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# A VISCOSITY METHOD WITH INERTIAL EFFECTS FOR SPLIT COMMON FIXED POINT PROBLEMS OF DEMICONTRACTIVE MAPPINGS

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**Abstract.** In this paper, we first propose a new algorithm for the split common fixed point problems of demicontractive mappings based on viscosity methods and inertial effects in Hilbert spaces. The algorithm is constructed in such a way that its step sizes are not related to the norm of a bounded linear operator. Then, we prove some strong convergence theorems under some suitable conditions. Finally, we provide a numerical example to show the effectiveness of our proposed algorithm. Our results generalize and improve some known results announced recently.

**Keywords.** Demicontractive mappings; Fixed point; Inertial effects; Split feasibility problem.

## 1. Introduction

Let $\mathscr{H}_1$ and $\mathscr{H}_2$ be real Hilbert spaces with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. Let $U : \mathscr{H}_1 \to \mathscr{H}_1$ and $V : \mathscr{H}_2 \to \mathscr{H}_2$ be nonlinear mappings. $Fix(U)$ and $Fix(V)$ denote the fixed points sets of $U$ and $V$, respectively, and $N_+$ denotes all positive integer set in this paper.

In 2009, Censor and Segal [5] first introduced the split common fixed points problem (SCFP) as follows:

$$\text{find } x \in Fix(U) \text{ such that } Ax \in Fix(V), \tag{1.1}$$

where $A : \mathscr{H}_1 \to \mathscr{H}_2$ is a bounded linear operator, and $U : \mathscr{H}_1 \to \mathscr{H}_1$ and $V : \mathscr{H}_2 \to \mathscr{H}_2$ are nonlinear mappings.

There are a number of real problems, such as intensity-modulated radiation therapy, image reconstruction, modeling inverse problems, and electron microscopy. In studying these problems, the SCFP plays a significant role. Therefore, many authors studied it and proposed some effective iterative algorithms; see, e.g., [12, 15, 16, 24, 27] and the references therein.

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In order to further study the solution of the SCFP (1.1), Censor and Segal [5] introduced the following fixed point equation

$$p = U(p - \tau A^*(I-V)Ap),\ \tau > 0,$$

and $p$ is a solution to the SCFP (1.1). They proposed the following iterative algorithm. For any initial point $x_1 \in \mathscr{H}_1$, define $\{x_n\}$ recursively by

$$x_{n+1} = U(x_n - \gamma A^*(I-V)Ax_n), \tag{1.2}$$

where $U$ and $V$ are directed operators, $A : \mathscr{H}_1 \to \mathscr{H}_2$ is a bounded linear operator with its adjoint operator $A^*$, and $\gamma \in (0,\ 2/\|A\|^2)$. They showed that the sequence generated by (1.2) is weakly convergent to a solution of the SCFP (1.1). Later, Moudafi [11] extended their results to demicontractive mappings. Following Moudafi's work, many scholars considered the SCFP (1.1) under various mappings; see, e.g., [7, 8, 17, 18, 21] and the references therein.

Note that, in algorithm (1.2), the step size depends on the operator norm $\| A \|$. However, computing the norm of the operator $\| A \|$ is usually complex. In order to overcome such difficulty, one popular method is to construct variable step sizes. In 2018, Wang and Xu [22] suggested the following variable step size

$$\tau_n = \frac{\rho_n}{\| x_n - Ux_n + A^*(I-V)Ax_n \|}, \tag{1.3}$$

where $U$ and $V$ are two nonexpansive mappings, and $\{\rho_n\} \subset (0,\ +\infty)$ is a sequence of real numbers satisfying

$$\sum_{k=0}^{\infty} \rho_k = \infty \text{ and } \sum_{k=0}^{\infty} \rho_k^2 < \infty. \tag{1.4}$$

They introduced the following iterative algorithm:

$$x_{n+1} = \alpha_n u + (1-\alpha_n)[x_n - \tau_n(x_n - Ux_n + A^*(I-V)Ax_n)], \tag{1.5}$$

where the step size $\tau_n$ is chosen as (1.3) with $\{\rho_n\}$ satisfying (1.4). They obtained a strong convergence result for the algorithm (1.5) under suitable conditions.

Recently, based on viscosity approximation methods, Wang *et al.* [23] proposed the following iterative scheme for more general demicontractive mappings $U$ and $V$:

$$x_{n+1} = \beta_n f(x_n) + (1-\beta_n)[x_n - \tau_n(x_n - Ux_n + A^*(I-V)Ax_n)], \tag{1.6}$$

where $f$ is a contractive mapping, and the step size $\tau_n$ is chosen as (1.3). They proved that the iterative sequence $\{x_n\}$ defined by (1.6) is strongly convergent to $p = P_\Omega \circ f(p)$.

On the other hand, the fast convergence of algorithms play a significant role in many practical applications. In 1964, Polyak [14] first proposed an inertial type extrapolation as an acceleration process. In recent years, many authors constructed various fast iterative algorithms by inertial extrapolation techniques, and proved that the inertial techniques play an important role in accelerating the convergence of original algorithms; see, e.g, [1, 2, 4, 6, 19] and the references therein.

Recently, Suparatulatorn *et al.* [20] suggested an accelerated algorithm for the SCFP (1.1) by combining viscosity approximation methods and inertial effects. Let $f : \mathscr{H}_1 \to \mathscr{H}_1$ be a contraction, and let $x_0,\, x_1 \in \mathscr{H}_1$ be arbitrary initial points. Their iterative sequence is as follows

$$\begin{aligned} x_{n+1} &= \alpha_n f(x_n+\theta_n(x_n-x_{n-1}))+(1-\alpha_n)[x_n+\theta_n(x_n-x_{n-1}) \\ &\quad -\rho_n(x_n+\theta_n(x_n-x_{n-1}) \\ &\quad -U(x_n+\theta_n(x_n-x_{n-1}))+A^*(I-V)A(x_n+\theta_n(x_n-x_{n-1})))],\ n\in\mathbb{N}, \end{aligned} \tag{1.7}$$

where $U : \mathscr{H}_1 \to \mathscr{H}_1$ and $V : \mathscr{H}_2 \to \mathscr{H}_2$ are demicontractive mappings, and $A : \mathscr{H}_1 \to \mathscr{H}_2$ is a bounded linear operator with its adjoint operator $A^*$. They proved that the sequence $\{x_n\}$ generated by (1.7) converges strongly to $z = P_\Omega \circ f(z)$ provided.

Inspired by the above works, based on variable step size (1.3), we construct a new algorithm by combing viscosity approximation methods and inertial effects for the SCFP (1.1) of demicontractive mappings. We prove that the sequence defined by our proposed algorithm converges to a solution to problem (1.1). Finally, we provide a numerical example to show the effectiveness of our proposed algorithm.

## 2. Preliminaries

Let $\mathscr{H}$ be a real Hilbert space, and let $C$ be a nonempty, closed, and convex subset of $\mathscr{H}$. $\{x_n\} \subset \mathscr{H}$ and $x \in \mathscr{H}, x_n \to x$ ($x_n \rightharpoonup x$) denote the strong (weak) convergence of $\{x_n\}$. $Fix(V)$ denotes the fixed points set of the mapping $V$.

For $x \in \mathscr{H}$, define the metric projection $P_C x$ from $\mathscr{H}$ onto $C$ by $P_C x := \arg\min_{y\in C} \|x-y\|^2$. It is well known that $P_C$ can be characterized by the following inequality: for $x \in \mathscr{H}$,

$$\langle x-P_C x,\, y-P_C x\rangle \le 0, \quad \forall y \in C.$$

The following equality is trivial. $2\langle x,\, y\rangle = \|x\|^2+\|y\|^2-\|x-y\|^2$.

**Definition 2.1.** A mapping $V : C \to C$ is said to be

(1) contraction if there exists $k \in (0,1)$ such that $\|Vx-Vy\| \le k\|x-y\|$, $\forall x,y \in C$;
(2) nonexpansive if $\|Vx-Vy\| \le \|x-y\|$, $\forall x,y \in C$;
(3) quasi-nonexpansive if $Fix(V) \neq \emptyset$ and $\|Vx-Vx^*\| \le \|x-x^*\|$, $\forall x \in C, x^* \in Fix(V)$;
(4) directed if $Fix\,(V) \neq \emptyset$ and $\|Vx-x^*\|^2 \le \|x-x^*\|^2-\|x-Vx\|^2$, $\forall x \in C, x^* \in Fix(V)$;
(5) $k$-demicontractive if there exists a constant $k \in [0,1)$ such that

$$\|Vx-x^*\|^2 \le \|x-x^*\|^2+k\|x-Vx\|^2, \quad \forall x \in C, x^* \in Fix(V);$$

or equivalently,

$$\langle x-Vx,\, x-x^*\rangle \ge \frac{1-k}{2}\|x-Vx\|^2, \quad \forall x \in C, x^* \in Fix(V). \tag{2.1}$$

**Definition 2.2.** [20] Let $V : C \to \mathscr{H}$ be an operator. $V$ is said to be demiclosed at $y \in \mathscr{H}$ if $Vx_n \to y$ implies $Vx = y$ for any sequence $\{x_n\}$ in $C$ with $x_n \rightharpoonup x \in C$.

**Remark 2.3.** Let $\mathscr{H} = l_2$ and $V : l_2 \to l_2$ be defined by $Vx = -kx$ for $\forall\, x \in l_2$, where $k > 1$ (see [25, Example 2.5]). Then $V$ is $\frac{k-1}{k+1}$-demicontractive but not quasi-nonexpansive. However, $I-V$ is demiclosed at 0. In fact, by assuming that $\{x_n\}$ is any sequence in $l_2$ such that $x_n \rightharpoonup x \in l_2$ and $\| x_n - Vx_n \| \to 0$, we can obtain $x = 0 \in Fix(V)$.

Let us now state some important facts, which will be used to prove our main results later.

**Lemma 2.4.** [9, 26] *Let $\{a_n\}$ and $\{c_n\}$ be nonnegative real number sequences such that*

$$a_{n+1} \leq (1-\delta_n)\, a_n + b_n + c_n, \;\; n \in \mathbb{N},$$

*where $\{\delta_n\}$ is a real sequence in $(0,1)$ and $\{b_n\}$ is a real sequence. Assume $\sum_{n=1}^{\infty} c_n < \infty$. Then the following results hold:*

*$(i)$ If $b_n \leq \delta_n M$ for some $M \geq 0$, then $\{a_n\}$ is a bounded sequence.*

*$(ii)$ If $\sum_{n=1}^{\infty} \delta_n = \infty$ and $\limsup\limits_{n\to\infty} \frac{b_n}{\delta_n} \leq 0$, then $\lim\limits_{n\to\infty} a_n = 0$.*

**Lemma 2.5.** [10] *Assume that $C$ is a closed and convex subset of a Hilbert space $\mathscr{H}$. Let $V$ be a self-mapping of $C$. If $V$ is $\tau$-demicontractive (which is also said to be $\tau$-quasi-strict pseudo-contractive in the work of* [10]*), then $Fix(V)$ is closed and convex.*

**Lemma 2.6.** [13] *(The Demiclosedness Principle of Nonexpansive Mappings). If $V : \mathscr{H} \to \mathscr{H}$ is a nonexpansive mapping, then $I - V$ is demiclosed at* 0.

## 3. Main Results

Let us assume that the following conditions hold:

$(A_1)$ $\mathscr{H}_1$ and $\mathscr{H}_2$ are two real Hilbert spaces, and $f : \mathscr{H}_1 \to \mathscr{H}_1$ is a contraction with constant $\alpha \in (0, \frac{1}{\sqrt{2}})$;

$(A_2)$ $U : \mathscr{H}_1 \to \mathscr{H}_1$ and $V : \mathscr{H}_2 \to \mathscr{H}_2$ are demicontractive mappings with coefficients $\beta \in [0,1)$ and $\mu \in [0,1)$, respectively, and both $I-U$ and $I-V$ are demiclosed at zero, where $I$ is the identity operator;

$(A_3)$ $A : \mathscr{H}_1 \to \mathscr{H}_2$ is a bounded linear operator with its adjoint operator $A^*$;

$(A_4)$ The problem (1.1) is consistent, i.e., its solution set $\Omega$ is nonempty.

Now, we construct the following inertial algorithm by using the viscosity approximation method and inertial effects.

**Algorithm 3.1.** *Initialization: Let $x_0,\, x_1 \in \mathscr{H}_1$ be arbitrary initial points. $x_{n+1}$ is computed recursively via*

*Step 1: $\omega_n = x_n + \theta_n(x_n - x_{n-1})$.*

*Step 2: If $\|\omega_n - U(\omega_n) + A^*(I-V)A\omega_n\| = 0$, then stop; else, go to Step* 3.

*Step 3:*

$$y_n = \omega_n - U(\omega_n) + A^*(I-V)A\omega_n,$$
$$x_{n+1} = \beta_n f(\omega_n) + (1-\beta_n)[\omega_n - \tau_n y_n],$$
$$\tau_n = \frac{\rho_n}{\|\omega_n - U(\omega_n) + A^*(I-V)A\omega_n\|},$$

*with $\theta_n \in [0,\ \theta] \subset [0,\ 1)$ and $\beta_n \in (0,\ 1)$, $n \in N_+$.*

**Theorem 3.2.** *Suppose both $(A_1) \sim (A_4)$ and the following conditions hold:*

$(C_1)$ $\lim\limits_{n\to\infty} \frac{\beta_n}{\rho_n} = 0$;

$(C_2)$ $\lim\limits_{n\to\infty} \frac{\theta_n}{\beta_n} \|x_n - x_{n-1}\| = 0$;

$(C_3)$ $\lim\limits_{n\to\infty} \beta_n = 0,\ \sum\limits_{n=1}^{\infty} \beta_n = \infty$*;*

$(C_4)$ $\sum_{n=1}^{\infty} \rho_n^2 < \infty$.

*Then the sequence $\{x_n\}$ generated by Algorithm 3.1 converges strongly to $x^* = P_\Omega \circ f(x^*)$.*

*Proof.* From Lemma 2.5, we find that $Fix(U)$ and $Fix(V)$ are both closed convex sets. Since $A$ is a bounded linear operator, that $A^{-1}(Fix(V))$ is closed convex. Thus, $P_\Omega$ is well defined. It is easy to know that $P_\Omega \circ f : \mathscr{H}_1 \to \mathscr{H}_1$ is a contraction. From the Banach fixed point theorem, there exists a unique element $x^* \in \mathscr{H}_1$, satisfying $x^* = P_\Omega \circ f(x^*) \in \Omega$. Choose

$$\tau = \frac{\min\{1-\beta, 1-\mu\}}{4\max\{1, \| A \|^2\}}.$$

By (2.1) and Algorithm 3.1, we have

$$\begin{aligned}
\langle y_n,\, \omega_n - x^*\rangle &= \langle \omega_n - U\omega_n,\, \omega_n - x^*\rangle + \langle A^*(I-V)A\omega_n,\, \omega_n - x^*\rangle \\
&= \langle \omega_n - U\omega_n,\, \omega_n - x^*\rangle + \langle (I-V)A\omega_n,\, A\omega_n - Ax^*\rangle \\
&\geq \frac{1-\beta}{2}\|\omega_n - U\omega_n\|^2 + \frac{1-\mu}{2}\|(I-V)A\omega_n\|^2 \\
&\geq \frac{1-\beta}{2}\|\omega_n - U\omega_n\|^2 + \frac{1-\mu}{2\|A\|^2}\|A^*(I-V)A\omega_n\|^2 \\
&\geq 2\tau(\|\omega_n - U\omega_n\|^2 + \|A^*(I-V)A\omega_n\|^2) \\
&\geq \tau(\|\omega_n - U\omega_n\| + \|A^*(I-V)A\omega_n\|)^2 \\
&\geq \tau\|y_n\|^2, \qquad (3.1)
\end{aligned}$$

It follows from (3.1) and $\tau_n \|y_n\| = \rho_n$ that

$$\begin{aligned}
\|\omega_n - \tau_n y_n - x^*\|^2 &= \|\omega_n - x^*\|^2 - 2\tau_n \langle y_n,\, \omega_n - x^*\rangle + \tau_n{}^2\|y_n\|^2 \\
&\leq \|\omega_n - x^*\|^2 - 2\tau_n\tau\|y_n\|^2 + \tau_n^2\|y_n\|^2 \\
&= \|\omega_n - x^*\|^2 - 2\tau\rho_n\|y_n\| + \rho_n^2 \qquad (3.2) \\
&\leq \|\omega_n - x^*\|^2 + \rho_n^2. \qquad (3.3)
\end{aligned}$$

From Algorithm 3.1 and (3.3), we obtain

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \|\beta_n(f(\omega_n) - x^*) + (1-\beta_n)[(\omega_n - \tau_n y_n) - x^*]\|^2 \\
&\leq \beta_n \|f(\omega_n) - x^*\|^2 + (1-\beta_n)\|\omega_n - \tau_n y_n - x^*\|^2 \\
&\leq 2\beta_n[\alpha^2\|\omega_n - x^*\|^2 + \|f(x^*) - x^*\|^2] + (1-\beta_n)(\|\omega_n - x^*\|^2 + \rho_n^2) \\
&\leq (1-\beta_n(1-2\alpha^2))\|\omega_n - x^*\|^2 + 2\beta_n\|f(x^*) - x^*\|^2 + \rho_n{}^2 \\
&\leq 2(1-\beta_n(1-2\alpha^2))\|x_n - x^*\|^2 + 2\beta_n\|f(x^*) - x^*\|^2 \\
&\quad + 2\theta_n^2(1-\beta_n(1-2\alpha^2))\|x_n - x_{n-1}\|^2 + \rho_n{}^2 \\
&= 2(1-\beta_n(1-2\alpha^2))\|x_n - x^*\|^2 + 2\beta_n(1-2\alpha^2)\varepsilon_n + \rho_n{}^2,
\end{aligned}$$

where

$$\varepsilon_n := \frac{(1-\beta_n(1-2\alpha^2))}{1-2\alpha^2}\frac{\theta_n^2}{\beta_n}\|x_n - x_{n-1}\|^2 + \frac{\|f(x^*) - x^*\|^2}{1-2\alpha^2}$$

for all $n \in N_+$. By our assumptions $(C_2)$ and $(C_3)$, we see that $\lim_{n\to\infty} \varepsilon_n = \frac{\|f(x^*)-x^*\|^2}{1-2\alpha^2}$ implies that the sequence $\{\varepsilon_n\}$ is bounded. Letting $M^* := \sup_{n\in N_+} \{\varepsilon_n\}$, for $n \geq n_0$, we have

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &\leq 2[(1-\beta_n(1-2\alpha^2))\|x_n-x^*\|^2+\beta_n(1-2\alpha^2)M^*]+{\rho_n}^2\\ &\leq 2\max\{\|x_n-x^*\|^2,\, M^*\}+\rho_n^2.\end{aligned}$$

It follows that

$$\begin{aligned}\|x_{n+1}-x^*\|^2 \leq\ & 2\max\{M^*,\ \|x_1-x^*\|^2,\ \|x_2-x^*\|^2,\ldots\ \|x_{n_0-1}-x^*\|^2,\ \|x_{n_0}-x^*\|^2\}\\ &+\sum_{i=n_0}^{n}\rho_i^2.\end{aligned}$$

Hence, by the above inequality and $(C_4)$, we assert that $\{x_n-x^*\}$ is bounded. This follows that $\{x_n\}$ is bounded. Moreover, by $(C_2)$ and $(C_3)$, we have

$$\lim_{n\to\infty}\|\omega_n-x_n\| = \lim_{n\to\infty}\theta_n\|x_n-x_{n-1}\| = 0.$$

So, $\{\omega_n\}$ is bounded.

Our next goal is to create an environment where Lemma 2.4 can be applied. To do so, let us define $a_n := \|x_n-x^*\|^2$ for $n\in N_+$ and set $R := \sup_{n\in N_+}\|x_n-x^*\|$. By Algorithm 3.1 and (3.2), we have that

$$\begin{aligned}a_{n+1} &= \langle x_{n+1}-x^*,\, x_{n+1}-x^*\rangle\\ &= \beta_n\langle f(\omega_n)-f(x^*),\, x_{n+1}-x^*\rangle+\beta_n\langle f(x^*)-x^*,\, x_{n+1}-x^*\rangle\\ &\quad +(1-\beta_n)\langle \omega_n-\tau_n y_n-x^*,\, x_{n+1}-x^*\rangle\\ &\leq \beta_n\alpha\|\omega_n-x^*\|\,\|x_{n+1}-x^*\|+\beta_n\langle f(x^*)-x^*,\, x_{n+1}-x^*\rangle\\ &\quad +(1-\beta_n)\|\omega_n-\tau_n y_n-x^*\|\,\|x_{n+1}-x^*\|\\ &\leq \frac{1}{2}\beta_n\alpha(\|\omega_n-x^*\|^2+a_{n+1})+\beta_n\langle f(x^*)-x^*,\, x_{n+1}-x^*\rangle\\ &\quad +\frac{1}{2}(1-\beta_n)(\|\omega_n-\tau_n y_n-x^*\|^2+a_{n+1})\\ &\leq \frac{1}{2}\beta_n\alpha(\|\omega_n-x^*\|^2+a_{n+1})+\beta_n\langle f(x^*)-x^*,\, x_{n+1}-x^*\rangle\\ &\quad +\frac{1}{2}(1-\beta_n)(\|\omega_n-x^*\|^2-2\tau\rho_n\|y_n\|+\rho_n^2+a_{n+1})\\ &= \frac{1}{2}(1-\beta_n(1-\alpha))(\|\omega_n-x^*\|^2+a_{n+1})+\beta_n\langle f(x^*)-x^*,\, x_{n+1}-x^*\rangle\\ &\quad -(1-\beta_n)\tau\rho_n\|y_n\|+\frac{1}{2}(1-\beta_n)\rho_n^2,\end{aligned}$$

which gives

$$\begin{aligned} a_{n+1} &\le \frac{1-\beta_n(1-\alpha)}{1+\beta_n(1-\alpha)}\|\omega_n - x^*\|^2 + \frac{2\beta_n}{1+\beta_n(1-\alpha)}\langle f(x^*)-x^*,\, x_{n+1}-x^*\rangle \\ &\quad -\frac{2}{1+\beta_n(1-\alpha)}(1-\beta_n)\tau\rho_n\|y_n\| + \frac{1}{1+\beta_n(1-\alpha)}\rho_n^2 \\ &\le (1-\frac{2\beta_n(1-\alpha)}{1+\beta_n(1-\alpha)})(a_n + 2R\theta_n\|x_n-x_{n-1}\| + \theta_n^2\|x_n-x_{n-1}\|^2) \\ &\quad +\frac{2\beta_n}{1+\beta_n(1-\alpha)}\langle f(x^*)-x^*,\, x_{n+1}-x^*\rangle \\ &\quad -\frac{2}{1+\beta_n(1-\alpha)}(1-\beta_n)\tau\rho_n\|y_n\| + \frac{1}{1+\beta_n(1-\alpha)}\rho_n^2. \end{aligned}$$

Define

$$\begin{aligned} b_n: &= 2R\theta_n\|x_n-x_{n-1}\| + \theta_n^2\|x_n-x_{n-1}\|^2 + \frac{2\beta_n}{1+\beta_n(1-\alpha)}\langle f(x^*)-x^*,\, x_{n+1}-x^*\rangle \\ &\quad -\frac{2}{1+\beta_n(1-\alpha)}(1-\beta_n)\tau\rho_n\|y_n\|, \\ c_n &= \frac{1}{1+\beta_n(1-\alpha)}\rho_n^2 \end{aligned}$$

and

$$\delta_n := \frac{2\beta_n(1-\alpha)}{1+\beta_n(1-\alpha)}$$

for all $n \in N_+$. It is easily seen that $\delta_n \in (0,1)$ for all $n \in N_+$ and $\sum\limits_{n=1}^{\infty}\delta_n = \infty$. This implies

$$a_{n+1} \le (1-\delta_n)a_n + b_n + c_n. \tag{3.4}$$

From $(C_4)$, we know that $\sum_{n=1}^{\infty} c_n^2 < \infty$. In order to use Lemma 2.4, it remains to show that $\limsup\limits_{n\to\infty}\frac{b_n}{\delta_n} \le 0$. Observe that

$$\begin{aligned} b_n &\le 2R\theta_n\|x_n-x_{n-1}\| + \theta_n^2\|x_n-x_{n-1}\|^2 + \frac{2\beta_n}{1+\beta_n(1-\alpha)}\langle f(x^*)-x^*,\, x_{n+1}-x^*\rangle \\ &\le 2R\theta_n\|x_n-x_{n-1}\| + \theta_n^2\|x_n-x_{n-1}\|^2 + \frac{2R\beta_n}{1+\beta_n(1-\alpha)}\|f(x^*)-x^*\|. \end{aligned}$$

This follows that $\limsup\limits_{n\to\infty}\frac{b_n}{\delta_n} < \infty$. In fact, $\limsup\limits_{n\to\infty}\frac{b_n}{\delta_n} \ge -1$. Suppose that $\limsup\limits_{n\to\infty}\frac{b_n}{\delta_n} < -1$. Then, there exists an $m_0 \in N_+$, such that $\frac{b_n}{\delta_n} \le -1$ for all $n \ge m_0$. We derive from (3.4), for $n \ge m_0$, that

$$a_{n+1} \le (1-\delta_n)a_n - \delta_n + c_n \le a_n - \delta_n + c_n.$$

By mathematical induction, it is not hard to see that

$$a_{n+1} \le a_{m_0} - \sum_{i=m_0}^{n}\delta_i + \sum_{i=m_0}^{n} c_i.$$

This, in turn, yields

$$\limsup_{n\to\infty} a_n \le a_{m_0} - \lim_{n\to\infty}\sum_{i=m_0}^{n}\delta_i + \sum_{i=m_0}^{\infty} c_i = -\infty,$$

which is a contradiction. Thus, $\limsup_{n\to\infty}\frac{b_n}{\delta_n}$ exists and is at least $-1$. We then take a subsequence $\{\frac{b_{n_k}}{\delta_{n_k}}\}$ such that

$$\limsup_{n\to\infty}\frac{b_n}{\delta_n} = \lim_{k\to\infty}\frac{b_{n_k}}{\delta_{n_k}}.$$

Without loss of generality, by the boundedness of $\{x_n\}$, we may assume

$$\limsup_{n\to\infty}\langle f(x^*)-x^*,\, x_{n+1}-x^*\rangle = \lim_{k\to\infty}\left\langle f(x^*)-x^*,\, x_{n_k+1}-x^*\right\rangle$$

exists. This implies that

$$\lim_{k\to\infty}\frac{(1-\beta_{n_k})\rho_{n_k}\tau\,\|y_{n_k}\|}{\beta_{n_k}}$$

also exists. Thus, by the conditions $(C_1)$ and $(C_4)$, we immediately obtain

$$\lim_{k\to\infty}\|y_{n_k}\| = \lim_{k\to\infty}\frac{\beta_{n_k}}{\rho_{n_k}}\frac{\rho_{n_k}\tau(1-\beta_{n_k})}{\beta_{n_k}}\|y_{n_k}\|\frac{1}{\tau(1-\beta_{n_k})} = 0. \tag{3.5}$$

From (3.1) and (3.5), we have

$$\begin{aligned}\|y_{n_k}\|\,\|\omega_{n_k}-x^*\| &\ge \langle y_{n_k},\, \omega_{n_k}-x^*\rangle \\ &\ge \frac{1-\beta}{2}\|\omega_{n_k}-U\omega_{n_k}\|^2 + \frac{1-\mu}{2}\|(I-V)A\omega_{n_k}\|^2,\end{aligned}$$

which implies that

$$\lim_{k\to\infty}\|\omega_{n_k}-U\omega_{n_k}\| = \lim_{k\to\infty}\|(I-V)A\omega_{n_k}\| = 0. \tag{3.6}$$

Since $I-U$ and $I-V$ are demiclosed at zero, any weak cluster point of $\{\omega_{n_k}\}$ belongs to $\Omega$. In view of

$$\|x_{n+1}-\omega_n\| \le \beta_n\|f(\omega_n)-\omega_n\| + (1-\beta_n)\tau_n\|y_n\|,$$

condition $(C_3)$, and (3.5), we obtain that

$$\lim_{k\to\infty}\left\|x_{n_k+1}-\omega_{n_k}\right\| = 0.$$

This implies that any weak cluster point of $\{x_{n_k+1}\}$ is also an element of $\Omega$. Without loss of generality, we may assume that $\{x_{n_k+1}\}$ converges weakly to $p^*\in\Omega$. Hence,

$$\begin{aligned}\limsup_{n\to\infty}\frac{b_n}{\delta_n} &\le \frac{1}{1-\alpha}\lim_{k\to\infty}\left\langle f(x^*)-x^*,\, x_{n_k+1}-x^*\right\rangle \\ &= \frac{1}{1-\alpha}\langle f(x^*)-x^*,\, p^*-x^*\rangle \le 0.\end{aligned}$$

Finally, by Lemma 2.4$(ii)$, we have

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\|x_n-x^*\|^2 = 0,$$

which concludes that $x_n\to x^*$ as $n\to\infty$. This completes the proof. □

**Remark 3.3.** Choose $\rho_n = \frac{1}{(n+1)^r}$, $\beta_n = \frac{1}{(n+1)^l}$, and $\frac{2}{3} < r < l \leq 1$, $n \in N_+$. The sequences $\{\rho_n\}$ and $\{\beta_n\}$ satisfy the conditions $(C_1) - (C_4)$ in Theorem 3.2.

**Remark 3.4.** For a special choice, the parameter $\theta_n$ in Algorithm 3.1 can be chosen as:

$$0 \leq \theta_n \leq \bar{\theta}_n, \quad \bar{\theta}_n = \begin{cases} \min\left\{\frac{\xi_n}{\|x_n - x_{n-1}\|}, \frac{n-1}{n+\eta-1}\right\}, & if\ x_n \neq x_{n-1}, \\ \frac{n-1}{n+\eta-1}, & otherwise, \end{cases}$$

where $n \in N_+$, $\eta \geq 3$ and $\{\xi_n\}$ is a positive sequence such that $\lim_{n\to\infty} \frac{\xi_n}{\beta_n} = 0$. This idea is from the recent inertial extrapolated step introduced in [1, 3].

If $U$ and $V$ are nonexpansive, then $U$ and $V$ are demicontractive. By Lemma 2.6, we know $I - U$ and $I - V$ are demiclosed at zero. Hence, by Theorem 3.2, we have the result below.

**Corollary 3.5.** *Let $\mathscr{H}_1$, $\mathscr{H}_2$, $f$, and $A$ satisfy the conditions $(A_1)$ and $(A_2)$. Let $U : \mathscr{H}_1 \to \mathscr{H}_1$ and $V : \mathscr{H}_2 \to \mathscr{H}_2$ be two nonexpansive mappings and $\Omega \neq \emptyset$. Assume that sequences $\{\rho_n\}$, $\{\beta_n\}$ and $\{\theta_n\}$ satisfy the conditions $(C_1) - (C_4)$ in Theorem 3.2. Then, the sequence $\{x_n\}$ generated by Algorithm 3.1 converges strongly to a solution $x^* = P_\Omega \circ f(x^*)$ of the SCFP (1.1).*

If $U$ and $V$ are quasi-nonexpansive with $Fix(U) \neq \emptyset$ and $Fix(V) \neq \emptyset$, then $U$ and $V$ are 0-demicontractive. Hence, by Theorem 3.2, we have the following result.

**Corollary 3.6.** *Let $\mathscr{H}_1$, $\mathscr{H}_2$, $f$, and $A$ satisfy the conditions $(A_1)$ and $(A_2)$. Let $U : \mathscr{H}_1 \to \mathscr{H}_1$ and $V : \mathscr{H}_2 \to \mathscr{H}_2$ be two quasi-nonexpansive mappings and $\Omega \neq \emptyset$. Let $I - U$ and $I - V$ be demiclosed at zero. Assume that sequences $\{\rho_n\}$, $\{\beta_n\}$ and $\{\theta_n\}$ satisfy the conditions $(C_1) - (C_4)$ in Theorem 3.2. Then, the sequence $\{x_n\}$ generated by Algorithm 3.1 converges strongly to a solution $x^* = P_\Omega \circ f(x^*)$ of the SCFP (1.1).*

## 4. Numerical Example

In this section, in order to demonstrate the effectiveness, realization, and convergence of Algorithm 3.1, we consider the following example in $(\mathbb{R}, |\cdot|)$.

**Example 4.1.** Let $\mathscr{H}_1 = \mathscr{H}_2 = \mathbb{R}$. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \frac{1}{2}x$. Let $A$, $U$, and $V : \mathbb{R} \to \mathbb{R}$ be defined by $Ax = 2x$, $U(x) = -2x$ and $V(x) = -3x$, respectively. Choose $\theta_n = \frac{1}{2}\bar{\theta}_n$, $\xi_n = \frac{1}{(n+1)^2}$, $\rho_n = \frac{1}{(n+1)^{\frac{3}{4}}}$, and $\beta_n = \frac{1}{n+1}$, $n \in N_+$, where the parameter $\bar{\theta}_n$ is taken from Remark 3.4. Then, the sequence $\{x_n\}$ generated by Algorithm 3.1 converges strongly to a solution of the SCFP (1.1).

It is easy to see that $Fix(U) = \{0\} = Fix(V)$ and $A^{-1}0 = 0$. Thus the solution set $\Omega = \{0\}$ of the SCFP (1.1) is not empty. Also $f$ is a contraction with constant $\frac{1}{2}$. $A$ is a bounded linear operator and its adjoint operator $A^*x = Ax = 2x$. From Remark 2.3, we obtain that $U$ and $V$ are $\frac{1}{3}$-demicontractive mapping and $\frac{1}{2}$-demicontractive mapping, respectively. Both $I - U$ and $I - V$ are demiclosed at zero. It can be observed that all the assumptions of $(A_1) \sim (A_4)$ and all the conditions $(C_1) \sim (C_4)$ are satisfied.

Algorithm 3.1 is reduced to the following: Let $x_0,\ x_1 \in \mathscr{H}_1$ be arbitrary initial points.

Step 1: $\omega_n = x_n + \frac{1}{2}\bar{\theta}_n(x_n - x_{n-1})$.

Step 2: If $\|\omega_n\| = 0$, then stop; else, go to Step 3.
Step 3:

$$\begin{cases} y_n = 19\omega_n, \\ x_{n+1} = (\frac{2n+1}{2(n+1)} - \frac{19n}{n+1}\tau_n)\omega_n, \\ \tau_n = \frac{(n+1)^{-3/4}}{19\|\omega_n\|}, \end{cases}$$

for $n \in N_+$. Hence, from Theorem 3.2, the sequence $\{x_n\}$ generated above converges strongly to $0 \in \Omega = \{0\}$.

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