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ON PERTURBATIONS OF ACCRETIVE OPERATORS IN LOCALLY CONVEX SPACES

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Abstract. Let E be a complete locally convex Hausdorff space, $A : D(A) \subset E \to E$ be an accretive operator satisfying the range condition, $T : D(T) \subset E \to E$ be a continuous compact mapping, and $p \in E$. In this paper, we prove the existence of solutions to the operator equation $p \in Ax + Tx$ and give a sufficient condition for the operator A to satisfy the range condition.

Keywords. Accretive operator; Complete locally convex Hausdorff space; Compact mapping; Topological degree.

1. Introduction and preliminaries

In this paper, let E be a real Hausdorff topological vector space generated by a countable increasing family of semi-norms $\{p_i\}_{i=1}^{+\infty}$, that is, E is a locally convex space. In this paper, we also assume that E is complete. Next, we list some complete locally convex spaces.

Example 1.1. Let $C_0^\infty[0,1]$ be the set of infinitely differentiable functions $x(\cdot)$ with $x^{(i)}(0) = x^{(i)}(1) = 0, i \geq 0\}$ and $p_i(x(\cdot)) = \max\{|x^{(j)}(t)| : j \leq i, t \in [0,1]\}, i \geq 0\}$. Then $C_0^\infty[0,1]$ is a locally convex space generated by semi-norms $\{p_i\}_{i=0}^\infty$, and it is complete.

Example 1.2. Let $C(R) = \{x(\cdot) : R \to R$ is continuous$\}$ and $p_i(x\cdot) = \max\{|x(t)| : t \in [-i,i]\}, i \geq 1\}$. Then $C(R)$ is a locally convex space generated by semi-norms $\{p_i\}_{i=1}^\infty$.

Example 1.3. Let $L^2(R) = \{x(\cdot) : R \to R$ is locally Lebesgue integrable$\}$ and $p_i(x(\cdot)) = (\int_{-i}^i x^2(t)\, dt)^{\frac{1}{2}}$ for each $i \geq 1$. Then $L^2(R)$ is a locally convex space generated by semi-norms $\{p_i\}_{i=1}^\infty$, and it is also complete.

The theory of nonlinear operators in locally convex spaces has been studied by many authors; see, e.g., [10, 11, 13, 14, 15, 16, 17]. In particular, accretive operators in locally convex spaces were extensively studied in [1] and [8].

Now, we recall some definitions for our main results.

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Definition 1.4. Let $A : D(A) \subset E \to 2^E$ be a nonlinear operator. Then
A is said to be *accretive* if, for any $x, y \in E$,

$$p_i(x-y) \le p_i(x-y+\lambda(u-v))$$

for all $u \in Ax$, $v \in Ay$, $\lambda > 0$ and $i \ge 1$;
A is said to be *maximal accretive* if it is accretive and

$$p_i(x-y_0) \le p_i(x-y_0+\lambda(u-v_0)),$$

for all $x \in D(A)$, $\lambda > 0$, $u \in Ax$ and $i \ge 1$, implies that $y_0 \in D(A)$ and $v_0 \in Ay_0$;
An accretive operator A is said to be *m-accretive* if

$$R(I+\lambda A) = E$$

for any $\lambda > 0$.

Example 1.5. Let $C_0^\infty[0,1]$ be same as in Example 1.1 and $A : C_0^\infty[0,1] \to C_0^\infty[0,1]$ be a mapping defined by

$$Ax(\cdot) = x'(\cdot) + \lambda x(\cdot)$$

for all $x(\cdot) \in C_0^\infty[0,1]$, where $\lambda > 0$ is a constant. Then A is an accretive operator on $C_0^\infty[0,1]$.

Example 1.6. Let $L^2(R)$ be same as in Example 1.3 and $K(x,y) : R \times R$ be a continuous function satisfying the following conditions:
$|K(x,y)| \le M|y| + N$ for all $(x,y) \in R^2$, where $M, N > 0$ are constants;
$[K(z,x) - K(z,y)](x-y) \ge 0$ for all $x, y, z \in R$.

We define a mapping $A : L^2(R) \to L^2(R)$ as follows:

$$(Af)(x) = K(x, f(x))$$

for all $f \in L^2(R)$ and $x \in R$. Then A is an accretive operator on $L^2(R)$.

Also, we recall the following definitions of *semi-inner products* in locally convex spaces:

$$[x,y]_i^+ = \lim_{h \to 0^+} \frac{p_i(x+hy) - p_i(x)}{h}$$

and

$$[x,y]_i^- = \lim_{h \to 0^+} \frac{p_i(x) - p_i(x-hy)}{h}$$

for all $x, y \in E$.

For each $i \in I$, $(x,y)_i^+ = p_i(x)[x,y]_i^+$ is called the *upper semi-inner product* with respect to $i \in I$. Analogously, $(x,y)_i^- = p_i(x)[x,y]_i^-$ is said to be the *lower semi-inner product* with respect to $i \in I$. For properties of semi-inner products, we refer to [3].

For each $\lambda > 0$, let

$$J_\lambda x := (I+\lambda A)^{-1}x, \quad A_\lambda x := \frac{1}{\lambda}(x - J_\lambda x)$$

for all $x \in R(I+\lambda A)$, where I is the identity mapping on E.

Proposition 1.7. [8] *Let $A : D(A) \subset E \to 2^E$ be an accretive operator and $\lambda > 0$. Then,*
$p_i(J_\lambda x - J_\lambda y) \le p_i(x-y)$ for all $x, y \in R(I+\lambda A)$ and $i \ge 1$;
A_λ is accretive.

Proposition 1.8. [8] *The following statements are equivalent:*
$A : D(A) \subset E \to 2^E$ *is accretive;*
$[x-y, u-v]_i^+ \geq 0$ *for all* $x, y \in D(A)$, $u \in Ax$, $v \in Ay$ *and* $i \geq 1$*;*
$(x-y, u-v)_i^+ \geq 0$ *for all* $x, y \in D(A)$, $u \in Ax$, $v \in Ay$ *and* $i \geq 1$.

In this paper, we show the existence of solutions to the operator equation $p \in Ax + Tx$ and give a sufficient condition for the operator A to satisfy the range condition.

2. Main Results

2.1. **The existence results.** Suppose that E is a real Hausdorff topologoical vector space generated by a countable family of semi-norms $\{p_i\}_{i=1}^{+\infty}$, and E is complete. Let $A : D(A) \subset E \to E$ be an accretive operator, $C : D(C) \subset E \to E$ be a continuous compact mapping, (i.e., C is continuous and $\overline{C(D(C))}$ is compact), and $p \in E$.

In this section, we present several existence results for the operator equation $p \in Ax + Cx$. In fact, such type equations in Banach spaces has been studied by [2, 6, 7, 9] and [12].

First, we need the following result from [14] (see also [15]) for our main results.

Theorem 2.1. *Let $U \subset E$ be an open subset and $T : \overline{U} \to E$ be a continuous mapping such that $T\overline{U}$ is compact and $x \neq Tx$ for all $x \in \partial U$. Then there exists a topological degree $deg(I-T, U, 0)$ satisfying the following properties:*
(1) $deg(I, U, 0) = 1$ if and only if $0 \in U$.
(2) If $deg(I-T, U, 0) \neq 0$, then $Tx = x$ has a solution in U.
(3) Let $T_t : [0,1] \times \overline{U} \to E$ be a continuous compact operator and $T_t x \neq x$ for all $(t, x) \in [0,1] \times \partial U$. Then $deg(I-T_t, U, 0)$ does not depend on $t \in [0,1]$.
(4) Let U_1, U_2 be two disjoint open subsets of U and $0 \notin (I-T)(\overline{U \setminus U_1 \cup U_2})$. Then

$$deg(I-T, U, 0) = deg(I-T, U_1, 0) + deg(I-T, U_2), 0).$$

Theorem 2.2. [11] *Let $U \subset E$ be an open subset, $P \subset E$ be a cone, and $T : \overline{U} \cap P \to P$ be a continuous mapping such that $T\overline{U} \cap P$ is compact and $x \notin Tx$ for all $x \in \partial U \cap P$. Then there exists a fixed point index, $ind(T, \Omega \cap P)$, satisfying the following properties:*
(1) $ind(x_0, U \cap P) = 1$ if $x_0 \in U \cap P$.
(2) If $ind(T, U \cap P) \neq 0$, then $x = Tx$ has a solution in $U \cap P$;
(3) If $U_i \subset U$ for $i = 1, 2$, $U_1 \cap U_2 = \emptyset$ and $0 \notin (I-T)[(\overline{U} \setminus (U_1 \cup U_2)) \cap P]$, then

$$ind(T, U \cap P) = ind(T, U_1 \cap P) + ind(T, U_2 \cap P).$$

(4) If $H(t,x) : [0,1] \times \overline{U} \cap P \to P$ is a continuous compact mapping and $x \neq H(t,x)$ for all $(t,x) \in [0,1] \times \partial U \cap P$, then $ind(H(t,\cdot), U \cap P)$ does not depend on $t \in [0,1]$.

By using Theorems 2.1 and 2.2, we have the following theorem.

Theorem 2.3. *Let $A : D(A) \subset E \to 2^E$ be an accretive operator, $F \subseteq E$ be a closed subspace with $F = (I + \lambda A)(D(A))$ for all $\lambda > 0$, and $U \subset F$ be an open subset with $\overline{U} \cap D(A) \neq \emptyset$. Let $T : D(A) \cap \overline{U} \to F$ be an operator and $V = (I + k_0 A)(U \cap D(A))$, where $k_0 > 0$ is a constant and $p \in F$. Suppose that $(I - k_0 T)(I + k_0 A)^{-1}$ is continuous, $(I - k_0 T)(I + k_0 A)^{-1}\overline{V}$ is compact, and there exists $z \in D(A) \cap U$ such that*

$$(g_i, Tx + f - p) \geq 0$$

for all $g_i \in E^*$ with $g_i(x-z) = p_i^2(x-z)$ for each $i \geq 1$, $x \in D(A)\cap U$ and $f \in Ax$. Then

$$p \in (A+T)(D(A)\cap \overline{U}).$$

Proof. We may assume that $z = 0$, $p = 0$, and $0 \in A0$. Otherwise, we set $U' = U - z$ and $A'x = A(x+z) - a$ for $x \in D(A') = D(A) - z$, where $a \in Az$ is a fixed element and $T'x = T(x+z) + a - p$ for all $x \in U' \cap D(A')$. We note that $0 \in Ax + Tx$ has a solution $x \in D(A)\cap \overline{U}$ if and only if $y = (I - k_0T)(I + k_0A)^{-1}y$ has a solution $y \in \overline{V}$. Since $(I + k_0A)^{-1}$ is continuous, $V = (I+k_0A)(D(A)\cap U$ is an open subset of F. We may assume that $y \neq (I-k_0T)(I+k_0A)^{-1}y$ for all $y \in \partial V$.

Now, we claim that $y \neq t(I - k_0T)(I + k_0A)^{-1}y$ for all $t \in [0,1]$ and $y \in \partial V$. If this is not true, then there exist $t_0 \in [0,1]$ and $y_0 \in \partial V$ such that $y_0 = t_0(I - k_0T)(I + k_0A)^{-1}y_0$. Set $x_0 = (I + k_0A)^{-1}y_0$. Then, $x_0 \in \partial U \cap D(A)$ and

$$t_0(T - k_0T)x_0 \in (I + k_0A)x_0.$$

So there exists $u_0 \in Ax_0$ such that $t_0(x_0 - k_0Tx_0) = x_0 + k_0u_0$, i.e.,

$$(t_0 - 1)x_0 = t_0k_0(Tx_0 + u_0) + (1 - t_0)k_0u_0.$$

Now, we take $g_i \in E^*$ such that $g_i(x_0) = p_i^2(x_0)$ for each $i \geq 1$ and $g_i(u_0) = (u_0,x_0)_i^+ \geq 0$ for each $i \geq 1$. Such g_i exists due to the Hahn-Banach Theorem. It follows that

$$(t_0 - 1)p_i^2(x_0) = t_0k_0(Tx_0 + u_0, g_i) + (1 - t_0)g_i(u_0) \geq 0$$

for each $i \geq 1$. Therefore, $t_0 = 1$ and $0 \in Ax_0 + Tx_0$, which is a contradiction, so we proved the claim. By Theorem 2.1, we have

$$deg(I - (I - k_0T)(I + k_0A)^{-1}, V, 0) = deg(I, V, 0) = 1.$$

So, $y = (I - k_0T)(I + k_0A)^{-1}y$ has a solution in V, i.e., $0 \in Ax + Tx$ has a solution in $D(A)\cap U$. This completes the proof. □

Theorem 2.4. *Let $A : D(A) \subset E \to 2^E$ be an m-accretive operator, and U be an open subset with $0 \in U \cap D(A)$ and $0 \in A0$. Let $T : D(A) \to E$ be an operator and $V = (I+k_0A)(U\cap D(A))$, where $k_0 > 0$ is a constant. Suppose that $(I - k_0T)(I + k_0A)^{-1}$ is continuous, $(I - k_0T)(I + k_0A)^{-1}\overline{V}$ is compact and*

$$p_i((I - k_0T)y) \leq p_i(y)$$

for all $y \in D(A)\cap \partial U$ and $i \geq 1$. Then $0 \in (A+T)(D(A)\cap \overline{U})$ has a solution.

Proof. We may assume that $0 \notin (A+T)(D(A)\cap \partial U)$. Since $(I + k_0A)^{-1}$ is continuous, V is open in E.

Now, we claim that $y \neq t(I - k_0T)(I + k_0A)^{-1}y$ for all $t \in [0,1]$ and $y \in \partial V$. If this is not true, then there exist $t_0 \in [0,1]$ and $y_0 \in \partial V$ such that $y_0 = t_0(I - k_0T)(I + k_0A)^{-1}y_0$. Set $x_0 = (I + k_0A)^{-1}y_0$. Then $x_0 \in \partial U \cap D(A)$ and there exists $u_0 \in Ax_0$ such that $x_0 + k_0u_0 = t_0(x_0 - k_0Tx_0)$. Since A is accretive and $0 \in A0$, we have

$$p_i(x_0) \leq p_i(x_0 + k_0u_0) = t_0p_i(x_0 - k_0Tx_0))$$

for each $i \geq 1$. This implies that $t_0 = 1$, which is a contradiction. So

$$deg(I - (I - k_0T)(I + k_0A)^{-1}, V, 0) = deg(I, V, 0) = 1$$

and $x = (I - k_0T)(I + k_0A)^{-1}x$ has a solution in V, i.e., $0 \in (A+T)(D(A)\cap \overline{U}$ has a solution. This completes the proof. □

Theorem 2.5. *Let $A : D(A) \subset E \to 2^E$ be an m-accretive operator, U be an open subset with $0 \in U \cap D(A)$ and $0 \in A0$, and let $T : \overline{U} \to E$ be continuous and compact. Suppose that $p_i(Tx) \leq p_i(x)$ for all $x \in \partial U \cap D(A)$ and $i \geq 1$. Then $-Ax + Tx$ has a fixed point in $\overline{U} \cap D(A)$.*

Proof. Since $(I + A)(D(A)) = E$, it is easy to see that $x \in -Ax + Tx$ if and only if $x \in (I + A)^{-1}Tx$. We may also assume that $x \neq (I + A)^{-1}Tx$ for all $x \in \partial U$.

Now, we show that $x \neq t(I + A)^{-1}Tx$ for all $t \in [0, 1]$ and $x \in \partial U$. If this is not true, then there exist $t_0 \in [0, 1]$ and $x_0 \in \partial U$ such that $x_0 = t_0(I + A)^{-1}Tx_0$. Since $0 \in A(0)$, we have $(I + A)^{-1}(0) = 0$. Thus

$$p_i(x_0) = t_0 p_i((I + A)^{-1}Tx_0) \leq t_0 p_i(Tx_0)$$

for each $i \geq 1$. From the assumption, we must have $t_0 = 1$, which is a contradiction. Thus $deg(I - (I + A)^{-1}T, U, 0) = deg(I, U, 0) = 1$, and $x = (I + A)^{-1}Tx$ has a solution in U, i.e., $-Ax + Tx$ has a fixed point in $D(A) \cap U$. This completes the proof. □

Theorem 2.6. *Let $P \subset E$ be a cone, $A : D(A) \subseteq P \to P$ be an accretive operator with $P = (I + \lambda A)(D(A))$ for all $\lambda > 0$, U be an open subset of E with $0 \in U$, and $T : D(A) \to P$ be a mapping such that $(I - k_0 T)(I + k_0 A)^{-1}$ is continuous and compact on $\overline{V}$, where $V = (I + k_0 A)(U \cap D(A))$ and $v_0 \in P$. Suppose that the following conditions are satisfied:*
(1) $Tx \leq k_0^{-1}x$ for all $x \in D(A)$;
(2) $(u + Tx - v_0, g_i) \geq 0$ for all i, $x \in \partial U \cap D(A))$ and $g_i \in E^*$ with $g_i(x) = p_i^2(x)$.
Then $p \in Ax + Tx$ has a solution in $D(A) \cap \overline{U}$.

Proof. We may assume that $p \notin (A + T)(\partial U \cap D(A))$. Set $Sx = Tx - p$ for all $x \in D(A)$, and note that V is relatively open in P. Since $Tx \leq k_0^{-1}x$ for all $x \in D(A)$, we have an operator $I - k_0 T : D(A) \to P$, so $(I - k_0 S)(I + k_0 A)^{-1} : P \to P$ is an operator.

Now, we can easily check that $x \neq t(I - k_0 S)(I + k_0 A)^{-1}x$ for all $x \in \partial V$ and $t \in [0, 1]$. By Theorem 2.2, we have $ind((I - k_0 S)(I + k_0 A)^{-1}, V) = ind(0, V)$.

Recall that $(I + \lambda A)D(A) = P$ for all $\lambda > 0$, so there exists $x_0 \in D(A)$ such that $0 \in x_0 + Ax_0$. Since $x_0 \geq 0$ and $Ax_0 \subseteq P$, we must have $x_0 = 0$ and $0 \in A(0)$. Therefore, $0 \in V$. Moreover, $ind(0, V) = 1$. Thus $y = (I - k_0 S)(I + k_0 A)^{-1}y$ has a solution in V, i.e., $p \in Ax + Tx$ has a solution in $D(A) \cap \overline{U}$. This completes the proof. □

Theorem 2.7. *Let $P \subset E$ be a cone, $A : D(A) \subseteq P \to P$ be an accretive operator with $P = (I + \lambda A)(D(A))$ for all $\lambda > 0$, U be an open subset of E with $0 \in U$, and $T : D(A) \to E$ be a mapping such that $Tx \leq k_0^{-1}x$ and $(I - k_0 T)(I + k_0 A)^{-1}$ is continuous and compact on $\overline{V}$, where $V = (I + k_0 A)(U \cap D(A))$. Suppose that $Tx + u \in P$ for all $x \in D(A)$ and $u \in Ax$. Then $0 \in Ax + Tx$ has a solution $D(A) \cap \overline{U}$.*

Proof. We may assume that $0 \notin (A + T)(\partial U \cap D(A))$. Note that V is relatively open in P. Since $Tx \leq k_0^{-1}x$ for all $x \in D(A)$, we have $I - k_0 T : D(A) \to P$.

Now, we claim that $x \neq t(I - k_0 T)(I + k_0 A)^{-1}x$ for all $x \in \partial V$ and $t \in [0, 1]$. If this is not true, there exist $t_0 \in [0, 1]$ and $x_0 \in \partial V$ such that

$$x_0 = t_0(I - k_0 T)(I + k_0 A)^{-1}x_0.$$

Setting $y_0 = (I + k_0 A)^{-1}x_0$, we have $y_0 \in \partial U \cap D(A)$ and $t_0(y_0 - k_0 T y_0) \in y_0 + k_0 A y_0$, i.e.,

$$(t_0 - 1)y_0 \in k_0(t_0 T y_0 + A y_0).$$

Since $Ty_0+u\in P$ and $u\in P$ for all $u\in Ay_0$, we must have $t_0=1$, which is a contradiction. In view of Theorem 2.2, we have $ind((I-k_0T)(I+k_0A)^{-1},V)=ind(0,V)=1$. Consequently, $0\in Ax+Tx$ has a solution $D(A)\cap\overline{U}$. This completes the proof. □

Theorem 2.8. *Let $P\subset E$ be a cone, $A:D(A)\subseteq P\to P$ be an accretive operator with $P=(I+\lambda A)(D(A))$ for all $\lambda>0$, U be an open subset of E with $0\in U$, and $T:D(A)\to E$ be a mapping such that $k_0Tx\leq x$ for all $x\in D(A)\cap\overline{U}$, $(I-k_0T)(I+k_0A)^{-1}$ is continuous, and compact on $\overline{V}$ for some $k_0>0$, where $V=(I+k_0A)(U\cap D(A))$, and $p\in P$ is a given element. If $p_i(x-k_0Tx+k_0p)\leq p_i(x)$ for all i and $x\in D(A)\cap\partial U$ with $p_i(x)>0$, then $p\in Ax+Tx$ has a solution $D(A)\cap\overline{U}$.*

Proof. We may assume that $p\notin(A+T)(\partial U\cap D(A))$. Set $Sx=Tx-p$ for all $x\in D(A)$. It is easy to check that $x\neq t(I-k_0S)(I+k_0A)^{-1}x$ for all $x\in\partial V$ and $t\in[0,1]$. So, we have $ind((I-k_0S)(I+k_0A)^{-1},V)=ind(0,V)=1$ and $p\in Ax+Tx$ has a solution $D(A)\cap\overline{U}$. This completes the proof. □

Theorem 2.9. *Let $P\subset E$ be a cone, $A:D(A)\subseteq P\to P$ be an accretive operator with $P=(I+A)(D(A))$, U be an open subset of E with $0\in U$, and $T:P\cap\overline{U}\to P$ be a continuous mapping such that $T(P\cap\overline{U})$ is compact. Suppose that*

$$p_i(Tx)\leq p_i(x)$$

for all $x\in\partial U\cap D(A)$ and $i\geq 1$ with $p_i(x)>0$. Then $-A+T$ has a fixed point in $D(A)\cap\overline{U}$.

Proof. It is easy to see that $x\in -Ax+Tx$ if and only if $x=(I+A)^{-}Tx$. We may also assume that $x\neq(I+A)^{-1}Tx$ for all $x\in\partial U$. Then we can easily see that $x\neq t(I+A)^{-1}Tx$ for all $t\in[0,1]$ and $x\in\partial U$. Thus

$$ind((I+A)^{-1}T,U,0)=ind(U,0)=1$$

and $x=(I+A)^{-1}Tx$ has a solution in U, i.e., $-Ax+Tx$ has a fixed point in $D(A)\cap U$. This completes the proof. □

2.2. **A sufficient condition for the range of accretive operators.** We suppose that E is a real Hausdorff topological vector space generated by a countable family $\{p_i\}_{i=1}^{+\infty}$ of semi-norms, and E is complete. Let $A:D(A)\subseteq E\to E$ be an accretive operator.

In this subsection, we provide a sufficient condition for the operator A to satisfy the range condition $D(A)\subseteq(I+\lambda A)(D(A))$. For the range of accretive operators in Banach spaces, we refer to [4] and [5].

Theorem 2.10. *Let $P\subset E$ be a cone, and let $A:P\to P$ be an accretive operator satisfying $Ax\leq\beta x$ for all $x\in P$, where $\beta>0$ is a constant. Suppose that $p_i(Ax-Ay)\leq Lp_i(x-y)$ for all $x,y\in P$ and $i\geq 1$. Then $(I+\lambda A)(P)=P$ for all $\lambda>0$.*

Proof. We only need to prove that $(I+A)(P)=P$. It is obvious that $A(0)=0$ since $0\leq A(0)\leq\beta\times 0$. For each $p\in P$ with $p\neq 0$, we prove that $p\in(I+A)(P)$. Set $A'x=Ax-p$ for $x\in P$.

Consider the following Cauchy problem:

$$\begin{cases}x'(t)\in-(I+A')x(t),\ t\in(0,+\infty),\\ x(0)=y\in P.\end{cases}\tag{2.1}$$

It is easy to see that (2.1) is equivalent to the following equation $x(t)=y+pt-\int_0^t(I+A)x(s)ds$.

First, we assume that $y > 0$ and take $1 > \alpha > 0$ sufficiently small such that $\alpha\beta < 1$, $\alpha(1+L) < 1$, and $\alpha(1+\beta) < 1$ Set

$$C([0,\alpha],P) = \{y + pt - w(t) : [0,\alpha] \to P : w(t) : [0,\alpha] \to P \text{ is continuous and } w(0) = 0\}.$$

We define semi-norms p_i' on $C([0,\alpha],E) = \{x(t) : [0,\alpha] \to E \text{ is continuous}\}$ by

$$p_i'(x(\cdot)) = \max_{t\in[0,\alpha]} p_i(x(t))$$

for each $i \geq 1$. Then one can easily see that $C([0,\alpha],E)$ is a complete locally convex space, and $C([0,\alpha],P)$ is closed in $C([0,\alpha],E)$.

Now, we define an operator $K : C([0,\alpha],P) \to C([0,\alpha],E)$ by

$$Kx(t) = y + pt - \int_0^t (I+A)x(s)ds$$

for all $x(\cdot) \in C([0,\alpha],P)$ and $t \in [0,\alpha]$. Since $Ax \leq \beta x$, we have

$$Kx(t) \geq y + pt - \int_0^t (1+\beta)x(s)ds$$

and

$$\int_0^t (1+\beta)x(s)ds = (1+\beta)[yt + \frac{1}{2}pt^2 - \int_0^t w(s)ds].$$

Thus $Kx(t) \geq 0$ for all $t \in [0,\alpha]$, so $K : C([0,\alpha],P) \to C([0,\alpha],P)$.

Now, we have

$$p_i'(Kx(\cdot) - Kz(\cdot)) \leq (1+L)\alpha p_i'(x(\cdot) - z(\cdot))$$

for all $x(\cdot), z(\cdot) \in C([0,\alpha],P)$ for $i \geq 1$. By using the standard Picard method, we know that K has a unique fixed point in $C([0,\alpha],P)$, which is the unique solution of the following problem:

$$\begin{cases} x'(t) = -(I+A')x(t), \ t \in (0,\alpha], \\ x(0) = y. \end{cases} \tag{2.2}$$

By the standard extension method, we can extend the solution of (2.2) to $(0,\infty)$, which is the unique solution to problem (2.1).

For each $i \geq 1$, let $m_i(t) = p_i(x(t;y) - x(t;z))$, where $x(t,y)$ and $x(t,z)$ are solutions to problem (2.1) corresponding to the initial values $y > 0$ and $z > 0$, respectively. Then

$$D^- m_i(t) = \lim_{h\to 0^+} \frac{m_i(t) - m_i(t-h)}{h} = [x(t,y) - x(t,z), x'(t,y) - x'(t,z)]_i^-,$$

which implies that

$$p_i(x(t,y) - x(t,z)) \leq e^{-t} p_i(y - z) \tag{2.3}$$

for all $y,z > 0$, $t \in (0,\infty)$, and $i \geq 1$. Take $y_n > 0$ for each $n \geq 1$ and $y_n \to 0$. By (2.3), we know that $x(t) = \lim_{n\to\infty} x(t,y_n)$ exists for all $t \geq 0$. Thus we can easily see that $x(t)$ is the unique solution to the problem (2.1) with the initial value $x(0) = 0$.

Finally, for each $T > 0$, we define a mapping $B_T : P \to P$ by

$$B_T y = x(T,y)$$

for all $y \in P$, where $x(t,y)$ is the unique solution of the problem (2.1) with the initial value $x(0) = y$. By the same reason as in (2.3), we have

$$p_i(B_T y - B_T z) \leq e^{-T} p_i(y - z)$$

for all $y,z \in P$ and $i \geq 1$. So B_T has a unique fixed point $y_0 \in P$. Since $B_T B_S = B_S B_S$ for all $T,S > 0$, by the uniqueness of solutions of problem (2.1), it follows that $B_t y_0 = x(t,y_0) = y_0$ for all $t > 0$. So $0 \in (I+A)x_0 - p$ and the conclusion of Theorem 2.10 is proved. □

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