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STRONG CONVERGENCE OF AN EXPLICIT EXTRAGRADIENT-LIKE METHOD FOR A SPLIT FEASIBILITY PROBLEM

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Abstract. The purpose of this paper is to introduce a new explicit extragradient method for finding minimum-norm solutions of a split feasibility problem in real Hilbert spaces. The advantage of the proposed method is that it uses the less number of iterations to obtain its strong convergence. Numerical experiments illustrate the performances of our new algorithm and provide a comparison with related algorithms.

Keywords. Split feasibility problem; explicit extragradient-like method; strong convergence; Hilbert space; iterative algorithm.

1. INTRODUCTION

Let H_1, H_2, and H_3 be three real Hilbert spaces with inner product $\langle\cdot,\cdot\rangle$ and induce norm $\|\cdot\|$. We use $Fix(T)$ to denote the set of fixed points of the mapping T.

In this paper, we consider the classical split feasibility problem (SFP for short). The SFP was first introduced by Censor and Elfving [7] for some modeling inverse problems that arise from n medical image reconstruction and phase retrievals [2]. The SFP can be formulated as finding a point x^* in $\mathbb{R}^n$ such that

$$x^* \in C \text{ and } Ax^* \in Q, \tag{1.1}$$

where C and Q are nonempty, closed, and convex subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, and A is a $m\times n$ matrix. The SFP, which models the intensity-modulated radiation therapy, has been extensively investigated by iterative method; see e.g., [10, 11, 13, 15, 16] and the references therein.

Recently SFP (1.1) was further studied in the infinite dimensional spaces; see, e.g., [4–6, 18]. One of the popular methods for solving the SFP is Byrne's CQ algorithm [2,3], which generates a sequence $\{x_n\}$ as follows

$$x_{n+1} = P_C(x_n - \tau_n A^*(I-P_Q)Ax_n), \tag{1.2}$$

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where C and Q are nonempty closed convex subsets of H_1 and H_2, respectively, and the step size τ_n is located in the interval $(0, 2/\|A\|^2)$, where A^* is the adjoint of A, P_C and P_Q are the metric projections onto C and Q, respectively.

In [17], Wang and Xu gave the following iterative algorithm for solving the split feasibility problem

$$x_{n+1} = P_C[(1-\alpha_n)(I - \tau_n A^*(I - P_Q)A)]x_n, \tag{1.3}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ such that

(1) $\lim_{n\to\infty} \alpha_n = 0$;
(2) $\Sigma_{n=1}^{\infty} \alpha_n = \infty$;
(3) $\Sigma_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ *or* $\lim_{n\to\infty} |\alpha_{n+1} - \alpha_n|/\alpha_n = 0$.

They proved the strong convergence of the iterative algorithm (1.3). We remark that Wang and Xu used the Tychonov regularization method and obtained a net of solutions for some minimization problems in [17].

Denote the solution set of the SFP by Γ, i.e., $\Gamma = \{x \in C, Ax \in Q\}$, and assume that Γ is closed, convex, and nonempty.

In order to solve the SFP (1.1), we consider the following minimization problem:

$$\min_{x\in C} h(x) = \frac{1}{2}\|(I - P_Q)Ax\|^2, \tag{1.4}$$

The minimization problem is in general ill-posed. A classical way to deal with such a possibly ill-posed problem is the well-known Tychonov regularization, which approximates a solution of problem (1.4) by the unique minimizer of the regularized problem:

$$\min_{x\in C} h_\alpha(x) = \frac{1}{2}\|(I - P_Q)Ax\|^2 + \frac{1}{2}\alpha\|x\|^2, \tag{1.5}$$

where $\alpha > 0$ is the regularization parameter. Denote the unique solution of (1.5) by x_α.

For any $\alpha > 0$, the solution x_α of (1.5) is uniquely defined. x_α is characterized by the inequality $\langle \bigtriangledown A^*(I - P_Q)Ax_\alpha + \alpha x_\alpha, x - x_\alpha\rangle \geq 0$, $\forall x \in C$. Finally, x_α converges strongly as $\alpha \to 0$ to the minimum-norm solution $\bar{x}$ of SFP (1.1).

Lemma 1.1. *[11, 18] Given $x^* \in H$, where H is a Hilbert space, x^* solves the SFP iff x^* solves the variational inequality $\langle A^*(I - P_Q)Ax^*, x - x^*\rangle \geq 0$ for all $x \in C$.*

We recall that the classical variational inequality problem is formulated as follows: Finding a point $x^* \in C$ such that $\langle Ax^*, x - x^*\rangle \geq 0$, $\forall x \in C$, where C is a nonempty, closed, and convex subset in a real Hilbert space H, and $A : H \to H$ is a single-valued mapping.

Korpelevich [9] introduced the so-called extragradient method for finding a solution of the variational inequality problem, which is expressed as follows:

$$x_0 \in C,\ y_n = P_C(x_n - \lambda F x_n),\ x_{n+1} = P_C(x_n - \lambda F y_n), \quad \forall n \geq 0, \tag{1.6}$$

where P_C is the metric projection of H onto C, $F : H \to H$ is monotone and L-Lipschitz continuous on C, and $\lambda \in (0, \frac{1}{L})$.

Motivated by the extragradient method, the following problem is raised naturally.

Question 1.1. Can we modify iterative scheme (1.6) to solve the split feasibility problem?

The purpose of this paper is to construct an explicit extragradient-like method for the SFP so that strong convergence is guaranteed. The paper is organized as follows. In Section 2, we use the Tychonov regularization to obtain a net of solutions for some minimization problem and introduce a method to obtain the minimum-norm solution of the SFP. In Section 3, we introduce a new algorithm and prove the strong convergence of the algorithm. In Section 4, some numerical experiments are provided to illustrate the performances of our new algorithm, and a comparison with related algorithms is carried out.

2. Preliminaries

In this section, we give some concepts and elementary facts which will be used in the proof of the main results. Using the Tychonov regularization, we obtain a net of solutions for some minimization problem such that it converges strongly to the minimum-norm solutions of the SFP (1.1).

Recall that [17] an element $\bar{x} \in \Gamma$ is said to be the minimum-norm solution of SFP (1.1) if $\|\bar{x}\| = \inf_{x\in\Gamma} \|x\|$.

Let T be an operator on a Hilbert space H. Recall that

T is said to be L-Lipschitz continuous with $L > 0$ if, for any x and y in H, $\|Tx-Ty\| \leq L\|x-y\|$.

T is said to be monotone if, for any x and y in H, $\langle Tx-Ty, x-y\rangle \geq 0$.

T is said to be pseudomonotone if, for any x and y in H, $\langle Tx, y-x\rangle \geq 0 \Rightarrow \langle Ty, y-x\rangle \geq 0$.

T is said to be nonexpansive if, for any x and y in H, $\|Tx-Ty\| \leq \|x-y\|$.

T is said to be firmly nonexpansive if, for any x and y in H, $\langle x-y, Tx-Ty\rangle \geq \|Tx-Ty\|^2$.

It is well known that T is firmly nonexpansive iff $2T-I$ is nonexpansive. Equivalently, $T = (I+S)/2$, where $S : H \to H$ is a nonexpansive mapping.

Assume that S is a nonempty, closed, and convex subset of the Hilbert space H, and let P_S denote the projection from H to H, that is, $P_S(y) = \{x \in S, \min_{x\in S} \|x-y\|\}$. It is well known that $P_S(y)$ is characterized by the inequality $\langle y-P_S(y), x-P_S(y)\rangle \leq 0$, $\forall x \in S$, and P_S *and* $(I-P_S)$ are nonexpansive and firmly nonexpansive.

Next, we collect some elementary facts which will be used in the proof of our main results.

Lemma 2.1. *[8, 19] Let X be a Banach space. Let C be a closed and convex subset of X, and let $T : C \to C$ be a nonexpansive mapping with $Fix(T) \neq \emptyset$. If $\{x_n\}$ is a sequence in C weakly converging to x and $\{(I-T)x_n\}$ converges strongly to y, then $(I-T)x = y$.*

Lemma 2.2. *[1] Let $\{s_n\}$ be a sequence of nonnegative real number. Let $\{\alpha_n\}$ be a sequence of real numbers in $[0,1]$ with $\Sigma_{n=1}^{\infty} \alpha_n = \infty$. Let $\{u_n\}$ be a sequence of nonnegative real numbers with $\Sigma_{n=1}^{\infty} u_n < \infty$. Let $\{t_n\}$ be a sequence of real number with $\limsup_n t_n \leq 0$. Suppose that $s_{n+1} = (1-\alpha_n)s_n + \alpha_n t_n + u_n$, $\forall n \in N$. Then $\lim_{n\to\infty} s_n = 0$.*

The condition $\limsup_{n\to\infty} t_n \leq 0$ in Lemma 2.2 can be slightly relaxed as follows.

Lemma 2.3. *[12] Let $\{s_n\}$ be a sequence of nonnegative real number. Let $\{\alpha_n\}$ be a sequence of real number in $[0,1]$ with $\Sigma_{n=1}^{\infty} \alpha_n = \infty$. Suppose that $s_{n+1} = (1-\alpha_n)s_n + \alpha_n t_n$, $\forall n \leq 1$. If $\limsup_{n\to\infty} t_n \leq 0$ for every subsequence $\{s_{n_k}\}$ of $\{s_n\}$ satisfying $\liminf_{k\to\infty}(s_{n_k+1} - s_{n_k}) \geq 0$, then $\lim_{n\to\infty} s_n = 0$.*

Lemma 2.4. *[14] Let $\{w_n\}$ and $\{z_n\}$ be bounded sequences in a Banach space, and let $\{\beta_n\}$ be a sequence in $[0,1]$ such that $0<\liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n<1$. If $w_{n+1}=(1-\beta_n)w_n+\beta_n z_n$ and $\limsup_{n\to\infty}(\|z_{n+1}-z_n\|-\|w_{n+1}-w_n\|)\le 0$, then $\lim_{n\to\infty}\|z_n-w_n\|=0$.*

Next, we introduce a new method.

Definition 2.1. Let $S=P_C(I-\gamma A^*(I-P_Q)A)$, $W=P_{C_n}(I-\gamma A^*(I-P_Q)AS)$, and $T=I-\frac{\beta_n}{1-\alpha_n}(I-S)$ with $0<\gamma<\min\{1,\|A\|\}$, where $C_n:=\{x\in H:\langle x_n-\gamma A^*(I-P_Q)Ax_n-Sx_n, x-Sx_n\rangle\le 0\}$, $\{\alpha_n\}$ and $\{\beta_n\}$ are two real sequences in $(0,1)$ such that $\{\beta_n\}\subset(a,1-\alpha_n)$ for some $a>0$, $\lim_{n\to\infty}\alpha_n=0$, $\Sigma_{n=1}^{\infty}\alpha_n=\infty$ and $\Sigma_{n=1}^{\infty}|\alpha_{n+1}-\alpha_n|<\infty$ or $\lim_{n\to\infty}|\alpha_{n+1}-\alpha_n|/\alpha_n=0$.

Remark 2.1. Take $\gamma\in(0,\min\{1,\|A\|\})$. For $\alpha_n\in(0,\frac{\gamma}{\|A\|+1})$, we define a mapping:

$$X_\sigma x:=(1-\alpha_n\gamma)x-\beta_n(I-W)x. \tag{2.1}$$

It is easy to check that X_σ is contractive. So, X_σ has a unique fixed point, denoted by x_σ, that is,

$$x_\sigma=(1-\alpha_n\gamma)x_\sigma-\beta_n(I-W)x_\sigma. \tag{2.2}$$

Lemma 2.5. *Let $\{x_n\}$ be a sequence generated by, $x_0\in H$, $x_{n+1}=(1-\alpha_n\gamma)x_n-\beta_n(I-W)x_n$. If there exists a subsequence $\{x_{n_k}\}$ convergent weakly to $\bar{x}\in H$ and $\lim_{k\to\infty}\|x_{n_k}-Sx_{n_k}\|=0$, then $\bar{x}\in\Gamma$.*

Proof. Observe that $\langle x_{n_k}-\gamma A^*(I-P_Q)Ax_{n_k}-Sx_{n_k}, x-Sx_{n_k}\rangle\le 0$, $\forall x\in C$, i.e., $\frac{1}{\gamma}\langle x_{n_k}-Sx_{n_k}, x-Sx_{n_k}\rangle\le\langle A^*(I-P_Q)Ax_{n_k}, x-Sx_{n_k}\rangle$, $\forall x\in C$. Equivalently, we have

$$\frac{1}{\gamma}\langle x_{n_k}-Sx_{n_k},x-Sx_{n_k}\rangle+\langle A^*(I-P_Q)Ax_{n_k},Sx_{n_k}-x_{n_k}\rangle\le\langle A^*(I-P_Q)Ax_{n_k},x-x_{n_k}\rangle,\ \forall x\in C. \tag{2.3}$$

Observe that $\{x_{n_k}\}$ is weakly convergent, By the Lipschitz continuity of $A^*(I-P_Q)A$, we have that $\{A^*(I-P_Q)Ax_{n_k}\}$ is bounded. Since $\lim_{k\to\infty}\|x_{n_k}-Sx_{n_k}\|=0$, then $\{Sx_{n_k}\}$ is also bounded. Let $k\to\infty$ in (2.3), we obtain

$$\liminf_{k\to\infty}\langle A^*(I-P_Q)Ax_{n_k},x-x_{n_k}\rangle\ge 0,\quad \forall x\in C. \tag{2.4}$$

Moreover, we have

$$\begin{aligned}\langle A^*(I-P_Q)ASx_{n_k},x-Sx_{n_k}\rangle &= \langle A^*(I-P_Q)ASx_{n_k}-A^*(I-P_Q)Ax_{n_k},x-x_{n_k}\rangle\\ &\quad+\langle A^*(I-P_Q)Ax_{n_k},x-x_{n_k}\rangle\\ &\quad+\langle A^*(I-P_Q)ASx_{n_k},x_{n_k}-Sx_{n_k}\rangle.\end{aligned} \tag{2.5}$$

Since $\lim_{k\to\infty}\|x_{n_k}-Sx_{n_k}\|=0$ and $A^*(I-P_Q)A$ is Lipschitz continuous on H, we have

$$\lim_{k\to\infty}\|A^*(I-P_Q)Ax_{n_k}-A^*(I-P_Q)ASx_{n_k}\|=0,$$

which together with (2.4) and (2.5) implies that $\liminf_{k\to\infty}\langle A^*(I-P_Q)ASx_{n_k},x-Sx_{n_k}\rangle\ge 0$, $\forall x\in C$. Next, we show that $\bar{x}\in\Gamma$. We choose a sequence $\{\eta_k\}$ of positive numbers such that $\{\eta_k\}$ is decreasing and convergent to 0. For each $k\ge 1$, we denote by n_{N_k} the smallest positive integer such that

$$\langle A^*(I-P_Q)ASx_{n_j},x-Sx_{n_j}\rangle+\eta_k\ge 0,\quad \forall j\ge n_{N_k}. \tag{2.6}$$

Since $\{\eta_k\}$ is decreasing, it is easy to see that $\{n_{N_k}\}$ is increasing. Furthermore, for each $k \geq 1$, since $\{Sx_{n_{N_k}}\} \subset C$, we have $A^*(I-P_Q)ASx_{n_{N_k}} \neq 0$. Setting

$$v_{n_{N_k}} = \frac{A^*(I-P_Q)ASx_{n_{N_k}}}{\|A^*(I-P_Q)ASx_{n_{N_k}}\|^2},$$

we have $\langle A^*(I-P_Q)ASx_{n_{N_k}}, v_{n_{N_k}} \rangle = 1$ for each $k \geq 1$. Now, for each $k \geq 1$, it follows from (2.6) that $\langle A^*(I-P_Q)ASx_{n_{N_k}}, x+\eta_k v_{n_{N_k}} - Sx_{n_{N_k}} \rangle \geq 0$. Since $A^*(I-P_Q)A$ is pseudomonotone on H, we have $\langle A^*(I-P_Q)A(x+\eta_k v_{n_{N_k}}), x+\eta_k v_{n_{N_k}} - Sx_{n_{N_k}} \rangle \geq 0$, which implies that

$$\begin{aligned}\langle A^*(I-P_Q)Ax, x - Sx_{n_{N_k}} \rangle \geq \langle A^*(I-P_Q)Ax - A^*(I-P_Q)A(x+\eta_k v_{n_{N_k}}), x+\eta_k v_{n_{N_k}} \\ - Sx_{n_{N_k}} \rangle - \eta_k \langle A^*(I-P_Q)Ax, v_{n_{N_k}} \rangle.\end{aligned} \tag{2.7}$$

Next, we show that $\lim_{k\to\infty} \eta_k v_{n_{N_k}} = 0$. Indeed, since $x_{n_k} \rightharpoonup \bar{x}$ and $\lim_{k\to\infty} \|x_{n_k} - Sx_{n_k}\| = 0$, we have $Sx_{n_k} \rightharpoonup \bar{x}$ as $k \to \infty$. Since $\{Sx_n\} \subset C$, we have $\bar{x} \in C$. We assume that $A^*(I-P_Q)A\bar{x} \neq 0$ (otherwise, $\bar{x}$ is a solution). Note that $A^*(I-P_Q)A$ satisfies $0 < \|A^*(I-P_Q)A\bar{x}\| \leq \liminf_{k\to\infty} \|A^*(I-P_Q)ASx_{n_k}\|$. Since $\{Sx_{n_{N_k}}\} \subset \{Sx_{n_k}\}$ and $\lim_k \eta_k = 0$, we obtain

$$0 \leq \limsup_{k\to\infty} \|\eta_k v_{n_{N_k}}\| = \limsup_{k\to\infty} \frac{\eta_k}{\|A^*(I-P_Q)ASx_{n_k}\|} \leq \frac{\limsup_{k\to\infty} \eta_k}{\liminf_{k\to\infty} \|A^*(I-P_Q)ASx_{n_k}\|} = 0,$$

which implies that $\lim_{k\to\infty} \eta_k v_{n_{N_k}} = 0$. Now, the right hand side of (2.7) tends to zero since $A^*(I-P_Q)A$ is Lipschitz continuous. Thus $\liminf_{k\to\infty} \langle A^*(I-P_Q)Ax, x - Sx_{n_{N_k}} \rangle \geq 0$. Hence, for all $x \in C$, $\langle A^*(I-P_Q)Ax, x-\bar{x} \rangle = \liminf_{k\to\infty} \langle A^*(I-P_Q)Ax, x - Sx_{n_{N_k}} \rangle \geq 0$. Therefore, $\bar{x} \in \Gamma$. This completes the proof. □

Theorem 2.1. *Let x_σ be given as (2.2). Then x_σ converges strongly as $\sigma \to 0$ to the minimum-norm solution $\breve{x}$ of the SFP.*

Proof. Since $I - \beta_n/(1-\alpha_n\gamma_)(I-W)$ is nonexpansive, we have

$$\begin{aligned}\|x_\sigma - \breve{x}\| &= \|(1-\alpha_n\gamma)x_\sigma - \beta_n(I-W)x_\sigma - (\breve{x} - \beta_n(I-W)\breve{x})\| \\ &= \|(1-\alpha_n\gamma)[x_\sigma - \beta_n/(1-\alpha_n\gamma)(I-W)x_\sigma] \\ &\quad - (1-\alpha_n\gamma)[\breve{x} - \beta_n/(1-\alpha_n\gamma)(I-W)\breve{x}] - \alpha_n\gamma\breve{x}\| \\ &\leq (1-\alpha_n\gamma)\|(x_\sigma - \beta_n/(1-\alpha_n\gamma)(I-W)x_\sigma) \\ &\quad - (\breve{x} - \beta_n/(1-\alpha_n\gamma)(I-W)\breve{x})\| + \alpha_n\gamma\|\breve{x}\| \\ &\leq (1-\alpha_n\gamma)\|x_\sigma - \breve{x}\| + \alpha_n\gamma\|\breve{x}\|\end{aligned}$$

Hence, $\|x_\sigma - \breve{x}\| \leq \|\breve{x}\|$. This shows that $\{x_\sigma\}$ is bounded. From (2.1), we have $\|x_\sigma - (x_\sigma - \beta_n(I-W)x_\sigma)\| = \alpha_n\|\gamma x_\sigma\| \to 0$. Assume that $\{\sigma_n\} \subseteq (0, \frac{2-\gamma_n\|A^*A\|}{2\gamma_n})$ is such that $\sigma_n \to 0^+$ as $n \to \infty$. Putting $x_n := x_{\sigma_n}$, we have $x_{n+1} = X_n x_n = (1-\alpha_n\gamma)x_n - \beta_n(I-W)x_n$, and

$$\|x_n - (x_n - \beta_n(I-W)x_n)\| = \alpha_n\|\gamma x_n\| \to 0. \tag{2.8}$$

Next, we prove that $\lim_{n\to\infty}\|x_n-\breve{x}\|=0$. Since $\breve{x}=P_\Gamma 0$, then $\breve{x}\in C\in C_n$, and

$$\begin{aligned}\|Wx_n-\breve{x}\|^2 &\le \langle Wx_n-\breve{x}, x_n-\gamma A^*(I-P_Q)ASx_n-\breve{x}\rangle\\ &=\frac{1}{2}\|Wx_n-\breve{x}\|^2+\frac{1}{2}\|x_n-\gamma A^*(I-P_Q)ASx_n-\breve{x}\|^2\\ &\quad-\frac{1}{2}\|Wx_n-x_n+\gamma A^*(I-P_Q)ASx_n\|^2\\ &=\frac{1}{2}\|Wx_n-\breve{x}\|^2+\frac{1}{2}\|x_n-\breve{x}\|^2+\frac{1}{2}\gamma\|A^*(I-P_Q)ASx_n\|^2\\ &\quad-\langle x_n-\breve{x},\gamma A^*(I-P_Q)ASx_n\rangle-\frac{1}{2}\|Wx_n-x_n\|^2\\ &\quad-\frac{1}{2}\gamma\|A^*(I-P_Q)ASx_n\|^2-\langle Wx_n-x_n,\gamma A^*(I-P_Q)ASx_n\rangle\\ &=\frac{1}{2}\|Wx_n-\breve{x}\|^2+\frac{1}{2}\|x_n-\breve{x}\|^2-\frac{1}{2}\|Wx_n-x_n\|^2\\ &\quad-\langle Wx_n-\breve{x},\gamma A^*(I-P_Q)ASx_n\rangle,\end{aligned}$$

which implies that

$$\begin{aligned}\|Wx_n-\breve{x}\|^2 &\le \|x_n-\breve{x}\|^2-\|Wx_n-x_n\|^2-2\langle Wx_n-\breve{x},\gamma A^*(I-P_Q)ASx_n\rangle\\ &\le \|x_n-\breve{x}\|^2-\|Wx_n-x_n\|^2+2\langle\gamma A^*(I-P_Q)ASx_n,\breve{x}-Sx_n\rangle\\ &\quad+2\langle\gamma A^*(I-P_Q)ASx_n,Sx_n-Wx_n\rangle.\end{aligned}$$

Since $\breve{x}$ is the solution of the SFP, we have $\langle A^*(I-P_Q)A\breve{x},x-\breve{x}\rangle\ge 0$ for all $x\in C$. By the pseudomontonicity of A on C, we have $\langle A^*(I-P_Q)Ax,x-\breve{x}\rangle\ge 0$ for all $x\in C$. Taking $x:=Sx_n\in C$, we obtain $\langle A^*(I-P_Q)ASx_n,\breve{x}-Sx_n\rangle\le 0$. Thus,

$$\begin{aligned}\|Wx_n-\breve{x}\|^2 &\le \|x_n-\breve{x}\|^2-\|Wx_n-x_n\|^2+2\langle\gamma A^*(I-P_Q)ASx_n,Sx_n-Wx_n\rangle\\ &=\|x_n-\breve{x}\|^2-\|Wx_n-Sx_n\|^2-\|Sx_n-x_n\|^2\\ &\quad-2\langle Wx_n-Sx_n,Sx_n-x_n\rangle+2\langle\gamma A^*(I-P_Q)ASx_n,Sx_n-Wx_n\rangle\\ &=\|x_n-\breve{x}\|^2-\|Wx_n-Sx_n\|^2-\|Sx_n-x_n\|^2\\ &\quad+2\langle x_n-\gamma A^*(I-P_Q)ASx_n-Sx_n,Wx_n-Sx_n\rangle.\end{aligned}$$

Since $Sx_n=P_{C_n}(x_n-\gamma A^*(I-P_Q)Ax_n)$ and $Wx_n\in C_n$, we have

$$\begin{aligned}&2\langle x_n-\gamma A^*(I-P_Q)ASx_n-Sx_n,Wx_n-Sx_n\rangle\\ &=2\langle x_n-\gamma A^*(I-P_Q)Ax_n-Sx_n,Wx_n-Sx_n\rangle\\ &\quad+2\gamma\langle A^*(I-P_Q)Ax_n-A^*(I-P_Q)ASx_n,Wx_n-Sx_n\rangle\\ &\le 2\gamma\langle A^*(I-P_Q)Ax_n-A^*(I-P_Q)ASx_n,Wx_n-Sx_n\rangle\\ &\le|\gamma|^2(\|A^*(I-P_Q)Ax_n-A^*(I-P_Q)ASx_n\|^2+\|Wx_n-Sx_n\|^2)\\ &\le|\gamma_n|^2(\|A\|^2\|x_n-Sx_n\|^2+\|Wx_n-Sx_n\|^2)\end{aligned}$$

Thus,

$$\begin{aligned}\|Wx_n-\breve{x}\|^2 &\le \|x_n-\breve{x}\|^2-\left(1-\frac{|\gamma|^2}{\|A\|^2}\right)\|Wx_n-Sx_n\|^2-(1-|\gamma|^2)\|Sx_n-x_n\|^2\\ &\le \|x_n-\breve{x}\|^2.\end{aligned}$$

Note that

$$\begin{aligned}&a\left(1-\frac{|\gamma|^2}{\|A\|^2}\right)\|Wx_n-Sx_n\|^2+a(1-|\gamma|^2)\|Sx_n-x_n\|^2\\ &\le \|x_n-\breve{x}\|^2-\|x_{n+1}-\breve{x}\|^2+\alpha_n\gamma\|\breve{x}\|^2.\end{aligned}$$

Indeed,

$$\begin{aligned}&\|x_{n+1}-\breve{x}\|^2\\ &=\|(1-\alpha_n\gamma-\beta_n)(x_n-\breve{x})+\beta_n(Wx_n-\breve{x})+\alpha_n\gamma\breve{x}\|^2\\ &=(1-\alpha_n\gamma-\beta_n)\|x_n-\breve{x}\|^2+\beta_n\|Wx_n-\breve{x}\|^2+\alpha_n\gamma\|\breve{x}\|^2\\ &\quad-\beta_n(1-\alpha_n\gamma-\beta_n)\|x_n-Wx_n\|^2-\alpha_n\gamma(1-\alpha_n\gamma-\beta_n)\|x_n\|^2-\alpha_n\gamma\beta_n\|Wx_n\|^2\\ &\le(1-\alpha_n\gamma-\beta_n)\|x_n-\breve{x}\|^2+\beta_n\|Wx_n-\breve{x}\|^2+\alpha_n\gamma\|\breve{x}\|^2\\ &\le(1-\alpha_n\gamma-\beta_n)\|x_n-\breve{x}\|^2+\beta_n\|x_n-\breve{x}\|^2-\beta_n\left(1-\frac{|\gamma|^2}{\|A\|^2}\right)\|Wx_n-Sx_n\|^2\\ &\quad-\beta_n(1-|\gamma|^2)\|Sx_n-x_n\|^2+\alpha_n\gamma\|\breve{x}\|^2\\ &=(1-\alpha_n\gamma)\|x_n-\breve{x}\|^2-\beta_n\left(1-\frac{|\gamma|^2}{\|A\|^2}\right)\|Wx_n-Sx_n\|^2\\ &\quad-\beta_n(1-|\gamma|^2)\|Sx_n-x_n\|^2+\alpha_n\gamma\|\breve{x}\|^2\\ &\le\|x_n-\breve{x}\|^2-\beta_n\left(1-\frac{|\gamma|^2}{\|A\|^2}\right)\|Wx_n-Sx_n\|^2-\beta_n(1-|\gamma|^2)\|Sx_n-x_n\|^2+\alpha_n\gamma\|\breve{x}\|^2.\end{aligned}$$

Thus,

$$\begin{aligned}&\beta_n\left(1-\frac{|\gamma|^2}{\|A\|^2}\right)\|Wx_n-Sx_n\|^2+\beta_n(1-|\gamma|^2)\|Sx_n-x_n\|^2\\ &\le \|x_n-\breve{x}\|^2-\|x_{n+1}-\breve{x}\|^2+\alpha_n\gamma\|\breve{x}\|^2.\end{aligned}$$

Moreover, since $\beta_n\ge a$ for all $n\ge 1$, we obtain

$$\begin{aligned}&a\left(1-\frac{|\gamma|^2}{\|A\|^2}\right)\|Wx_n-Sx_n\|^2+a(1-|\gamma|^2)\|Sx_n-x_n\|^2\\ &\le \|x_n-\breve{x}\|^2-\|x_{n+1}-\breve{x}\|^2+\alpha_n\gamma\|\breve{x}\|^2.\end{aligned}$$

Note that $\|x_{n+1}-\breve{x}\|^2\le(1-\alpha_n\gamma)^2\|x_n-\breve{x}\|^2+\alpha_n\gamma[2\beta_n\|x_n-Wx_n\|\|x_{n+1}-\breve{x}\|+2\langle\breve{x},\breve{x}-x_{n+1}\rangle]$, $\forall n\ge n_0$. Indeed, setting $t_n=(1-\beta_n)x_n+\beta_nWx_n$ for each $n\ge 1$, we have

$$\|t_n-\breve{x}\|\le(1-\beta_n)\|x_n-\breve{x}\|+\beta_n\|Wx_n-\breve{x}\|\le\|x_n-\breve{x}\|,\forall n\ge n_0, \tag{2.9}$$

and $\|t_n - x_n\| = \beta_n \|x_n - Wx_n\|$. Using (2.9), we have

$$\begin{aligned}&\|x_{n+1}-\breve{x}\|^2\\&=\|(1-\alpha_n\gamma)(t_n-\breve{x})-\alpha_n\gamma(x_n-t_n)-\alpha_n\gamma\breve{x}\|^2\\&\le(1-\alpha_n\gamma)\|t_n-\breve{x}\|^2-2\langle\alpha_n\gamma(x_n-t_n)-\alpha_n\gamma\breve{x},x_{n+1}-\breve{x}\rangle\\&\le(1-\alpha_n\gamma)\|x_n-\breve{x}\|^2+\alpha_n\gamma[2\beta_n\|x_n-Wx_n\|\|x_{n+1}-\breve{x}\|\\&\quad+2\langle\breve{x},\breve{x}-x_{n+1}\rangle],\forall n\ge n_0.\end{aligned} \tag{2.10}$$

Set $a_n := \|x_n - \breve{x}\|^2$ and $b_n = 2\beta_n\|x_n - Wx_n\|\|x_{n+1} - \breve{x}\| + 2\langle\breve{x}, \breve{x} - x_{n+1}\rangle$. Then, (2.10) can be rewritten as $a_{n+1} \le 1 - \alpha_n\gamma a_n + \alpha_n\gamma b_n$. In view of Lemma 2.3, it is sufficient to show that $\limsup_{k\to\infty} b_{n_k} \le 0$ for every subsequence $\{a_{n_k}\}$ of $\{a_n\}$ satisfying $\liminf_{k\to\infty}(a_{n_k+1} - a_{n_k}) \ge 0$, i.e., $\liminf_{k\to\infty}(\|x_{n_k+1} - \breve{x}\|^2 - \|x_{n_k} - \breve{x}\|^2) \ge 0$. Suppose that $\{\|x_{n_k} - \breve{x}\|\}$ is a subsequence of $\{\|x_n - \breve{x}\|\}$ such that $\liminf_{k\to\infty}(\|x_{n_k+1} - \breve{x}\|^2 - \|x_{n_k} - \breve{x}\|^2) \ge 0$. From (2.8), we obtain

$$\begin{aligned}\limsup_{k\to\infty} a\left(1-\frac{|\gamma|^2}{\|A\|^2}\right)&\|Wx_n-Sx_n\|^2+a(1-|\gamma|^2)\|Sx_n-x_n\|^2\\&\le\limsup_{k\to\infty}[\|x_{n_k}-\breve{x}\|^2-\|x_{n_k+1}-\breve{x}\|^2+\alpha_n\gamma\|\breve{x}\|^2]\\&=-\liminf_{k\to\infty}[\|x_{n_k+1}-\breve{x}\|^2-\|x_{n_k}-\breve{x}\|^2]\le 0.\end{aligned}$$

This implies that $\lim_{k\to\infty}\|x_{n_k} - Sx_{n_k}\| = 0$ and $\lim_{k\to\infty}\|Wx_{n_k} - Sx_{n_k}\| = 0$. Thus, $\|x_{n_k} - Wx_{n_k}\| \le \|x_{n_k} - Sx_{n_k}\| + \|Wx_{n_k} - Sx_{n_k}\| \to 0$ *as* $k \to \infty$. Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$, which converges weakly to a point $\bar{x}$. From (2.2), we have $\|\bar{x} - (\bar{x} - \beta_n(I - W)\bar{x})\| = \alpha_n\|\gamma\bar{x}\| \to 0$, i.e., $\bar{x} \in Fix(T)$. From $\lim_{k\to\infty}\|x_{n_k} - Sx_{n_k}\| = 0$, we conclude that $\bar{x} \in \Gamma$. On the other hand, we have $\lim_{k\to\infty}\|x_{n_k+1} - x_{n_k}\| = \lim_{k\to\infty}[\alpha_{n_k}\gamma\|x_{n_k}\| + \beta_{n_k}\|x_{n_k} - Wx_{n_k}\|] = 0$, and $\limsup_{k\to\infty}\langle\breve{x}, \breve{x} - x_{n_k}\rangle = \lim_{k\to\infty}\langle\breve{x}, \breve{x} - x_{n_k}\rangle = \langle\breve{x}, \breve{x} - \bar{x}\rangle$. From the definition of $\breve{x} = P_\Gamma 0$, we have $\limsup_{k\to\infty}\langle\breve{x}, \breve{x} - x_{n_k}\rangle = \langle\breve{x}, \breve{x} - \bar{x}\rangle \le 0$. Combining (2.6) and (2.7), we have

$$\limsup_{k\to\infty}\langle\breve{x},\breve{x}-x_{n_k+1}\rangle=\limsup_{k\to\infty}\langle\breve{x},\breve{x}-x_{n_k}\rangle=\langle\breve{x},\breve{x}-\bar{x}\rangle\le 0.$$

Thus, $\limsup b_n = \limsup 2\beta_n\|x_n - Wx_n\|\|x_{n+1} - \breve{x}\| + 2\langle\breve{x}, \breve{x} - x_{n+1}\rangle \le 0$. Hence, it follows from Lemma 2.3 that $\lim_{n\to\infty}\|x_n - \breve{x}\| = 0$. So, $x_\sigma \to \breve{x}$ $(\sigma \to 0)$, the minimum-norm solution of the SFP. □

3. Main Results

In this section, we introduce a new extragradient algorithm for solving the split feasibility problem. Under mild assumptions, the sequence generated by the proposed algorithm converges strongly to $\breve{x} \in \Gamma$, where $\|\breve{x}\| = \min\{\|y\| : y \in \Gamma\}$. First, the following conditions are assumed for the convergence of the algorithm:

Assumption 3.1. Assume that $\{\alpha_n\}$ and $\{\beta_n\}$ are two real sequences in $(0,1)$ such that $\{\beta_n\} \subset (a, 1-\alpha_n)$ for some $a > 0$, $\lim_{n\to\infty}\alpha_n = 0$, $\Sigma_{n=1}^{\infty}\alpha_n = \infty$ and $\Sigma_{n=1}^{\infty}|\alpha_{n+1} - \alpha_n| < \infty$ or $lim_{n\to\infty}|\alpha_{n+1} - \alpha_n|/\alpha_n = 0$.

Now, we present our algorithm.

Algorithm 3.1. **Initialization:** Give $\tau \in (0,1)$ and $\gamma \in (0, \frac{2}{\|A\|^2})$. Let x_0 be an arbitrary vector in H.

Iterative Steps: Calculate x_{n+1} as follows:
Step 1. Compute $y_n = P_C(x_n - \gamma A^*(I-P_Q)Ax_n)$.
Step 2. Compute $z_n = P_{C_n}(x_n - \gamma A^*(I-P_Q)Ay_n)$, where

$$C_n := \{x \in H : \langle x_n - \gamma A^*(I-P_Q)Ax_n - y_n, x - y_n\rangle \leq 0\}.$$

Step 3. Compute $x_{n+1} = (1-\alpha_n-\beta_n)x_n + \beta_n z_n$.
Set $n := n+1$ and go to **Step 1.**

Theorem 3.1. *The sequence $\{x_n\}$ generated by Algorithm 3.1 converges strongly to the minimum-norm solution $\breve{x}$ of the SFP (1.1).*

Proof. Let R_n and R be defined by

$$R_n x := (1-\alpha_n)[I - \frac{\beta_n}{1-\alpha_n}(I-W)]x = (1-\alpha_n)Tx,$$

and

$$Rx := [I - \frac{\beta_n}{1-\alpha_n}(I-W)]x = Tx,$$

where $T = I - \frac{\beta_n}{(1-\alpha_n)}(I-W)$. By Lemma 2.1, it is easy to see that R_n is a contraction with contractive constant $1-\alpha_n$, and Algorithm 3.1 can be written as $x_{n+1} = R_n x_n$. For any $\breve{x} \in \Gamma$, we have $\|R_n\breve{x} - \breve{x}\| = \|(1-\alpha_n)T\breve{x} - \breve{x}\| = \alpha_n\|T\breve{x}\| = \alpha_n\|\breve{x}\|$. Hence,

$$\begin{aligned}\|x_{n+1} - \breve{x}\| &\leq \|R_n x_n - R_n\breve{x}\| + \|R_n\breve{x} - \breve{x}\| \\ &\leq (1-\alpha_n)\|x_n - \breve{x}\| + \alpha_n\|\breve{x}\| \\ &\leq \max\{\|x_n - \breve{x}\|, \|\breve{x}\|\}.\end{aligned}$$

It follows that $\|x_n - \breve{x}\| \leq \max\{\|x_0 - \breve{x}\|, \|\breve{x}\|\}$, so $\{x_n\}$ is bounded.

Next, we prove that $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$. Indeed,

$$\begin{aligned}\|x_{n+1} - x_n\| &= \|R_n x_n - R_{n-1}x_{n-1}\| \\ &\leq \|R_n x_n - R_n x_{n-1}\| + \|R_n x_{n-1} - R_n x_{n-1}\| \\ &\leq (1-\alpha_n)\|x_n - x_{n-1}\| + \|R_n x_{n-1} - R_n x_{n-1}\|.\end{aligned}$$

Notice that $\|R_n x_{n-1} - R_n x_{n-1}\| = \|(1-\alpha_n)Tx_{n-1} - (1-\alpha_{n-1})Tx_{n-1}\| \leq |\alpha_n - \alpha_{n-1}|\|x_{n-1}\|$. Hence, $\|x_{n+1} - x_n\| \leq (1-\alpha_n)\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|x_{n-1}\|$. By virtue of Assumption 3.1 and Lemma 2.2, we have $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$. Therefore,

$$\|x_n - Rx_n\| \leq \|x_{n+1} - x_n\| + \|R_n x_n - Rx_n\| \leq \|x_{n+1} - x_n\| + \alpha_n\|x_n\| \to 0.$$

The demiclosedness principle ensures that each weak limit point of $\{x_n\}$ is a fixed point of the nonexpansive mapping $R = T$, that is, a point in the solution set Γ of the SFP (1.1).

Finally, we prove that $lim_{n\to\infty}\|x_n - \breve{x}\| = 0$. Choose $0 < \tau < 1$ such that $\gamma_n/(1-\tau) < 2/\rho(A^*A)$. Then, $T = I - \beta_n(I-W) = \tau I + (1-\tau)V$, where $V = I - \beta_n/(1-\tau)(I-W)$ is a non-expansive mapping. Taking $y \in \Gamma$, we deduce that

$$\begin{aligned}\|x_{n+1}-y\|^2 &= \|(1-\alpha_n)(Tx_n-y)-\alpha_n y\|^2\\ &\le (1-\alpha_n)\|(Tx_n-y)\|+\alpha_n\|y\|^2\\ &\le \|\tau(x_n-y)+(1-\tau)(Vx_n-y)\|^2+\alpha_n\|y\|^2\\ &\le \tau^2\|x_n-y\|^2+(1-\tau)^2\|(Vx_n-y)\|^2-\tau(1-\tau)\|x_n-Vx_n\|^2+\alpha_n\|y\|^2\\ &\le \|x_n-y\|^2-\tau(1-\tau)\|x_n-Vx_n\|^2+\alpha_n\|y\|^2.\end{aligned}$$

It follows that

$$\begin{aligned}\tau(1-\tau)\|x_n-Vx_n\|^2 &\le \|x_n-y\|^2-\|x_{n+1}-y\|^2+\alpha_n\|y\|^2\\ &\le (\|x_n-y\|+\|x_{n+1}-y\|)(\|x_n-y\|-\|x_{n+1}-y\|)+\alpha_n\|y\|^2\\ &\le (\|x_n-y\|+\|x_{n+1}-y\|)\|x_n-x_{n+1}\|+\alpha_n\|y\|^2 \to 0.\end{aligned}$$

In view of $T=I-\beta_n(I-S)=\tau I+(1-\tau)V$, we have that $\lim_{n\to\infty}\|Tx_n-x_n\|=0$. Take a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\limsup_{n\to\infty}\langle x_n-\breve{x},-\breve{x}\rangle = lim_{k\to\infty}\langle x_{n_k}-\breve{x},-\breve{x}\rangle$. By virtue of the boundedness of $\{x_n\}$, we may further assume, with no loss generality, that $x_{n_k}\rightharpoonup \bar{x}$. Using $\|Rx_n-x_n\|\to 0$ and the demiclosedness principle, we know that $\bar{x}\in\Gamma$. Since $\breve{x}$ is the projection of the origin onto Γ, we have that $\limsup_{n\to\infty}\langle x_n-\breve{x},-\breve{x}\rangle=\lim_{k\to\infty}\langle x_{n_k}-\breve{x},-\breve{x}\rangle = \langle \bar{x}-\breve{x},-\breve{x}\rangle\le 0$.

Finally, we compute

$$\begin{aligned}\|x_{n+1}-\breve{x}\|^2 &= \|(1-\alpha_n)(Tx_n-\breve{x})+\alpha_n(-\breve{x})\|^2\\ &= (1-\alpha_n)^2\|(Tx_n-\breve{x})\|^2+\alpha_n^2\|\breve{x}\|^2+2\alpha_n(1-\alpha_n)\langle Tx_n-\breve{x},-\breve{x}\rangle\\ &\le (1-\alpha_n)\|(Tx_n-\breve{x})\|^2+\alpha_n[\alpha_n\|\breve{x}\|^2+2(1-\alpha_n)\langle Tx_n-\breve{x},-\breve{x}\rangle].\end{aligned}$$

Since $\limsup_{n\to\infty}\langle x_n-\breve{x},-\breve{x}\rangle\le 0$ and $\|x_n-Tx_n\|\to 0$, we know that $\limsup_{n\to\infty}(\alpha_n\|\breve{x}\|^2+2(1-\alpha_n)\langle Tx_n-\breve{x},-\breve{x}\rangle)\le 0$. By Lemma 2.2, we conclude that $\lim_{n\to\infty}\|x_n-\breve{x}\|=0$. This completes the proof. □

4. Numerical Experiments

We provide a numerical example to illustrate the effectiveness of our algorithm. The program was written in Matlab and all results are carried out on a personal DELL computer with Intel(R) Core(TM)i5-5200 CPU @ 2.20GHz, and RAM 4.00 GB. We take $error = 10^{-5}, 10^{-7}, 10^{-10}, 10^{-12}$, and 10^{-15}, respectively. In Algorithm 3.1, we consider the split feasibility problem (1.1) with $H_1=\mathbb{R}$, $H_2=\mathbb{R}$, d $C=[-20,20]$, $Q=[0,0]$, $Ax=x$, and $\gamma=\frac{1}{4}$. Take $\alpha_n=\frac{1}{n+1}$, $\beta_n=\frac{1}{3}$, and an initial point $x_1=18$. Obviously, $x^*=0$ is a solution of the problem. In Algorithm 3.1, we have

$$\begin{cases} y_n = P_C(x_n-\frac{1}{4}x_n);\\ z_n = P_{C_n}(x_n-\frac{1}{4}y_n);\\ x_{n+1} = (\frac{2}{3}-\frac{1}{n+1})x_n+\frac{1}{3}z_n.\end{cases}$$

For (1.2), we take $H_1=\mathbb{R}$, $H_2=\mathbb{R}$, $C=[-20,20]$, $Q=[0,0]$, and $A_1x=x$. Take $\tau_n=\frac{1}{4}$, and an initial point $x=18$. Obviously, $x^*=0$ is a solution of the problem. Then $x_{n+1}=P_C(x_n-\frac{1}{4}x_n)$. From (1.3), we take $H_1=\mathbb{R}$, $H_2=\mathbb{R}$, $C=[-20,20]$, $Q=[0,0]$, and $Ax=x$. Take $\tau_n=\frac{1}{4}$,

$\alpha_n = \frac{1}{n+1}$, and an initial point $x = 18$. Obviously, $x^* = 0$ is a solution of the problem. Then $x_{n+1} = P_C[\frac{n}{n+1}(x_n - \frac{1}{4}x_n)]$

Iterative Method	Error	Number of Iterations	Time/s
(3.1)	10^{-5}	27	0.015625
(1.2)	10^{-5}	51	0.03125
(1.3)	10^{-5}	38	0.03125
(3.1)	10^{-7}	39	0.03125
(1.2)	10^{-7}	67	0.046875
(1.3)	10^{-7}	53	0.03125
(3.1)	10^{-10}	58	0.03125
(1.2)	10^{-10}	91	0.046875
(1.3)	10^{-10}	76	0.046875
(3.1)	10^{-12}	70	0.0625
(1.2)	10^{-12}	107	0.0625
(1.3)	10^{-12}	91	0.0625
(3.1)	10^{-15}	89	0.0625
(1.2)	10^{-15}	131	0.0625
(1.3)	10^{-15}	114	0.0625

From the table, it is easy to see that our iterative algorithm converges faster than the others.

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