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AN EXTRAGRADIENT-LIKE ITERATIVE PROCESS FOR COCOCERCIVE MAPPINGS AND STRICTLY PSEUDOCONTRACTIVE MAPPINGS

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Abstract. An extragradient-like iterative process is introduced and investigated for two cocercive mappings and a strictly pseudocontractive mapping. A convergence theorem of common solutions of a fixed point problem of a strictly pseudocontractive mapping and of two variational inequalities is established under appropriate conditions on control sequences.

Keywords. Cocercive mappings; Fixed points; Strictly pseudocontractive mappings; Variational inequality problem.

1. Introduction-Preliminaries

Fixed points of nonlinear operators are under the spotlight of research in the communities of nonlinear functional analysis and optimization theory. Fixed points of nonlinear operators provides and guarantees the existence of nonlinear equations, differential and integral. In addition to the existence of fixed points, the approximation of fixed point is also important from viewpoint of real applications. Indeed, many problems arising in signal processing, image recovery, medicine, transportation and so on can be modelled and solved by fixed point methods; see, e.g., [1, 2, 3, 8, 9].

Next, we give some definitions of nonlinear operators. They were extensively studied due to their wide applications in various equations. Let T be a nonlinear operator with the domain $Dom(T)$ and the range $Ran(T)$, and let $Fix(T)$ denote its fixed point set. One recalls that T is said to be contraction if and only if

$$\|Tx - Ty\| \le \lambda \|x - y\|, \quad \forall x, y \in Dom(T).$$

It is also called a λ-contractive mapping. Let X be a complete metric space. It is known that Picard iteration, $x_1 \in C$, $x_{n+1} = Tx_n$, $n \ge 1$, converges to the unique fixed point of contractive mappings, the Banach fixed point theorem, which is fundamental and essential for a lot of the existence problems of various nonlinear equations.

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One recalls that T is said to be nonexpansive if and only if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in Dom(T),$$

that is, the Lipschtiz mapping with the constant 1. The theory of nonexpansive mappings (existence and approximation) attached much attention due to their wide application; see, e.g., [7, 23, 24, 28, 33]. The simple Picard cannot convergence to fixed points of nonlinear mappings even that they have fixed points. Let C be a convex and closed subset of a Hilbert space H. Let T be a nonexpansive mapping with fixed points. The Krasnoselski–Mann iterative process (normal Mann iterative process) has been extensively studied due to its power in dealing with fixed points of nonexpansive mappings. It reads

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_n,$$

where $\{\alpha_n\}$ is a positive real number sequence in $(0,1)$ and T is a nonexpansive mapping on C with fixed point. Krasnoselski–Mann iteration is desirable for solving fixed points of nonexpansive, however, the convergence is weak, that is, the iterative sequence converges weakly (converges in weak topology) to a fixed point of a nonexpansive mapping. For an example, we refer to [11]. As we know that a number of real problems happen in infinite dimensional spaces, the strong convergence is better than the weak convergence in many aspects. So, how to improve the Krasnoselski–Mann iteration to reach the strong convergence is an interesting and important problem. For obtaining the strong convergence of the Krasnoselski–Mann iteration, there two kinds of regularization methods: projection-based methods and contraction-based methods; see, e.g., [16, 17, 21, 26, 32] and the references therein.

For projection-based methods, CQ methods are popular and efficient, which need to calculate the metric projection. The drawback is that it is not easy to implement these method since metric projection is not easy to calculate in a number of situations. In this paper, we focus on contraction-based methods. First, we have to mention the classical Halpern iteration

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)T x_n,$$

where $\{\alpha_n\}$ is sequence in $(0,1)$ and u is a fixed vector in subset C. This iteration uses contractive mappings, that is, the convex combination of nonexpansvie mapping T and fixed vector U is a contractive mapping. We remark here that this iteration is really powerful (1) no metric projections are involved, (2) strong convergence is guaranteed, (3) Weak conditions are required only, and (4) both Hilbert spaces and Banach spaces are valid; see, e.g, [15, 22, 30] and references cited therein. Along the line of the Halpern iteration, one of the improvement is the Moudafi viscosity, i.e.,

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)T x_n,$$

where $\{\alpha_n\}$ is sequence in $(0,1)$ and f is a fixed contractive mapping. Indeed, the convex combination of f, the contractive mapping and T, the nonexpansive mapping is also contractive. For convergence theorems of the Moudafi viscosity, we refer to [18, 25, 34].

Next, one recalls another important mappings, strictly pseudocontractive mappings. T is said to be strictly pseudocontractive if only if

$$\|Tx - Ty\|^2 \leq \xi \|x - Tx + Ty - y\|^2 + \|x - y\|^2, \quad \forall x, y \in H,$$

where ξ denotes the real number in $[0,1)$. Obvious, if $\xi = 0$, then it reduces to a nonexpansive mapping. If $\xi = 1$, then it reduces a pseudocontractive mapping. Strictly pseudocontractive mappings were introduced by Browder and Petryshyn in 1967 [4]. It is an important extension of a nonexpansive mapping. Its fixed point has been extensively studied by Halpern and Moudafi iterations. The nonlinear term $\xi\|x-TxTy-y\|^2$ cases the difficult in dealing this nonlinear operator.

Moudafi iteration turn our attention to variational inequalities since the fixed point got by Moudafi iteration is also a solution to a variational inequality with the contractive mapping f. For both Halpern iteration or Moudafi iteration, $\alpha_n \to 0$ as $n\to\infty$ and $\sum_{n=1}^{\infty}\alpha_n=\infty$ are necessary for the convergence analysis.

Let A be a nonlinear operator on H with some monotonicity. One now recalls the following variational inequality problem is to find an $x\in C$ with $\langle y-x, Ax\rangle \geq 0$, $\forall y\in C$. The symbol $Sol(A,C)$ is borrowed to denote the solution set of the variational inequality. Projection gradient method is popular to study the variational inequality due to equivalence between the inequality and fixed points. That is, ξ is a a fixed point of $Proj_C^H(Id-tA)$, where Id stands for the identity mapping and t stands for a positive real constant, if and only if ξ is a solution of the variational inequality. Many results were obtained by using this equivalence for the variational inequality in finite dimensional and infinite dimensional spaces, respectively, see, e.g, [5, 6, 12, 19, 20] and the references therein.

With some monotonicity on A, $Proj_C^H(Id-tA)$, the resolvent operator, is nonexpansive. Recalling the monotone operator theory, we list the following definitions. A mapping A is said to be cocoercive if and only if there exists a positive real number α such that $\langle Ax-Ay, x-y\rangle \geq \alpha\|Ax-Ay\|^2$, $\forall x,y\in H$. It is called also the α-inverse-strongly mapping. A mapping A is said to be a monotone mapping if $\langle Ax-Ay, x-y\rangle \geq 0$, $\forall x,y\in H$; Obvious, cocoercive mappings are monotone. Indeed, they are also continuous.

In 2005, Iiduka and Takahahsi [13] studied the variational inequality with a cocoercive mapping and a fixed point of nonexpansive mapping. In 2007, Yao and Yao [36] further studied the same problem by a new extragradient-like method a cocoercive mapping and a fixed point of nonexpansive mapping. Recently, various contraction-based methods were extensively studied by many authors; see, e.g., [10, 14, 31].

Finally, we need and list some important tools for our solution theorem.

Let $Proj_C^H$ be the metric projection onto C: $Proj_C^H(x) := \arg\min\{\|x-y\|, y\in C\}$. It has the following very well-known properties:

$$\langle x-Proj_C^H(x), y-Proj_C^H(x)\rangle \leq 0, \quad \forall x\in H, y\in C.$$

$$\|Proj_C^H(x)-Proj_C^H(y)\|^2 \leq \langle Proj_C^H(x)-Proj_C^H(y), x-y\rangle, \quad \forall x\in H, y\in H.$$

The last property is commonly called the firm nonexpansivity property.

Lemma 1.1. [37] *Let C be a convex and closed subset of a Hilbert space. Let $T: C\to C$ be a strict pseducontraction. It yields that $Fix(T)$, the fixed point set of T, is convex and closed. Besides, $I-T$ is demiclosed.*

Remark 1.2. Since nonexpansive mappings are also strictly pseudocontractive, we have that the fixed point set of a nonexpansive mapping also is convex and closed, and $I-T$ is demiclosed.

Lemma 1.3. [37] *Let C be a convex and closed subset of a Hilbert space H. Let $T: C\to C$ be a κ-strictly pseudocontractive mapping. Let T be a mapping defined by $T^t x = tx+(1-t)Tx$ for each $x\in C$. Then S^t is nonexpansive such that $Fix(T^t)=Fix(T)$ provided $t\in[\kappa,1)$.*

The following lemma is a result of [29] in Hilbert spaces.

Lemma 1.4. *Let H be a real Hilbert space. Let $\{y_n\}$ and $\{x_n\}$ be bounded vector sequences in space H. Let $\varepsilon_n \in (0,1)$ be a sequence such that $0 < \liminf_{n\to\infty} \varepsilon_n \le \limsup_{n\to\infty} \varepsilon_n < 1$. Put $x_{n+1} = (1-\varepsilon_n)x_n + \varepsilon_n y_n$. If*

$$\limsup_{n\to\infty}(\|\varepsilon_n - \varepsilon_{n+1}\| - \|x_n - x_{n+1}\|) = 0,$$

then $\lim_{n\to\infty} \|x_n - \varepsilon_n\| = 0$.

Lemma 1.5. [35] *Let $\{x_n\}$ be a nonnegative real number sequence with the relation $x_{n+1} \le (1-t_n)x_n + y_n + z_n$, $\forall n \ge 0$, where $\{z_n\}$ is a nonnegative real number sequence such that $\sum_{n=0}^{\infty} z_n < \infty$, $\{y_n\}$ is a real number sequence such that $\limsup_{n\to\infty} \frac{y_n}{t_n} \le 0$, and $\{t_n\} \subset (0,1)$ such that $\sum_{n=0}^{\infty} t_n = \infty$. Then $\lim_{n\to\infty} x_n = 0$.*

Lemma 1.6. [27] *Let C be a convex and closed subset of a Hilbert space H. Let $N_C v$ denote the normal cone to C at $v \in C$, and $B : C \to H$ a monotone mapping,*

$$N_C v = \{w \in H : \langle v-u, w\rangle \ge 0, \quad \forall u \in C\}.$$

Further, give a mapping X on C by

$$Xv = \begin{cases} \emptyset, & v \notin C, \\ Bv + N_C v, & v \in C. \end{cases}$$

Then X is maximal monotone and $0 \in Xv$ if and only $\langle u - v, Bv\rangle \ge 0$, $\forall u \in C$.

2. Main Results

Theorem 2.1. *Let C be a convex, closed, and nonempty subset of a Hilbert space H. Let $Proj_C^H$ be the nearest point projection from space H onto its subset C. Let A be an α-cocercive mapping, and let B be β-cocercive mapping. Let $\{t_n\}$ and $\{s_n\}$ be two nonnegative real number sequences. Let $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ be three real sequence in $(0,1)$ such that $\alpha_n + \beta_n + \gamma_n = 1$. Let f be a λ-contractive self-mapping on C with coefficient $\lambda \in (0,1)$, and let T be a strictly pseudo-contractive self-mapping with coefficient κ on C. Let $\{x_n\}$ be a vector sequence defined by the following iterative process*

$$\begin{cases} z_n = Proj_C^H(Id - s_nB)x_n, \\ y_n = Proj_C^H(Id - t_nA)z_n \\ x_{n+1} = \alpha_n f(y_n) + (1-\lambda_n)\gamma_n Ty_n + \lambda_n\gamma_n y_n + \beta_n x_n. \end{cases}$$

Assume that the control sequence satisfy the following restrictions $0 < s' \le s_n \le s'' < 2\beta$, $0 < t' \le t_n \le t'' < 2\beta$, $\alpha_n \to 0$ as $n \to \infty$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\kappa \le \lambda_n < 1$, $\lim_{n\to\infty} |\lambda_n - \lambda_{n+1}| = 0$, $\lim_{n\to\infty} |s_n - s_{n+1}| = 0$, $\lim_{n\to\infty} |t_n - t_{n+1}| = 0$, and $0 < \liminf_{n\to\infty} \beta \le \limsup_{n\to\infty} \beta < 1$. If $\Upsilon = Sol(A,C) \cap Sol(B,C) \cap Fix(T) \ne \emptyset$, then the sequence $\{x_n\}$, generated via the above iteration, converges strongly to a special solution x^*, which is the unique solution to the following variational inequality: $\langle f(x^*) - x^*, x^* - y\rangle \ge 0$, $\forall y \in \Upsilon$.

Proof. To show the vector sequence $\{x_n\}$ is bounded, we need show that both $Id - s_nB$ and $Id - t_nA$ are nonexpansive. From the cocoerciveness of the two mappings, we have

$$\begin{aligned}\|(Id-t_nA)\bar{x}-(Id-t_nA)\hat{x}\|^2 &= \|\bar{x}-\hat{x}\|^2+t_n^2\|A\bar{x}-A\hat{x}\|^2-2t_n\langle \bar{x}-\hat{x}, A\bar{x}-A\hat{x}\rangle \\ &\leq \|\bar{x}-\hat{x}\|^2+t_n^2\|A\bar{x}-A\hat{x}\|^2-2t_n\alpha\|A\bar{x}-A\hat{x}\|^2 \\ &= \|\bar{x}-\hat{x}\|^2+t_n(t_n-2\alpha)\|A\bar{x}-A\hat{x}\|^2, \quad \forall \bar{x},\hat{x}\in C,\end{aligned}$$

and

$$\begin{aligned}\|(Id-s_nB)\bar{x}-(Id-s_nB)\hat{x}\|^2 &= \|\bar{x}-\hat{x}\|^2+s_n^2\|B\bar{x}-B\hat{x}\|^2-2s_n\langle \bar{x}-\hat{x}, B\bar{x}-B\hat{x}\rangle \\ &\leq \|\bar{x}-\hat{x}\|^2+s_n^2\|B\bar{x}-B\hat{x}\|^2-2s_n\beta\|B\bar{x}-B\hat{x}\|^2 \\ &= \|\bar{x}-\hat{x}\|^2+s_n(s_n-2\beta)\|B\bar{x}-B\hat{x}\|^2, \quad \forall \bar{x},\hat{x}\in C.\end{aligned}$$

From the condition on $\{t_n\}$ and $\{s_n\}$, we have that $Id - t_nA$ and $Id - s_nB$ are nonexpansive. By putting $T^{\lambda_n} := \lambda_n Id + (1-\lambda_n)T$, we see from Lemma 1.3 that T^{λ_n} is a nonexpansive mapping. Besides, $Fix(T^{\lambda_n}) = Fix(T)$ for all n. Let $\theta \in \Upsilon$ be a common solution. Then

$$\begin{aligned}\|z_n-\theta\|^2 &\leq \|(Id-s_nB)x_n-(Id-s_nB)\theta\|^2 \\ &\leq \|x_n-\theta\|^2-s_n(2\beta-s_n)\|Bx_n-B\theta\|^2 \\ &\leq \|x_n-\theta\|^2,\end{aligned}$$

and then

$$\begin{aligned}\|y_n-\theta\|^2 &\leq \|(Id-t_nA)z_n-(Id-t_nA)\theta\|^2 \\ &\leq \|z_n-\theta\|^2-t_n(2\alpha-t_n)\|Az_n-A\theta\|^2 \\ &\leq \|z_n-\theta\|^2 \leq \|x_n-\theta\|^2.\end{aligned}$$

It implies from the nonexpansivity of T^{λ_n} that

$$\begin{aligned}\|x_{n+1}-\theta\| &\leq \gamma_n\|T^{\lambda_n}y_n-T^{\lambda_n}\theta\|+\alpha_n\|f(y_n)-f(\theta)\|+\alpha_n\|f(\theta)-\theta\|+\beta_n\|x_n-\theta\| \\ &\leq \gamma_n\|y_n-\theta\|+\alpha_n\lambda\|y_n-\theta\|+\alpha_n\|f(\theta)-\theta\|+\beta_n\|x_n-\theta\| \\ &\leq \gamma_n\|x_n-\theta\|+\alpha_n\lambda\|x_n-\theta\|+\alpha_n\|f(\theta)-\theta\|+\beta_n\|x_n-\theta\| \\ &= (1-\lambda)\alpha_n\frac{\|f(\theta)-\theta\|}{1-\lambda}+(1-\alpha_n(1-\lambda))\|x_n-\theta\|.\end{aligned}$$

By a simple mathematical induction, one asserts that $\{x_n\}$ is a bounded vector sequence in C. Since $Proj_C^H$ is nonexpansive, one also asserts

$$\begin{aligned}\|z_{n+1}-z_n\| &= \|Proj_C^H(Id-s_{n+1}B)x_{n+1}-Proj_C^H(Id-s_nB)x_n\| \\ &\leq \|(Id-s_{n+1}B)x_{n+1}-(Id-s_nB)x_n\| \\ &\leq \|(Id-s_nB)x_{n+1}-(Id-s_{n+1}B)x_{n+1}\|+\|(Id-s_nB)x_n-(Id-s_nB)x_{n+1}\| \\ &\leq |s_{n+1}-s_n|\|Bx_{n+1}\|+\|x_n-x_{n+1}\|,\end{aligned}$$

and

$$\begin{aligned}\|y_{n+1}-y_n\| &= \|Proj_C^H(Id-t_{n+1}A)z_{n+1}-Proj_C^H(Id-t_nA)z_n\|\\ &\le \|(Id-t_{n+1}A)z_{n+1}-(Id-t_nA)z_n\|\\ &\le \|(Id-t_nA)z_{n+1}-(Id-t_{n+1}A)z_{n+1}\|+\|(Id-t_nA)z_n-(Id-t_nA)z_{n+1}\|\\ &\le |t_{n+1}-t_n|\|Az_{n+1}\|+\|z_n-z_{n+1}\|.\end{aligned}$$

The two estimations above show that

$$\|y_n-y_{n+1}\| \le |t_{n+1}-t_n|\|Az_{n+1}\|+|s_{n+1}-s_n|\|Bx_{n+1}\|+\|x_n-x_{n+1}\|.$$

Observe that

$$\|T^{\lambda_{n+1}}y_{n+1}-T^{\lambda_n}y_{n+1}\| \le |\lambda_{n+1}-\lambda_n|\|y_{n+1}-Ty_{n+1}\|.$$

This further shows that

$$\begin{aligned}\|T^{\lambda_{n+1}}y_{n+1}-T^{\lambda_n}y_n\| &\le \|T^{\lambda_{n+1}}y_{n+1}-T^{\lambda_n}y_{n+1}\|+\|T^{\lambda_n}y_{n+1}-T^{\lambda_n}y_n\|\\ &\le \|T^{\lambda_{n+1}}y_{n+1}-T^{\lambda_n}y_{n+1}\|+\|y_n-y_{n+1}\|\\ &\le |\lambda_n-\lambda_{n+1}|(\|Ty_{n+1}\|+\|y_{n+1}\|)+\|y_n-y_{n+1}\|\\ &\le \|x_n-x_{n+1}\|+(|t_{n+1}-t_n|+|\lambda_n-\lambda_{n+1}|+|s_{n+1}-s_n|)M,\end{aligned}$$

where M is an appropriate constant. To use Lemma 1.4, we put $\varepsilon_n=\frac{x_{n+1}-\beta_nx_n}{1-\beta_n}$. Hence,

$$\varepsilon_{n+1}-\varepsilon_n=\frac{\alpha_{n+1}f(y_{n+1})+\gamma_{n+1}T^{\lambda_{n+1}}y_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_nf(y_n)+\gamma_nT^{\lambda_n}y_n}{1-\beta_n}.$$

We estimate as follows

$$\begin{aligned}&\|\varepsilon_{n+1}-\varepsilon_n\|\\ &\le \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|f(y_{n+1})-T^{\lambda_{n+1}}y_{n+1}\|+\|T^{\lambda_{n+1}}y_{n+1}-T^{\lambda_n}y_n\|+\frac{\alpha_n}{1-\beta_n}\|f(y_n)-T^{\lambda_n}y_n\|\\ &\le \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|f(y_{n+1})-T^{\lambda_{n+1}}y_{n+1}\|+\frac{\alpha_n}{1-\beta_n}\|f(y_n)-T^{\lambda_n}y_n\|+\|x_n-x_{n+1}\|\\ &\quad +M(|t_{n+1}-t_n|+|\lambda_{n+1}-\lambda_n|+|s_n-s_{n+1}|),\end{aligned}$$

that is,

$$\begin{aligned}&\|\varepsilon_{n+1}-\varepsilon_n\|-\|x_{n+1}-x_n\|\\ &\le \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|f(y_{n+1})-T^{\lambda_{n+1}}y_{n+1}\|+\frac{\alpha_n}{1-\beta_n}\|f(y_n)-T^{\lambda_n}y_n\|\\ &\quad +M(|s_{n+1}-s_n|+|t_{n+1}-t_n|+|\lambda_n-\lambda_{n+1}|).\end{aligned}$$

Since the above vector sequences all are bounded, one immediately obtains

$$\limsup_{n\to\infty}(\|\varepsilon_n-\varepsilon_{n+1}\|-\|x_n-x_{n+1}\|)=0.$$

Lemma 1.4 indicates that $\|\varepsilon_n - x_n\| \to 0$ as $n \to \infty$. It follows that $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$ due to $x_{n+1} - x_n = (1-\beta_n)(\varepsilon - x_n)$. On the other hand, we have

$$\begin{aligned}
&\|x_{n+1} - \theta\|^2 \\
&= \|\alpha_n(f(y_n) - \theta) + \beta_n(x_n - \theta) + \gamma_n(T^{\lambda_n} y_n - \theta)\|^2 \\
&\le \alpha_n\|f(y_n) - \theta\|^2 + \beta_n\|x_n - \theta\|^2 + \gamma_n\|Proj_C^H(Id - t_nA)z_n - Proj_C^H(Id - t_nA)\theta\|^2 \\
&\le \alpha_n\|f(y_n) - \theta\|^2 + \beta_n\|x_n - \theta\|^2 + \gamma_n\|(Id - s_nB)x_n - (Id - s_nB)\theta\|^2 \\
&\le \alpha_n\|f(y_n) - \theta\|^2 + \|x_n - \theta\|^2 + s_n(s_n - 2\beta)\gamma_n\|Bx_n - B\theta\|^2,
\end{aligned}$$

which indicates

$$s_n(2\beta - s_n)\gamma_n\|Bx_n - B\theta\|^2 \le \alpha_n\|f(y_n) - \theta\|^2 + \|x_n - x_{n+1}\|(\|x_n - \theta\| + \|x_{n+1} - \theta\|).$$

The restrictions on $\{\alpha_n\}$, $\{\beta_n\}$, and $\{s_n\}$ yield that $\lim_{n\to\infty}\|Bx_n - B\theta\| = 0$. Note that $Proj_C^H$ is firmly nonexpansive. This shows that

$$\begin{aligned}
\|z_n - \theta\|^2 &= \langle Proj_C^H(Id - s_nB)x_n - Proj_C^H(Id - s_nB)\theta, z_n - \theta\rangle \\
&\le \langle (Id - s_nB)x_n - (Id - s_nB)\theta, z_n - \theta\rangle \\
&= \frac{1}{2}(\|x_n - \theta\|^2 + \|z_n - \theta\|^2 - \|(x_n - z_n) - s_n(Bx_n - B\theta)\|^2) \\
&= \frac{1}{2}(\|x_n - \theta\|^2 + \|z_n - \theta\|^2 - \|x_n - z_n\|^2 + 2\|x_n - z_n\|\|Bx_n - B\theta\|),
\end{aligned}$$

that is,

$$\|z_n - \theta\|^2 \le \|x_n - \theta\|^2 + 2\|x_n - z_n\|\|Bx_n - B\theta\| - \|x_n - z_n\|^2.$$

Note that

$$\begin{aligned}
\|x_{n+1} - \theta\|^2 &\le \alpha_n\|f(y_n) - \theta\|^2 + \beta_n\|x_n - \theta\|^2 + \gamma_n\|T^{\lambda_n}y_n - T^{\lambda_n}\theta\|^2 \\
&\le \alpha_n\|f(y_n) - \theta\|^2 + \beta_n\|x_n - \theta\|^2 + \gamma_n\|Proj_C^H(Id - t_nA)z_n - \theta\|^2 \\
&\le \alpha_n\|f(y_n) - \theta\|^2 + \beta_n\|x_n - \theta\|^2 + \gamma_n\|z_n - \theta\|^2.
\end{aligned}$$

With a substitution, we arrive at

$$\|x_{n+1} - \theta\|^2 \le \alpha_n\|f(y_n) - \theta\|^2 + \|x_n - \theta\|^2 - \gamma_n\|x_n - z_n\|^2 + 2\gamma_n\|x_n - z_n\|\|Bx_n - B\theta\|,$$

that is,

$$\begin{aligned}
&\gamma_n\|x_n - z_n\|^2 \\
&\le \alpha_n\|f(y_n) - \theta\|^2 + \|x_n - \theta\|^2 - \|x_{n+1} - \theta\|^2 + 2\gamma_n\|x_n - z_n\|\|Bx_n - B\theta\|, \\
&\le \alpha_n\|f(y_n) - \theta\|^2 + (\|x_n - \theta\| + \|x_{n+1} - \theta\|)\|x_n - x_{n+1}\| + 2\|x_n - z_n\|\|Bx_n - B\theta\|.
\end{aligned}$$

Using the conditions on $\{\alpha_n\}$ and $\{\beta_n\}$, we obtain that $\|z_n - x_n\| \to 0$. Observe that

$$\begin{aligned}
\|x_{n+1} - \theta\|^2 &\le \alpha_n\|f(y_n) - \theta\|^2 + \beta_n\|x_n - \theta\|^2 + \gamma_n\|T^{\lambda_n}y_n - T^{\lambda_n}\theta\|^2 \\
&\le \alpha_n\|f(y_n) - \theta\|^2 + \beta_n\|x_n - \theta\|^2 + \gamma_n\|(Id - t_nA)z_n - (Id - t_nA)\theta\|^2 \\
&\le \alpha_n\|f(y_n) - \theta\|^2 + \|x_n - \theta\|^2 + t_n(t_n - 2\alpha)\gamma_n\|Az_n - A\theta\|^2,
\end{aligned}$$

so

$$t_n(2\alpha - t_n)\gamma_n\|Az_n - A\theta\|^2 \le \alpha_n\|f(y_n) - \theta\|^2 + (\|x_n - \theta\| + \|x_{n+1} - \theta\|)\|x_n - x_{n+1}\|$$

Using the restrictions on $\{\alpha_n\}$, $\{\beta_n\}$, and $\{t_n\}$, one asserts $\|Az_n - A\theta\| \to 0$ as $n \to \infty$. Since $Proj_C^H$ is firmly nonexpansive, we also have $2\|y_n - \theta\|^2 \le \|x_n - \theta\|^2 + \|y_n - \theta\|^2 - \|y_n - z_n\|^2 + 2\|y_n - z_n\|\|Az_n - A\theta\|$, that is,

$$\|y_n - \theta\|^2 \le \|x_n - \theta\|^2 - \|y_n - z_n\|^2 + 2\|y_n - z_n\|\|Az_n - A\theta\|.$$

With a substitution, we have

$$\begin{aligned}\|x_{n+1} - \theta\|^2 &\le \alpha_n\|f(y_n) - \theta\|^2 + \beta_n\|x_n - \theta\|^2 + \gamma_n\|T^{\lambda_n} y_n - \theta\|^2 \\ &\le \alpha_n\|f(y_n) - \theta\|^2 + \|x_n - \theta\|^2 - \gamma_n\|y_n - z_n\|^2 + 2\|y_n - z_n\|\|Az_n - A\theta\|,\end{aligned}$$

which is equivalent to

$$\begin{aligned}\gamma_n\|y_n - z_n\|^2 \le \alpha_n\|f(y_n) - \theta\|^2 &+ (\|x_n - \theta\| + \|x_{n+1} - \theta\|)\|x_n - x_{n+1}\| \\ &+ 2\|y_n - z_n\|\|Az_n - A\theta\|.\end{aligned}$$

This clearly indicates $\|y_n - z_n\| \to 0$ as $n \to \infty$. In view of $x_{n+1} = \alpha_n f(y_n) + \gamma_n T^{\lambda_n} y_n + \beta_n x_n$ and $\|x_n - x_{n+1}\| \to 0$ as $n \to \infty$, we have $x_n - T^{\lambda_n} y_n \to 0$ as $n \to \infty$. Observe that

$$\begin{aligned}\|y_n - Ty_n\| &\le \|Ty_n - T^{\lambda_n} y_n\| + \|y_n - T^{\lambda_n} y_n\| \\ &\le \lambda_n\|Ty_n - y_n\| + \|x_n - T^{\lambda_n} y_n\| + \|x_n - z_n\| + \|z_n - y_n\|.\end{aligned}$$

This proves that $\|y_n - Ty_n\| \to 0$ as $n \to \infty$. Since $\{y_n\}$ is a bounded vector sequence, we may directly assume that it converges weakly to some point, say q. By Lemma 1.1, we have that $q \in Fix(T)$.

Next, we show that $\limsup_{n\to\infty}\langle x_{n+1} - x^*, f(x^*) - x^*\rangle \ge 0$. Let U be the maximally monotone mapping defined by:

$$Ux = \begin{cases} Bx + N_C x, & x \in C, \\ \emptyset, & x \notin C. \end{cases}$$

We have $y - Bx \in N_C x$, where $(x, y) \in Graph(U)$. So, $\langle x - m, y - Bx\rangle \ge 0$, for all $m \in C$. From $z_n = Proj_C(x_n - s_n Bx_n)$, one obtains $\langle \frac{z_n - x_n}{s_n} + Bx_n, x - z_n\rangle \ge 0$. Since $z_n \in C$, one has

$$\begin{aligned}\langle x - z_n, y\rangle &\ge \langle x - z_n, Bx\rangle \\ &\ge \langle x - z_n, Bx\rangle - \langle \frac{z_n - x_n}{s_n} + Bx_n, x - z_n\rangle \\ &= \langle x - z_n, Bx - Bz_n\rangle + \langle x - z_n, Bz_n - Bx_n\rangle - \langle \frac{z_n - x_n}{s_n}, x - z_n\rangle \\ &\ge \langle x - z_n, Bz_n - Bx_n\rangle - \langle x - z_n, \frac{z_n - x_n}{s_n}\rangle.\end{aligned}$$

Since $\{z_n\}$ also weakly converges to q, then $\langle x - q, y\rangle \ge 0$. By the fact that U is maximally monotone, one has that $q \in T^{-1}(0)$, so $q \in Sol(B, C)$. In the same way, we can give $q \in Sol(B, C)$. Hence, $q \in \Upsilon$. From the projection and the inner produce, we have that

$$\limsup_{n\to\infty}\langle x_{n+1} - x^*, f(x^*) - x^*\rangle \le 0.$$

Finally, we observe that

$$\begin{aligned}
&\|x_{n+1}-x^*\|^2 \\
&\le \langle x_{n+1}-x^*, \alpha_n(f(y_n)-x^*)+\beta_n(x_n-x^*)+\gamma_n(T^{\lambda_n}y_n-x^*)\rangle \\
&\le \alpha_n\langle x_{n+1}-x^*, f(y_n)-x^*\rangle+\beta_n\langle x_{n+1}-x^*, x_n-x^*\rangle+\gamma_n\langle x_{n+1}-x^*, y_n-x^*\rangle \\
&\le \alpha_n\langle x_{n+1}-x^*, f(x^*)-x^*\rangle+\alpha_n\langle x_{n+1}-x^*, f(y_n)-f(x^*)\rangle \\
&\quad +\beta_n\langle x_{n+1}-x^*, x_n-x^*\rangle+\gamma_n\langle x_{n+1}-x^*, T^{\lambda_n}y_n-x^*\rangle \\
&\le \alpha_n\langle x_{n+1}-x^*, f(x^*)-x^*\rangle+\alpha_n\|x_{n+1}-x^*\|\|f(y_n)-f(x^*)\| \\
&\quad +\beta_n\|x_{n+1}-x^*\|\|x_n-x^*\|+\gamma_n\|x_{n+1}-x^*\|\|T^{\lambda_n}y_n-x^*\| \\
&\le \alpha_n\langle x_{n+1}-x^*, f(x^*)-x^*\rangle+\frac{\alpha_n}{2}(\|x_{n+1}-x^*\|^2+\|f(y_n)-f(x^*)\|^2) \\
&\quad +\frac{\beta_n}{2}(\|x_{n+1}-x^*\|^2+\|x_n-x^*\|^2)+\frac{\gamma_n}{2}(\|x_{n+1}-x^*\|^2+\|y_n-x^*\|^2).
\end{aligned}$$

It follows that

$$\begin{aligned}
\|x_{n+1}-x^*\|^2 &\le 2\alpha_n\langle x_{n+1}-x^*, f(x^*)-x^*\rangle+\alpha_n\|f(y_n)-f(x^*)\|^2 \\
&\quad +\beta_n\|x_n-x^*\|^2+\gamma_n\|y_n-x^*\|^2 \\
&\le 2\alpha_n\langle x_{n+1}-x^*, f(x^*)-x^*\rangle+\alpha_n\lambda\|y_n-x^*\|^2 \\
&\quad +\beta_n\|x_n-x^*\|^2+\gamma_n\|y_n-x^*\|^2. \\
&\le 2\alpha_n\langle x_{n+1}-x^*, f(x^*)-x^*\rangle+(1-\alpha_n(1-\lambda))\|x_n-x^*\|^2.
\end{aligned}$$

By using Lemma 1.5, we concluding $x_n \to x^*$ as $n\to\infty$. This completes the proof. □

From Theorem 2.1, we have the following results on two variational inequality problems immediately.

Corollary 2.2. *Let C be a convex, closed, and nonempty subset of a Hilbert space H. Let $Proj_C^H$ be the nearest point projection from space H onto its subset C. Let A be an α-cocercive mapping, and let B be β-cocercive mapping. Let $\{t_n\}$ and $\{s_n\}$ be two nonnegative real number sequences. Let $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ be three real sequence in $(0,1)$ such that $\alpha_n+\beta_n+\gamma_n=1$. Let f be a λ-contractive self-mapping on C with coefficient $\lambda\in(0,1)$. Let $\{x_n\}$ be a vector sequence defined by the following iterative process*

$$\begin{cases} z_n = Proj_C^H(Id-s_nB)x_n, \\ y_n = Proj_C^H(Id-t_nA)z_n \\ x_{n+1}=\alpha_n f(y_n)+\gamma_n y_n+\beta_n x_n. \end{cases}$$

Assume that the control sequence satisfy the following restrictions $0<s'\le s_n\le s''<2\beta$, $0<t'\le t_n\le t''<2\beta$, $\alpha_n\to 0$ as $n\to\infty$, $\sum_{n=1}^{\infty}\alpha_n=\infty$, $\lim_{n\to\infty}|s_n-s_{n+1}|=0$, $\lim_{n\to\infty}|t_n-t_{n+1}|=0$, and $0<\liminf_{n\to\infty}\beta\le\limsup_{n\to\infty}\beta<1$. If $\Upsilon=Sol(A,C)\cap Sol(B,C)\neq\emptyset$, then the sequence $\{x_n\}$, generated via the above iteration, converges strongly to a special solution x^*, which is the unique solution to the following variational inequality: $\langle f(x^*)-x^*, x^*-y\rangle\ge 0$, $\forall y\in\Upsilon$.

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