J. Nonlinear Funct. Anal. 2022 (2022) 20 https://doi.org/10.23952/jnfa.2022.20

MATHRES

Journal of Nonlinear Functional Analysis

Available online at http://jnfa.mathres.org

EXISTENCE AND STABILITY OF SOLUTIONS FOR COUPLED FRACTIONAL DELAY Q-DIFFERENCE SYSTEMS

CHUNPING TIAN, HAIBO GU*, NING LI

School of Mathematics Science, Xinjiang Normal University, Urumqi 830017, China

Abstract. In this paper, we study the existence and stability of solutions to a class of coupled fractional delay q-difference systems and obtain some similar results in a generalized Banach space. Finally, two examples are provided to illustrate the results.

Keywords. Coupled fractional q-difference system; Generalized Banach space; q-Gronwall inequality; Ulam stability.

1. INTRODUCTION

As an active direction, fractional calculus has broad prospects and has been successfully applied in many fields, such as physics, stochastic process, anomalous diffusion abd so on. q-Calculus as an important bridge connecting mathematics and physics plays an important role in quantum mechanics, nuclear and high energy physics, generation mathematics, and computer field. With the development of the theory, many real-world phenomena can be modelled with a high accuracy by different fractional boundary value problems. q-Calculuses were first investigated in [3, 8, 11, 17, 18]. In recent years, q-Calculus was mainly applied in engineering and economics. Ferreira [10] considered the boundary value problems of the fractional q-difference equations with the properties of the Green's function and the fixed point theorem. In [5, 12, 16, 22], the existence results of solutions for fractional q-difference equations were studied. Subsequently, many researchers focused on fractional q-Calculus systems in various cases, including multi-point boundary value, integral boundary value, different fractional differential operators with the delay variables, and composition operators problems; see, e.g., [6, 14, 15, 19, 20, 21] and the references therein. But the stability of coupled fractional q-difference equations were relatively less explored. In [7, 23], the stability of fractional difference about non-autonomous systems and quantum equations of Euler type was discussed, respectively. In [1, 2, 4, 9, 13], the existence and finite-time stability of a class of fractional

*Corresponding author.

E-mail address: hbgu_math@163.com (H. Gu).

Received November 16, 2021; Accepted April 26, 2022.

q-difference time-delay systems by the q-Gronwall inequality and the fixed point theorem were studied.

In [13], Ma et al. discussed the stability conditions of the coupled systems of a fractional q-difference Lotka-Volterra model by the fixed point theorem:

$$\begin{cases} {}^{c}D_q^{\alpha}u(t) = c(t)f(t,u(t),v(t)), & t \in (0,T], \\ {}^{c}D_q^{\beta}v(t) = d(t)g(t,u(t),v(t)), & t \in (0,T], \\ u(0) = \gamma, \quad D_q u(T) = 0, \\ v(0) = \delta, \quad D_q v(T) = 0, \end{cases}$$

where ${}^{c}D_q^{\alpha}$, ${}^{c}D_q^{\beta}$ is the Caputo fractional q-difference derivative of order $1 < \alpha, \beta \leq 2$, $c,d \in C([0,T],R)$, $f,g \in C([0,T] \times R^2, R)$, and $T = q^{n_0}$.

In [9], Butt et al. considered the Ulam stability of the time-delay equation by the fractional q-Gronwall inequality:

$$\begin{cases} {}_{a}^{c}D_q^{\alpha}u(t) = F(t,u(t),u(\wp(\tau t))), & t \in T_a, \\ u(t) = \Phi(t), & t \in I_\tau, \end{cases}$$

where $u : T_{\tau a} \to R$, $F : T_{\tau a} \times R^2 \to R$, $\wp : T_a \to T_a$, $\Phi : I_\tau \to R$, $T_a = [a,\infty)_q$, $\tau = q^d$, $d \in N_0$, and ${}_{a}^{c}D_q^{\alpha}$ denotes the Caputo fractional q-difference operator of order $0 < \alpha < 1$.

In [2], Abbas et al. investigated the coupled implicit Caputo fractional q-difference systems:

$$\begin{cases} ({}^{c}D_q^{\alpha_1}u_1)(t) = f_1(t,u_1(t),u_2(t),({}^{c}D_q^{\alpha_1}u_1)(t)), & t \in [0,T], \\ ({}^{c}D_q^{\alpha_2}u_2)(t) = f_2(t,u_1(t),u_2(t),({}^{c}D_q^{\alpha_2}u_2)(t)), & t \in [0,T], \\ (u_1(0),u_2(0)) = (u_{01},u_{02}), \end{cases}$$

where ${}^{c}D_q^{\alpha_i}$ is the Caputo fractional q-difference derivative of order α_i, $0 < q < 1$, $T > 0$, $\alpha_i \in (0,1]$, $f_i : I \times R^3 \to R$ $(i = 1,2)$ are given continuous functions. It deals with the existence, and Ulam stability results for a coupled fractional initial value q-difference system on continuous intervals by fixed point theorem.

Based on the above work, in this paper, we discuss the existence and Ulam stability of the following Caputo coupled fractional delay q-difference system:

$$\begin{cases} {}_{a}^{c}D_q^{\alpha_1}(u_1(t) - \lambda_1 u_1(\Im(\tau t))) = f_1(t,u_1(t),u_2(t),{}_{a}I_q^{\alpha_1}u_1(\Im(\tau t))), & t \in T_a, \\ {}_{a}^{c}D_q^{\alpha_2}(u_2(t) - \lambda_2 u_2(\Im(\tau t))) = f_2(t,u_1(t),u_2(t),{}_{a}I_q^{\alpha_2}u_2(\Im(\tau t))), & t \in T_a, \\ u_1(t) = \Phi_1(t), & t \in I_\tau, \\ u_2(t) = \Phi_2(t), & t \in I_\tau, \end{cases} \tag{1.1}$$

where ${}_{a}^{c}D_q^{\alpha_i}$ is the Caputo fractional q-derivative operator of order $0 < \alpha_i \leq 1$ from the starting point $t = a$, $0 < q < 1$, the parameter $0 < \lambda_i < 1$, $\Im : T_a \to T_a$ is a operator, the continuous maps $u_i(t) : T_{\tau a} \to R$, $\Phi_i(t) : I_\tau \to R$, $f_i : T_a \times R^3 \to R$, where $i = 1,2$.

The time scale T_q is defined as $T_q = \{q^n : n \in \mathbb{Z}\} \cup \{0\}$, where $\mathbb{Z}$ is the set of integers. For $a = q^{n_0}$, $n_0 \in Z$, we denote $T_a = [a,\infty)_q = \{q^{-i}a : i = 0,1,2,\cdots\}$. Further define $T_{\tau a} = [\tau a,\infty)_q = \{\tau a, q^{-1}\tau a, q^{-2}\tau a, \cdots\}$, where $\tau = q^d \in T_q$, $d \in N_0$, $N_0 = \{0,1,2,\cdots\}$. $I_\tau = \{\tau a, q^{-1}\tau a, q^{-2}\tau a, \cdots, q^{-i}\tau a, \cdots, a\}$ is a bounded time scale as $i \leq d$ $(i \in N_0)$, and $I_\tau = \{a\}$ is the non-delay special case as $d = 0$.

This article is composed as follows. In Section 2, some basic definitions and lemmas are given to prove the main results. In Section 3, we discuss the existence and uniqueness of solutions to system (1.1). In Section 4, the Ulam-Hyers stability and Ulam-Hyers-Rassias stability are investigated by the q-Gronwall inequality. In Section 5, it demonstrates the existence and stability results of solution in the Generalized Banach space. Finally, examples are given to illustrate the conclusions in Section 6.

2. Preliminaries

In this section, we provide some definitions and lemmas for fractional q-Calculus which will be used in what follows. Let $N=\{1,2,3,\cdots\}$, and let $\mathbb{C}$ be the complex set.

Definition 2.1. [22] The q-difference factorial function in $\mathbb{C}$ is given by

$$(t-s)_q^{\alpha}=t^{\alpha}\prod_{k=0}^{\infty}\frac{t-q^ks}{t-q^{k+\alpha}s},\ 0\le s\le t,\ \alpha\in\mathbb{C}\backslash\{\pm n\},\ n\in N_0.$$

Definition 2.2. [22] For $\mid q\mid<1$, q-Gamma function is defined as

$$\Gamma_q(\alpha)=\frac{(1-q)^{1-\alpha}}{(1-q)_q^{1-\alpha}},\ \alpha\in\mathbb{C}\setminus\{-n:n\in N_0\}.$$

In particular,

$$\Gamma_q(\alpha+1)=[\alpha]_q\Gamma_q(\alpha)=\frac{1-q^{\alpha}}{1-q}\Gamma_q(\alpha),\ \Gamma_q(1)=1,\ \alpha>0.$$

Definition 2.3. [11] For $u:T_q\to R$, the q-integral of u is defined as

$$\int_0^t u(s)d_qs=(1-q)t\sum_{i=0}^{\infty}q^iu(tq^i),$$

for $0\le a\in T_q,\ \int_a^t u(s)d_qs=\int_0^t u(s)d_qs-\int_0^a u(s)d_qs.$

Definition 2.4. [11] The q-derivative of function $f:T_q\to R$ is defined as

$$(D_qf)(x)=\frac{f(x)-f(qx)}{(1-q)x},\quad (D_qf)(0)=\lim_{x\to 0}(D_qf)(x),$$

and q-derivative of higher order by $(D_q^0f)(x)=f(x),\ (D_q^nf)(x)=D_q(D_q^{n-1}f)(x),\ n\in N.$

Definition 2.5. [5, 17] Let $f:T_q\to R,\ \alpha\neq -k,\ k\in N$, then the fractional q-integral of order α is defined as

$${}_aI_q^{\alpha}f(t)=\frac{1}{\Gamma_q(\alpha)}\int_a^t(t-qs)_q^{\alpha-1}f(s)d_qs.$$

In particular, ${}_aI_q^{\alpha}(1)=\frac{(t-a)_q^{\alpha}}{\Gamma_q(\alpha+1)}$.

Definition 2.6. [8] Let $f:T_q\to R,\ \alpha>0$, and the $n=\lceil\alpha\rceil$ is a minimum integer greater than or equal to the α. Then the Caputo fractional q-derivative of order α of function f is defined as

$${}_a^cD_q^{\alpha}f(t)={}_aI_q^{n-\alpha}D_q^nf(t)=\frac{1}{\Gamma_q(n-\alpha)}\int_a^t(t-qs)_q^{n-\alpha-1}D_q^nf(s)d_qs.$$

Lemma 2.7. [5] *Let $\alpha \in R^{+}\backslash N$, and the following equation holds:*

$$({}_aI_q^{\alpha}{}^c_aD_q^{\alpha} f)(t) = f(t) - \sum_{k=0}^{n-1} \frac{(t-a)_q^k}{\Gamma_q(k+1)} (D_q^k f)(a).$$

If $\alpha \in (0,1]$, then $({}_aI_q^{\alpha}{}^c_aD_q^{\alpha} f)(t) = f(t) - f(a)$.

Lemma 2.8. [4] *Let $\alpha > 0$, $u(t)$ and $v(t)$ be nonnegative functions, and $w(t)$ be a nonnegative and nondecreasing function such that $w(t) \leq M$, $t \in T_a$, where M is a constant. If $u(t) \leq v(t) + w(t)_aI_q^{\alpha}u(t)$, then $u(t) \leq v(t) + \sum\limits_{k=1}^{\infty} (w(t)\Gamma_q(\alpha))^k {}_aI_q^{k\alpha} v(t)$.*

Corollary 2.9. [4] *Under the conditions of Lemma 2.8, further assume that $v(t)$ is a nondecreasing function for $t \in T_a$. Then, $u(t) \leq v(t)_qE_{\alpha}(w(t)\Gamma_q(\alpha), t-a)$, $t \in T_a$, where ${}_qE_{\alpha}(\lambda, t-a) = \sum\limits_{k=0}^{\infty} \lambda^k \frac{(t-a)_q^{k\alpha}}{\Gamma_q(k\alpha+1)}$ is the q-Mittag-Leffler function.*

Definition 2.10. [18] Any subset of sequences in l_0^{∞} is called uniformly Cauchy (or equi Cauchy) if, for every $\varepsilon > 0$, there exists an integer N such that, for any sequence $x = \{x(n)\}$ and $i, j > N$, $\mid x(i) - x(j) \mid < \varepsilon$. (Where l_0^{∞} is the set of all real sequences $x = \{x(t)\}_{t=0}^{\infty}$ from the starting point $t = 0$.)

Lemma 2.11. [18] *(Discrete Arzela-Ascoli Theorem) Any subset of sequences l_0^{∞}, which is bounded and uniformly Cauchy, is called relatively compact.*

Lemma 2.12. [23] *(Schauder Fixed Point Theorem) Let S be a non-empty, closed, and convex subset of a Banach space Ω, and $\mathrm{T} : S \to S$ a continuous mapping such that $\mathrm{T}(S)$ is a relatively compact subset of Ω. Then T has at least one fixed point in S. That is, there exists $x \in S$ such that $\mathrm{T}x = x$.*

Next, setting $X = \{l_{\infty}(T_{\tau a})$ of bounded functions (sequences) on $T_{\tau a}\}$ from the starting point $t = a$, we consider the Banach space $C := X \times X = \{(u_1, u_2) | u_i : T_{\tau a} \to R\}$ equipped with the norm $\|(u_1, u_2)\|_C = \|u_1\|_{\infty} + \|u_2\|_{\infty}$, where $\|u_i\|_{\infty} = \sup\limits_{t \in T_{\tau a}} |u_i(t)|$, and $\|(u_1, u_2) - (v_1, v_2)\|_C = \|(u_1 - v_1), (u_2 - v_2)\|_C$.

Now, let $\varepsilon > 0$, $\varphi : T_a \to R^{+}$ be a non-decreasing continuous function, and consider the following inequalities:

$$\begin{cases} \left|{}^c_aD_q^{\alpha_1}(u_1(t) - \lambda_1 u_1(\Im(\tau t))) - f_1(t, u_1(t), u_2(t), {}_aI_q^{\alpha_1} u_1(\Im(\tau t)))\right| \leq \varepsilon, \\ \left|{}^c_aD_q^{\alpha_2}(u_2(t) - \lambda_2 u_2(\Im(\tau t))) - f_2(t, u_1(t), u_2(t), {}_aI_q^{\alpha_2} u_2(\Im(\tau t)))\right| \leq \varepsilon, \end{cases} \quad t \in T_a, \tag{2.1}$$

$$\begin{cases} \left|{}^c_aD_q^{\alpha_1}(u_1(t) - \lambda_1 u_1(\Im(\tau t))) - f_1(t, u_!(t), u_2(t), {}_aI_q^{\alpha_1} u_1(\Im(\tau t)))\right| \leq \varphi(t), \\ \left|{}^c_aD_q^{\alpha_2}(u_2(t) - \lambda_2 u_2(\Im(\tau t))) - f_2(t, u_1(t), u_2(t), {}_aI_q^{\alpha_2} u_2(\Im(\tau t)))\right| \leq \varphi(t), \end{cases} \quad t \in T_a, \tag{2.2}$$

and

$$\begin{cases} \left|{}^c_aD_q^{\alpha_1}(u_1(t) - \lambda_1 u_1(\Im(\tau t))) - f_1(t, u_1(t), u_2(t), {}_aI_q^{\alpha_1} u_1(\Im(\tau t)))\right| \leq \varepsilon\varphi(t), \\ \left|{}^c_aD_q^{\alpha_2}(u_2(t) - \lambda_2 u_2(\Im(\tau t))) - f_2(t, u_1(t), u_2(t), {}_aI_q^{\alpha_2} u_2(\Im(\tau t)))\right| \leq \varepsilon\varphi(t), \end{cases} \quad t \in T_a. \tag{2.3}$$

The following definitions can be derived from [2, 3].

Definition 2.13. System (1.1) is said to be Ulam-Hyers stable if there exists a real number $c_{f_1,f_2} > 0$ such that, for all $\varepsilon > 0$ and for each solution $(u_1,u_2) \in C$ of (2.1), there exists a solution $(v_1,v_2) \in C$ of system (1.1) with

$$\|(u_1(t)-v_1(t),u_2(t)-v_2(t))\| \leq \varepsilon c_{f_1,f_2}, \quad t \in T_a.$$

Definition 2.14. System (1.1) is said to be Ulam-Hyers-Rassias stable with respect to φ if there exists a real number $c_{f_1,f_2,\varphi} > 0$ such that, for all $\varepsilon > 0$ and for each solution $(u_1,u_2) \in C$ of (2.3), there exists a solution $(v_1,v_2) \in C$ of system (1.1) with

$$\|(u_1(t)-v_1(t),u_2(t)-v_2(t))\| \leq \varepsilon c_{f_1,f_2,\varphi}\varphi(t), \quad t \in T_a.$$

Definition 2.15. System (1.1) is said to be Generalized Ulam-Hyers stable if there exists c_{f_1,f_2} : $C(R^+,R^+)$ with $c_{f_i}(0)=0,\ i=1,2$ such that, for all $\varepsilon > 0$ and for each solution $(u_1,u_2) \in C$ of (2.1), there exists a solution $(v_1,v_2) \in C$ of system (1.1) with

$$\|(u_1(t)-v_1(t),u_2(t)-v_2(t))\| \leq c_{f_1,f_2}(\varepsilon), \quad t \in T_a.$$

Definition 2.16. System (1.1) is Generalized Ulam-Hyers-Rassias stable with respect to φ if there exists a real number $c_{f_1,f_2,\varphi} > 0$ such that, for all $\varepsilon > 0$ and for each solution $(u_1,u_2) \in C$ of (2.2), there exists a solution $(v_1,v_2) \in C$ of system (1.1) with

$$\|(u_1(t)-v_1(t),u_2(t)-v_2(t))\| \leq c_{f_1,f_2,\varphi}\varphi(t), \quad t \in T_a.$$

3. Existence and Uniqueness of Solutions

In this section, we compute and derive the equivalent solution of coupled integral equations corresponding to coupled system (1.1), and further present required criteria for the existence of solutions.

The following hypotheses will be used:

(H_1) there exist constants $r_i, p_i, s_i \geq 0$ such that

$$\|f_i(t,u_1,v_1,w_1)-f_i(t,u_2,v_2,w_2)\| \leq r_i\|u_1-u_2\|+p_i\|v_1-v_2\|+s_i\|w_1-w_2\|,\ i=1,2;$$

(H_2) the map $\Im$ remains in the delay time scale I_τ.

(H_3) for the non-decreasing continuous function $\varphi : T_a \to R$ presented above, there exists $\beta_i \in R^+$ such that ${}_aI_q^{\alpha_i}\varphi(t) \leq \beta_i\varphi(t)$, $i=1,2$.

Remark 3.1. For the continuous function $f : T_q \to R$, ${}_aI_q^{\alpha_i}f(t) = \frac{1}{\Gamma_q(\alpha_i)}\int_a^t (t-qs)_q^{\alpha_i-1}f(s)d_qs \leq \frac{(t-a)_q^{\alpha_i}}{\Gamma_q(\alpha_i+1)}\|f\| \leq \frac{(T-a)_q^{\alpha_i}}{\Gamma_q(\alpha_i+1)}\|f\|$, $h_i \triangleq \frac{(T-a)_q^{\alpha_i}}{\Gamma_q(\alpha_i+1)}$, $i=1,2,\ t<T$.

Set $\hat{\lambda}=\max\{\lambda_1,\lambda_2\}$, $\hat{r}=\max\{r_1,r_2\}$, $\hat{p}=\max\{p_1,p_2\}$, $\hat{s}=\max\{s_1,s_2\}$, and $\hat{h}=\max\{h_1,h_2\}$.

Lemma 3.2. *The solution of system (1.1) is equivalent to the solution of the integral equation:*

$$u_i(t) = \begin{cases} \Phi_i(t), \quad t \in I_\tau, \\ \Phi_i(a)-\lambda_i\Phi_i(\Im(\tau a))+\lambda_i u_i(\Im(\tau t))+{}_aI_q^{\alpha_i}f_i(t,u_1(t),u_2(t),{}_aI_q^{\alpha_i}u_i(\Im(\tau t))),\ t \in T_a, \end{cases}$$

where $i=1,2$.

Proof. For $t \in I_\tau$, $u_i(t) = \Phi_i(t)$ is obvious. For $t \in T_a$, by applying ${}_aI_q^{\alpha_i}$ on both sides of problem (1.1), we have

$$\begin{cases} {}_aI_q^{\alpha_1}{}_a^cD_q^{\alpha_1}(u_1(t) - \lambda_1 u_1(\Im(\tau t))) = {}_aI_q^{\alpha_1} f_1(t, u_1(t), u_2(t), {}_aI_q^{\alpha_1} u_1(\Im(\tau t))), \\ {}_aI_q^{\alpha_2}{}_a^cD_q^{\alpha_2}(u_2(t) - \lambda_2 u_2(\Im(\tau t))) = {}_aI_q^{\alpha_2} f_2(t, u_1(t), u_2(t), {}_aI_q^{\alpha_2} u_2(\Im(\tau t))), \end{cases} \quad t \in T_a,$$

which leads to

$$u_i(t) = \Phi_i(a) + \lambda_i u_i(\Im(\tau t)) - \lambda_i \Phi_i(\Im(\tau a)) + {}_aI_q^{\alpha_i} f_i(t, u_1(t), u_2(t), {}_aI_q^{\alpha_i} u_i(\Im(\tau t))), i = 1, 2.$$

□

Theorem 3.3. *If $(H_1) - (H_2)$ and the assumption $\hat{\lambda} + \hat{h}(\hat{r} + \hat{p} + \hat{s}\hat{h}) < 1$ hold, then the coupled system (1.1) under finite time-delay scale has at least one solution.*

Proof. The proof is divided into the following step.

Step 1. Define the operators $A_i : X \to X$ by

$$(A_1u)(t) =$$
$$\begin{cases} \Phi_1(t), & t \in I_\tau, \\ \Phi_1(a) - \lambda_1 \Phi_1(\Im(\tau a)) + \lambda_1 u_1(\Im(\tau t)) + {}_aI_q^{\alpha_1} f_1(t, u_1(t), u_2(t), {}_aI_q^{\alpha_1} u_1(\Im(\tau t))), & t \in T_a, \end{cases}$$

and

$$(A_2u)(t) =$$
$$\begin{cases} \Phi_2(t), & t \in I_\tau, \\ \Phi_2(a) - \lambda_2 \Phi_2(\Im(\tau a)) + \lambda_2 u_2(\Im(\tau t)) + {}_aI_q^{\alpha_1} f_2(t, u_1(t), u_2(t), {}_aI_q^{\alpha_1} u_2(\Im(\tau t))), & t \in T_a, \end{cases}$$

and consider the continuous operator $A : C \to C$ defined by $(A(u_1, u_2))(t) = ((A_1u)(t), (A_2u)(t))$. Let us choose

$$\Lambda \geq 2\frac{(1 - \hat{\lambda})\hat{\rho} + \hat{h}\hat{\theta}}{1 - \hat{\lambda} - \hat{h}(\hat{r} + \hat{p} + \hat{s}\hat{h})},$$

where $\rho_i = \sup_{t \in I_\tau} \Phi_i(t), \hat{\rho} = \max\{\rho_1, \rho_2\}$, $\theta_i = \max_{t<T}\{|f_i(t, 0, 0, 0)|\}$, and $\hat{\theta} = \max\{\theta_1, \theta_2\}$, and set $\Sigma = \{(u_1, u_2) \in C : \| (u_1, u_2) \|_C \leq \Lambda\}$ which is nonempty, closed, and convex. For $t \in I_\tau$, $\|A(u_1, u_2)(t)\| = \|A_1u(t)\| + \|A_2u(t)\| \leq \rho_1 + \rho_2 \leq 2\hat{\rho} \leq \Lambda$. For $t \in T_a$, we have

$$\begin{aligned} \|A_1(u)\| \leq\ & \rho_1 - \lambda_1\rho_1 + \lambda_1\Lambda + \tfrac{1}{\Gamma_q(\alpha_1)} \textstyle\int_a^t (t - qs)_q^{\alpha_1 - 1} \left|f_1(t, u_1(t), u_2(t), {}_aI_q^{\alpha_1} u_1(\Im(\tau t)))\right| d_qs \\ \leq\ & (1 - \lambda_1)\rho_1 + \lambda_1\Lambda + \tfrac{1}{\Gamma_q(\alpha_1)} \textstyle\int_a^t (t - qs)_q^{\alpha_1 - 1} (\left|f_1(t, u_1(t), u_2(t), {}_aI_q^{\alpha_1} u_1(\Im(\tau t))) - \right. \\ & \left. f_1(t, 0, 0, 0)\right| + |f_1(t, 0, 0, 0)|) d_qs \\ \leq\ & (1 - \lambda_1)\rho_1 + \lambda_1\Lambda + \tfrac{1}{\Gamma_q(\alpha_1)} \textstyle\int_a^t (t - qs)_q^{\alpha_1 - 1} (\theta_1 + r_1 |u_1(t)| + p_1 |u_2(t)| + \\ & s_1h_1 |u_1(\Im(\tau t))|) d_qs \\ \leq\ & (1 - \lambda_1)\rho_1 + \lambda_1\Lambda + h_1(\theta_1 + r_1\Lambda + p_1\Lambda + s_1h_1\Lambda) \ \leq\ \Lambda, \end{aligned}$$

and

$$\begin{aligned}
\|A_2(u)\| \leq\ & \rho_2 - \lambda_2\rho_2 + \lambda_2\Lambda + \tfrac{1}{\Gamma_q(\alpha_2)} \textstyle\int_a^t (t-qs)_q^{\alpha_2-1} \left| f_2(t,u_1(t),u_2(t),{}_aI_q^{\alpha_2}u_2(\Im(\tau t)))\right| d_q s \\
\leq\ & (1-\lambda_2)\rho_2 + \lambda_2\Lambda + \tfrac{1}{\Gamma_q(\alpha_2)} \textstyle\int_a^t (t-qs)_q^{\alpha_2-1} (\left| f_2(t,u_1(t),u_2(t),{}_aI_q^{\alpha_2}u_2(\Im(\tau t))) - \right. \\
& \left. f_2(t,0,0,0)\right| + |f_2(t,0,0,0)|) d_q s \\
\leq\ & (1-\lambda_2)\rho_2 + \lambda_2\Lambda + \tfrac{1}{\Gamma_q(\alpha_2)} \textstyle\int_a^t (t-qs)_q^{\alpha_2-1} (\theta_2 + r_2|u_1(t)| + p_2|u_2(t)| + \\
& s_2h_2|u_2(\Im(\tau t))|)\, d_q s \\
\leq\ & (1-\lambda_2)\rho_2 + \lambda_2\Lambda + h_2(\theta_2 + r_2\Lambda + p_2\Lambda + s_2h_2\Lambda) \ \leq\ \Lambda.
\end{aligned}$$

Hence,

$$\|A(u_1,u_2)\| = \|A_1(u)\| + \|A_2(u)\| \leq 2\left[(1-\hat\lambda)\hat\rho + \hat\lambda\Lambda + \hat h(\hat\theta + \hat r\Lambda + \hat p\Lambda + \hat s\hat h\Lambda)\right] \leq \Lambda.$$

which means that A maps Σ into Σ.

Step 2. A is continuous.

Let $\{u_{1n}\},\{u_{2n}\} \in C$ $(n \in N)$ be two sequences such that $(u_{1n},u_{1n}) \to (u_1,u_2)$ $(n\to\infty)$ in Σ. This implies that $u_{1n}(t) \to u_1(t), u_{2n}(t) \to u_2(t), u_{1n}(\Im(\tau t)) \to u_1(\Im(\tau t))$, and $u_{2n}(\Im(\tau t)) \to u_2(\Im(\tau t))$. Let

$$f_{1n}(t) \triangleq f_1(t,u_{1n}(t),u_{2n}(t),{}_aI_q^{\alpha_1}u_{1n}(\Im(\tau t))),$$

$$f_{2n}(t) \triangleq f_2(t,u_{1n}(t),u_{2n}(t),{}_aI_q^{\alpha_2}u_{2n}(\Im(\tau t))),$$

$$f_1(t) \triangleq f_1(t,u_1(t),u_2(t),{}_aI_q^{\alpha_1}u_1(\Im(\tau t))),$$

and

$$f_2(t) \triangleq f_2(t,u_1(t),u_2(t),{}_aI_q^{\alpha_2}u_2(\Im(\tau t))).$$

By (H_1), we have $\|f_{1n}(t) - f_1(t)\| \to 0$ and $\|f_{2n}(t) - f_2(t)\| \to 0$ as $n\to\infty$. Due to the Lebesgue dominated convergence theorem, we obtain ${}_aI_q^{\alpha_i}\|f_{in}(t) - f_i(t)\| \to 0$, $i = 1,2$, as $n\to\infty$. So, for $t \in I_\tau$, since $\Phi_i(t)$ is continuous, we have that A is continuous. For each $t \in T_a$, we have

$$\|A_1(u_{1n}) - A_1(u_1)\| \leq \lambda_1\|u_{1n}(\Im(\tau t)) - u_1(\Im(\tau t))\| + {}_aI_q^{\alpha_1}\|f_{1n}(t) - f_1(t)\| \to 0,\ n\to\infty,$$

and

$$\|A_2(u_{2n}) - A_2(u_2)\| \leq \lambda_2\|u_{2n}(\Im(\tau t)) - u_2(\Im(\tau t))\| + {}_aI_q^{\alpha_2}\|f_{2n}(t) - f_2(t)\| \to 0,\ n\to\infty.$$

Hence

$$\|A(u_{1n},u_{2n}) - A(u_1,u_2)\| = \|A_1(u_{1n}) - A_1(u_1)\| + \|A_2(u_{2n}) - A_2(u_2)\| \to 0,\ n\to\infty.$$

Step 3. Show that A is a relatively compact operator.

It is clear that $A(\Sigma)$ is bounded. Now we have to show that A is uniformly Cauchy. For $t \in I_\tau$, it is true. For $t_1, t_2 \in T_a$ such that $t_1 > t_2 > a$, let $(u_1, u_2) \in \Sigma$. It follows that

$$\begin{aligned}
& |(A_1u)(t_1) - (A_1u)(t_2)| \\
= \ & \lambda_1 |u_1(\Im(\tau t_1)) - u_1(\Im(\tau t_2))| + \left|\frac{1}{\Gamma_q(\alpha_1)}\int_a^{t_1}(t_1-qs)_q^{\alpha_1-1}f_1(s)d_qs\right| \\
& -\left|\frac{1}{\Gamma_q(\alpha_1)}\int_a^{t_2}(t_2-qs)_q^{\alpha_1-1}f_1(s)d_qs\right| \\
\leq \ & \lambda_1 |u_1(\Im(\tau t_1)) - u_1(\Im(\tau t_2))| + \frac{1}{\Gamma_q(\alpha_1)}\int_{t_2}^{t_1}\left|(t_1-qs)_q^{\alpha_1-1}f_1(s)\right|d_qs \\
& +\frac{1}{\Gamma_q(\alpha_1)}\int_a^{t_2}\left|[(t_1-qs)_q^{\alpha_1-1}-(t_2-qs)_q^{\alpha_1-1}]f_1(s)\right|d_qs,
\end{aligned}$$

and

$$\begin{aligned}
& |(A_2u)(t_1) - (A_2u)(t_2)| \\
\leq \ & \lambda_2 |u_2(\Im(\tau t_1)) - u_2\Im(\tau t_2))| + \frac{1}{\Gamma_q(\alpha_2)}\int_{t_2}^{t_1}\left|(t_1-qs)_q^{\alpha_2-1}f_2(s)\right|d_qs \\
& +\frac{1}{\Gamma_q(\alpha_2)}\int_a^{t_2}\left|[(t_1-qs)_q^{\alpha_2-1}-(t_2-qs)_q^{\alpha_2-1}]f_2(s)\right|d_qs.
\end{aligned}$$

By (H_2), $u_1(\Im) = \Phi_1(t), u_2(\Im) = \Phi_2(t)$, $t \in I_\tau$. As $t_1 \to t_2$, we have $|u_1(\Im(\tau t_1)) - u_1(\Im(\tau t_2))|$ $= |\Phi_1(t_1) - \Phi_1(t_2)|_{I_\tau} \to 0$ and $|u_2(\Im(\tau t_1)) - u_2(\Im(\tau t_2))| = |\Phi_2(t_1) - \Phi_2(t_2)|_{I_\tau} \to 0$. It follows that

$$|(A_1u)(t_1) - (A_1u)(t_2)| \to 0, \quad |(A_2u)(t_1) - (A_2u)(t_2)| \to 0,$$

and then

$$|A(u_1,u_2)(t_1) - A(u_1,u_2)(t_2)| \to 0, \quad t_1 \to t_2.$$

In view of Lemma 2.12, the Schauder fixed point theorem, there exits at least one fixed point of A in Σ. □

Theorem 3.4. *Suppose that $(H_1)-(H_2)$ hold. If $(u_1(t), u_2(t))$ and $(v_1(t), v_2(t))$ both satisfy the solutions of problem (1.1) under the boundary conditions, then $(u_1(t), u_2(t)) = (v_1(t), v_2(t))$.*

Proof. Letting $z(t) = (u_1(t), u_2(t)) - (v_1(t), v_2(t))$, and $z_i(t) = u_i(t) - v_i(t)$, $i = 1, 2$, we have $z(t) = z_1(t) + z_2(t)$. We prove that $z(t) = 0$, for $t \in I_\tau$. For $t \in T_a$, we have

$$z_i(t) = \lambda_i z_i(\Im(\tau t)) + {}_aI_q^{\alpha_i} f_i(t, u_1(t), u_2(t), {}_aI_q^{\alpha_i} u_i(\Im(\tau t))) - {}_aI_q^{\alpha_i} f_i(t, v_1(t), v_2(t), {}_aI_q^{\alpha_i} v_i(\Im(\tau t))).$$

For $t \in I_{\tau^{-1}} = \{a, q^{-1}a, q^{-2}a, \cdots, \tau^{-1}a\}$, by (H_2), $z_i(\Im(\tau t)) = 0$. Letting $\bar{z}(t) = \max\{\|z_1\|, \|z_2\|\}$, we have

$$\begin{aligned}
\|z_i(t)\| &= {}_aI_q^{\alpha_i} f_i(t, u_1(t), u_2(t), {}_aI_q^{\alpha_i} u_i(\Im(\tau t))) - {}_aI_q^{\alpha_i} f_i(t, v_1(t), v_2(t), {}_aI_q^{\alpha_i} v_i(\Im(\tau t))) \\
&\leq {}_aI_q^{\alpha_i} \left\| f_i(t, u_1(t), u_2(t), {}_aI_q^{\alpha_i} u_i(\Im(\tau t))) - f_i(t, v_1(t), v_2(t), {}_aI_q^{\alpha_i} v_i(\Im(\tau t)))\right\| \\
&\leq {}_aI_q^{\alpha_i}(r_i\|z_1(t)\| + p_i\|z_2(t)\| + s_ih_i\|z_i(\Im(\tau t))\|) \\
&\leq {}_aI_q^{\alpha_i}(\hat{r}\|z_1(t)\| + \hat{p}\|z_2(t)\|).
\end{aligned}$$

It follows that $\bar{z}(t) \leq (\hat{r} + \hat{p}){}_aI_q^{\alpha_i}\bar{z}(t)$. From Corollary 2.9, $z_i(t) \leq \bar{z}(t) \leq 0 \cdot {}_qE_{\alpha_i}(\hat{r}\Gamma_q(\alpha_i), t-a)$. Hence, $z(t) = 0$ for $t \in I_{\tau^{-1}}$. For $t \in [\tau^{-1}a, \infty)_q$, letting $\hat{z}_i(t) = \sup\limits_{\theta \in I_\tau} \|z_i(\Im(\tau\theta))\|$ and $\hat{z}(t) =$

$\max\{\|\hat{z}_1\|,\|\hat{z}_2\|\}$, we see from (H_1) that

$$
\begin{aligned}
&\|z_i(t)\| \\
&\le \lambda_i\|z_i(\Im(\tau t))\| + {}_aI_q^{\alpha_i}\left\|f_i(t,u_1(t),u_2(t),{}_aI_q^{\alpha_i}u_i(\Im(\tau t))) - f_i(t,v_1(t),v_2(t),{}_aI_q^{\alpha_i}v_i(\Im(\tau t)))\right\| \\
&\le \hat{\lambda}\hat{z}_i(t) + {}_aI_q^{\alpha_i}(\hat{r}\|z_1(t)\| + \hat{p}\|z_2(t)\| + \hat{s}\hat{h}\|z_i(\Im(\tau t))\|),
\end{aligned}
$$

which implies that $\hat{z}(t) \le \hat{\lambda}\hat{z}(t) + (\hat{r}+\hat{p}+\hat{s}\hat{h}){}_aI_q^{\alpha_i}\hat{z}(t)$, and hence $\hat{z}(t) \le \frac{\hat{r}+\hat{p}+\hat{s}\hat{h}}{1-\hat{\lambda}}{}_aI_q^{\alpha_i}\hat{z}(t)$. From Corollary 2.9, $z_i(t) \le \hat{z}(t) \le 0\cdot {}_qE_{\alpha_i}(\frac{\hat{r}+\hat{p}+\hat{s}\hat{h}}{1-\hat{\lambda}}\Gamma_q(\alpha_i), t-a)$, so there exists $\hat{z}(t)=0$, $t\in[\tau^{-1}a,\infty)_q$. From the above, $(u_1(t),u_2(t)) = (v_1(t),v_2(t))$, $t\in T_a$. □

4. Ulam Stability

In this section, we deal with the Ulam-Hyers stability and Ulam-Hyers-Rassias stability of coupled system (1.1).

Lemma 4.1. *If the coupled function $(u_1(t),u_2(t))$ defined on the $T_{\tau a}$ is the solution to (2.1), then the following inequalities hold:*

$$
\begin{cases}
\left|u_1(t)-u_1(a)-\lambda_1u_1(\Im(\tau t))+\lambda_1u_1(\Im(\tau a)) - {}_aI_q^{\alpha_1}f_1(t,u_1(t),u_2(t),{}_aI_q^{\alpha_1}u_1(\Im(\tau t)))\right| \le \frac{\varepsilon(t-a)_q^{\alpha_1}}{\Gamma_q(\alpha_1+1)}, \\
\left|u_2(t)-u_2(a)-\lambda_2u_2(\Im(\tau t))+\lambda_2u_2(\Im(\tau a)) - {}_aI_q^{\alpha_2}f_2(t,u_1(t),u_2(t),{}_aI_q^{\alpha_2}u_2(\Im(\tau t)))\right| \le \frac{\varepsilon(t-a)_q^{\alpha_2}}{\Gamma_q(\alpha_2+1)}.
\end{cases}
$$

Proof. Notice that $(u_1(t),u_2(t))$ is a solution to (2.1) if and only if there exists a function $h(t)$ such that $|h(t)|\le\varepsilon$ and

$$
\begin{cases}
{}_a^cD_q^{\alpha_1}(u_1(t)-\lambda_1u_1(\Im(\tau t))) - f_1(t,u_1(t),u_2(t),{}_aI_q^{\alpha_1}u_1(\Im(\tau t))) = h(t), \\
{}_a^cD_q^{\alpha_2}(u_2(t)-\lambda_2u_2(\Im(\tau t))) - f_2(t,u_1(t),u_2(t),{}_aI_q^{\alpha_2}u_2(\Im(\tau t))) = h(t).
\end{cases} \tag{4.1}
$$

By applying the operator ${}_aI_q^{\alpha_i}$ to both sides of (4.1), we have

$$
\begin{cases}
u_1(t)-u_1(a)-\lambda_1u_1(\Im(\tau t))+\lambda_1u_1(\Im(\tau a)) - {}_aI_q^{\alpha_1}f_1(t,u_1(t),u_2(t),{}_aI_q^{\alpha_1}u_1(\Im(\tau t))) = {}_aI_q^{\alpha_1}h(t), \\
u_2(t)-u_2(a)-\lambda_2u_2(\Im(\tau t))+\lambda_2u_2(\Im(\tau a)) - {}_aI_q^{\alpha_2}f_2(t,u_1(t),u_2(t),{}_aI_q^{\alpha_2}u_2(\Im(\tau t))) = {}_aI_q^{\alpha_2}h(t),
\end{cases}
$$

which implies that

$$
\begin{aligned}
&\left|u_1(t)-u_1(a)-\lambda_1u_1(\Im(\tau t))+\lambda_1u_1(\Im(\tau a)) - {}_aI_q^{\alpha_1}f_1(t,u_1(t),v_1(t),{}_aI_q^{\alpha_1}u_1(\Im(\tau t)))\right| \\
&\le {}_aI_q^{\alpha_1}|h(t)| \le \varepsilon\,{}_aI_q^{\alpha_1}(1) = \frac{\varepsilon(t-a)_q^{\alpha_1}}{\Gamma_q(\alpha_1+1)},
\end{aligned}
$$

and

$$
\begin{aligned}
&\left|u_2(t)-u_2(a)-\lambda_2u_2(\Im(\tau t))+\lambda_2u_2(\Im(\tau a)) - {}_aI_q^{\alpha_2}f_2(t,u_1(t),u_2(t),{}_aI_q^{\alpha_2}u_2(\Im(\tau t)))\right| \\
&\le {}_aI_q^{\alpha_2}|h(t)| \le \varepsilon\,{}_aI_q^{\alpha_2}(1) = \frac{\varepsilon(t-a)_q^{\alpha_2}}{\Gamma_q(\alpha_2+1)}.
\end{aligned}
$$

□

Theorem 4.2. *If $(H_1)-(H_2)$ hold, then system (1.1) is Ulam-Hyers stable.*

Proof. Assume that $(u_1(t),u_2(t))$ satisfy (2.1), $(v_1(t),v_2(t))$ is the solution of system (1.1) and $(u_1(t),u_2(t))=(v_1(t),v_2(t))$ for $t\in I_\tau$. We conclude that

$$v_1(t)=\begin{cases}u_1(t), & t\in I_\tau,\\ u_1(a)-\lambda_1u_1(\Im(\tau a))+\lambda_1v_1(\Im(\tau t))+{}_aI_q^{\alpha_1}f_1(t,v_1(t),v_2(t),{}_aI_q^{\alpha_1}v_1(\Im(\tau t))), & t\in T_a.\end{cases}$$

and

$$v_2(t)=\begin{cases}u_2(t), & t\in I_\tau,\\ u_2(a)-\lambda_2u_2(\Im(\tau a))+\lambda_2v_2(\Im(\tau t))+{}_aI_q^{\alpha_2}f_2(t,v_1(t),v_2(t),{}_aI_q^{\alpha_2}v_2(\Im(\tau t))), & t\in T_a.\end{cases}$$

For $t\in I_\tau$, it is apparent that $\|(u_1(t)-v_1(t)),(u_2(t)-v_2(t))\|=0$. For $t\in I_{\tau^{-1}}=\{a,q^{-1}a,q^{-2}a,\cdots,\tau^{-1}a\}$, we have $u_1(\Im(\tau t))-v_1(\Im(\tau t))=0$, $u_2(\Im(\tau t))-v_2(\Im(\tau t))=0$. By (H_1) and Lemma 4.1, we have

$$\begin{aligned}
&\|v_1(t)-u_1(t)\|\\
=\;&\left\|u_1(t)-u_1(a)+\lambda_1u_1(\Im(\tau a))-\lambda_1v_1(\Im(\tau t))-{}_aI_q^{\alpha_1}f_1(t,v_1(t),v_2(t),{}_aI_q^{\alpha_1}v_1(\Im(\tau t)))\right\|\\
\le\;&\left\|u_1(t)-u_1(a)+\lambda_1u_1(\Im(\tau a))-\lambda_1v_1(\Im(\tau t))-{}_aI_q^{\alpha_1}f_1(t,u_1(t),v_2(t),{}_aI_q^{\alpha_1}u_1(\Im(\tau t)))\right\|\\
&+\left\|{}_aI_q^{\alpha_1}f_1(t,u_1(t),v_2(t),{}_aI_q^{\alpha_1}u_1(\Im(\tau t)))-{}_aI_q^{\alpha_1}f_1(t,v_1(t),v_2(t),{}_aI_q^{\alpha_1}v_1(\Im(\tau t)))\right\|\\
\le\;&\left\|u_1(t)-u_1(a)+\lambda_1u_1(\Im(\tau a))-\lambda_1v_1(\Im(\tau t))-{}_aI_q^{\alpha_1}f_1(t,u_1(t),v_2(t),{}_aI_q^{\alpha_1}u_1(\Im(\tau t)))\right\|\\
&+{}_aI_q^{\alpha_1}\left\|f_1(t,u_1(t),v_2(t),{}_aI_q^{\alpha_1}u_1(\Im(\tau t)))-f_1(t,v_1(t),v_2(t),{}_aI_q^{\alpha_1}v_1(\Im(\tau t)))\right\|\\
\le\;&\tfrac{\varepsilon(t-a)_q^{\alpha_1}}{\Gamma_q(\alpha_1+1)}+r_1{}_aI_q^{\alpha_1}\|(u_1(t)-v_1(t))\|.
\end{aligned}$$

Similarly, one has $\|v_2(t)-u_2(t)\|\le\varepsilon\frac{(t-a)_q^{\alpha_2}}{\Gamma_q(\alpha_2+1)}+p_2{}_aI_q^{\alpha_2}\|u_2(t)-v_2(t)\|$. Since $\frac{\varepsilon(t-a)_q^{\alpha_i}}{\Gamma_q(\alpha_i+1)}$ is nonnegative and nondecreasing function about t for each $t\in I_{\tau^{-1}}$, Corollary 2.9 yields that

$$\|u_1(t)-v_1(t)\|\le\left[h_1\cdot{}_qE_{\alpha_1}(r_1\Gamma_q(\alpha_1),T-a)\right]\triangleq c_{f_1}$$

and $\|u_2(t)-v_2(t)\|\le\left[h_2\cdot{}_qE_{\alpha_2}(p_2\Gamma_q(\alpha_2),T-a)\right]\triangleq c_{f_2}$. It follows that

$$\begin{aligned}
&\|(u_1(t)-v_1(t),u_2(t)-v_2(t)\|\\
&=\|u_1(t)-v_1(t)\|+\|u_2(t)-v_2(t)\|\\
&\le\varepsilon\left[h_1\cdot{}_qE_{\alpha_1}(r_1\Gamma_q(\alpha_1),T-a)+h_2\cdot{}_qE_{\alpha_2}(p_2\Gamma_q(\alpha_2),T-a)\right]\triangleq\varepsilon c_{f_1f_2},\quad\forall t\in I_{\tau^{-1}}.
\end{aligned}$$

For $[\tau^{-1}a,\infty)_q$, let $\hat z_1(t)=\sup_{\theta\in I_\tau}\|(u_1(\Im(\theta t))-v_1(\Im(\theta t))\|$ and $\hat z_2(t)=\sup_{\theta\in I_\tau}\|(u_2(\Im(\theta t))-v_2(\Im(\theta t))\|$. Observe

$$\begin{aligned}
&\|u_1(t)-v_1(t)\|\\
\le\;&\left\|u_1(t)-u_1(a)+\lambda_1u_1(\Im(\tau a))-\lambda_1v_1(\Im(\tau t))-{}_aI_q^{\alpha_1}f_1(t,u_1(t),v_2(t),{}_aI_q^{\alpha_1}u_1(\Im(\tau t)))\right\|\\
&+{}_aI_q^{\alpha_1}\left\|f_1(t,u_1(t),v_2(t),{}_aI_q^{\alpha_1}u_1(\Im(\tau t)))-f_1(t,v_1(t),v_2(t),{}_aI_q^{\alpha_1}v_1(\Im(\tau t)))\right\|\\
\le\;&\tfrac{\varepsilon(t-a)_q^{\alpha_1}}{\Gamma_q(\alpha_1+1)}+{}_aI_q^{\alpha_1}\left(r_1\|(u_1(t)-v_1(t))\|+s_1h_1\|u_1(\Im(\tau t))-v_1(\Im(\tau t))\|\right)\\
\le\;&\tfrac{\varepsilon(t-a)_q^{\alpha_1}}{\Gamma_q(\alpha_1+1)}+(r_1+s_1h_1){}_aI_q^{\alpha_1}\hat z_1(t),
\end{aligned}$$

and

$$\|u_2(t)-v_2(t)\| \le \frac{\varepsilon(t-a)_q^{\alpha_2}}{\Gamma_q(\alpha_2+1)} + (p_2+s_2h_2)_a I_q^{\alpha_2}\hat{z}_2(t).$$

From Corollary 2.9, we have

$$\|u_1(t)-v_1(t)\| \le \hat{z}_1(t) \le \varepsilon\left[h_1\cdot {}_qE_{\alpha_1}((r_1+s_1h_1)\Gamma_q(\alpha_1), T-a)\right] \triangleq \varepsilon c_{f_1},$$

and

$$\|u_2(t)-v_2(t)\| \le \hat{z}_2(t) \le \varepsilon\left[h_2\cdot {}_qE_{\alpha_2}((p_2+s_2h_2)\Gamma_q(\alpha_2), T-a)\right] \triangleq \varepsilon c_{f_2}.$$

Thus

$$\begin{aligned}&\|(u_1(t)-v_1(t), u_2(t)-v_2(t)\| = \|u_1(t)-v_1(t)\| + \|u_2(t)-v_2(t)\|\\ &\le \varepsilon\left[h_1\cdot {}_qE_{\alpha_1}((r_1+s_1h_1)\Gamma_q(\alpha_1), T-a) + h_2\cdot {}_qE_{\alpha_2}((p_2+s_2h_2)\Gamma_q(\alpha_2), T-a)\right] \triangleq \varepsilon c_{f_1f_2}.\end{aligned}$$

Hence the proof is completed by Definition 2.13. □

Corollary 4.3. *Under hypothesis $(H_1)-(H_2)$, system (1.1) is Generalized Ulam-Hyers stable.*

Proof. From Definition 2.15 and Theorem 4.2, we can conclude the desired conclusion immediately. □

Theorem 4.4. *If hypothesis $(H_1)-(H_3)$ hold, then system (1.1) is Ulam-Hyers-Rassias stable about φ.*

Proof. Let $(u_1(t), u_2(t))$ be the solution to (2.3). Then

$$\begin{cases} -\varphi(t) \le {}_a^cD_q^{\alpha_1}(u_1(t)-\lambda_1u_1(\Im(\tau t))) - f_1(t, u_1(t), u_2(t), {}_aI_q^{\alpha_1}u_1(\Im(\tau t))) \le \varphi(t),\\ -\varphi(t) \le {}_a^cD_q^{\alpha_2}(u_2(t)-\lambda_2u_2(\Im(\tau t))) - f_2(t, u_1(t), u_2(t), {}_aI_q^{\alpha_2}u_2(\Im(\tau t))) \le \varphi(t),\end{cases}$$

which implies that

$$\begin{cases} \left|u_1(t)-u_1(a)-\lambda_1u_1(\Im(\tau t))+\lambda_1u_1(\Im(\tau a)) - {}_aI_q^{\alpha_1}f_1(t,u_1(t),u_2(t),{}_aI_q^{\alpha_1}u_1(\Im(\tau t)))\right| \le {}_aI_q^{\alpha_1}\varphi(t),\\ \left|u_2(t)-u_2(a)-\lambda_2u_2(\Im(\tau t))+\lambda_2u_2(\Im(\tau a)) - {}_aI_q^{\alpha_2}f_2(t,u_1(t),u_2(t),{}_aI_q^{\alpha_2}u_2(\Im(\tau t)))\right| \le {}_aI_q^{\alpha_2}\varphi(t),\end{cases}$$

In view of (H_3), we have

$$\begin{cases} \left|u_1(t)-u_1(a)-\lambda_1u_1(\Im(\tau t))+\lambda_1u_1(\Im(\tau a)) - {}_aI_q^{\alpha_1}f_1(t,u_1(t),u_2(t),{}_aI_q^{\alpha_1}u_1(\Im(\tau t)))\right| \le \varepsilon\beta_1\varphi(t),\\ \left|u_2(t)-u_2(a)-\lambda_2u_2(\Im(\tau t))+\lambda_2u_2(\Im(\tau a)) - {}_aI_q^{\alpha_2}f_2(t,u_1(t),u_2(t),{}_aI_q^{\alpha_2}u_2(\Im(\tau t)))\right| \le \varepsilon\beta_2\varphi(t),\end{cases}$$

For $t\in I_\tau$, $\|(u_1(t)-v_1(t), u_2(t)-v_2(t)\| = 0$. For $t\in I_{\tau^{-1}}$, $u_1(\Im(\tau t))-v_1(\Im(\tau t)) = 0$, $u_2(\Im(\tau t)) - v_2(\Im(\tau t)) = 0$, $\|u_1(t)-v_1(t)\| \le \varepsilon\beta_1\varphi(t) + r_{1a}I_q^{\alpha_1}\|u_1(t)-u(t)\|$, and $\|u_2(t)-v_2(t)\| \le \varepsilon\beta_2\varphi(t) + p_{2a}I_q^{\alpha_2}\|u_2(t)-v_2(t)\|$. Using Corollary 2.9, we have $\|u_1(t)-v_1(t)\| \le \varepsilon\beta_1\varphi(t)\cdot {}_qE_{\alpha_1}(r_1\Gamma_q(\alpha_1), T-a) \triangleq \varepsilon c_{f_1,\varphi}\varphi(t)$, $\|u_2(t)-v_2(t)\| \le \varepsilon\beta_2\varphi(t)\cdot {}_qE_{\alpha_2}(p_2\Gamma_q(\alpha_2), T-a) \triangleq \varepsilon c_{f_2,\varphi}\varphi(t)$, and

$$\begin{aligned}&\|(u_1(t)-v_1(t), u_2(t)-v_2(t)\|\\ &\le \varepsilon\varphi(t)\left[\beta_1\cdot {}_qE_{\alpha_1}(r_1\Gamma_q(\alpha_1), T-a) + \beta_2\cdot {}_qE_{\alpha_2}(p_2\Gamma_q(\alpha_2), T-a)\right]\\ &\triangleq \varepsilon c_{f_1f_2,\varphi}\varphi(t).\end{aligned}$$

For $t \in [\tau^{-1}a, \infty)_q$, we have

$$\begin{aligned}\|u_1(t)-v_1(t)\| \le \hat{z}_1(t) &\le \varepsilon\beta_1\varphi(t)+(r_1+s_1h_1)_a I_q^{\alpha_1}\hat{z}_1(t)\\ &\le \varepsilon\varphi(t)\left[\beta_1\cdot {}_qE_{\alpha_1}((r_1+s_1h_1)\Gamma_q(\alpha_1),T-a)\right] \triangleq \varepsilon c_{f_1,\varphi}\varphi(t),\end{aligned}$$

and

$$\begin{aligned}\|u_2(t)-v_2(t)\| \le \hat{z}_2(t) &\le \varepsilon\beta_2\varphi(t)+(p_2+s_2h_2)_a I_q^{\alpha_2}\hat{z}_v(t)\\ &\le \varepsilon\varphi(t)\left[\beta_2\cdot {}_qE_{\alpha_2}((p_2+s_2h_2)\Gamma_q(\alpha_2),T-a)\right] \triangleq \varepsilon c_{f_2,\varphi}\varphi(t).\end{aligned}$$

Hence, we have

$$\begin{aligned}&\|(u_1(t)-v_1(t),u_2(t)-v_2(t)\|\\ \le\ & \varepsilon\varphi(t)\left[\beta_1\cdot {}_qE_{\alpha_1}((r_1+s_1h_1)\Gamma_q(\alpha_1),T-a)+\beta_2\cdot {}_qE_{\alpha_2}((p_2+s_2h_2)\Gamma_q(\alpha_2),T-a)\right]\\ \triangleq\ & \varepsilon c_{f_1,f_2,\varphi}\varphi(t), \quad \forall t\in[\tau^{-1}a,\infty)_q.\end{aligned}$$

Hence system (1.1) is Ulam-Hyers-Rassias stable as mentioned above. □

Corollary 4.5. *If assumptions $(H_1)-(H_3)$ hold, then system (1.1) is Generalized Ulam-Hyers-Rassias stable about φ.*

Proof. In view of Definition 2.16 and Theorem 4.4, the desired conclusion can be obtained easily. □

5. Results In Generalized Banach Spaces

Now, we consider the uniqueness and Ulam-Hyers-Rassias of coupled system (1.1) in generalized Banach space. Let $C = X\times X = \{(u,v)|u,v : T_{\tau a}\to R^n\}$ be bounded functions vectors. $C^2 = C\times C$ is a generalized Banach space with the norm $\|(u,v)\|_{C^2} = \|u\|_\infty+\|v\|_\infty$. For any $u,v\in R^n$ with $u=(u_1,u_2,\cdots,u_n)$, $v=(v_1,v_2,\cdots,v_n)$, $\|u_i\| = \sup_{t\in T_{\tau a}}|u_i(t)|$, $\|u\|_\infty = \max\{\|u_1\|,\|u_2\|,\cdots,\|u_n\|\}$, and $\max(u,v) = (\max(u_1,v_1),\max(u_2,v_2),\ldots,\max(u_n,v_n))$. If $m\in R$, then $u\le m$ means $u_i\le m$, $i=1,2,\cdots,m$. Let C is nonempty set. Recall from [1] that (C,d) is called a generalized metric space with $d(u,v) = (d_1(u,v),d_2(u,v),\cdots,d_n(u,v))^T$, and a vector-valued metric map $d : C\times C\to R^n$ with the following properties for all $u,v\in C$: $(i)d(u,v)\ge 0$, and $d(u,v)=0$ if and only if $u=v$; $(ii)d(u,v)=d(v,u)$; $(iii)d(u,w)\le d(u,v)+d(v,w)$. Notice that d is a generalized metric space on C if and only if $d_i(i=1,2,\cdots,n)$ are metric on C. A square matrix D of real numbers is said to be convergent to zero if and only if its spectral radius $\rho(D)$ is strictly less than 1. That is to say that all the eigenvalues of D are in the open unit disc, i.e., $|\lambda|<1$ for every $\lambda\in\mathbb{C}$ with $\det(D-\lambda I)=0$, where I denotes the unit matrix of $D_{n\times n}(R)$. Let (C,d) be a generalized metric space. An operator $N : C\to C$ is said to be contractive [1] if there exists a matrix M convergent to zero such that $d(N(u),N(v))\le Md(u,v)$, $\forall u,v\in C$.

Lemma 5.1. [1] *The matrix $A = \begin{pmatrix} a & b\\ c & d\end{pmatrix}$ converges to zero in the following cases:*
(1) $b=c=0$, $a,d>0$, *and* $\max\{a,d\}<1$.
(2) $c=0$, $a,d>0$, $a+d<1$, *and* $-1<b<0$.
(3) $a+b=c+d=0$, $a>1, c>0$, *and* $|a-c|<1$.

Lemma 5.2. [16] *Let (C,d) be a complete generalized metric space, and let $N: C \to C$ be a contractive operator with Lipschitz matrix M. Then N has a unique fixed point u_0, and, for each $u \in C$,*

$$d(N^k(u),u_0) \leq M^k(M)^{-1}d(u,N(u)),\ for\ all\ k \in \mathbb{N}.$$

Theorem 5.3. *Suppose that $(H_1)-(H_2)$ hold under $u_i,v_i,w_i \in R^n$, $t \in T_a$. If*

$$\begin{pmatrix} \lambda_1+\mathrm{h}_1(r_1+s_1h_1) & p_1\mathrm{h}_1 \\ r_2\mathrm{h}_2 & \lambda_2+\mathrm{h}_2(p_2+s_2h_2) \end{pmatrix}$$

converges to 0, *then system (1.1) exists a unique solution.*

Proof. Define the operators $F_1,F_2: C^2 \to C$, $i=1,2$, by

$$F_i(u_1,u_2)(t) = \begin{cases} \Phi_i(t), & t \in I_\tau, \\ \Phi_i(a)-\lambda_i\Phi_i(\Im(\tau a))+\lambda_i u_i(\Im(\tau t))+{}_aI_q^{\alpha_i}f_i(t,u_1(t),u_2(t),{}_aI_q^{\alpha_i}u_i(\Im(\tau t))), & t \in T_{\tau a}, \end{cases}$$

and consider the operator $F: C^2 \to C^2$ defined by $(F(u_1,u_2))(t) = ((F_1(u_1,u_2))(t),(F_2(u_1,u_2))(t))$.

Next, in order to verify the conditions, taking any $(u_1,u_2),(v_1,v_2) \in C^2$, for $t \in I_\tau$, we obtain $\|F(u_1,u_2)(t)-F(v_1,v_2)(t)\| = 0$. For $t \in T_a$, we see that

$$\begin{aligned} &\|F_i(u_1,u_2)(t)-F_i(v_1,v_2)(t)\| \\ &\leq \lambda_i\|u_i(\Im(\tau t))-v_i(\Im(\tau t))\| \\ &+\left\|{}_aI_q^{\alpha_i}f_i(t,u_1(t),u_2(t),{}_aI_q^{\alpha_i}u_i(\Im(\tau t)))-{}_aI_q^{\alpha_i}f_i(t,v_1(t),v_2(t),{}_aI_q^{\alpha_i}v_i(\Im(\tau t))\right\| \\ &\leq \frac{r_i(t-a)_q^{\alpha_i}}{\Gamma_q(\alpha_i+1)}\|u_1-v_1\|+\frac{p_i(t-a)_q^{\alpha_i}}{\Gamma_q(\alpha_i+1)}\|u_2-v_2\|+(\frac{s_ih_i(t-a)_q^{\alpha_i}}{\Gamma_q(\alpha_i+1)}+\lambda_i)\|u_i(\Im(\tau t))-v_i(\Im(\tau t))\|, \end{aligned}$$

For $t \in I_{\tau^{-1}}$, $\|u_i(\Im(\tau t))-v_i(\Im(\tau t))\| = 0$. Letting

$$M_0 = \begin{pmatrix} r_1h_1 & p_1h_1 \\ r_2h_2 & p_2h_2 \end{pmatrix},$$

we have $d(F(u_1,u_2)(t),F(v_1,v_2)(t)) \leq M_0 d((u_1,u_2),(v_1,v_2))$. When $t \in [\tau^{-1}a,\infty)_q$, we have

$$\|F_1(u_1,u_2)(t)-F_1(v_1,v_2)(t)\| \leq (\frac{(r_1+s_1h_1)(t-a)_q^{\alpha_1}}{\Gamma_q(\alpha_1+1)}+\lambda_1)\|u_1-v_1\|+\frac{p_1(t-a)_q^{\alpha_1}}{\Gamma_q(\alpha_1+1)}\|u_2-v_2\|,$$

and

$$\|F_2(u_1,u_2)(t)-F_2(v_1,v_2)(t)\| \leq \frac{r_2(t-a)_q^{\alpha_2}}{\Gamma_q(\alpha_2+1)}\|u_1-v_1\|+(\frac{(p_2+s_2h_2)(t-a)_q^{\alpha_2}}{\Gamma_q(\alpha_2+1)}+\lambda_2)\|u_2-v_2\|.$$

Letting

$$M = \begin{pmatrix} \lambda_1+\mathrm{h}_1(r_1+s_1h_1) & p_1\mathrm{h}_1 \\ r_2\mathrm{h}_2 & \lambda_2+\mathrm{h}_2(p_2+s_2h_2) \end{pmatrix},$$

we have $d(F(u_1,u_2)(t),F(v_1,v_2)(t)) \leq Md((u_1,u_2),(v_1,v_2))$, where

$$d(F(u_1,u_2),F(v_1,v_2)) = \begin{pmatrix} \|F_1(u_1,u_2)-F_1(v_1,v_2)\| \\ \|F_2(u_1,u_2)-F_2(v_1,v_2)\| \end{pmatrix},$$

and

$$d((u_1,u_2),(v_1,v_2)) = \begin{pmatrix} \|u_1-v_1\| \\ \|u_2-v_2\| \end{pmatrix}.$$

Since M converges to 0, we conclude from Lemma 5.2 that system (1.1) has a unique solution. $\square$

Let $\varphi : T_a \to R^n$ be a non-decreasing continuous function. There exist constant matrixes $\varepsilon, \beta_i > 0$ such that ${}_aI_q^{\alpha_i}\varphi(t) \le \beta_i\varphi(t)$. On the other hand, $(v_1(t),v_2(t)) \in C$ satisfy the following inequalities

$$\begin{cases} \left|{}_a^cD_q^{\alpha_1}(v_1(t)-\lambda_1 v_1(\Im(\tau t))) - f_1(t,v_1(t),v_2(t),{}_aI_q^{\alpha_1}v_1(t)(\Im(\tau t)))\right| \le \varphi(t), \\ \left|{}_a^cD_q^{\alpha_2}(v_2(t)-\lambda_2 v_2(\Im(\tau t))) - f_2(t,v_1(t),v_2(t),{}_aI_q^{\alpha_2}v_2(\Im(\tau t)))\right| \le \varphi(t), \end{cases} \quad t \in T_a. \tag{5.1}$$

Theorem 5.4. *Suppose $(H_1)-(H_3)$ hold under $u_i,v_i,w_i \in R^n$, $t \in T_a$. Then system (1.1) is Ulam-Hyers-Rassias stable about φ in the generalized Banach space.*

Proof. Let $(v_1(t),v_2(t))$ be a solution to (5.1). Let $(u_1(t),u_2(t))$ be a solution to system (1.1) and $(v_1(t),v_2(t)) = (u_1(t),u_2(t))$ for $t \in I_\tau$. Define

$$u_i(t) = \begin{cases} v_i(t),\ t \in I_\tau, \\ v_i(a) - \lambda_i v_i(\Im(\tau a)) + \lambda_i u_i(\Im(\tau t)) - {}_aI_q^{\alpha_i} f_i(t,u_1(t),u_2(t),{}_aI_q^{\alpha_i}u_i(\Im(\tau t))),\ t \in T_a. \end{cases}$$

From (5.1), we have $\left|{}_a^cD_q^{\alpha_i}(v_i(t)-\lambda_i v_i(\Im(\tau t))) - f_i(t,v_1(t),v_2(t),{}_aI_q^{\alpha_i}v_i(\Im(\tau t)))\right| \le \varphi(t)$, $i = 1,2$, and

$$\begin{aligned} &\left|v_i(t) - v_i(a) + \lambda_i v_i(\Im(\tau a)) - \lambda_i v_i(\Im(\tau t)) - {}_aI_q^{\alpha_i} f_i(t,v_1(t),v_2(t),{}_aI_q^{\alpha_i}v_i(\Im(\tau t)))\right| \\ &\le {}_aI_q^{\alpha_i}\varphi(t) \le \beta_i\varphi(t), \end{aligned}$$

which yields that

$$\begin{aligned} &\|v_i(t)-u_i(t)\| \\ \le\ & \left\|v_i(t) - v_i(a) + \lambda_i v_i(\Im(\tau a)) - \lambda_i u_i(\Im(\tau t)) - {}_aI_q^{\alpha_i} f_i(t,v_1(t),v_2(t),{}_aI_q^{\alpha_i}v_i(\Im(\tau t)))\right\| \\ &+ \left\|{}_aI_q^{\alpha_i} f_i(t,v_1(t),v_2(t),{}_aI_q^{\alpha_i}v_i(\Im(\tau t))) - {}_aI_q^{\alpha_i} f_i(t,u_1(t),u_2(t),{}_aI_q^{\alpha_i}u_i(\Im(\tau t)))\right\| \\ \le\ & \varepsilon\beta_i\varphi(t) + {}_aI_q^{\alpha_i}(r_i\|v_1(t)-u_1(t)\| + p_2\|v_2(t)-u_2(t)\| + s_ih_i\|v_i(\Im(\tau t)) - u_i(\Im(\tau t))\|). \end{aligned}$$

For $t \in I_\tau$, it is clear. For $t \in I_{\tau^{-1}}$, $\|u_i(\Im(\tau t)) - v_i(\Im(\tau t))\| = 0$. Letting $\mathbb{B} = \begin{pmatrix} r_1 & p_1 \\ r_2 & p_2 \end{pmatrix}$, we have $d((v_1(t),v_2(t)),(u_1(t),u_2(t))) \le \varepsilon \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} \varphi(t) + \mathbb{B}{}_aI_q^{\alpha_i} d((v_1(t),v_2(t)),(u_1(t),u_2(t)))$. From Corollary 2.9, we obtain

$$d((v_1(t),v_2(t)),(u_1(t),u_2(t))) \le \varepsilon\varphi(t)\left[\begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} {}_aE_q(\mathbb{B}\Gamma_q(\alpha),T-a)\right] \triangleq \varepsilon c_{f_1f_2}\varphi(t)$$

For $t \in [\tau^{-1}a,\infty)_q$, let $\hat{z}_i(t) = \sup_{\theta\in I_\tau} \|v_i(\Im(\tau t)) - u_i(\Im(\tau t))\|$, $\mathbb{G} = \begin{pmatrix} r_1+s_1h_1 & p_1 \\ r_2 & p_2+s_2h_2 \end{pmatrix}$, we can also deduce

$$d((v_1(t),v_2(t)),(u_1(t),u_2(t))) \le \begin{pmatrix} \hat{z}_1(t) \\ \hat{z}_2(t) \end{pmatrix} \le \varepsilon \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} \varphi(t) + \mathbb{G}\,{}_aI_q^{\alpha_i} \begin{pmatrix} \hat{z}_1(t) \\ \hat{z}_2(t) \end{pmatrix}.$$

By Corollary 2.9, we have

$$\begin{pmatrix} \hat{z}_1(t) \\ \hat{z}_2(t) \end{pmatrix} \le \varepsilon\varphi(t) \left[\begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} {}_aE_q(\mathbb{F}\Gamma_q(\alpha), T-a) \right] \triangleq \varepsilon c_{f_1 f_2} \varphi(t),$$

which yields that $d((v_1(t),v_2(t)),(u_1(t),u_2(t))) \le \varepsilon c_{f_1 f_2}\varphi(t)$. Therefore system (1.1) is Ulam-Hyers-Rassias stable about φ. $\square$

6. Examples

Example 6.1. Consider the following coupled system of fractional q-difference equation:

$$\begin{cases} {}^c_a\mathrm{D}^{1/2}_{1/4}(u_1(t) - \lambda_1 u_1(\Im(\tau t))) = f_1(t,u_1(t),u_2(t),{}_aI^{1/2}_{1/4}u_1(\Im(\tau t))), \quad t\in[\frac{1}{16},\infty)_q, \\ {}^c_a\mathrm{D}^{1/3}_{1/4}(u_2(t) - \lambda_2 u_2(\Im(\tau t))) = f_2(t,u_1(t),u_2(t),{}_aI^{1/3}_{1/4}u_2(\Im(\tau t))), \quad t\in[\frac{1}{16},\infty)_q, \\ \Phi_1(t) = e^{\log t}, \quad t\in I_\tau, \\ \Phi_2(t) = \log e^t, \quad t \in I_\tau. \end{cases}$$

where

$$\begin{cases} f_1(t,u_1(t),u_2(t),w(t)) = \frac{\sin u_1(t) + \cos u_2(t) + sin w(\tau t)}{300}, \\ f_2(t,u_1(t),u_2(t),w(t)) = \frac{\cos u_1(t) + \sin u_2(t) + cos w(\tau t)}{300}, \end{cases}$$

$\alpha_1 = \frac{1}{2},\ \alpha_2 = \frac{1}{3},\ q = \frac{1}{4},\ \lambda_1 = \frac{1}{30},\ \lambda_2 = \frac{1}{12},\ \Im(\tau t) = \tau t,\ T = q^{-2}.$

$$\begin{aligned} &|f_i(t,u_1(t),u_2(t),w_1(t)) - f_i(t,v_1(t),v_2(t),w_2(t))| \\ &\qquad \le \tfrac{1}{300}(|u_1(t)-v_1(t)| + |u_2(t)-v_2(t)| + |w_1(\tau t) - w_1(\tau t)|) \end{aligned}$$

since $\Gamma_q(\alpha)$ is increasing about α and $\Gamma_q(\alpha) > 1 (\alpha > 0)$, $\hat{h} = \frac{(T-a)_q^{\alpha_1}}{\Gamma_q(\alpha_2+1)} < 8$, $\hat{\lambda} + \hat{h}(\hat{r} + \hat{p} + \hat{s}\hat{h}) = \frac{1}{12} + 8\times(\frac{1}{150} + \frac{1}{300^2}) < 1$. Assume that $\varphi(t) = \mathrm{e}^{\arctan t}$ for each $t \in T_a$, there exists a real number $0 < \varepsilon_i < 1$ such that ${}_aI_q^{\alpha_i}\varphi(t) \le \frac{\mathrm{e}^{\arctan t}}{\varepsilon_i^2(1+q+q^2)} \le \frac{1}{\varepsilon_i^2}\varphi(t) \le \beta_i\varphi(t)$. So we have the results.

Example 6.2. Consider the following new coupled fractional q-difference equation:

$$\begin{cases} {}^c_a\mathrm{D}^{1/2}_{1/4}(u_1(t) - \lambda_1 u_1(\Im(\tau t))) = f_1(t,u_1(t),u_2(t),{}_aI^{1/2}_{1/4}u_1(\Im(\tau t))), \quad t\in[\frac{1}{16},\infty)_q, \\ {}^c_a\mathrm{D}^{1/3}_{1/4}(u_2(t) - \lambda_2 u_2(\Im(\tau t))) = f_2(t,u_1(t),u_2(t),{}_aI^{1/3}_{1/4}u_2(\Im(\tau t))), \quad t\in[\frac{1}{16},\infty)_q, \\ \Phi_1(t) = e^{\log t}, \quad t\in I_\tau, \\ \Phi_2(t) = \log e^t, \quad t \in I_\tau. \end{cases}$$

where $u_1(t), u_2(t) \in R^n$, $f_i : T_a \times R^n \times R^n \times R^n \to R^n$,

$$\begin{cases} f_1(t,u_1(t),u_2(t),w(t)) = \frac{kt^2}{1+|u_1(t)|+|u_2(t)|+|\omega(t)|}(e^{-8}+\frac{1}{e^{t+3}})(t^2u_1(t)+w(t)), \\ f_2(t,u_1(t),u_2(t),w(t)) = \frac{kt^2}{e^{t+3}(1+|u_1(t)|+|u_2(t)|+|\omega(t)|)}(tu_2(t)+w(t)), \end{cases} \quad t \in [\frac{1}{16},\infty)_q,$$

and $k > 0$. The hypothesis of Theorem 5.3 is satisfied with $r_1 = \sup_t \left|(e^{-8}+\frac{1}{e^{t+3}})kt^5\right|$, $p_1 = 0$, $s_1 = \sup_t \left|(e^{-8}+\frac{1}{e^{t+3}})kt^3\right|$, $r_2 = 0$, $p_2 = \sup_t \left|\frac{kt^4}{e^{t+3}}\right|$, and $s_2 = \sup_t \left|\frac{kt^3}{e^{t+3}}\right|$. Due to the general selectivity of the constant k, the matrix

$$M = \begin{pmatrix} \lambda_1 + (r_1 + s_1 h_1)h_1 & 0 \\ 0 & \lambda_2 + (p_2 + s_2 h_2)h_2 \end{pmatrix}$$

converges to 0. Hence, system (1.1) has a unique solution in the generalized Banach space.

Funding

This research was supported by Outstanding Young Science and technology Training program of Xinjiang (2019Q022), National Natural Science Foundation of China (11961069), Natural Science Foundation of Xinjiang (2019D01A71), and Scientific Research Programs of Colleges in Xinjiang (XJEDU2018Y033).

References

[1] S. Abbas, M. Benchohra, N. Laledj, Y. Zhou, Existence and Ulam stability for implicit fractional q-difference equations, Adv. Difference Equ. 2019 (2019) 480.
[2] S. Abbas, M. Benchohra, M. Samat, Y. Zhou, Coupled implicit Caputo fractional q-difference systems, Adv. Difference Equ. 2019 (2019) 527.
[3] S. Abbas, M. Benchohra, G.M. N'Guérékata, Topics in Fractional Differential Equations, Springer, New York, 2012.
[4] T. Abdeljawad, J. Alzabut, D. Baleanu, A generalized q-fractional Gronwall inequality and its applications to nonlinear delay q-fractional difference systems, J. Inequal. Appl. 2016 (2016) 240.
[5] T. Abdeljawad, D. Baleanu, Caputo q-fractional initial value problems and a q-analogue Mittag-Leffler function, Commun. Nonlinear Sci. Numer. Simul. 16 (2011) 4682-4688.
[6] B. Ahmad, S.K. Ntouyas, Boundary Value Problems for q -Difference Equations and Inclusions with Nonlocal and Integral Boundary Conditions, Math. Modelling Anal. 19 (2014) 647-663.
[7] D.R. Anderson, M. Onitsuka, Hyers-Ulam stability for quantum equations of Euler type, Discrete Dyn. Nat. Soc. 2020 (2020) 5626481.
[8] M.H. Annaby, Z.S. Mansour, q-Fractional Calculus and Equations, Lecture Notes in Mathematics, Vol. 2056, Springer Berlin Heidelberg, 2012.
[9] R.I. Butt, T. Abdeljawad, M.A. Alqudah, M.U. Rehman, Ulam stability of Caputo q-fractional delay difference equation: q-fractional Gronwall inequality approach, J. Inequal. Appl. 2019 (2019) 305.
[10] R.A.C. Ferreira, Nontrivial solutions for fractional q-difference boundary value problems, Electron. J. Qual. Theory Differ. Equ. 70 (2010) 1-10.
[11] V. Kac, P. Cheung, Quantum Calculus, Springer, New York, 2000.
[12] K.S. Kostadinov, J.V. Manojlović, Existence of positive strongly decaying solutions of second-order nonlinear q-difference equations, J. Difference Equ. Appl. 26 (2020) 729-752.
[13] K. Ma, Z. Han, Y. Zhang, Stability conditions of a coupled system of fractional q-difference Lotka-Volterra model, Int. J. Dyn. Sys. Differential Equ. 6 (2016) 305-317.
[14] K. Ma, Z. Han, S. Sun, Existence and uniqueness of solutions for fractional q-difference Schrödinger equations, J. Appl. Math. Comput. 62 (2020) 611-620.

[15] S.K. Ntouyas, J.D. Tariboon, J.X. Yin, Nonlocal boundary value problems for q-difference equations and inclusions, Int. J. Equ. 2015 (2015) 203715.
[16] I.R. Petre, A. Petrusel, Krasnoselskii's theorem in generalized Banach spaces and applications, Electron. J. Qual. Theory Differ. Equ. 2012 (2012) 85.
[17] P.M. Rajkovic, S.D. Marinkovic, M.S. Stankovic, Fractional integrals and derivatives in q-Calculus, Appl. Anal. Discrete Math. 1 (2007) 311-323.
[18] H.L. Royden, P.M. Fitzpatrick, Real Analysis, China Machine Press, 2009.
[19] S. Stević, On q -difference asymptotic solutions of a system of nonlinear functional differential equations, Appl. Math. Comput. 219 (2013) 8295-8301.
[20] D. Vivek, K. Kanagarajan, S. Harikrishnan, Existence of Solutions for Impulsive Fractional q-difference Equations with Nonlocal Condition, J. Appl. Nonlinear Dyn. 6 (2017) 479-486.
[21] G.T. Wang, Twin iterative positive solutions of fractional q-difference Schrödinger equations, Appl. Math. Lett. 76 (2018) 103-109.
[22] T. Zhang, Q. Guo, The solution theory of the nonlinear q-fractional differential equations, Appl. Math. Lett. 104 (2020) 106282.
[23] L. Zhang, Y. Zhou, Existence and attractivity of solutions for fractional difference equations, Adv. Difference Equ. 2018 (2018) 191.