)\|x'-f(x_n)\|^2\\ &\quad-(2c-r_n)r_n(1-\beta_n)\|Ax'-Aw_n\|^2+\beta_n\|x'-Err_n\|^2,\end{aligned}$$

and

$$\begin{aligned}&\|x'-x_{n+1}\|^2\\ &\leq\gamma_n\|x'-x_n\|^2+(1-\gamma_n)\|x'-y_n\|^2\\ &\leq\gamma_n\|x'-x_n\|^2+(1-\gamma_n)(1-\alpha_n)(1-\beta_n)(1-\gamma_n)\|x'-x_n\|^2\\ &\quad+\alpha_n(1-\beta_n)\|x'-f(x_n)\|^2\\ &\quad-(2c-r_n)r_n(1-\beta_n)(1-\gamma_n)\|Ax'-Aw_n\|^2+\beta_n\|x'-Err_n\|^2\\ &\leq\|x'-x_n\|^2+\alpha_n\|x'-f(x_n)\|^2\\ &\quad-(2c-r_n)r_n(1-\beta_n)(1-\gamma_n)\|Ax'-Aw_n\|^2+\beta_n\|x'-Err_n\|^2.\end{aligned}$$

So,

$$\begin{aligned}&(2c-r_n)r_n(1-\beta_n)(1-\gamma_n)\|Ax'-Aw_n\|^2\\ &\leq(\|x'-x_n\|+\|x'-x_{n+1}\|)\|x_n-x_{n+1}\|\\ &\quad+\alpha_n\|x'-f(x_n)\|^2+\beta_n\|x'-Err_n\|^2.\end{aligned}$$

From the parameters on the control sequences, one asserts that $\lim_{n\to\infty}\|Aw_n - Ax'\| = 0$. From the property of projection P_C (i.e., firmly nonexpansive), one arrives at

$$\begin{aligned}2\|x'-z_n\|^2 &\le 2\langle x'-z_n, (Id-r_nA)x'-(Id-r_nA)q_n\rangle \\ &\le \|x'-z_n\|^2 + \|(Id-r_nA)x'-(Id-r_nA)w_n\|^2 \\ &\quad - \|z_n-w_n+r_nAw_n-r_nAx'\|^2,\end{aligned}$$

that is,

$$\|x'-z_n\|^2 \le \|x'-w_n\|^2 - \|z_n-w_n\|^2 + 2\|z_n-w_n\|\|r_nAw_n-r_nAx'\|. \tag{2.9}$$

It holds that

$$\|x'-w_n\|^2 \le (1-\alpha_n)\|x'-x_n\|^2 + \alpha_n\|x'-f(x_n)\|^2.$$

These imply that

$$\begin{aligned}&\|x'-z_n\|^2 \\ &\le \|x'-w_n\|^2 - \|z_n-w_n\|^2 + 2\|z_n-w_n\|\|r_nAw_n-r_nAx'\| \\ &\le (1-\alpha_n)\|x'-x_n\|^2 + \alpha_n\|x'-f(x_n)\|^2 - \|z_n-w_n\|^2 + 2\|z_n-w_n\|\|r_nAw_n-r_nAx'\|.\end{aligned} \tag{2.10}$$

On the other hand,

$$\begin{aligned}\|x'-y_n\|^2 &\le (1-\beta_n)\|Tx'-Tz_n\|^2 + \beta_n\|x'-Err_n\|^2 \\ &\le (1-\beta_n)\|x'-z_n\|^2 + \beta_n\|x'-Err_n\|^2.\end{aligned} \tag{2.11}$$

Combing (2.10) and (2.11), one has

$$\begin{aligned}\|x'-y_n\|^2 \le (1-\alpha_n)(1-\beta_n)\|x'-x_n\|^2 + \alpha_n\|x'-f(x_n)\|^2 - (1-\beta_n)\|z_n-w_n\|^2 \\ + 2\|z_n-w_n\|\|r_nAw_n-r_nAx'\| + \beta_n\|x'-Err_n\|^2.\end{aligned}$$

So, one has

$$\begin{aligned}\|x'-x_{n+1}\|^2 &\le \gamma_n\|x'-x_n\|^2 + (1-\gamma_n)\|x'-y_n\|^2 \\ &\le \gamma_n\|x'-x_n\|^2 + (1-\alpha_n)(1-\beta_n)(1-\gamma_n)\|x'-x_n\|^2 + \alpha_n\|x'-f(x_n)\|^2 \\ &\quad - (1-\beta_n)(1-\gamma_n)\|z_n-w_n\|^2 + 2\|z_n-w_n\|\|r_nAw_n-r_nAx'\| + \beta_n\|x'-Err_n\|^2,\end{aligned}$$

which sends us to

$$\begin{aligned}(1-\beta_n)(1-\gamma_n)\|z_n-w_n\|^2 \le \|x'-x_n\|^2 - \|x'-x_{n+1}\|^2 + \alpha_n\|x'-f(x_n)\|^2 \\ + 2\|z_n-w_n\|\|r_nAw_n-r_nAx'\| + \beta_n\|x'-Err_n\|^2.\end{aligned}$$

By the condition on the parameters, one has $z_n - w_n \to \infty$ when $n \to \infty$ and $x_n - w_n \to \infty$ when $n \to \infty$, respectively. One also has $y_n - Tz_n \to \infty$ when $n \to \infty$. From the generation of x_{n+1}, that is, $x_{n+1} = \gamma_n x_n + (1-\gamma_n)y$, we assert $y_n - x_n \to \infty$ when $n \to \infty$. In view of P_Ω and f, we obtain a contractive mapping $P_\Omega f$. Next, we denote the unique fixed point of the mapping $P_\Omega f$ by $\bar{x}$, and show that $\limsup_{n\to\infty}\langle f(\bar{x})-\bar{x}, w_n-\bar{x}\rangle \le 0$. Observe that $\{w_n\}$ is a bounded vector sequence, one may pick a subsequence $\{w_{n_m}\}$ of $\{w_n\}$ such that

$$\limsup_{n\to\infty}\langle f(\bar{x})-\bar{x}, w_n-\bar{x}\rangle = \lim_{m\to\infty}\langle f(\bar{x})-\bar{x}, w_{n_m}-\bar{x}\rangle.$$

Furthermore, by the boundedness of $\{w_{n_m}\}$, we may pick a subsequence $\{w_{n_{m_e}}\}$, which converges weakly some point t. Indeed, we may assume that w_{n_m} converges weakly to t. Let

$$Mw = \begin{cases} \emptyset, & w \notin C, \\ N_C w + Aw, & w \in W. \end{cases}$$

So, M is a maximally monotone operator. Let $(w,w') \in Graph(M)$. Since $z_n \in C$ and $w' - Aw \in N_C w$, one gets $\langle Aw - w', w - z_n \rangle \le 0$. Observe from the generation of z_n that

$$\langle (Id - r_n A)w_n - z_n, z_n - w \rangle \ge 0,$$

which holds that

$$\begin{aligned}
\langle w - z_{n_m}, w' \rangle &\ge \langle w - z_{n_m}, Aw \rangle \\
&\ge \langle w - z_{n_m}, Aw \rangle - \langle \frac{z_{n_m} - w_{n_m}}{r_{n_m}} + Aw_{n_m}, w - z_{n_m} \rangle \\
&= \langle w - z_{n_m}, Aw - Az_{n_m} \rangle + \langle w - z_{n_m}, Az_{n_m} - Aw_{n_m} - \frac{z_{n_m} - w_{n_m}}{r_{n_m}} \rangle \\
&= \langle w - z_{n_m}, Aw - Az_{n_m} \rangle + \langle w - z_{n_m}, Az_{n_m} - Aw_{n_m} \rangle - \langle w - z_{n_m}, \frac{z_{n_m} - w_{n_m}}{r_{n_m}} \rangle \\
&\ge \langle w - z_{n_m}, Az_{n_m} - Aw_{n_m} \rangle - \langle w - z_{n_m}, \frac{z_{n_m} - w_{n_m}}{r_{n_m}} \rangle.
\end{aligned}$$

Since A is Lipschitz continuous, and as $n \to \infty$, $z_n - w_n \to 0$, we get $\langle w - t, w' - 0 \rangle \ge 0$, that is, $t \in M^{-1}(0)$. This shows from Lemma 1.4 that $t \in VI(C,A)$

On the other hand, T enjoys the demiclosed principle, we easily conclude that t is a fixed point of T, that is, $t = Tt$. From now on, we show that $\{x_n\}$ converges strongly to $\bar{x}$. Observe that Observe that

$$\begin{aligned}
\|\bar{x} - y_n\|^2 &\le (1-\beta_n)\|T\bar{x} - Tz_n\|^2 + \beta_n \|\bar{x} - Err_n\|^2 \\
&\le (1-\beta_n)\|P_C(Id - r_n A)\bar{x} - P_C(Id - r_n A)w_n\|^2 + \beta_n \|\bar{x} - Err_n\|^2 \\
&\le (1-\beta_n)\|(Id - r_n A)\bar{x} - (Id - r_n A)w_n\|^2 + \beta_n \|\bar{x} - Err_n\|^2 \\
&\le (1-\beta_n)\|\bar{x} - w_n\|^2 + \beta_n \|\bar{x} - Err_n\|^2.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\|\bar{x} - w_n\|^2 &\le (1-\alpha_n)\langle \bar{x} - x_n, \bar{x} - w_n \rangle + \alpha_n \langle \bar{x} - f(\bar{x}), \bar{x} - w_n \rangle + \alpha_n \langle f(\bar{x}) - f(x_n), \bar{x} - w_n \rangle \\
&\le (1-\alpha_n)\|\bar{x} - x_n\|\|\bar{x} - w_n\| + \alpha_n \langle \bar{x} - f(\bar{x}), \bar{x} - w_n \rangle + \alpha_n \|f(\bar{x}) - f(x_n)\|\|\bar{x} - w_n\|
\end{aligned}$$

and hence

$$2\|\bar{x} - w_n\|^2 \le (1 - \alpha_n(1-\xi))\|(\bar{x} - x_n\|^2 + \|\bar{x} - w_n\|^2) + 2\alpha_n \langle \bar{x} - f(\bar{x}), \bar{x} - w_n \rangle.$$

By using the fact

$$\|\bar{x} - w_n\|^2 \le (1 - \alpha_n(1-\xi))\|\bar{x} - x_n\|^2 + 2\alpha_n \langle \bar{x} - f(\bar{x}), \bar{x} - w_n \rangle,$$

we get

$$\begin{aligned}\|\bar{x}-y_n\|^2 &\le (1-\beta_n)\|T\bar{x}-Tz_n\|^2+\beta_n\|\bar{x}-Err_n\|^2\\ &\le (1-\beta_n)\|P_C(Id-r_nA)\bar{x}-P_C(Id-r_nA)w_n\|^2+\beta_n\|\bar{x}-Err_n\|^2\\ &\le (1-\alpha_n(1-\xi))(1-\beta_n)\|\bar{x}-x_n\|^2+2\alpha_n(1-\beta_n)\langle\bar{x}-f(\bar{x}),\bar{x}-w_n\rangle+\beta_n\|\bar{x}-Err_n\|^2.\end{aligned}$$

So, it follows that

$$\begin{aligned}\|\bar{x}-x_{n+1}\|^2 &\le \gamma_n\|\bar{x}-x_n\|^2+(1-\gamma_n)\|\bar{x}-y_n\|^2\\ &\le [1-\alpha_n(1-\xi)(1-\beta_n)(1-\gamma_n)]\|\bar{x}-x_n\|^2+2\alpha_n(1-\beta_n)(1-\gamma_n)\langle\bar{x}-f(\bar{x}),\bar{x}-w_n\rangle\\ &\quad+\beta_n\|\bar{x}-Err_n\|^2\end{aligned}$$

Lemma 1.3 is applicable to the above inequality, and thus, we have the desired conclusion immediately. This finishes the proof.

□

From Theorem 2.1, we have the following result immediately.

Corollary 2.1. *Let H be a real Hilbert space. Let T be a nonexpansive mapping with a nonempty fixed point set on C, where C is a convex, closed, and nonempty subset of space H Let $A: C\to H$ be a c-cocoercive mapping. Let $f: C\to C$ be a ξ-contractive mapping. Let $\{x_n\}$ be the iterative sequence generated in the following process: $x_0\in C$ and*

$$\begin{cases} w_n=(1-\alpha_n)x_n+\alpha_n f(x_n),\\ x_{n+1}=\gamma_n x_n+(1-\gamma_n)TP_C(Id-r_nA)w_n, \quad \forall n\ge 0,\end{cases}$$

where Id is the identity on H, $\{r_n\}$ is a positive real number sequence in $(0,2c)$, and $\{\alpha_n\}$ and $\{\gamma_n\}$ are real number sequences in $(0,1)$. Assumed that $0<r\le r_n\le r'<2c$ for some $r,r'\in(0,2c)$; $\sum_{n=0}^{\infty}\alpha_n=\infty$ with $\lim_{n\to\infty}\alpha_n=0$; $0<\gamma\le\gamma_n\le\gamma'<1$ for some $\gamma,\gamma\in(0,1)$, $\sum_{n=1}^{\infty}|\alpha_{n+1}-\alpha_n|<\infty$, and $\sum_{n=1}^{\infty}|\gamma_{n+1}-\gamma_n|<\infty$, $\sum_{n=1}^{\infty}|r_{n+1}-r_n|<\infty$. If $\Omega:=F(T)\cap Sol(A,C)$ is not empty, then $\{x_n\}$ converges strongly to a point $\bar{x}\in\Omega$, and $\bar{x}=P_\Omega f(\bar{x})$.

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