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THE POSITIVE SOLUTIONS TO THE BOUNDARY VALUE PROBLEM OF A NONLINEAR SINGULAR IMPULSIVE DIFFERENTIAL SYSTEM

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Abstract. In this paper, the boundary value problem of a class of nonlinear singular impulsive differential equations in Banach space is studied. By constructing a special cone and defining a special operator, the impulsive problem is transformed into a continuous problem. By using the fixed point theorem of cone extension and cone compression, the existence of multiple positive solutions is obtained.

Keywords. Boundary value problem; Impulsive differential system; Multiple positive solutions; Singular.

1. Introduction

In this paper, we assume that $(E, \|\cdot\|)$ is a real Banach space. If P is a normal cone in E, we say that the normal constant is 1. We consider the existence of two positive solutions to the boundary value problem of the following second order singular impulsive differential equations

$$\begin{cases} x^{''}(t)+f_1(t,x(t),y(t),x(t)+y(t))=\theta, t\in(0,1), t\neq t_k;\\ y^{''}(t)+f_2(t,x(t),y(t),x(t)+y(t))=\theta, t\in(0,1), t\neq t_k;\\ \Delta x=x(t_k^+)-x(t_k^-)=I_{1k}(x(t_k))t=t_k, k=1,2,\cdots,m;\\ \Delta y=y(t_k^+)-y(t_k^-)=I_{2k}(y(t_k))t=t_k, k=1,2,\cdots,m;\\ x(0)=x^{'}(1)=\theta;\\ y(0)=y^{'}(1)=\theta, \end{cases} \tag{1.1}$$

where θ is a zero element in E.

Let $J=[0,1]$ and $C[J\times J]=:\{(x,y)|(x,y):J\times J\to E\times E,\ x(t)\ and\ y(t)\ are\ continuous\ on\ J\}$. It is easy to know that $C[J\times J]$ becomes a Banach space under the norm $\| (x,y) \| = \| x \|_C + \|$

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$y \|_C$, where $\| x \|_C = \sup_{t\in J} \| x(t) \|$, $\| y \|_C = \sup_{t\in J} \| y(t) \|$. Let $PC[J\times J] =: \{(x,y)|(x,y) : J\times J \to E\times E,\ x(t)\ and\ y(t)\ are\ continuous\ at\ t \neq t_k,\ and\ on\ the\ left\ and\ the\ right\ limit\ exists\ at\ t = t_k, k = 1,2,\cdots,m\}$. It is easy to see that $PC[J\times J]$ becomes a Banach space under the norm $\|(x,y)\| =: \| x \|_{PC} + \| y \|_{PC}$, where $\| x \|_{PC} = \sup_{t\in J} \| x(t) \|$, $\| y \|_{PC} = \sup_{t\in J} \| y(t) \|$. In (1.1), $f_i \in C[(0,1)\times P\times P\times P\setminus\{\theta\},P]$, $I_{ik} \in C[P,P]$, for $i = 1,2$ and $k = 1,2,\cdots,m$. Problem (1.1) is the boundary value problem of the second-order nonlinear singular impulsive differential equation system. As a system widely existing in modern science, the impulse system is powerful to describe the models of population dynamics, ecology, ecosystem, biotechnology and so on. From the viewpoint of theory research, impulse differential equations have become one of the important branches of applied mathematics in recent years. Many authors studied impulsive differential equations, and their theoretical results have been widely applied to many fields; see, e.g., [3, 4, 5] and the references therein. We study the solutions of impulsive differential equations (1.1) by using the noncompact measure and the fixed point theorem on cones in [3, 4, 5]. There are many theorems on the solutions of impulsive differential equation in Banach spaces, such as the Leggett-Williams fixed point theorem, the Schauder fixed point theorem, the mixed monotone operator fixed point theorem, and the fixed-point theorem on the cones. Recently, the existence of at least three positive solutions to impulsive differential equation by using the Leggett-Williams fixed point theorem was studied (see, e.g., [2, 6, 11]), and the existence of a positive solution to impulsive differential equation by using the Schauder fixed point theorem and the fixed point theorems of mixed monotone operators was also studied (see, e.g., [10, 14, 15]). For differential equation or impulsive differential equation, there are many research results on the existence of solutions, but there are few results on the existence of solutions of differential equations, in particular, the impulsive differential equations. Tang and Wang [8] used the fixed point theorem of cone tensions and cone compressions and explored the existence of positive solutions of non-negative boundary value problems for second-order singular differential equations on infinite intervals. Wang and Tang [9] further studied the existence of positive solutions to the boundary value problems of nonlinear impulsive differential equations. On the other hand, many scholars studied singular differential equations or singular impulsive differential equations. We refer to [1, 7, 12] for the existence of positive solutions for boundary value problems of second-order singular differential equations. There are few cases that there are singularities in impulsive differential equations. In boundary value problem (1.1), $f_i(t,x(t),y(t),x(t)+y(t)) (i = 1,2)$ is singular at point $t = 0,1$. In order to overcome the difficulties caused by the singularity, we first consider an approximation to the boundary value problem (1.1) of the impulsive differential equations.

In [13], Yan and Liu constructed a special operator to overcome the obstacles caused by impulse when studying the singular impulse differential equation on the half-line, and transformed the impulse problem into a continuous problem. Based on this, in this paper, we first transform the impulse problem into a continuity problem by constructing a special operator, and then use the fixed point theorem of cone extensions and cone compressions to study the boundary value problem of the singular impulse differential equations, and the existence of two positive solutions is obtained.

2. Preliminaries

The positive solution to problem (1.1) is $(x,y) \in PC[J\times J]\bigcap C^2[J'\times J']$, where $J' = [0,1]\setminus \{t_1,t_2,\cdots,t_m\}$. It satisfies boundary value problem (1.1), and $x(t) > \theta,\ y(t) > \theta$, when $t \in [0,1]$. Let $x(t) : (0,1) \to E$ be continuous. If the limit $\lim\limits_{\varepsilon\to 0^+}\int_\varepsilon^1 x(t)dt$ exists, then the abstract generalized integral $\int_0^1 x(t)dt$ converges. Similarly, the convergence and divergence of other generalized integrals can be defined. Kuratowskii noncompact measures of space $E, C[J\times J]$, and space $PC[J\times J]$ are respectively represented by α, α_C, and α_{PC}. For the noncompact measures and their properties, we refer to [4, 5].

Lemma 2.1. [4] *Let $S\subset C[J,E]$ be bounded, and let S be equicontinuous on J. Then $\alpha_c(S) = \sup\limits_{t\in J}\alpha(S(t))$, where $S(t) = \{x(t) : x\in S\}(t\in J)$.*

Lemma 2.2. [4] *Let $D\subset PC[J\times E]$ be bounded, and let D be equicontinuous on $J_k(k = 0,1,\cdots,m; t_0 = 0, t_{m+1} = 1)$. Then $\alpha_{PC}(D) = \sup\limits_{t\in J}\alpha(D(t))$, where $J_0 = [0,t_1], J_k = (t_k,t_{k+1}]$.*

Lemma 2.3. [4] *Let S,T be a bounded open set of E. Then $\alpha(S+T)\le\alpha(S)+\alpha(T)$, where $S+T = \{x = y+z \mid y\in S, z\in T\}$.*

Lemma 2.4. [4] *(Fixed Point Theorem of Cone Expansion and Cone Compression) Let P be a cone in E and $P_{r,s} = \{x\in P, r\le\| x\|\le s\}(s>r>0)$. Let $A : P_{r,s}\to P$ be a strict set compression operator satisfing one of the following two conditions:*
(i) when $x\in P$, $\| x\| = r$, $\| Ax\|\ge\| x\|$, and when $x\in P$, $\| x\| = s$, $\| Ax\|\le\| x\|$;
(ii) when $x\in P$, $\| x\| = r$, $\| Ax\|\le\| x\|$, and when $x\in P$, $\| x\| = s$, $\| Ax\|\ge\| x\|$.
Then A has at least one fixed point in $P_{r,s}$.

3. Main Results

For the sake of convenience, we give the following conditions:

(H_1) Let $f_i(t,x,y,x+y)\le p_i(t)q_i(x+y)$, where $p_i\in C[(0,1),R^+]$ and $q_i\in C[P,P]$ satisfy $\int_0^1 sp_i(s)q_i[sr,R]ds < +\infty$ with $q_i[r,R] = \sup_{r\le\|(x,y)\|\le R}\| q_i(x+y)\| < +\infty, 0<r<R, i = 1,2$.

(H_2) For any $r>0$, $f_i(t,x,y,x+y)$ is uniformly continuous on $(0,1)\times B_E(0,r)\times B_E(0,r)\times B_E(0,r)\setminus\{\theta\}$, where $B_E(0,r) = \{x | x\in E, 0<\| x\|\le r\}, i = 1,2$.

(H_3) There exist $l_{ij}\ge 0$ $(i = 1,2; j = 1,2,3)$, for any $t\in(0,1)$ such that $\alpha(f_i(t,C_1,C_2,C_1+C_2))\le l_{i1}\alpha(C_1)+l_{i2}\alpha(C_2)+l_{i3}\alpha(C_1+C_2)$, where C_1, C_2 is a bounded set of E with $C_1+C_2 = \{x+y | x\in C_1, y\in C_2\}$. And there exist $L_{ik}\ge 0$ such that $\alpha(I_{ik}(C_i))\le L_{ik}\alpha(C_i)(i = 1,2; k = 1,2,\cdots,m)$. In addition,

$$l = \max\{4[(l_{i1}+l_{i3})(1+\sum_{k=1}^m L_{1k})+(l_{i2}+l_{i3})(1+\sum_{k=1}^m L_{2k})] < 1, i = 1,2\}.$$

(H_4) Let $I_{1k}(t) = \sum\limits_{0<t_k<t} I_{1k}(x(t_k)), I_{2k}(t) = \sum\limits_{0<t_k<t} I_{2k}(x(t_k))$. We assume that $\lim\limits_{\substack{x\to I_{1k}(t)\\ y\to I_{2k}(t)}}\frac{f_i(t,x,y,x+y)}{x}\ge m_i(t)(i = 1,2)$ are unanimously established with $t\in(0,1)$, where $m_i\in C[(0,1),R^+](i = 1,2)$, and there is $[a,b]\subset(0,1)$ satisfying $\int_a^b as(m_1(s)+m_2(s))ds > 1$.

(H$_5$) $\lim\limits_{\substack{x\to\infty\\ y\to\infty}} \frac{f_i(t,x,y,x+y)}{x+y} \geq n_i(t)(i=1,2)$ are unanimously established with $t\in(0,1)$, where $n_i \in C[(0,1),R^+]$, and there is $[a^*,b^*]\subset(0,1)$ satisfying $\int_{a^*}^{b^*} a^*s(n_1(s)+n_2(s))ds>1$.

To overcome the difficulties caused by singularity, we first consider an approximation to (1.1)

$$\begin{cases} x''(t)+f_1(t,x(t),y(t),x(t)+y(t)+\frac{e}{n})=\theta, t\in(0,1), t\neq t_k;\\ y''(t)+f_2(t,x(t),y(t),x(t)+y(t)+\frac{e}{n})=\theta, t\in(0,1), t\neq t_k, \end{cases} \tag{3.1}$$

where $e\in P$ with $\parallel e\parallel=1$, and its impulse condition and boundary value condition are the same as (1.1). We use the cone P in E to construct the following cones in $C[J\times J]$ and $PC[J\times J]$:

$$K=:\{(x,y)\in C[J\times J]:\forall t,s\in J, x(t)\geq tx(s), y(t)\geq ty(s)\},$$

and

$$Q=:\{(x,y)\in PC[J\times J]:\forall t\in J, x(t)\geq\theta, y(t)\geq\theta\}.$$

If $(x,y)\in PC[J\times J]$ is a solution to the following integral system

$$\begin{cases} x(t)=\int_0^1 G(t,s)f_1(s,x(s),y(s),x(s)+y(s)+\frac{e}{n})ds+\sum\limits_{0<t_k<t} I_{1k}(x(t_k));\\ y(t)=\int_0^1 G(t,s)f_2(s,x(s),y(s),x(s)+y(s)+\frac{e}{n})ds+\sum\limits_{0<t_k<t} I_{2k}(y(t_k)), \end{cases} \tag{3.2}$$

where $G(t,s)=:\min\{t,s\}$, then it is easy to prove that (x,y) is also a solution to boundary value problem (3.1).

Now, let us consider integral system (3.2). Let

$$\begin{cases} x_1(t)=x(t)-\sum\limits_{0<t_k<t} I_{1k}(x(t_k)), t\in(0,1);\\ y_1(t)=y(t)-\sum\limits_{0<t_k<t} I_{2k}(y(t_k)), t\in(0,1). \end{cases}$$

Then

$$\begin{cases} x(t)=x_1(t)+\sum\limits_{0<t_k<t} I_{1k}(x(t_k));\\ y(t)=y_1(t)+\sum\limits_{0<t_k<t} I_{2k}(y(t_k)). \end{cases} \tag{3.3}$$

For any $(x_1,y_1)\in K$, define the following operator

$$\begin{cases} (T_1x_1)(t)=x_1(t)+\sum\limits_{0<t_k<t} I_{1k}((T_1x_1)(t_k)), t\in(0,1);\\ (T_2y_1)(t)=y_1(t)+\sum\limits_{0<t_k<t} I_{2k}((T_2y_1)(t_k)), t\in(0,1). \end{cases} \tag{3.4}$$

Then integral system (3.2) is transformed into

$$\begin{cases} x_1(t)=\int_0^1 G(t,s)f_1(s,(T_1x_1)(s),(T_2y_1)(s),(T_1x_1)(s)+(T_2y_1)(s)+\frac{e}{n})ds;\\ y_1(t)=\int_0^1 G(t,s)f_2(s,(T_1x_1)(s),(T_2y_1)(s),(T_1x_1)(s)+(T_2y_1)(s)+\frac{e}{n})ds. \end{cases} \tag{3.5}$$

According to equation (3.5), the operator A_n is defined on K by

$$A_n(x_1,y_1)=:(A_{1n}(x_1,y_1),A_{2n}(x_1,y_1)),(x_1,y_1)\in K, \tag{3.6}$$

where

$$\begin{cases} A_{1n}(x_1,y_1)(t)=:\int_0^1 G(t,s)f_1(s,(T_1x_1)(s),(T_2y_1)(s),(T_1x_1)(s)+(T_2y_1)(s)+\frac{e}{n})ds, t\in(0,1);\\ A_{2n}(x_1,y_1)(t)=:\int_0^1 G(t,s)f_2(s,(T_1x_1)(s),(T_2y_1)(s),(T_1x_1)(s)+(T_2y_1)(s)+\frac{e}{n})ds, t\in(0,1), \end{cases}$$

and $(T_1x_1)(t),(T_2y_1)(t)$ are defined by equation (3.4).

It is easy to prove that the fixed point of (3.6) on K is also a solution to integral system (3.5). Through (3.3) and (3.4), the solution of integral system (3.2), that is, the solution of the boundary value problem (3.1), can be obtained. Let

$$\begin{aligned} K_1 &= : \{x \in C[J,P] : \forall t,s \in J, x(t) \geq tx(s)\}, \\ K_2 &= : \{y \in C[J,P] : \forall t,s \in J, y(t) \geq ty(s)\}, \\ Q_1 &= : \{x \in PC[J,P] : \forall t \in J, x(t) \geq \theta\}, \\ Q_2 &= : \{y \in PC[J,P] : \forall t \in J, y(t) \geq \theta\}. \end{aligned}$$

Lemma 3.1. *If $I_{ik} \in C[P,P]$ for each $i = 1,2; k = 1,2,\cdots,m$, then $T_i : K_i \to Q_i (i = 1,2)$ are continuously bounded.*

Proof. Obviously, the definition of T_i, $i = 1,2$, is reasonable, and, for $\forall t \in J, \forall x \in K_1$, the operator T_1 satisfies $(T_1x)(t) \geq x(t) \geq \theta$.

Let $\{x_n\} \subseteq K_1, x_0 \in K_1$, and $\lim\limits_{n\to\infty} \| x_n - x_0 \| = 0$. Obviously,

$$\sup_{t\in(0,t_1]} \| (T_1x_n)(t) - (T_1x_0)(t) \| = \sup_{t\in(0,t_1]} \| x_n(t) - x_0(t) \| \leq \| x_n - x_0 \| \to 0 (n \to \infty).$$

Let $t_0 = 0$ and $t_{m+1} = 1$. From the continuity of I_{1k}, $k = 1,2,\cdots,m$, we know that

$$\lim_{n\to\infty} \| (I_{1k}(T_1x_n)(t_k)) - I_{1k}((T_1x_0)(t_k)) \| = 0, k = 1,2,\cdots,m.$$

Therefore, it can be obtained from (3.4) that

$$\begin{aligned} &\sup_{t\in(t_k,t_{k+1}]} \| (T_1x_n)(t) - (T_1x_0)(t) \| \\ &\leq \sup_{t\in(t_k,t_{k+1}]} \| x_n(t) - x_0(t) \| + \sum_{i=1}^{k} \| (I_{1i}(T_1x_n)(t_i)) - I_{1i}((T_1x_0)(t_i)) \| \\ &\to 0 \ (as\ n \to \infty),\ k = 1,2,\cdots,m. \end{aligned}$$

Thus, $\| T_1x_n - T_1x_0 \|_{PC} \to 0$ as $n \to \infty$, so $T_1 : K_1 \to Q_1$ is continuous. Similarly, it can be proved that $T_2 : K_2 \to Q_2$ is continuous. Assume that $\Omega \subseteq K_1$ is bounded. Then there exists a constant $M > 0$ such that $\forall x \in \Omega, \| x \|_C \leq M$. It is easy to know from the continuity of $I_{1k}, k = 1,2,\cdots,m$, that $I_{1k}((T\Omega)(t_k))(k = 1,2,\cdots,m)$ are bounded. It can be obtained from (3.4) that

$$\sup_{x\in\Omega} \sup_{t\in(0,t_1]} \| (T_1x)(t) \| = \sup_{x\in\Omega} \sup_{t\in(0,t_1]} \| x(t) \| \leq \sup_{x\in\Omega} \| x(t) \| \leq M,$$

so $T_1(\Omega)_{(0,t_1]}$ is bounded. Similarly, it can be deduced that

$$\begin{aligned} \sup_{x\in\Omega} \sup_{t\in(t_k,t_{k+1}]} \| (T_1x)(t) \| &\leq \sup_{x\in\Omega} \sup_{t\in(t_k,t_{k+1}]} \| x(t) \| + \sum_{i=1}^{k} \| I_{1i}((T_1x)(t_i)) \| \\ &< +\infty, k = 1,2,\cdots,m. \end{aligned}$$

Therefore, $T_1(\Omega)$ is bounded, that is, $T_1 : K_1 \to Q_1$ is bounded. In the same way, we can show that $T_2 : K_2 \to Q_2$ is bounded. □

It is easy to derive the following lemma from the definition of the noncompactness measures.

Lemma 3.2. *Let S, T be the bounded set of E, where the noncompact measures in $E, F, E \times F$ are all denoted by $\alpha(\cdot)$. The norm in $E \times F$ is defined by $\| (x,y) \| = \| x \| + \| y \|$. Then $\alpha(S \times T) \leq \alpha(S) + \alpha(T)$, where $S \times T = \{(x,y) | x \in S, y \in T\}$.*

Let

$$K_r =: \{(x,y) : \sup_{(x,y) \in K} \| (x,y) \| \leq r\}$$

and

$$T_r := \sup_{(x,y) \in K_r} \| (T_1 x, T_1 y) \| .$$

Then it is easy to know from (3.4) that $T_r \geq r$.

Lemma 3.3. *If condition (H_1) and (H_2) are satisfied, then $A_n : K_r \to K$ is continuous.*

Proof. First of all, for any $(x_1, y_1) \in K_r$, we prove $A_n(x_1, y_1) \in K$. Obviously, $A_{in}(x_1, y_1)(t) \geq \theta, i = 1, 2, t \in (0,1)$. By (H_1) and Lemma 3.1, we have

$$\begin{aligned}
&\| f_i(s, (T_1 x_1)(s), (T_2 y_1)(s), (T_1 x_1)(s) + (T_2 y_1)(s) + \frac{e}{n}) \| \\
&\leq p_i(s) q_i[\frac{1}{n}, \| (T_1 x_1, T_2 y_1) \|_{PC} + 1] \\
&\leq p_i(s) q_i[\frac{s}{n}, \| (T_1 x_1, T_2 y_1) \|_{PC} + 1], s \in (0,1), i = 1, 2.
\end{aligned}$$

It follows from (3.6) that

$$\begin{aligned}
&\| A_{in}(x_1, y_1)(t) \| \\
&\leq \int_0^1 G(t,s) \| f_i(s, (T_1 x_1)(s), (T_2 y_1)(s), (T_1 x_1)(s) + (T_2 y_1)(s) + \frac{e}{n}) \| ds \\
&\leq \int_0^1 s p_i(s) q_i[\frac{s}{n}, \| (T_1 x_1, T_2 y_1) \|_{PC} + 1] ds \\
&< +\infty, \ i = 1, 2,
\end{aligned}$$

that is

$$\| A_n(x_1, y_1)(t) \| = \sup_{t \in (0,1)} \| A_{1n}(x_1, y_1)(t) \| + \sup_{t \in (0,1)} \| A_{2n}(x_1, y_1)(t) \| < +\infty.$$

Therefore, $A_n : K_r \to K$ is a bounded operator. In addition, for $\forall t, t' \in J$, we have

$$\begin{aligned}
&\| A_{in}(x_1, y_1)(t) - A_{in}(x_1, y_1)(t') \| \\
&\leq \| \int_0^1 G(t,s) f_i(s, (T_1 x_1)(s), (T_2 y_2)(s), (T_1 x_1)(s) + (T_2 y_1)(s) + \frac{e}{n}) ds \\
&- \int_0^1 G(t',s) f_i(s, (T_1 x_1)(s), (T_2 y_2)(s), (T_1 x_1)(s) + (T_2 y_1)(s) + \frac{e}{n}) ds \| \\
&\leq \int_0^1 | G(t,s) - G(t',s) | s p_i(s) q_i[\frac{s}{n}, \| (T_1 x_1, T_2 y_1) \|_{PC} + 1] ds \\
&\to 0 \ (t \to t'), i = 1, 2.
\end{aligned} \tag{3.7}$$

Therefore, $A_n(x_1, y_1) \in C[J \times J]$. For $\forall t, u, s \in J$, in view of

$$G(t,s) = \min\{t,s\} \geq \min\{ut,s\} \geq t \min\{u,s\} = tG(u,s),$$

for any $t \in (0,1)$, we have

$$\begin{aligned}
&A_{in}(x_1,y_1)(t)\\
&= \int_0^1 G(t,s) f_i(s,(T_1x_1)(s),(T_2y_1)(s),(T_1x_1)(s)+(T_2y_1)(s)+\frac{e}{n})ds\\
&\geq t\int_0^1 G(u,s) f_i(s,(T_1x_1)(s),(T_2y_1)(s),(T_1x_1)(s)+(T_2y_1)(s)+\frac{e}{n})ds\\
&= tA_{in}(x_1,y_1)(u), i=1,2.
\end{aligned}$$

Therefore, $A_n(K_r) \subseteq K$.

Now, we prove that $A_n : K_r \to K$ is continuous. Assume $\{(x_{1k},y_{1k})\} \subseteq K_r, (x_{10},y_{10}) \in K_r$, such that

$$\lim_{k\to\infty} \| (x_{1k},y_{1k}) - (x_{10},y_{10}) \|_C = 0.$$

In the following, we prove $\lim\limits_{k\to\infty} \| A_n(x_{1k},y_{1k}) - A_n(x_{10},y_{10}) \|_C = 0$. From the hypothesis, it is easy to know that $\{(x_{1k},y_{1k}) : k \in N\}$ (N is a natural number) is bounded, and then $\{x_{1k}\}$,$\{y_{1k}\}$ are bounded. According to Lemma 3.1, $T_i (i=1,2)$ are continuous and bounded. Therefore, $\lim\limits_{k\to\infty} \| T_1x_{1k} - T_1x_{10} \|_{PC} = 0$ and $\lim\limits_{k\to\infty} \| T_2y_{1k} - T_2y_{10} \|_{PC} = 0$. So, we know that $\{T_1x_{1k}\}$ and $\{T_2y_{1k}\}$ are bounded, namely, there exists $M' > 0$, for $\forall k \in N$, there exists $\| T_1x_{1k} \|_{PC} \leq \frac{M'}{2}, \| T_2y_{1k} \|_{PC} \leq \frac{M'}{2}$. As a result, for $\forall k \in N, s \in (0,1)$, we have

$$\| f_i(s,(T_1x_{1k})(s),(T_2y_{1k})(s),(T_1x_{1k})(s)+(T_2y_{2k})(s)+\frac{e}{n}) \| \leq p_i(s)q_i[\frac{s}{n},M'+1], i=1,2. \quad (3.8)$$

It can be seen from (H_1) and (3.8) that $\lim\limits_{k\to\infty} A_{in}(x_{1k},y_{1k})(t) = A_{in}(x_{10},y_{10})(t), \forall t \in (0,1), i=1,2$. Similar to the derivation of (3.7), it is easy to derive that $\{A_n(x_{1k},y_{1k})\}_{k\geq 1}$ is equicontinuous on $(0,1)$ due to (3.8). By (H_2), for $\forall \varepsilon > 0$, there exists $N_i \in N$ such that, $k > N_i$,

$$\begin{aligned}
&\| f_i(s,(T_1x_{1k})(s),(T_2y_{1k})(s),(T_1x_{1k})(s)+(T_2y_{1k})(s)+\frac{e}{n})\\
&\quad - f_i(s,(T_1x_{10})(s),(T_2y_{10})(s),(T_1x_{10})(s)+(T_2y_{10})(s)+\frac{e}{n}) \| < 2\varepsilon, i=1,2.
\end{aligned}$$

It is obtained by the Lebesgue Control Convergence Theorem that

$$\begin{aligned}
&\| A_{in}(x_{1k},y_{1k})(t) - A_{in}(x_{10},y_{10})(t) \|\\
&\leq \int_0^1 G(t,s) \| f_i(s,(T_1x_{1k})(s),(T_2y_{1k})(s),(T_1x_{1k})(s)+(T_2y_{1k})(s)+\frac{e}{n})\\
&\quad - f_i(s,(T_1x_{10})(s),(T_2y_{10})(s),(T_1x_{10})(s)+(T_2y_{10})(s)+\frac{e}{n}) \| ds\\
&\leq \int_0^1 s\cdot 2\varepsilon ds = \varepsilon, i=1,2.
\end{aligned}$$

Let $N = \max\{N_i, i=1,2\}$. When $k > N$, we have $\| A_n(x_{1k},y_{1k})(t) - A_n(x_{10},y_{10})(t) \| \leq \varepsilon$. Based on this, we have $\lim\limits_{k\to\infty} A_n(x_{1k},y_{1k})(t) = A_n(x_{10},y_{10})(t), t \in (0,1)$. Thus, the Ascholi-Arzela theorem tells us that $\{A_n(x_{1k},y_{1k})\}$ is relatively compact in K. Therefore, there should be $\lim\limits_{k\to\infty} \| A_n(x_{1k},y_{1k}) - A_n(x_{10},y_{10}) \|_C = 0$. In fact, if this is not true, then there are $\varepsilon_0 > 0$ and $\{(x_{1k_i},y_{1k_i})\} \subset \{x_{1k},y_{1k})\}$ satisfying $\| A_n(x_{1k_i},y_{1k_i}) - A_n(x_{10},y_{10}) \| \geq \varepsilon_0 (i=1,2,\cdots)$. Since $\{A_n(x_{1k},y_{1k})\}$ is relatively compact, then there exists a subsequence of $\{A_n(x_{1k_i},y_{1k_i})\}$, which converges to some (x_1,y_1) in K. Without loss of generality, let us still set $\lim\limits_{i\to\infty} A_n(x_{1k_i},y_{1k_i}) = (x_1,y_1)$, i.e., $\lim\limits_{i\to\infty} \| A_n(x_{1k_i},y_{1k_i})-(x_1,y_1) \|_C= 0$. So, there are $(x_1,y_1) = A_n(x_{10},y_{10})$. This is a contradiction. Hence, $A_n : K_r \to K$ is continuous. □

Lemma 3.4. *If condition $(H_1)-(H_3)$ are satisfied, then $A_n : K_r \to K$ is a strict set compression operator.*

Proof. Let $\Omega \subseteq K_r$ be bounded. From the proof of the bounded operator in Lemma 3.3, we know that $A_n(\Omega)$ is bounded. Now We only need to prove that

$$\alpha_C(A_n(\Omega)) \le l\alpha_C(\Omega),\ l<1. \tag{3.9}$$

In fact, let $\Omega_1 = \{x_1|(x_1,y_1)\in\Omega\}$ and $\Omega_2 = \{y_1|(x_1,y_1)\in\Omega\}$. By Lemma 3.1, there exists $M''\ge 0$ such that, for $\forall x_1\in\Omega_1$, $\| T_1x_1 \|_{PC}\le \frac{M''}{2}$ and, for $\forall y_1\in\Omega_2$, $\| T_2y_2 \|_{PC}\le \frac{M''}{2}$. From Lemma 3.2, there is

$$\alpha_C(A_n(\Omega)) \le \sup_{t\in(0,1)} \alpha(A_{1n}(\Omega)(t)) + \sup_{t\in(0,1)} \alpha(A_{2n}(\Omega)(t)), \tag{3.10}$$

where $A_{in}(\Omega)(t) = \{A_{in}(x_1,y_1)(t) \mid (x_1,y_1)\in\Omega\}, t\in(0,1)$. Let

$$D_{i\delta} := \{\int_\delta^{1-\delta} G(t,s)f_i(s,(T_1x_1)(s),(T_2y_1)(s),(T_1x_1)(s)+(T_2y_1)(s)+\frac{e}{n})ds, (x_1,y_1)\in\Omega\},$$

$\delta\in(0,\frac12)$ for each $i=1,2$. According to (H_1) and (H_2), for $\forall(x_1,y_1)\in\Omega, \forall t\in(0,1)$, we have

$$\begin{aligned}
&\| \int_\delta^{1-\delta} G(t,s)f_i(s,(T_1x_1)(s),(T_2y_1)(s),(T_1x_1)(s)+(T_2y_1)(s)+\frac{e}{n})ds \\
&\quad -\int_0^1 G(t,s)f_i(s,(T_1x_1)(s),(T_2y_1)(s),(T_1x_1)(s)+(T_2y_1)(s)+\frac{e}{n})ds \| \\
&\le \| \int_0^\delta G(t,s)f_i(s,(T_1x_1)(s),(T_2y_1)(s),(T_1x_1)(s)+(T_2y_1)(s)+\frac{e}{n})ds \| \\
&\quad + \| \int_{1-\delta}^1 G(t,s)f_i(s,(T_1x_1)(s),(T_2y_1)(s),(T_1x_1)(s)+(T_2y_1)(s)+\frac{e}{n})ds \| \\
&\le \int_0^\delta sp_i(s)ds\cdot q_i[\frac{1}{n},M''+1] + \int_{1-\delta}^1 sp_i(s)ds\cdot q_i[\frac{1}{n},M''+1] \\
&\le q_i[\frac{1}{n},M''+1](\int_0^\delta sp_i(s)ds + \int_{1-\delta}^1 sp_i(s)ds), i=1,2.
\end{aligned} \tag{3.11}$$

Therefore, it can be see from (3.11) that the Hausdorff distance between D_δ and $A_n(\Omega)$ approaches 0, namely, $d_H(D_\delta,A_n(\Omega))\to 0(\delta\to 0^+)$. Hence, there is

$$\alpha_C(A_n(\Omega)) = \lim_{\delta\to 0^+} \alpha_C(D_\delta) \tag{3.12}$$

Observe that

$$\begin{aligned}
&\int_\delta^{1-\delta} G(t,s)f_i(s,(T_1x_1)(s),(T_2y_1)(s),(T_1x_1)(s)+(T_2y_1)(s)+\frac{e}{n})ds \\
&\in (1-2\delta)\overline{CO}\{G(t,s)f_i(s,(T_1x_1)(s),(T_2y_1)(s),(T_1x_1)(s)+(T_2y_1)(s)+\frac{e}{n}), \\
&s\in[\delta,1-\delta]\}, i=1,2.
\end{aligned}$$

Let $I_\delta = [\delta, 1-\delta]$. For $\forall t \in (0,1)$, from (3.10), (H_2), (H_3), and [5, Equation (9.4.11)], we find that

$$\begin{aligned}
&\alpha(A_{in}(\Omega)(t)) \\
&= \alpha(\{\int_\delta^{1-\delta} G(t,s) f_i(s,(T_1x_1)(s),(T_2y_1)(s),(T_1x_1)(s)+(T_2y_1)(s)+\frac{e}{n})ds : (x_1,y_1)\in\Omega\}) \\
&\le (1-2\delta)\alpha(\overline{CO}\{G(t,s)f_i(s,(T_1x_1)(s),(T_2y_1)(s),(T_1x_1)(s) \\
&\quad +(T_2y_1)(s)+\frac{e}{n}) : s\in I_\delta,(x_1,y_1)\in\Omega\}) \\
&\le \alpha(\{G(t,s)f_i(s,(T_1x_1)(s),(T_2y_1)(s),(T_1x_1)(s)+(T_2y_1)(s)+\frac{e}{n}) : s\in I_\delta,(x_1,y_1)\in\Omega\}) \\
&\le \alpha(\{f_i(s,(T_1x_1)(s),(T_2y_1)(s),(T_1x_1)(s)+(T_2y_1)(s)+\frac{e}{n}) : s\in I_\delta,(x_1,y_1)\in\Omega\}) \\
&\le l_{i1}\max_{s\in I_\delta}\alpha((T_1\Omega_1)(s)) + l_{i2}\max_{s\in I_\delta}\alpha((T_2\Omega_2)(s)) + l_{i3}\max_{s\in I_\delta}\alpha((T_1\Omega_1)(s)+(T_2\Omega_2)(s)) \\
&\le 2(l_{i1}+l_{i3})\alpha_{PC}(T_1\Omega_1) + 2(l_{i2}+l_{i3})\alpha_{PC}(T_2\Omega_2) \\
&\le 2(l_{i1}+l_{i3})(1+\sum_{k=1}^m L_{1k})\alpha(\Omega_1) + 2(l_{i2}+l_{i3})(1+\sum_{k=1}^m L_{2k})\alpha(\Omega_2) \\
&\le 2[(l_{i1}+l_{i3})(1+\sum_{k=1}^m L_{1k}) + (l_{i2}+l_{i3})(1+\sum_{k=1}^m L_{2k})]\alpha_C(\Omega) \\
&< \frac{l}{2}\alpha_C(\Omega), i=1,2.
\end{aligned}$$

It is known from the arbitrariness of t that

$$\sup_{t\in(0,1)}\alpha(A_{1n}(\Omega)(t)) + \sup_{t\in(0,1)}\alpha(A_{2n}(\Omega)(t)) < l\alpha_C(\Omega),$$

namely, $\alpha_C(D_\delta) < l\alpha_C(\Omega)$. Combining (3.10) and (3.12), we see that equation (3.9) can be obtained immediately, namely, $\alpha_C(A_n(\Omega)) \le l\alpha_C(\Omega)$. Therefore, $A_n : K_r \to K$ is a strict set compression operator. □

Theorem 3.5. *If conditions (H_1)-(H_5) are satisfied, and there exists $R>0$ such that*

$$\int_0^1 sp_i(s)q_i[sR, T_R+1]ds < \frac{R}{2}, i=1,2. \tag{3.13}$$

Then A_n has at least two nontrivial fixed points in the cone K.

Proof. We only need to show that A_n satisfies Lemma 2.4 in cone K. First, for R in (3.13), there is

$$\| A_n(x_1,y_1) \| \le \| (x_1,y_1) \|, \forall (x_1,y_1)\in\partial K_R. \tag{3.14}$$

In fact, for $\forall(x_1,y_1)\in\partial K_R$ and $\forall t\in(0,1)$, it can be seen from (H_1) and equation (3.13) that

$$\begin{aligned}
&\| A_{in}(x_1,y_1)(t) \| \\
&= \| \int_0^1 G(t,s)f_i(s,(T_1x_1)(s),(T_2y_1)(s),(T_1x_1)(s)+(T_2y_1)(s)+\frac{e}{n})ds \| \\
&\le \int_0^1 sp_i(s)q_i[sR,T_R+1]ds < \frac{R}{2}, i=1,2.
\end{aligned}$$

It is known by the arbitrariness of t that

$$\| A_n(x_1,y_1) \| = \sup_{t\in(0,1)} \| (A_{1n}(x_1,y_1)(t)) \| + \sup_{t\in(0,1)} \| (A_{2n}(x_1,y_1)(t)) \| < R.$$

Namely, equation (3.14) is true.

Secondly, from (H_4), if we choose a sufficiently small $r(0<r<\min\{1,R\})$, then

$$\| A_n(x_1,y_1) \| \geq \| (x_1,y_1) \|, \forall (x_1,y_1) \in \partial K_r. \tag{3.15}$$

In fact, because $(x_1,y_1) \to (\theta,\theta)$ is equivalent to $(T_1x_1,T_2y_1) \to (I_{1k}(t)),I_{2k}(t)))$, there is $\delta_i > 0$ such that, when $\| (x_1,y_1) \| < \delta_i$,

$$\begin{aligned}
\| A_{in}(x_1,y_1) \|_C &\geq \sup_{t\geq a} \| \int_0^1 G(t,s)f_{in}(s,(T_1x_1)(s),(T_2y_1)(s),(T_1x_1)(s)+(T_2y_1)(s)+\frac{e}{n})ds \| \\
&\geq \| \int_0^1 G(a,s)m_i(s)((T_1x_1)(s)+(T_2y_1)(s))ds \| \\
&\geq \| \int_a^b G(a,s)m_i(s)(x_1(s)+y_1(s))ds \| \\
&\geq \int_a^b asm_i(s)ds\cdot \| (x_1,y_1) \|, i=1,2.
\end{aligned}$$

Therefore, when $\delta = \min\{\delta_1,\delta_2\}$ and $\| (x_1,y_1) \| < \delta$, there is

$$\begin{aligned}
\| A_n(x_1,y_1) \| &= \| A_{1n}(x_1,y_1) \|_C + \| A_{2n}(x,y) \|_C \\
&\geq (\int_a^b asm_1(s)ds + \int_a^b asm_2(s)ds)\cdot \| (x_1,y_1) \| \\
&\geq \| (x_1,y_1) \| .
\end{aligned}$$

This means that equation (3.15) is true.

Finally, if (H$_5$) holds, we can choose sufficiently large $R_1 > \max\{1,R\}$ to prove

$$\| A_n(x_1,y_1) \| \geq \| (x_1,y_1) \|, \forall (x_1,y_1) \in \partial K_{R_1} \tag{3.16}$$

In fact, there are $\overline{N}_i > 0$ such that, when $\| (x_1,y_1) \| > \overline{N}_i$,

$$\begin{aligned}
&\| A_{in}(x_1,y_1) \| \\
&\geq \sup_{t\geq a^*} \| \int_0^1 G(t,s)f_{in}(s,(T_1x_1)(s),(T_2y_1)(s),(T_1x_1)(s)+(T_2y_1)(s)+\frac{e}{n})ds \| \\
&\geq \| \int_{a^*}^{b^*} G(a^*,s)n_i(s)((T_1x_1)(s)+(T_2y_1)(s))ds \| \\
&\geq \| \int_{a^*}^{b^*} a^*n_i(s)\cdot(x_1(s)+y_1(s))ds \| \\
&\geq \int_{a^*}^{b^*} a^*sn_i(s)ds\cdot \| (x_1,y_1) \|, i=1,2.
\end{aligned}$$

If $\| (x_1,y_1) \| > \max\{\overline{N}_1,\overline{N}_2\}$, we see that there is

$$\begin{aligned} \| A_n(x_1,y_1) \| &= \| A_{1n}(x_1,y_1) \|_C + \| A_{2n}(x_1,y_1) \|_C \\ &\geq \int_{a^*}^{b^*} a^* s(n_1(s)+n_2(s))ds \cdot \| (x_1,y_1) \| > \| (x_1,y_1) \| . \end{aligned}$$

That is, equation (3.16) is true. By combining (3.14)- (3.16) and Lemma 2.4, we know that A_n has at least one non-trivial fixed point on $\overline{K}_{R_1} \backslash K_R$ and $\overline{K}_R \backslash K_r$, respectively. □

Theorem 3.6. *If all the conditions of Theorem 3.5 are satisfied, then there are at least two positive solutions to boundary value problem (1.1).*

Proof. We see that, when $n \geq n_0$, $\{(x_{1n},y_{1n})\}$ and $\{(\overline{x}_{1n},\overline{y}_{1n})\}$ are the sequences obtained by Theorem (3.5). Then $A_n(x_{1n},y_{1n}) = (x_{1n},y_{1n}), (x_{1n},y_{1n}) \in \overline{K}_{R_1} \setminus K_R$, and $A_n(\overline{x}_{1n},\overline{y}_{1n}) = (\overline{x}_{1n},\overline{y}_{1n}), (\overline{x}_{1n},\overline{y}_{1n}) \in \overline{K}_R \setminus K_r$. Let $D_1 =: \{(x_{1n},y_{1n}) \mid n \geq n_0\}$. Obviously, D_1 is uniformly bounded. In view of

$$\begin{cases} x_{1n}(t) = \int_0^1 G(t,s) f_1(s,(T_1x_{1n})(s),(T_2y_{1n})(s),(T_1x_{1n})(s)+(T_2y_{1n})(s)+\frac{e}{n})ds; \\ y_{1n}(t) = \int_0^1 G(t,s) f_2(s,(T_1x_{1n})(s),(T_2y_{1n})(s),(T_1x_{1n})(s)+(T_2y_{1n})(s)+\frac{e}{n})ds, \end{cases} \tag{3.17}$$

similar to the proof in Lemma 3.3, it can be obtained that D_1 is a relatively compact set in $\overline{K}_{R_1} \setminus K_R$, so there exists a subcolumn of $\{(x_{1n},y_{1n}) \mid n \geq n_0\}$ converging to $(x_1,y_1) \in \overline{K}_{R_1} \setminus K_R$. Let us say that $\{(x_{1n},y_{1n}) \mid n \geq n_0\}$ converges to (x_1,y_1), that is, $\lim_{n\to\infty} x_{1n}(t) = x_1(t)$, $\lim_{n\to\infty} y_{1n}(t) = y_1(t)$. According to Lebesgue's Control Convergence Theorem, letting $n \to \infty$ in (3.17), we have

$$\begin{cases} x_1(t) = \int_0^1 G(t,s) f_1(s,(T_1x_1)(s),(T_2y_1)(s),(T_1x_1)(s)+(T_2y_1)(s))ds; \\ y_1(t) = \int_0^1 G(t,s) f_2(s,(T_1x_1)(s),(T_2y_1)(s),(T_1x_1)(s)+(T_2y_1)(s))ds. \end{cases}$$

So we obtain a positive solution (x,y) to boundary value problem (1.1) satisfying

$$\begin{cases} x(t) = x_1(t) + \sum\limits_{0<t_k<t} I_{1k}(x(t_k)), \\ y(t) = y_1(t) + \sum\limits_{0<t_k<t} I_{2k}(y(t_k)). \end{cases}$$

Similarly, $\{(\overline{x}_{1n},\overline{y}_{1n}) \mid n \geq n_0\}$ has subcolumns converging to $(\overline{x}_1,\overline{y}_1)$. Obviously, $(\overline{x}_1,\overline{y}_1) \in \overline{K}_R \setminus K_r$, so $(\overline{x}(t),\overline{y}(t))$ satisfies

$$\begin{cases} \overline{x}(t) = \overline{x}_1(t) + \sum\limits_{0<t_k<t} I_{1k}(\overline{x}(t_k)), \\ \overline{y}(t) = \overline{y}_1(t) + \sum\limits_{0<t_k<t} I_{2k}(\overline{y}(t_k)). \end{cases}$$

Hence, $(\overline{x}(t),\overline{y}(t))$ is another positive solution to boundary value problem (1.1). Now let us prove $(x,y) \neq (\overline{x},\overline{y})$, namely $(x_1,y_1) \neq (\overline{x}_1,\overline{y}_1)$. We need only to prove that operator A_n has no fixed point on ∂K_R. In fact, suppose that $(x_1',y_1') \in \partial K_R$ is a fixed point of operator A_n, that is,

$$A_n(x_1',y_1')(t) = (A_{n1}(x_1',y_1')(t), A_{n2}(x_1',y_1')(t)),$$

similar to (3.14), it can be obtained that $R = \| (x_1',y_1') \| = \| x_1' \|_C + \| y_1' \|_C < R$. This is a contradiction, so A_n has no fixed point on K_R. □

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