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MULTIPLE SOLUTIONS OF KIRCHHOFF EQUATIONS WITH A SMALL PERTURBATIONS

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Abstract. In this paper, we consider the following Kirchhoff equation

$$\begin{cases} -\left(a+b\int_\Omega |\nabla u|^2\,\mathrm{d}x\right)\Delta u = f(x,u)+tg(x,u), & x\in\ \Omega,\\ u=0, & x\in\ \partial\Omega, \end{cases}$$

with the Dirichlet boundary value. Assuming that the main term $f(x,u)$ is sublinear and odd with respect to u, and the perturbation term is a any continuous function with a small coefficient, we establish the existence and multiplicity of nontrivial solutions for the problem. The approach relies on a combination of variational and minimization methods coupled with the reduction technique.

Keywords. Dirichlet boundary condition; Kirchhoff equation; Perturbation problem; Variant symmetric mountain pass lemma.

1. Introduction and Main Results

This paper was motivated by some recent results concerning the following Kirchhoff equation

$$\begin{cases} -\left(a+b\int_\Omega |\nabla u|^2\,\mathrm{d}x\right)\Delta u = f(x,u), & x\in\ \Omega,\\ u=0, & x\in\ \partial\Omega, \end{cases} \tag{1.1}$$

where $a>0, b>0$ are real constant, and Ω is bounded smooth domain in $\mathbb{R}^N$.

Problem (1.1) is nonlocal because of the presence of the term $\int_\Omega |\nabla u|^2\,\mathrm{d}x\Delta u$, which implies that the equation in (1.1) is no longer a pointwise identity. Nonlocal effect finds its applications in biological systems. This phenomenon provokes some mathematical difficulties, which make

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the study of such a class of problems particularly interesting. Indeed, problem (1.1) is related to the stationary analogue of the Kirchhoff equation

$$u_{tt} - (a + b\int_\Omega |\nabla_x u|^2 \,\mathrm{d}x)\Delta_x u = f(x,u), \tag{1.2}$$

which was proposed by Kirchhoff [11] in 1883. Some early classical studies on Kirchhoff equations can be seen in Pohožaev [17]. After the work of Lions [14], Problem (1.2) was extensively studied; see, e.g. [1, 2, 6] and the references therein.

Recently, more attention has been paid to problem (1.1) by variational methods. When the nonlinearity f is superlinear, Eq. (1.1) has been widely studied; see, e.g., [3, 15, 19, 20, 27]. For the case that the nonlinearity f is asymptotically linear, we refer to [4, 16, 21, 25] and the references therein. Under some weak assumptions on f, the existence of sign-changing solutions with the aid of some new analytical skills were proved in [22, 23].

In [5], the authors considered existence and bifurcation of positive solutions to Kirchhoff-type equations via topological degree argument and variational methods. Using Nehari manifold and variational methods, the authors [12, 24] considered two different kinds of equations with positive solutions.

To our knowledge, there are few results on the existence of multiple solutions to problem (1.1) with a small perturbations. In this paper, we consider the Dirichlet boundary value problem

$$\begin{cases} -\left(a + b\int_\Omega |\nabla u|^2 \,\mathrm{d}x\right)\Delta u = f(x,u) + tg(x,u) & x \in \Omega, \\ u = 0 & x \in \partial\Omega, \end{cases} \tag{1.3}$$

where $a > 0, b > 0$ are real constant, Ω is bounded smooth domain in $\mathbb{R}^N$, and t is a small parameter. In [8], Kajikiya considered the existence of positive solutions for $b = 0$ by using mountain pass lemma. In [9], Kajikiya obtained multiple solutions via symmetric mountain pass lemma. Under the motivation of Kajikiya [8], Lan [13] considered the existence of positive solutions of problem (1.3). We next show the multiplicity of solutions for problem (1.3) by borrowing the method introduced in [9, 10, 26].

We look for the weak solutions of (1.3), which are the same as the critical points of the functional $I_t : H_0^1(\Omega) \to \mathbb{R}$ defined by

$$I_t(u) = \frac{a}{2}\int_\Omega |\nabla u|^2 \,\mathrm{d}x + \frac{b}{4}\left(\int_\Omega |\nabla u|^2 \,\mathrm{d}x\right)^2 - \int_\Omega (F(x,u) + tG(x,u))\,\mathrm{d}x,$$

where $F(x,u) = \int_0^u f(x,s)\,\mathrm{d}s$ and $G(x,u) = \int_0^u g(x,s)\,\mathrm{d}s$. As usual, we equip $H_0^1(\Omega)$ with the norm $\|u\|^2 = \int_\Omega |\nabla u|^2 \,\mathrm{d}x$. I_t is of $C^1(H_0^1(\Omega),\mathbb{R})$ and $u, v \in H_0^1(\Omega)$ with the derivatives given by

$$\langle I_t'(u), v\rangle = (a + b\int_\Omega |\nabla u|^2 \,\mathrm{d}x)\int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x - \int_\Omega (f(x,u)v + tg(x,u)v)\,\mathrm{d}x.$$

Before stating this theorem, we first recall the notion of genus.

Let E be an infinite dimensional Banach space and A a subset of E. A is said to be symmetric if $x \in A$ implies $-x \in A$. For a closed symmetric set A, which does not contain the origin, we define a genus $\gamma(A)$ of A by the smallest integer k such that there exists an odd continuous mapping from A to $\mathbb{R}\backslash\{0\}$. If there does not exists such a k, we define $\gamma(A) = \infty$. Moreover, we set $\gamma(\emptyset) = 0$. For each $k \in \mathbb{N}$, let $\Gamma_k = \{A \in \Gamma | \gamma(A) \geq k\}$. We define $c_k = \inf_{A\in\Gamma_k}\sup_{u\in A} I_0(u)$.

Theorem A (Variant Symmetric Mountain Pass Lemma). Let E be an infinite dimensional Banach space and $I_0 \in C^1(E,\mathbb{R})$ an even functional with $I_0(0)=0$. Suppose that I_0 satisfies

(A1) I_0 is bounded from below and satisfies the (PS) condition;

(A2) if for any $k \in \mathbb{N}$, there exists an $A_k \in \Gamma_k$ such that $\sup_{u\in A_k} I_0(u) < 0$.

Then either (i) or (ii) below holds

(i) there exists a sequence of critical points $\{u_k\}$ satisfying $I_0(u_k)<0$ for all k and $\|u_k\| \to 0$ as $k \to \infty$;

(ii) there exists two critical point sequences $\{u_k\}$ and $\{v_k\}$ such that $I_0(u_k)=0$, $u_k \neq 0$, $\lim_{k\to\infty} u_k = 0$, $I_0(v_k)<0$, $\lim_{k\to\infty} I_0(v_k)=0$, and $\{v_k\}$ converges to a non-zero limit.

We are going to give suitable conditions and use the above Theorem to obtain multiple small solutions of problem (1.3). Let $f(x,u)$ and $g(x,u)$ be Hölder continuous functions defined on $\overline{\Omega}\times[-\varepsilon,\varepsilon]$ with an $\varepsilon>0$ and satisfy that

(B1) the function $f(x,u)$ is odd on u, and there exist $1<r<4$ and $C>0$ such that $|f(x,u)| \le C|u|^{r-1}$;

(B2) there exist an $\varepsilon>0$ such that

$$\begin{aligned} &uf(x,u)-2F(x,u)<0 \,\text{when}\, 0<|u|<\varepsilon \,\text{and}\, x\in\overline{\Omega},\\ &f(x,u)=g(x,u)=F(x,u)=0 \,\text{when}\, |u|\ge\varepsilon \,\text{and}\, x\in\overline{\Omega},\\ &F(x,u)>0 \,\text{when}\, 0<|u|<\varepsilon \,\text{and}\, x\in\overline{\Omega}; \end{aligned}$$

(B3) there exist an $x_0\in\overline{\Omega}$ and a constant $r_0>0$ such that

$$\liminf_{u\to 0}\left(\inf_{x\in B_{r_0}(x_0)} u^{-2}F(x,u)\right) > -\infty$$

and

$$\limsup_{u\to 0}\left(\inf_{x\in B_{r_0}(x_0)} u^{-2}F(x,u)\right) = +\infty,$$

where $B_{r_0}(x_0)$ is the ball in $\overline{\Omega}$ centered at x_0 with radius r_0.

The main results of this paper are as follows.

Theorem 1.1. *Suppose that (B1)-(B3) hold. If $t=0$, then (1.3) has a sequence of solutions whose $C^2(\overline{\Omega})$-norms converges to zero.*

Theorem 1.2. *Suppose that (B1)-(B3) hold for any $k\in\mathbb{N}$ and any $\varepsilon>0$. Then there exists an $\delta(k,\varepsilon)>0$ such that if $|t|\le\delta(k,\varepsilon)$, then (1.3) has at least k distinct solutions whose $C^2(\overline{\Omega})$-norms are less than ε.*

In the following discussions, we denote various positive constants as C or C_i $(i=0,1,2,3,\dots)$ for convenience.

2. Proofs of Main Results

Proof of the Theorem 1.1. We prove the Theorem 1 by Theorem A. It is easy to check that $I_0\in C^1(H_0^1(\Omega),\mathbb{R})$ is even, and $I_0(0)=0$. We next prove that I_0 satisfies (A1) and (A2) of Theorem A. We divide this proof into two steps.

Step 1. From (B1), for $u \in H_0^1(\Omega)$,

$$\begin{aligned} I_0(u) &= \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \int_\Omega F(x,u)\,\mathrm{d}x \\ &\geq \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - C_1\|u\|^r \end{aligned}$$

and then I_0 is coercive, which together with $I_0 \in C^1$ implies that I_0 is bounded below. Now let $\{u_n\}$ be a (PS)-sequence, that is,

$$|I_0(u_n)| \leq C_2 \ \text{ and } \ I_0'(u_n) \to 0 \ \text{ as } \ n \to \infty. \tag{2.1}$$

Note that I_0 is coercive. This together with (2.1) implies that $\{u_n\}$ is bounded. By usual arguments, we can assume that, up to a subsequence (denoted also by $\{u_n\}$), there exists $u \in H_0^1(\Omega)$ such that

$$\begin{aligned} &u_n \rightharpoonup u \text{ in } H_0^1(\Omega), \\ &u_n \to u \text{ in } L^s(\Omega), \text{ where } 2 \leq s < 2^*, \\ &u_n(x) \to u(x) \text{ a.e. } x \in \Omega. \end{aligned} \tag{2.2}$$

By (B1) and the Hölder inequality, we have

$$\begin{aligned} \left|\int_\Omega (f(x,u_n) - f(x,u))(u_n - u)\,\mathrm{d}x\right| &\leq \int_\Omega |(f(x,u_n) - f(x,u))(u_n - u)|\,\mathrm{d}x \\ &\leq \int_\Omega \left|C(|u_n|^{r-1} + |u|^{r-1})(u_n - u)\right|\,\mathrm{d}x \\ &\leq C[\|u_n\|_{L^r}^{r-1} + \|u\|_{L^r}^{r-1}]\|u_n - u\|_{L^r} \\ &\leq C_3\|u_n - u\|_{L^r} \to 0. \end{aligned} \tag{2.3}$$

As $I_0'(u_n) \to 0$ and $u_n \to u$ in $L^s(\Omega)$, we have

$$I_0'(u_n)(u_n - u) \to 0, \text{ and } I_0'(u)(u_n - u) \to 0.$$

Then, as $n \to \infty$,

$$\begin{aligned} o(1) &= \langle I_0'(u_n) - I_0'(u), u_n - u\rangle \\ &= a\|u_n - u\|^2 + b\int_\Omega |\nabla u_n|^2\,\mathrm{d}x \int_\Omega \nabla u_n \cdot \nabla(u_n - u)\,\mathrm{d}x - b\int_\Omega |\nabla u|^2\,\mathrm{d}x \int_\Omega \nabla u \cdot \nabla(u_n - u)\,\mathrm{d}x \\ &\quad - \int_\Omega (f(x,u_n) - f(x,u))\,(u_n - u)\,\mathrm{d}x \\ &= a\|u_n - u\|^2 + b\int_\Omega |\nabla u_n|^2\,\mathrm{d}x \int_\Omega |\nabla(u_n - u)|^2\,\mathrm{d}x - \int_\Omega (f(x,u_n) - f(x,u))\,(u_n - u)\,\mathrm{d}x \\ &\quad + b\left(\int_\Omega |\nabla u_n|^2\,\mathrm{d}x - \int_\Omega |\nabla u|^2\,\mathrm{d}x\right)\int_\Omega \nabla u \cdot \nabla(u_n - u)\,\mathrm{d}x. \end{aligned} \tag{2.4}$$

For any $v \in L^2(\Omega)$, by (2.2) and Hölder inequality, we obtain

$$\langle v, u_n - u\rangle = \int_\Omega \nabla v \cdot \nabla(u_n - u)\,\mathrm{d}x \leq \|\nabla v\|_{L^2}\|\nabla(u_n - u)\|_{L^2} \to 0,$$

which implies that $\nabla u_n \rightharpoonup \nabla u$ in $L^2(\Omega)$. Then it follows that

$$b\left(\int_\Omega |\nabla u_n|^2\,\mathrm{d}x - \int_\Omega |\nabla u|^2\,\mathrm{d}x\right)\int_\Omega \nabla u \cdot \nabla(u_n - u)\,\mathrm{d}x \to 0 \ \text{ as } n \to \infty. \tag{2.5}$$

Combining (2.2)-(2.5), we obtain

$$0 \le a\|u_n - u\|^2 = -b\int_\Omega |\nabla u_n|^2 \,\mathrm{d}x \int_\Omega |\nabla (u_n - u)|^2 \,\mathrm{d}x + o(1) \le o(1),$$

which implies that $u_n \to u$ in $H_0^1(\Omega)$. Therefore I_0 satisfies (PS) condition. Consequently, I_0 satisfies the condition (A1) of Theorem A.

Step 2. We follow the idea of the geometric construction introduce in [10]. By coordinate translation, we can assume $x_0 = 0$ in (B3). Let $\mathscr{C}$ denote the cube

$$\mathscr{C} := \{x = (x_1, x_2, ..., x_N) | -r_0/2 \le x_i \le r_0/2, i = 1, 2, ..., N\},$$

where r_0 is given in (B3). Evidently, $\mathscr{C} \subseteq B_{r_0}$. By (B3), there exist constants $\delta, \rho > 0$ and two sequences of positive numbers $\delta_n \to 0, M_n \to \infty$ as $n \to \infty$ such that

$$F(x,u) \ge -\rho u^2, \ \forall x \in \mathscr{C} \text{ and } |u| \le \delta \tag{2.6}$$

and

$$F(x, \pm\delta_n)/\delta_n^2 \ge M_n, \ \forall x \in \mathscr{C} \text{ and } n \in \mathbb{N}. \tag{2.7}$$

For any fixed $k \in \mathbb{N}$, let $m \in \mathbb{N}$ be the smallest positive integer satisfying $m^N \ge k$. We divide the cube $\mathscr{C}$ equally into m^N small cubes by planes parallel to each face of $\mathscr{C}$ and denote them by $\mathscr{C}_i$ with $1 \le i \le m^N$. Then the edge of each $\mathscr{C}_i$ has the length of $l := r_0/m$. For each $1 \le i \le k$, we make a cube $\mathscr{D}_i$ in $\mathscr{C}_i$ such that $\mathscr{D}_i$ has the same center as that of $\mathscr{C}_i$, the faces of $\mathscr{D}_i$ and $\mathscr{C}_i$ are parallel, and the edge of $\mathscr{D}_i$ has the length of $l/2$. Choose a function $\psi \in C_0^\infty(\mathbb{R}, \mathbb{R})$ such that

$$\begin{cases} \psi(t) \equiv 1 & \text{for } t \in [-l/4, l/4], \\ \psi(t) \equiv 0 & \text{for } t \in \mathbb{R}\backslash[-l/2, l/2], \\ 0 \le \psi(t) \le 1 & \text{for } t \in \mathbb{R}. \end{cases}$$

Define

$$\varphi(x) = \psi(x_1)\psi(x_2)\cdots\psi(x_N), \quad \forall x = (x_1, x_2, ..., x_N) \in \mathbb{R}^N.$$

For each $1 \le i \le k$, let $y_i \in \mathbb{R}^N$ by the center of both $\mathscr{C}_i$ and $\mathscr{D}_i$, and define $\varphi_i(x) = \varphi_i(x - y_i)$, $\forall x \in \mathbb{R}^N$. Then it is easy to see that

$$supp \varphi_i \subseteq \mathscr{C}_i, \tag{2.8}$$

and

$$\varphi_i(x) = 1, \ \forall x \in \mathscr{D}_i, \ 0 \le \varphi_i(x) \le 1, \ \forall x \in \mathbb{R}^N, \tag{2.9}$$

for all $1 \le i \le k$. Set

$$\mathscr{V}_k := \left\{ (s_1, s_2, ..., s_k) \in \mathbb{R}^k | \max_{1 \le i \le k} |s_i| = 1 \right\},$$

and

$$\mathscr{W}_k := \left\{ \sum_{i=1}^k s_i \varphi_i | (s_1, s_2, ..., s_k) \in \mathscr{V}_k \right\}.$$

Evidently, $\mathscr{V}_k$ is homeomorphic to the unit sphere in $\mathbb{R}^k$ by an odd mapping. Thus $\gamma(\mathscr{V}_k) = k$. If we define the mapping $\eta : \mathscr{V}_k \to \mathscr{W}_k$ by

$$\eta(s_1, s_2, ..., s_k) = \sum_{i=1}^k s_i \varphi_i, \ \forall (s_1, s_2, ..., s_k) \in \mathscr{V}_k,$$

then η is odd and homeomorphic. Therefore $\gamma(\mathscr{W}_k)=\gamma(\mathscr{V}_k)=k$. Moreover, it is evident that $\mathscr{W}_k$ is compact and hence there is a constant $C_k>0$ such that

$$\|u\|\le C_k,\ \ \forall u\in\mathscr{W}_k. \tag{2.10}$$

For each $\delta_n\in(0,\delta)$ given by (2.6) and $u=\sum_{i=1}^k s_i\varphi_i\in\mathscr{W}_k$, due to (2.8) and (2.9), there holds

$$\begin{aligned} I_0(\delta_n u) &= \frac{a}{2}\|\delta_n u\|^2+\frac{b}{4}\left(\int_\Omega|\nabla(\delta_n u)|^2\,\mathrm{d}x\right)^2-\int_\Omega F(x,\delta_n\sum_{i=1}^k s_i\varphi_i)\,\mathrm{d}x \\ &\le \frac{a\delta_n^2}{2}\|u\|^2+\frac{b\delta_n^4}{4}\|u\|^4-\sum_{i=1}^k\int_{\mathscr{C}_i}F(x,\delta_n s_i\varphi_i)\,\mathrm{d}x. \end{aligned} \tag{2.11}$$

By the definition of $\mathscr{V}_k$, there exists some integer $1\le i_u\le k$ such that $|s_{i_u}|=1$. Then

$$\begin{aligned} \sum_{i=1}^k\int_{\mathscr{C}_i}F(x,\delta_n s_i\varphi_i)\,\mathrm{d}x &= \int_{\mathscr{D}_i}F(x,\delta_n s_{i_u}\varphi_{i_u})\,\mathrm{d}x+\int_{\mathscr{C}_i\setminus\mathscr{D}_i}F(x,\delta_n s_{i_u}\varphi_{i_u})\,\mathrm{d}x \\ &\quad +\sum_{i\ne i_u}\int_{\mathscr{C}_i}F(x,\delta_n s_i\varphi_i)\,\mathrm{d}x. \end{aligned} \tag{2.12}$$

By (2.6) and (2.9), we obtain

$$\int_{\mathscr{C}_i\setminus\mathscr{D}_i}F(x,\delta_n s_{i_u}\varphi_{i_u})\,\mathrm{d}x+\sum_{i\ne i_u}\int_{\mathscr{C}_i}F(x,\delta_n s_i\varphi_i)\,\mathrm{d}x\ge-\rho r_0^N\delta_n^2. \tag{2.13}$$

Here we use the fact that the volume of cube $\mathscr{C}$ in $\mathbb{R}^N$ is r_0^N. Combining (2.7) and (2.10)-(2.13), we have

$$\begin{aligned} I_0(\delta_n u) &\le \frac{aC_k^2\delta_n^2}{2}+\frac{bC_k^4\delta_n^4}{4}+\rho r_0^N\delta_n^2-\int_{\mathscr{D}_i}F(x,\delta_n s_{i_u}\varphi_{i_u})\,\mathrm{d}x \\ &\le \delta_n^2\left(\frac{aC_k^2}{2}+\frac{bC_k^4\delta_n^2}{4}+\rho r_0^N-\frac{l^NM_n}{2^N}\right). \end{aligned} \tag{2.14}$$

Since $M_n\to\infty$ as $n\to\infty$, we can choose $n_0\in\mathbb{N}$ large enough such that, for $n\ge n_0$, the right-hand side of (2.14) is negative. We construct an $A_k\in\Gamma_k$ satisfying (A2). Define $A_k:=\{\delta_{n_0}u|\,u\in\mathscr{W}_k\}$. Then $\gamma(A_k)=\gamma(\mathscr{W}_k)=k$ and $\sup_{u\in A_k}I_0(u)<0$. Therefore I_0 satisfies the condition (A2) of Theorem A. By Theorem A, we obtain a sequence of nontrivial critical points $\{u_k\}$ of I_0 satisfying $I_0(u_k)\le 0$ for all $k\in\mathbb{N}$ and $u_k\to 0$ in $H_0^1(\Omega)$ as $k\to\infty$. The Ascoli-Arzelà theorem guarantee that u_k converges to zero in $C(\overline{\Omega})$, and hence in $C^2(\overline{\Omega})$ by the elliptic regularity theorem (see [7]). The proof is complete.

Lemma 2.1. *For any $\varepsilon>0$, there exists a $\delta>0$ such that $|t|\le\delta, I_t'(u)=0$ and $|I_t(u)|\le\delta$. Then $\|u\|_{C^2(\overline{\Omega})}\le\varepsilon$.*

Proof. For any $k\in\mathbb{N}$, there exists a sequence $\{t_k\}\to 0$ such that u_k satisfies $I_{t_k}'(u_k)=0, I_{t_k}(u_k)\to 0$. Since $f(x,u)$ and $g(x,u)$ are bounded on $\overline{\Omega}\times\mathbb{R}$, the elliptic regularity theorem gives a constant C_4 such that $\|u_k\|\le C_4$ for $k\in\mathbb{N}$. Let $\{v_k\}$ be any subsequence of $\{u_k\}$. Then Ascoli-Arzelà theorem yields a subsequence $\{w_k\}$ of $\{v_k\}$, which converges to a certain limit w in the

$C^1(\overline{\Omega})$-space. Consequently, we have

$$\begin{cases} -\left(a+b\int_\Omega |\nabla w_k|^2\,\mathrm{d}x\right)\Delta w_k = f(x,w_k) & x\in\ \Omega, \\ w_k = 0 & x\in\ \partial\Omega, \end{cases} \tag{2.15}$$

and

$$I_0(w_k) = \frac{a}{2}\int_\Omega |\nabla w_k|^2\,\mathrm{d}x + \frac{b}{4}(\int_\Omega |\nabla w_k|^2\,\mathrm{d}x)^2 - \int_\Omega F(x,w_k)\,\mathrm{d}x = 0. \tag{2.16}$$

We multiple the first equation of (2.15) by w_k and integrate it over Ω to see

$$(a+b\int_\Omega |\nabla w_k|^2\,\mathrm{d}x)\int_\Omega |\nabla w_k|^2\,\mathrm{d}x = \int_\Omega w_k f(x,w_k)\,\mathrm{d}x. \tag{2.17}$$

Then we combine (2.16) with (2.17) to obtain

$$I_0(w_k) = \int_\Omega (\frac{1}{2}w_k f(x,w_k) - F(x,w_k))\,\mathrm{d}x - \frac{b}{4}(\int_\Omega |\nabla w_k|^2\,\mathrm{d}x)^2.$$

Letting $k\to\infty$, we have

$$\lim_{k\to\infty} I_0(w_k) = I_0(w) = \int_\Omega (\frac{1}{2}wf(x,w) - F(x,w))\,\mathrm{d}x - \frac{b}{4}(\int_\Omega |\nabla w|^2\,\mathrm{d}x)^2 = 0.$$

It follows that

$$\int_\Omega (\frac{1}{2}wf(x,w) - F(x,w))\,\mathrm{d}x = \frac{b}{4}(\int_\Omega |\nabla w|^2\,\mathrm{d}x)^2.$$

In view of $\frac{b}{4}(\int_\Omega |\nabla w|^2\,\mathrm{d}x)^2 \geq 0$ and $\frac{1}{2}wf(x,w) - F(x,w) < 0$ by (C3), we can obtain $w\equiv 0$. Therefore, u_k convergence to zero in $C^1(\overline{\Omega})$. Moreover, the elliptic regularity theorem guarantees that the convergence is valid in the $C^2(\overline{\Omega})$-sense. The proof is complete. □

Proof of the Theorem 1.2. There exists constants t_{k+1} such that $0 < t_{k+1} \leq 1$ and a functional $\psi \in C([0,1],\mathbb{R})$ such that $\psi(0) = 0, c_{k+1} < -\psi(t)$ for $t\in[0,t_{k+1}]$. Since $G(x,u)$ is bounded on $\overline{\Omega}\times\mathbb{R}$, we have

$$|I_t(u) - I_0(u)| = |\int_\Omega tG(x,u)\,\mathrm{d}x| \leq |t|\int_\Omega |G(x,u)|\,\mathrm{d}x \leq C_4|t| = \psi(t), \tag{2.18}$$

where $C_4 > 0$ is a constant and independent of u and t. Determine the constant $r > 0$ small enough that $c_k + r < c_{k+1}$. We choose $t_{k+1}\in(0,1]$ so small that

$$c_k + r + 2\psi(t) < c_{k+1},\ \ c_{k+1} + \psi(t) < 0\ \ \text{for } t\in[0,t_{k+1}].$$

Fix $t\in[0,t_{k+1}]$ arbitrarily and define

$$d_{k+1}(t) = \inf_{A\in\Gamma_{k+1}} \sup_{u\in A} I_t(u). \tag{2.19}$$

Combining (2.18) with (2.19), we have

$$\begin{aligned} d_{k+1}(t) &\leq \inf_{A\in\Gamma_{k+1}} \sup_{u\in A}(I_0(u) + \psi(t)) \\ &= \inf_{A\in\Gamma_{k+1}} \sup_{u\in A} I_0(u) + \psi(t) \\ &= c_{k+1} + \psi(t). \end{aligned}$$

and

$$\begin{aligned} d_{k+1}(t) &\geq \inf_{A\in\Gamma_{k+1}} \sup_{u\in A}(I_0(u)-\psi(t)) \\ &= \inf_{A\in\Gamma_{k+1}} \sup_{u\in A} I_0(u)-\psi(t) \\ &= c_{k+1}-\psi(t). \end{aligned}$$

Therefore, we obtain $c_{k+1}-\psi(t) \leq d_{k+1}(t) \leq c_{k+1}+\psi(t)$.

Next, we show that $d_{k+1}(t)$ is a critical value of $I_t(\cdot)$. Let us argue indirectly. If $d_{k+1}(t)$ is a regular value of $I_t(\cdot)$, w use deformation lemma with $c = d_{k+1}(t), c-\bar{\varepsilon} = c_k + r + \psi(t)$. For the lemma, the readers can refer to [18, p.82, Theorem A.4]. We take an $\varepsilon > 0$ and an odd mapping $\eta \in C(E,E)$ satisfying the conditions below.

(H1) If $d_{k+1}(t)$ is a regular value of I_t and $I_t(u) \leq d_{k+1}(t)+\varepsilon$, then $I_t(\eta(u)) \leq d_{k+1}(t)-\varepsilon$.

(H2) If $I_t(u) \leq c_k + r + \psi(t)$, then $\eta(u) = u$.

By the definition of d_{k+1}, there exist an $A_1 \in \Gamma_{k+1}$ such that $\sup_{u\in A_1} I_t(u) < d_{k+1}(t)+\varepsilon$. By (H1), we have

$$\sup_{u\in A_1} I_t(\eta(u)) \leq d_{k+1}(t)-\varepsilon. \tag{2.20}$$

By the definition of c_k, we have $\sup_{u\in A_1} I_0(u) < c_k + r$, and

$$\sup_{u\in A_1} I_t(u) \leq \sup_{u\in A_1} I_0(u)+\psi(t) < c_k + r + \psi(t).$$

The inequality above with (H2) implies that $\eta(u) = u \in A_1$. However, (2.20) contradicts the definition of $d_{k+1}(t)$. Consequently, $d_{k+1}(t)$ is a critical value. This shows that $I_t(\cdot)$ has a critical value in the interval $[c_{k+1}-\psi(t), c_{k+1}+\psi(t)]$. For any $k \in \mathbb{N}, \delta > 0$, we choose increasing $p(i) > 0$ with $1 \leq i \leq k$ such that $-\delta < c_{p(1)}$ and $c_{p(i)} < c_{p(i+1)}$ for $1 \leq i \leq k$. There exist $\varepsilon > 0$ small enough such that $d_{p(i)}(t)$ with $1 \leq i \leq k$ are defined for $t \in [0,\varepsilon]$ and

$$-\delta < c_{p(1)}-\psi(t), \quad c_{p(i)}+\psi(t) < c_{p(i+1)}-\psi(t) \ \text{ on } \ [0,\varepsilon].$$

This means that $d_{p(i)}(t) < d_{p(i+1)}(t)$. Then, for $t \in [0,\varepsilon]$, $I_t(\cdot)$ has at least k critical values

$$-\delta < d_{p(1)}(t) < d_{p(2)}(t) < \ldots < d_{p(k)}(t) < 0.$$

In view of Lemma 2.1 it is enough to obtain that $I_t(\cdot)$ has at least k solutions whose $C^2(\overline{\Omega})$-norms are less than ε. Therefore the proof is complete.

Remark 2.2. Here we give a weaker condition (C1)-(C3): Let $f(x.u)$ and $g(x,u)$ be Hölder continuous functions defined on $\overline{\Omega}\times\mathbb{R}$ and satisfy the conditions below

(B1) the function $f(x,u)$ is odd in u, and there exist $1 < r < 4, C > 0$ such that $|f(x,u)| \leq C|u|^{r-1}$;

(B2) $uf(x,u)-2F(x,u) < 0$, where $F(x,u)$ is defined by

$$F(x,u) := \int_0^u f(x,t)\,\mathrm{d}t;$$

(B3) there exist an $x_0 \in \overline{\Omega}$ and a constant $r_0 > 0$ such that

$$\liminf_{u\to 0}\left(\inf_{x\in B_{r_0}(x_0)} u^{-2}F(x,u)\right) > -\infty,$$

and

$$\limsup_{u\to 0}\left(\inf_{x\in B_{r_0}(x_0)} u^{-2}F(x,u)\right)=+\infty.$$

Next, we explain that the weaker condition (C1)-(C3) gives the condition (A1)-(A3). Without loss of generality, we choose a function $\phi\in C_0^\infty(\mathbb{R},\mathbb{R})$ such that

$$\begin{aligned}
&0\le\phi(u)\le 1 \text{ for } u\in\mathbb{R},\\
&\phi(u)=1 \text{ for } |u|\le\frac{\varepsilon}{2},\\
&\phi(u)>0 \text{ for } |u|<\varepsilon,\\
&\phi(u)=0 \text{ for } |u|\ge\varepsilon.
\end{aligned}$$

If $\phi(u)$ is even in $\mathbb{R}$ and strictly decreasing in $(\varepsilon/2,\varepsilon)$, then we have $\phi'(u)<0$. We define $\widetilde{f}(x,u),\widetilde{F}(x,u),\widetilde{g}(x,u)$ by

$$\begin{aligned}
&\widetilde{f}(x,u)=\phi(u)f(x,u),\\
&\widetilde{g}(x,u)=\phi(u)g(x,u),\\
&\widetilde{F}(x,u)=\int_0^u\widetilde{f}(x,u)\,\mathrm{d}s=\phi(u)F(x,u).
\end{aligned}$$

It is clear that $\widetilde{f}$ satisfies (C1) and (C3). We verify (C2). By the definition, $\widetilde{f}(x,u),\widetilde{g}(x,u),\widetilde{F}(x,u)$ vanish when $|u|\ge\varepsilon$ and $x\in\overline{\Omega}$. Moreover, $\widetilde{F}(x,u)>0$ when $0<|u|<\varepsilon$ and $x\in\overline{\Omega}$. Observe the relation,

$$\frac{\partial}{\partial u}(u^{-2}\widetilde{F}(x,u)) = \frac{u^2\widetilde{f}(x,u)-2u\widetilde{F}(x,u)}{u^4}=u^{-3}(u\widetilde{f}(x,u)-2\widetilde{F}(x,u)).$$

Using the definition of $\widetilde{F}(x,u)$ with (B2), we obtain

$$\begin{aligned}
\frac{\partial}{\partial u}(u^{-2}\widetilde{F}(x,u)) &= \frac{\partial}{\partial u}(u^{-2}\phi(u)F(x,u))\\
&= \phi'(u)u^{-2}F(x,u)+\phi(u)\frac{\partial}{\partial u}(u^{-2}F(x,u))\\
&= \phi'(u)u^{-2}F(x,u)+\phi(u)u^{-3}(uf(x,u)-2F(x,u))<0
\end{aligned}$$

provided that $0<|u|<\varepsilon$ and $x\in\overline{\Omega}$. We see that $u\widetilde{f}(x,u)-2\widetilde{F}(x,u)<0$. Therefore $\widetilde{f}(x,u)$ satisfies (C2). It is enough to prove Theorem 1.1 and Theorem 1.2 with f and g replaced by $\widetilde{f}$ and $\widetilde{g}$ under (C1)-(C3), respectively because $\widetilde{f}(x,u)=f(x,u)$ and $\widetilde{g}(x,u)=g(x,u)$ for $|u|$ sufficiently small.

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