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HYBRID VISCOSITY APPROXIMATION METHODS WITH GENERALIZED CONTRACTIONS FOR ZEROS OF MONOTONE OPERATORS AND FIXED POINT PROBLEMS

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Abstract. In this paper, we construct implicit and explicit hybrid viscosity iterative algorithms with generalized contractions for approximating a common solution of the zero point problem with an inverse-strongly monotone mapping and a maximal monotone operator and the fixed point problem of an infinite family of nonexpansive mappings. Under suitable conditions, we obtain two strong convergence theorems in Hilbert space. Some applications and numerical examples are provided to support our main results.

Keywords. Implicit iterative algorithms; Inverse-strongly monotone operator; Maximal monotone operator; Meir-Keeler viscosity approximation method; Nonexpansive mapping.

1. Introduction

Let H be an infinite dimensional real Hilbert space with inner product and norm denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, respectively. Let C be a nonempty, closed, and convex subset of H. Let P_C be the metric projection from H onto C, and let $T: C\to C$ be a mapping. The set of fixed points of T is denoted by $F(T)$, that is, $F(T)=\{x\in C: Tx=x\}$. Recall that T is said to be L-Lipschitzian if $\|Tx-Ty\|\le L\|x-y\|$, $\forall x,y\in C$. If $0<L<1$, then T is said to be a L-contraction; if $L=1$, then T is said to be a nonexpansive mapping. From [22, 30], we know that $F(T)$ is closed and convex. Let $F: C\to H$ be a single-valued mapping. Recall that F is said to be α-inverse strongly monotone (for short, α-ism) if there exists $\alpha>0$ such that $\langle x-y, Fx-Fy\rangle\ge\alpha\|Fx-Fy\|^2$, $\forall x,y\in C$. Obviously, if F is an α-inverse-strongly monotone mapping, then it is $\frac{1}{\alpha}$-Lipschitz continuous. Also, if $0<r\le 2\alpha$, then $I-rF$ is nonexpansive; see, e.g., [11] and [19].

Let B be a mapping of H into 2^H and $dom(B)$ denote the effective domain of B, that is, $dom(B)=\{x\in H: Bx\neq\emptyset\}$. Recall that B is said to be a monotone operator on H iff $\langle x-y, u-v\rangle\ge 0$ for all $x,y\in dom(B)$, $u\in Bx$, and $v\in By$. A monotone operator B on H is said to be maximal iff its graph is not properly contained in the graph of any other monotone operator on H. For a maximal monotone operator B on H and $r>0$, a single-valued operator $J_r=$

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$(I+rB)^{-1}: H \to dom(B)$ is called the resolvent of B for r. Let $B^{-1}(0) = \{x \in H : 0 \in Bx\}$. It is well known that $B^{-1}0 = F(J_r)$ for all $r > 0$. We denote by $A_r = \frac{1}{r}(I - J_r)$ the Yosida approximation of B for $r > 0$. From [20], we know that $A_r x \in BJ_r x, \ \forall x \in H, \ r > 0$.

We know that the theory of nonexpansive mappings in a Hilbert space is important because it can be applied to convex optimization, the theory of nonlinear evolution equations, and others. In 2000, Moudafi [10] first proposed the viscosity approximation method for approximating the fixed points of a nonexpansive mapping (see [12, 26] for further developments in Banach spaces). His method reads $x_{n+1} = \sigma_n f(x_n) + (1-\sigma_n)Tx_n$, $\forall n \geq 0$, where f is a contraction on H, and σ_n is a sequence in $(0,1)$. Under suitable conditions, he proved that the sequence generated above converges strongly to a point $x^* \in F(T)$, which is also the unique solution of the variational inequality $\langle (I-f)x^*, x - x^* \rangle \geq 0$, $\forall x \in F(T)$.

In 2006, Marino and Xu [8] further proposed the following viscosity approximation iterative algorithms: $x_{n+1} = \alpha_n \gamma f(x_n) + (1-\alpha_n A)Tx_n$, $\forall n \geq 0$, where f is a k-contraction, and A is a strongly positive bounded linear self-adjoint operator. Under some condition imposed on $\{\alpha_n\}$, they proved that the sequence $\{x_n\}$ generated by their algorithm converges strongly to a point $x^* \in F(T)$, which is also the unique solution of the variational inequality $\langle (A-\gamma f)x^*, x - x^* \rangle \geq 0$, $\forall x \in F(T)$. Indeed, it is also the optimality condition for the minimization problem $\min_{x \in F(T)} \frac{1}{2}\langle Ax, x\rangle - h(x)$, where h is a potential function for γf (i.e., $h'(x) = \gamma f(x)$ for $x \in H$).

Let $\{T_i\}_{i=1}^{\infty} : C \to C$ be an infinite countable family of nonexpansive mappings, and let $\lambda_1, \lambda_2, \dots$ be real numbers such that $0 \leq \lambda_i \leq 1$ for every $i \geq 1$. For $\forall n \geq 1$, the mapping W_n is defined by

$$\begin{cases} U_{n,n+1} = I, \\ U_{n,n} = \lambda_n T_n U_{n,n+1} + (1-\lambda_n)I, \\ U_{n,n-1} = \lambda_{n-1} T_{n-1} U_{n,n} + (1-\lambda_{n-1})I, \\ \vdots \\ U_{n,k} = \lambda_k T_k U_{n,k+1} + (1-\lambda_k)I, \\ U_{n,k-1} = \lambda_{k-1} T_{k-1} U_{n,k} + (1-\lambda_{k-1})I, \\ \vdots \\ U_{n,2} = \lambda_2 T_2 U_{n,3} + (1-\lambda_2)I, \\ W_n = U_{n,1} = \lambda_1 T_1 U_{n,2} + (1-\lambda_1)I, \end{cases}$$

where I is the identity operator on H. Such a mapping W_n is the *W-mapping* generated by $T_n, T_{n-1}, \dots, T_1$ and $\lambda_n, \lambda_{n-1}, \dots, \lambda_1$ (see [15]).

For approximating the common fixed point of an infinite countable family of nonexpansive mappings $\{T_i\}_{i=1}^{\infty}$, Yao, Liou and Chen [27] proposed the following viscosity approximation iterative algorithm $x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A)W_n x_n$, $\forall n \geq 0$, where f is a k-contraction, and A is a strongly positive bounded linear operator. Under certain assumptions imposed on the parameters, they proved that the sequence $\{x_n\}$ generated above converges strongly to $x^* \in \bigcap_{i=1}^{\infty} F(T_i)$, which is also the unique solution of the variational inequality $\langle (A - \gamma f)x^*, x - x^* \rangle \geq 0$, $\forall x \in \bigcap_{i=1}^{\infty} F(T_i)$.

In 2013, Takahashi [21] proposed the following viscosity implicit and explicit composite iterative algorithms for finding a solution of monotone inclusion problems

$$x_n = \alpha_n \gamma g(x_n) + (1-\alpha_n G) J_{\lambda_n}(I - \lambda_n A) T_{r_n} x_n, \quad \forall n \geq 1$$

and

$$x_{n+1} = \alpha_n \gamma g(x_n) + (1 - \alpha_n G) J_{\lambda_n} (I - \lambda_n A) T_{r_n} x_n, \quad \forall n \geq 1.$$

where g is a contraction with the constant $k \in (0,1)$, G is a strongly positive bounded linear self-adjoint operator, A is an α-inverse-strongly monotone mapping, and J_λ and T_r are the resolvents of maximal monotone operators B and F, respectively. Under certain assumptions imposed on the parameters $\{r_n\}$, $\{\alpha_n\}$, and $\{\lambda_n\}$, the sequence $\{x_n\}$ generated by the above implicit and explicit algorithm both converges strongly to a point $q \in (A+B)^{-1}(0) \bigcap F^{-1}(0)$, which also solves the variational inequality $\langle (G - \gamma g)q, p - q\rangle \geq 0$, $\forall p \in (A+B)^{-1}(0) \bigcap F^{-1}(0)$.

Recently, Sunthrayuth, Cho and Kumam [16] constructed a viscosity iterative algorithm based on Meir-Keeler contraction for approximating the zeros of accretive operators in Banach spaces, and Yao et al. [28] introduced a projected algorithm with a Meir-Keeler contraction for finding the fixed points of the pseudocontractive mappings.

Motivated by the above related results in this field, the purpose of this paper is to construct viscosity implicit and explicit composite iterative algorithms based on generalized contractions (Meir-Keeler contractions or (ψ, L)-contractions) for approximating a common solution of the zero point prblem of an inverse-strongly monotone mapping and a maximal monotone operator and the set of fixed point problem of an infinite countable family of nonexpansive mappings in Hilbert space.

This paper is organized as follows. In Section 2, we give some basic definitions, propositions, and lemmas which are used in proving our main results. In Section 3, we present the hybrid viscosity implicit and explicit composite iterative algorithms with the generalized contractions for approximating the common solution, and establish two strong convergence theorems. In Section 4, we apply our main results to variational inequality problem, equilibrium problem, constrained convex minimization problem, and generalized split feasibility problem. In Section 5, we give two numerical examples to support our main results.

2. Preliminaries

Throughout the paper, Let N and R^+ be the set of all positive integers and all positive real numbers, respectively. $\rightarrow$ and $\rightharpoonup$ denote the strong convergence and weak convergence, respectively. In addition, $F(T)$ and $\omega_w(x_n)$ denote the fixed point set of T and the weak ω-limit set of x_n, respectively, that is, $F(T) = \{x \in C : Tx = x\}$ and $\omega_w(x_n) = \{u : \exists x_{n_j} \rightharpoonup u\}$. Below we gather some basic definitions and results which are needed in the subsequent sections.

It is known that in a real Hilbert space H, the following inequality holds: $\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle$, $\forall x, y \in H$. Recall that a mapping $T : H \to H$ is said to be firmly nonexpansive if $2T - I$ is nonexpansive, or equivalently, $\langle x - y, Tx - Ty\rangle \geq \|Tx - Ty\|^2$, $x, y \in H$. Alternatively, T is firmly nonexpansive if and only if T can be expressed as $T = \frac{I+S}{2}$, where $S : H \to H$ is nonexpansive. Recall that a mapping $T : H \to H$ is said to be an averaged mapping if it can be written as the average of the identity I and a nonexpansive mapping, that is,

$$T = (1 - \alpha)I + \alpha S. \tag{2.1}$$

where $\alpha \in (0,1)$ and $S : H \to H$ is nonexpansive. More precisely, we also say that T is α-averaged (for short, α-av). Clearly, a firmly nonexpansive mapping (in particular, the projection) is a $\frac{1}{2}$-averaged mapping.

Proposition 2.1. (Basic properties of averaged mappings [3]) *For given operators $S,T,V : H \to H$, we have the following facts.*

(i) If $T = (1-\alpha)S + \alpha V$ for some $\alpha \in (0,1)$ and S is averaged and V is nonexpansive, then T is averaged.
(ii) T is firmly nonexpansive if and only if the complement $I - T$ is firmly nonexpansive.
(iii) If $T = (1-\alpha)S + \alpha V$ for some $\alpha \in (0,1)$, S is firmly nonexpansive, and V is nonexpansive, then T is averaged.
(iv) The composite of finitely many averaged mappings is averaged. That is, if each of the mappings $\{T_i\}_{i=1}^N$ is averaged, so is the composite $T_1 \dots T_N$. In particular, if T_1 is α_1-averaged and T_2 is α_2-averaged, where $\alpha_1, \alpha_2 \in (0,1)$, then the composite T_1T_2 is α-averaged, where $\alpha = \alpha_1 + \alpha_2 - \alpha_1\alpha_2$.
(v) If the mappings $\{T_i\}_{i=1}^N$ are averaged and have a common fixed point, then $\bigcap_{i=1}^N Fix(T_i) = Fix(T_1 \dots T_N)$.
(vi) If T is α-averaged, then $\|Tx - z\|^2 \leq \|x - z\|^2 - \frac{1-\alpha}{\alpha}\|Tx - x\|^2$, $\forall x \in H$, $z \in Fix(T)$.

The following proposition summarizes some results on the relationship between the averaged mappings and the inverse-strongly monotone operators.

Proposition 2.2. [3] *Let $T : H \to H$ be an operator from H to itself.*

(i) T is nonexpansive if and only if the complement $I - T$ is $\frac{1}{2}$-ism.
(ii) If T is ν-ism, then for $\gamma > 0$, γT is $\frac{\nu}{\gamma}$-ism.
(iii) T is averaged if and only if the complement $I - T$ is ν-ism for some $\nu > \frac{1}{2}$. Indeed, for $\alpha \in (0,1)$, T is α-averaged if and only if $I - T$ is $\frac{1}{2\alpha}$-ism.

Recall that P_C is the metric projection from H into C if, for each point $x \in H$, the unique point $P_Cx \in C$ satisfies the property: $\|x - P_Cx\| = \inf_{y \in C} \|x - y\| =: d(x,C)$.

Lemma 2.3. [22] *For a given $x \in H$,*

(i) $z = P_Cx$ if and only if $\langle x - z, z - y\rangle \geq 0, \forall y \in C$;
(ii) $z = P_Cx$ if and only if $\|x - z\|^2 \leq \|x - y\|^2 - \|y - z\|^2$;
(iii) $\langle P_Cx - P_Cy, x - y\rangle \geq \|P_Cx - P_Cy\|^2, \forall x, y \in H$.

A mapping $\psi : R^+ \to R^+$ is said to be a *L-function* if $\psi(0) = 0$, $\psi(t) > 0$ for each $t > 0$, and, for every $s > 0$, there exists $u > s$ such that $\psi(t) \leq s$ for each $t \in [s,u]$. As a consequence, every L-function ψ satisfies $\psi(t) < t$ for each $t > 0$.

Definition 2.4. Let (X,d) be a metric space. A mapping $f : X \to X$ is said to be

(i) a (ψ, L)-contraction if $\psi : R^+ \to R^+$ is said to be a L-function and $d(f(x), f(y)) < \psi(d(x,y))$ for all $x, y \in X$, $x \neq y$;
(ii) a Meir-Keeler type mapping if, for each $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that, for each $x, y \in X$ with $\varepsilon \leq d(x,y) < \varepsilon + \delta$, $d(f(x), f(y)) < \varepsilon$.

Proposition 2.5. [7] *Let (X,d) be a metric space and $f : X \to X$ be a mapping. The following assertions are equivalent:*

(i) f is a Meir-Keeler type mapping;
(ii) there exists a L-function $\psi : R^+ \to R^+$ such that f is a (ψ, L)-contraction.

Proposition 2.6. [17] *Let C be a convex subset of a Banach space X and $f : C \to C$ be a Meir-Keeler type mapping. Then, for each $\varepsilon > 0$, there exists $k \in (0,1)$ such that*

$$\|x-y\| \geq \varepsilon \text{ implies } \|f(x)-f(y)\| \leq k\|x-y\|.$$

Lemma 2.7. [9] *A Meir-Keeler contraction defined on a complete metric space has a unique fixed point.*

Lemma 2.8. [17] *Let C be a convex subset of a Banach space E. Let T be a nonexpansive mapping on C, and let f be a Meir-Keeler contraction on C. Then the following hold:*

(i) Tf is a Meir-Keeler contraction on C;
(ii) for each $\alpha \in (0,1)$, $(1-\alpha)T + \alpha f$ is a Meir-Keeler contraction on C.

Throughout this paper, generalized contraction mappings refers to Meir-Keeler type mappings or (ψ, L)-contractions. And we assume that the L-function from the definition of (ψ, L)-contraction is continuous, strictly increasing, and $\lim_{t\to\infty} \eta(t) = \infty$, where $\eta(t) = t - \psi(t)$ for all $t \in R^+$. η is a bijection on R^+.

Concerning the mapping W_n, the following lemmas are important to prove our results.

Lemma 2.9. [15] *Let C be a nonempty, closed, and convex subset of a strictly convex Banach space E. Let $T_1, T_2, \ldots$ be nonexpansive mappings of C into itself such that $\bigcap_{i=1}^{\infty} Fix(T_i)$ is nonempty, and let $\lambda_1, \lambda_2, \ldots$ be real numbers such that $0 < \lambda_i \leq b < 1$ for any $i \geq 1$. Then, for every $x \in C$ and $k \in N$, $\lim_{n\to\infty} U_{n,k}x$ exists.*

Using Lemma 2.9, one can define mapping W of C into itself as follows: $Wx = \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1}x$, $\forall x \in C$. Throughout this paper, we assume that $0 < \lambda_i \leq b < 1$ for any $i \geq 1$. It is obvious that nonexpansivity of each T_i ensures the nonexpansivity of W_n. Since W_n is nonexpansive, then W is also nonexpansive.

Lemma 2.10. [15] *Let C be a nonempty, closed, and convex subset of a strictly convex Banach space E. Let $T_1, T_2, \ldots$ be nonexpansive mappings of C into itself such that $\bigcap_{i=1}^{\infty} Fix(T_i)$ is nonempty, and let $\lambda_1, \lambda_2, \ldots$ be real numbers such that $0 < \lambda_i \leq b < 1$ for any $i \geq 1$. Then $F(W) = \bigcap_{i=1}^{\infty} F(T_i)$.*

To obtain the main results of this paper, we also need the following lemmas.

Lemma 2.11. *(Demiclosedness principle [6]) Let $T : C \to C$ be a nonexpansive mapping with $Fix(T) \neq \emptyset$. If $\{x_n\}$ is a sequence in C that converges weakly to x and if $\{(I-T)x_n\}$ converges strongly to y, then $(I-T)x = y$. In particular, if $y = 0$, then $x \in Fix(T)$.*

Lemma 2.12. [22] *Let H be a Hilbert space and B be a maximal monotone mapping on H. Let J_r be the resolvent of B defined by $J_r = (I + rB)^{-1}$ for each $r > 0$.*

(i) For each $r > 0$, J_r is single-valued and firmly nonexpansive;
(ii) $D(J_r) = H$ and $Fix(J_r) = \{x \in D(B) : 0 \in Bx\}$.

Lemma 2.13. [23] *Let H be a real Hilbert space and let B be a maximal monotone operator on H. For $r > 0$ and $x \in H$, define the resolvent $J_r x$. Then $\frac{s-t}{s}\langle J_s x - J_t x, J_s x - x\rangle \geq \|J_s x - J_t x\|^2$ for all $s,t > 0$, and $x \in H$.*

From Lemma 2.13, we have $\|J_\lambda x - J_\mu x\| \leq (|\lambda - \mu|/\lambda)\|x - J_\lambda x\|$ for all $\lambda, \mu > 0$ and $x \in H$; see also [5, 22].

Lemma 2.14. [13] *Let C be a nonempty, closed, and convex subset of H, $A: C \to H$ be a mapping, and $B: H \to 2^H$ be a maximal monotone operator. Then $F(J_\lambda(I-\lambda A)) = (A+B)^{-1}(0)$.*

We know that a linear bounded operator $A: H \to H$ is called *strongly positive* if and only if there exists $\overline{\gamma} > 0$ such that $\langle Ax, x\rangle \geq \overline{\gamma}\|x\|^2$ for all $x \in H$. And we call such A a strongly positive operator with coefficient $\overline{\gamma}$.

Lemma 2.15. [8] *Let H be a Hilbert space and let A be a strongly positive bounded linear self-adjoint operator on H with coefficient $\overline{\gamma} > 0$. If $0 < \delta \leq \|A\|^{-1}$, then $\|I - \delta A\| \leq 1 - \delta\overline{\gamma}$.*

Lemma 2.16. [18] *Let $\{x_n\}$ and $\{z_n\}$ be bounded sequences in a Hilber space X and let $\{\tau_n\}$ be a sequence in $[0,1]$ with $0 < \liminf_{n\to\infty} \tau_n \leq \limsup_{n\to\infty} \tau_n < 1$. If $x_{n+1} = \tau_n z_n + (1-\tau_n)x_n$ for all integers $n \geq 0$ and $\limsup_{n\to\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0$, then $\lim_{n\to\infty}\|z_n - x_n\| = 0$.*

Lemma 2.17. [26] *Let $\{a_n\}$ be a sequence of nonnegative real numbers such that $a_{n+1} \leq (1-b_n)a_n + c_n$, where b_n is a sequence in $(0,1)$ and $\{c_n\}$ is a sequence such that*

(i) $\sum_{n=1}^{\infty} b_n = \infty$;
(ii) $\limsup_{n\to\infty} \frac{c_n}{b_n} \leq 0$ *or* $\sum_{n=1}^{\infty} |c_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

3. Main Results

Lemma 3.1. *Let H be an infinite dimensional real Hilbert space, $f: H \to H$ be a Meir-Keeler-type contraction, and A be a strongly positive bounded linear self-adjoint operator on H with coefficient $\overline{\gamma} > 0$. For any nonempty closed convex subset D of H, if $\|A\| \leq 1$ and constant $\gamma \leq \bar{\gamma}$, then $P_D(I - A + \gamma f)$ has a unique fixed point in D. Or equivalently, the following variational inequality: $\langle (A - \gamma f)x, z - x\rangle \geq 0$, $\forall z \in D$. has a unique solution in D.*

Proof. Since f is a Meir-Keeler-type contraction, then, for any $\|x - y\| \leq \varepsilon + \delta$, $\|f(x) - f(y)\| \leq \varepsilon$. Observe that $\|(I - A + \gamma f)x - (I - A + \gamma f)y\| \leq \|(I - A)(x - y)\| + \gamma\|f(x) - f(y)\|$.

Case 1. $\|x - y\| \leq \varepsilon$. From Lemma 2.15, we have

$$\begin{aligned}\|(I-A+\gamma f)x-(I-A+\gamma f)y\| &\leq \|I-A\|\|x-y\|+\gamma\psi(\|x-y\|)\\ &\leq (1-\bar{\gamma})\|x-y\|+\gamma\|x-y\|\\ &\leq \|x-y\| \leq \varepsilon.\end{aligned}$$

Case 2. $\varepsilon + \delta \geq \|x - y\| > \varepsilon$. From Lemma 2.15 and Proposition 2.6, we have

$$\begin{aligned}\|(I-A+\gamma f)x-(I-A+\gamma f)y\| &\leq \|I-A\|\|x-y\|+\gamma k_\varepsilon\|x-y\|\\ &\leq (1-\bar{\gamma})\|x-y\|+\gamma k_\varepsilon\|x-y\|\\ &\leq (1-\bar{\gamma}+\gamma k_\varepsilon)(\varepsilon+\delta).\end{aligned}$$

Taking $\delta = \frac{(\bar{\gamma} - \gamma k_\varepsilon)\varepsilon}{1 - \bar{\gamma} + \gamma k_\varepsilon}$, we obtain that $\|(I - A + \gamma f)x - (I - A + \gamma f)y\| \leq \varepsilon$. Therefore, $I - A + \gamma f$ is a Meir-Keeler-type contraction on H. From Lemma 2.8, we have that $P_D(I - A + \gamma f)$ is a Meir-Keeler-type contraction from H onto D. It follows from Lemma 2.7 that $P_D(I - A + \gamma f)$ has a unique fixed point in D. By Lemma 2.3, we have that $\langle (A - \gamma f)x, z - x\rangle \geq 0$, $\forall z \in D$. has a unique solution in D. $\square$

Lemma 3.2. *Let H be an infinite dimensional real Hilbert space and C be a nonempty closed convex subset of H. Let $\{T_i\}_{i=1}^{\infty} : C \to C$ be an infinite family of nonexpansive mappings and $f : H \to H$ be a Meir-Keeler-type contraction. Let A be a strongly positive bounded linear self-adjoint operator on H with coefficient $\overline{\gamma} > 0$ and F be an α-inverse-strongly monotone mapping of C into H. Let B and G be maximal monotone mappings on H such that the domains of B and G are included in C. Let $J_\rho = (I+\rho B)^{-1}$ and $T_\eta = (I+\eta G)^{-1}$ for $\rho > 0$ and $\eta > 0$. Define the mapping G_n on H by $G_n x = \alpha_n \gamma f(x) + (I-\alpha_n A)W_n J_{\rho_n}(I-\rho_n F)T_{\eta_n}x$, where W_n is the W-mapping generated by $T_n, T_{n-1}, \ldots, T_1$ and $\lambda_n, \lambda_{n-1}, \ldots, \lambda_1$. Asumme that $0 < \rho_n < 2\alpha$, $0 < \alpha_n \le \|A\|^{-1}$, and constant $\gamma \le \bar{\gamma}$. Then G_n is a Meir-Keeler-type contractions on H.*

Proof. Since F is an α-inverse-strongly monotone mapping and $\rho_n \subset (0,2\alpha)$, then $I-\rho_n F$ is nonexpansive. By Lemma 2.12, we have that $W_n J_{\rho_n}(I-\rho_n F)T_{\eta_n}$ is also nonexpansive. For $\forall x, y \in H$, it follows from Lemma 2.15 that

$$\begin{aligned}\|G_n x - G_n y\| &\le \alpha_n \gamma \|f(x)-f(y)\| + \|I-\alpha_n A\| \|W_n J_{\rho_n}(I-\rho_n F)T_{\eta_n}x - W_n J_{\rho_n}(I-\rho_n F)T_{\eta_n}y\| \\ &\le \alpha_n \gamma \|f(x)-f(y)\| + (1-\alpha_n \overline{\gamma})\|x-y\|.\end{aligned}$$

Since f is a Meir-Keeler-type contraction, then, for any $\|x-y\| \le \varepsilon + \delta$, $\|f(x)-f(y)\| \le \varepsilon$.

Case 1. $\|x-y\| \le \varepsilon$. From constant $\gamma \le \bar{\gamma}$, we have that

$$\begin{aligned}\|G_n x - G_n y\| &\le \alpha_n \gamma \psi(\|x-y\|) + (1-\alpha_n \overline{\gamma})\|x-y\| \\ &\le \alpha_n \gamma \|x-y\| + (1-\alpha_n \overline{\gamma})\|x-y\| \\ &\le (1-\alpha_n \overline{\gamma} + \alpha_n \gamma)\varepsilon \\ &\le \varepsilon.\end{aligned}$$

Case 2. $\varepsilon + \delta \ge \|x-y\| > \varepsilon$. From Proposition 2.6, there exist k_ε such that

$$\begin{aligned}\|G_n x - G_n y\| &\le \alpha_n \gamma k_\varepsilon \|x-y\| + (1-\alpha_n \overline{\gamma})\|x-y\| \\ &\le \alpha_n \gamma k_\varepsilon \|x-y\| + (1-\alpha_n \overline{\gamma})\|x-y\| \\ &\le (1-\alpha_n \overline{\gamma} + \alpha_n \gamma k_\varepsilon)(\varepsilon + \delta).\end{aligned}$$

Taking $\delta = \frac{(\alpha_n \overline{\gamma} - \alpha_n \gamma k_\varepsilon)\varepsilon}{1-\alpha_n \overline{\gamma} + \alpha_n \gamma k_\varepsilon}$, we obtain that $\|G_n x - G_n y\| \le \varepsilon$. Hence, G_n is a Meir-Keeler-type contractions on H. □

Now we state and prove our main results.

Theorem 3.3. *Let H be an infinite dimensional real Hilbert space and C be a nonempty closed convex subset of H. Let $\{T_i\}_{i=1}^{\infty} : C \to C$ be an infinite family of nonexpansive mappings and $f : H \to H$ be a Meir-Keeler-type contraction. Let A be a strongly positive bounded linear self-adjoint operator on H with coefficient $\overline{\gamma} > 0$ and F be an α-inverse-strongly monotone mapping of C into H. Let B and G be maximal monotone mappings on H such that the domains of B and G are included in C. Let $J_\rho = (I+\rho B)^{-1}$ and $T_\eta = (I+\eta G)^{-1}$ for $\rho > 0$ and $\eta > 0$. Assume that $\Gamma = \bigcap_{i=1}^{\infty} F(T_i) \bigcap (F+B)^{-1}0 \bigcap G^{-1}0 \neq \emptyset$. For an arbitrary $x_1 \in H$, Let $\{x_n\}$ be a sequence generated by the following algorithm:*

$$\begin{cases} y_n = J_{\rho_n}(I-\rho_n F)T_{\eta_n}x_n, \\ x_n = \alpha_n \gamma f(x_n) + (I-\alpha_n A)W_n y_n. \end{cases} \tag{3.1}$$

where W_n is the W-mapping generated by $T_n, T_{n-1}, \ldots, T_1$ and $\lambda_n, \lambda_{n-1}, \ldots, \lambda_1$, constant $\gamma \le \bar{\gamma}$ and $\|A\| \le 1$. Assumme that $\{\alpha_n\}$, $\{\rho_n\}$ and $\{\eta_n\}$ satisfying the following conditions:

(i) $\{\alpha_n\} \subset (0,1)$, $\lim_{n\to\infty} \alpha_n = 0$;
(ii) $0 < a \le \rho_n \le b < 2\alpha$;
(iii) $\eta_n \subset (0,\infty)$, $\liminf_{n\to\infty} \eta_n > 0$.

Then $\{x_n\}$ *converges strongly to a point* $\omega \in \Gamma$*, which is also the unique solution of the variational inequality:*

$$\langle (A-\gamma f)x, z-x\rangle \ge 0, \quad \forall z \in \Gamma. \tag{3.2}$$

Proof. First, we show that $\{x_n\}$ is well defined. Consider the mapping G_n on H defined by $G_n x = \alpha_n \gamma f(x) + (I-\alpha_n A)W_n J_{\rho_n}(I-\rho_n F)T_{\eta_n} x$. We see from Lemma 3.2 and lemma 2.7 that G_n has a unique fixed point $x_n^f \in H$ such that $x_n^f = \alpha_n \gamma f(x_n^f) + (I-\alpha_n A)W_n J_{\rho_n}(I-\rho_n F)T_{\eta_n} x_n^f$. For simplicity, we write x_n for x_n^f if no confusion occurs.

Second, we show that $\{x_n\}$ is bounded. Pick any $p \in \Gamma$. From Lemma 2.12 and Lemma 2.14, we have that $\|y_n - p\| = \|J_{\rho_n}(I-\rho_n F)T_{\eta_n}x_n - J_{\rho_n}(I-\rho_n F)T_{\eta_n}p\| \le \|x_n - p\|$. Fixed ε_0 for $\forall n \ge 1$.

Case 1. $\|x_n - p\| < \varepsilon_0$. It is obvious that $\{x_n\}$ is bounded.

Case 2. $\|x_n - p\| \ge \varepsilon_0$. By Proposition 2.6, there exists $k_{\varepsilon_0} \in (0,1)$ such that $\|f(x_n) - f(p)\| \le k_{\varepsilon_0}\|x_n - p\|$. Then

$$\begin{aligned}
\|x_n - p\| &= \|\alpha_n(\gamma f(x_n) - Ap) + (I-\alpha_n A)(W_n y_n - p)\| \\
&\le \alpha_n\|\gamma f(x_n) - Ap\| + \|I-\alpha_n A\|\|W_n y_n - p\| \\
&\le \alpha_n\gamma\|f(x_n) - f(p)\| + \alpha_n\|\gamma f(p) - Ap\| + (1-\alpha_n\overline{\gamma})\|y_n - p\| \\
&\le (1-\alpha_n\overline{\gamma} + \alpha_n\gamma k_{\varepsilon_0})\|x_n - p\| + \alpha_n\|\gamma f(p) - Ap\|.
\end{aligned}$$

Hence $\|x_n - p\| \le \frac{1}{\overline{\gamma}-\gamma k_{\varepsilon_0}}\|\gamma f(p) - Ap\|$. Obviously, $\{x_n\}$ is bounded, and then $\{y_n\}$ is also bounded.

Third, we show that $x_{n_i} \rightharpoonup \omega \in \Gamma$, where $\{x_{n_i}\}$ a subsequence of $\{x_n\}$. Since F is a α-ism, and $\rho_n \in (0,2\alpha)$, it follows from Proposition 2.2 that $\rho_n F$ is $\frac{\alpha}{\rho_n}$-ism and $I-\rho_n F$ is $\frac{\rho_n}{2\alpha}$-averaged. We know that J_{ρ_n} and T_{η_n} are $\frac{1}{2}$-averaged. By Propsition 2.1, we obtain that $J_{\rho_n}(I-\rho_n F)T_{\eta_n}$ is $\frac{3}{4} + \frac{\rho_n}{8\alpha}$-av. Hence,

$$\begin{aligned}
\|y_n - p\|^2 &= \|J_{\rho_n}(I-\rho_n)FT_{\eta_n}x_n - p\|^2 \\
&\le \|x_n - p\|^2 - \frac{1-(\frac{3}{4}+\frac{\rho_n}{8\alpha})}{\frac{3}{4}+\frac{\rho_n}{8\alpha}}\|y_n - x_n\|^2 \\
&= \|x_n - p\|^2 - \frac{\frac{1}{4}-\frac{\rho_n}{8\alpha}}{\frac{3}{4}+\frac{\rho_n}{8\alpha}}\|y_n - x_n\|^2.
\end{aligned} \tag{3.3}$$

From Lemma 2.15, we have

$$\begin{aligned}
\|x_n - p\|^2 &= \|\alpha_n(\gamma f(x_n) - Ap) + (I-\alpha_n A)(W_n y_n - p)\|^2 \\
&\le \|(I-\alpha_n A)(W_n y_n - p)\|^2 + 2\alpha_n\langle \gamma f(x_n) - Ap, x_n - p\rangle \\
&\le (1-\alpha_n\overline{\gamma})^2\|y_n - p\|^2 + 2\alpha_n\langle \gamma f(x_n) - Ap, x_n - p\rangle \\
&\le (1-\alpha_n\overline{\gamma})^2\|y_n - p\|^2 + 2\alpha_n\|\gamma f(x_n) - Ap\|\|x_n - p\|.
\end{aligned} \tag{3.4}$$

Substituting (3.3) into (3.4), we obtain that

$$(I-\alpha_n\overline{\gamma})^2\frac{\frac{1}{4}-\frac{\rho_n}{8\alpha}}{\frac{3}{4}+\frac{\rho_n}{8\alpha}}\|y_n-x_n\|^2 \le [(1-\alpha_n\overline{\gamma})^2-1]\|x_n-p\|^2+2\alpha_n\|\gamma f(x_n)-Ap\|\|x_n-p\|.$$

Since $\alpha_n \to 0$ and $0<a\le\rho_n\le b<2\alpha, \forall n\ge 1$, we obtain that

$$\lim_{n\to\infty}\|y_n-x_n\|=0. \tag{3.5}$$

Observe that

$$\begin{aligned}\|x_n-W_nx_n\| &= \|\alpha_n\gamma f(x_n)-\alpha_nAW_nx_n+(I-\alpha_nA)(W_ny_n-W_nx_n)\|\\ &\le \alpha_n\|\gamma f(x_n)-AW_nx_n\|+\|I-\alpha_nA\|\|W_ny_n-W_nx_n\|\\ &\le \alpha_n\|\gamma f(x_n)-AW_nx_n\|+(I-\alpha_n\overline{\gamma})\|y_n-x_n\|\to 0,(n\to\infty),\end{aligned}$$

and $\|x_n-Wx_n\|\le\|x_n-W_nx_n\|+\|W_nx_n-Wx_n\|\to 0$ as $n\to\infty$. Since $\{x_n\}$ is bounded, then there exist a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i}\rightharpoonup\omega$. By (3.5), we see that $y_{n_i}\rightharpoonup\omega$. Since $\{y_{n_i}\}\subset C$ and C is closed and convex, we have $\omega\in C$. From Lemma 2.11 and Lemma 2.10, we obtain $\omega\in F(W)=\bigcap_{i=1}^{\infty}F(T_i)$.

Next, we show that $\omega\in(F+B)^{-1}(0)\bigcap G^{-1}(0)$. Put $u_n=T_{\eta_n}x_n$. Since $\{x_n\}$ is bounded, then $\{y_n\}$ and $\{u_n\}$ are also bounded. Observe that

$$\|y_n-p\|=\|J_{\rho_n}(I-\rho_nF)u_n-J_{\rho_n}(I-\rho_nF)p\|\le\|u_n-p\|, \tag{3.6}$$

Since T_{η_n} is firmly nonexpansive, we have

$$2\|u_n-p\|^2\le 2\langle x_n-p,u_n-p\rangle=\|x_n-p\|^2+\|u_n-p\|^2-\|u_n-x_n\|^2.$$

Then

$$\|u_n-p\|^2\le\|x_n-p\|^2-\|u_n-x_n\|^2. \tag{3.7}$$

Substituting (3.6) and (3.7) into (3.4), we obtain that

$$(1-\alpha_n\overline{\gamma})^2\|u_n-x_n\|^2\le[(1-\alpha_n\overline{\gamma})^2-1]\|x_n-p\|^2+2\alpha_n\|\gamma f(x_n)-Ap\|\|x_n-p\|.$$

Since $\lim_{n\to\infty}\alpha_n=0$, we have that $\lim_{n\to\infty}\|u_n-x_n\|=0$. In view of (3.5), we have $\lim_{n\to\infty}\|u_n-y_n\|=0$. Hence there exist a subsequence $\{u_{n_i}\}$ of u_n and a subsequence $\{\rho_{n_i}\}$ of $\{\rho_n\}$ such that $u_{n_i}\rightharpoonup\omega\in C$ and $\rho_{n_i}\to\rho_0$ for some $\rho_0\in[a,b]$. Put $v_n=(I-\rho_nF)u_n$. It follows from (3.1) and Lemma 2.13 that

$$\begin{aligned}\|J_{\rho_0}(I-\rho_0F)u_{n_i}-y_{n_i}\| &\le \|J_{\rho_0}(I-\rho_0F)u_{n_i}-J_{\rho_0}(I-\rho_{n_i}F)u_{n_i}\|+\|J_{\rho_0}(I-\rho_{n_i}F)u_{n_i}-y_{n_i}\|\\ &\le \|(I-\rho_0F)u_{n_i}-(I-\rho_{n_i}F)u_{n_i}\|+\|J_{\rho_0}v_{n_i}-J_{\rho_{n_i}}v_{n_i}\|\\ &\le |\rho_{n_i}-\rho_0|\|Fu_{n_i}\|+\tfrac{|\rho_{n_i}-\rho_0|}{\rho_0}\|J_{\rho_0}v_{n_i}-v_{n_i}\|\to 0,(n\to\infty).\end{aligned}$$

Again from $\lim_{n\to\infty}\|u_{n_i}-y_{n_i}\|=0$, we have $\lim_{n\to\infty}\|J_{\rho_0}(I-\rho_0F)u_{n_i}-u_{n_i}\|=0$. Observe that $J_{\rho_0}(I-\rho_0F)$ is nonexpansive. By Lemma 2.11, we have $\omega=J_{\rho_0}(I-\rho_0F)\omega$ and hence $\omega\in(F+B)^{-1}(0)$. Now we show that $\omega\in G^{-1}(0)$. Since G is a maximal monotone operator, then $A_{\eta_{n_i}}x_{n_i}\in GT_{\eta_{n_i}}x_{n_i}$, where A_η is the Yosida approximation of G for $\eta>0$. Furthermore, for any $(\mu,\upsilon)\in G$, we have $\langle\mu-u_{n_i},\upsilon-\frac{x_{n_i}-u_{n_i}}{\eta_{n_i}}\rangle\ge 0$. Since $\liminf_{n\to\infty}\eta_n>0$, $u_{n_i}\rightharpoonup\omega$, and $\|x_{n_i}-u_{n_i}\|\to 0$, we have $\langle\mu-\omega,\upsilon\rangle\ge 0$. Since G is a maximal monotone operator, we have $0\in G\omega$ and hence $\omega\in G^{-1}(0)$. Thus $\omega\in\Gamma$.

Fourth, we show that $x_{n_i} \to \omega$. Assume that x_{n_i} does not converge to ω. Then there exist a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ such that $\|x_{n_{i_j}} - \omega\| > \varepsilon$ for some $\varepsilon > 0$. By Proposition 2.6, $\|f(x_{n_{i_j}}) - f(\omega)\| \leq k_\varepsilon \|x_{n_{i_j}} - \omega\|$. Then

$$\begin{aligned}
\|x_{n_{i_j}} - \omega\|^2 &= \|\alpha_{n_{i_j}}\gamma f(x_{n_{i_j}}) - \alpha_{n_{i_j}}\gamma f(\omega) + \alpha_{n_{i_j}}\gamma f(\omega) - \alpha_{n_{i_j}}A\omega \\
&\quad -(I - \alpha_{n_{i_j}}A)\omega + (I - \alpha_{n_{i_j}}A)W_{n_{i_j}}y_{n_{i_j}}\|^2 \\
&= \alpha_{n_{i_j}}\gamma\langle f(x_{n_{i_j}}) - f(\omega), x_{n_{i_j}} - \omega\rangle + \alpha_{n_{i_j}}\langle \gamma f(\omega) - A\omega, x_{n_{i_j}} - \omega\rangle \\
&\quad + \langle (I - \alpha_{n_{i_j}}A)(W_{n_{i_j}}y_{n_{i_j}} - \omega), x_{n_{i_j}} - \omega\rangle \\
&\leq (1 - \alpha_{n_{i_j}}\overline{\gamma} + \alpha_{n_{i_j}}\gamma k_\varepsilon)\|x_{n_{i_j}} - \omega\|^2 + \alpha_{n_{i_j}}\langle \gamma f(\omega) - A\omega, x_{n_{i_j}} - \omega\rangle.
\end{aligned}$$

Obviously $\|x_{n_{i_j}} - \omega\|^2 \leq \frac{1}{\overline{\gamma} - \gamma k_\varepsilon}\langle \gamma f(\omega) - A\omega, x_{n_{i_j}} - \omega\rangle$. Since $x_{n_i} \rightharpoonup \omega$, then $x_{n_{i_j}} \to \omega$ as $j \to \infty$. The contradiction permits us to conclude that $x_{n_i} \to \omega$.

Finally, we show that $x_n \to \omega$, which is also the unique solution of variational inequality (3.2). Observe that $x_n = \alpha_n \gamma f(x_n) + (I - \alpha_n A)W_n J_{\rho_n}(I - \rho_n F)T_{\eta_n}x_n$. Hence, we conclude that

$$(A - \gamma f)x_n = -\frac{1}{\alpha_n}(I - W_n J_{\rho_n}(I - \rho_n F)T_{\eta_n})x_n + A(I - W_n J_{\rho_n}(I - \rho_n F)T_{\eta_n})x_n.$$

Since $W_n J_{\rho_n}(I - \rho_n F)T_{\eta_n}$ is nonexpansive, we have that $I - W_n J_{\rho_n}(I - \rho_n F)T_{\eta_n}$ is monotone. Note that, for any given $z \in \Gamma$,

$$\begin{aligned}
\langle (A - \gamma f)x_n, x_n - z\rangle &= -\frac{1}{\alpha_n}(\langle (I - W_n J_{\rho_n}(I - \rho_n F)T_{\eta_n})x_n - (I - W_n J_{\rho_n}(I - \rho_n F)T_{\eta_n})z, x_n - z\rangle) \\
&\quad + \langle A(I - W_n J_{\rho_n}(I - \rho_n F)T_{\eta_n})x_n, x_n - z\rangle \\
&\leq \langle A(I - W_n J_{\rho_n}(I - \rho_n F)T_{\eta_n})x_n, x_n - z\rangle \\
&\leq \|A\|\|x_n - W_n y_n\|\|x_n - z\|.
\end{aligned} \tag{3.8}$$

Also

$$\begin{aligned}
\|x_n - W_n y_n\| &\leq \|x_n - W_n x_n\| + \|W_n x_n - W_n y_n\| \\
&\leq \|x_n - W_n x_n\| + \|x_n - y_n\| \to 0 \ (n \to \infty).
\end{aligned}$$

Now, replacing n with n_i in (3.8) and letting $i \to \infty$, we have

$$\langle (A - \gamma f)\omega, \omega - z\rangle = \lim_{i\to\infty}\langle (A - \gamma f)x_{n_i}, x_{n_i} - z\rangle \leq 0.$$

From the arbitrariness of $z \in \Gamma$, we have that $\omega \in \Gamma$ is a solution of (3.2). Further, it is easy to verify that Γ is closed and convex. By Lemma 3.1, we know that (3.2) has a unique solution. Hence we conclude that $x_n \to \omega$ as $n \to \infty$. □

Lemma 3.4. *Let H be an infinite dimensional real Hilbert space, and A be a strongly positive bounded linear self-adjoint operator on H with coefficient $\overline{\gamma} > 0$. If $\{\beta_n\} \subset (0,1)$ and $\alpha_n \leq (1 - \beta_n)\|A\|^{-1}$ for $\forall n \geq 1$, then $\|(1 - \beta_n)I - \alpha_n A\| \leq 1 - \beta_n - \alpha_n\overline{\gamma}$, $\forall n \geq 1$.*

Proof. From condition $\alpha_n \leq (1 - \beta_n)\|A\|^{-1}$ for $\forall x \in H$, we have

$$\langle ((1 - \beta_n)I - \alpha_n A)x, x\rangle = (1 - \beta_n)\|x\|^2 - \alpha_n\langle Ax, x\rangle \geq (1 - \beta_n - \alpha_n\|A\|)\|x\|^2 \geq 0.$$

That is, $(1-\beta_n)I-\alpha_n A$ is positive operator on H. Since A is a strongly positive bounded linear self-adjoint operator on H with coefficient $\overline{\gamma}>0$, we have

$$\begin{aligned} \|(1-\beta_n)I-\alpha_n A\| &= \sup\{\langle((1-\beta_n)I-\alpha_n A)x,x\rangle : x\in H, \|x\|=1\} \\ &= \sup\{1-\beta_n-\alpha_n\langle Ax,x\rangle : x\in H, \|x\|=1\} \\ &\le 1-\beta_n-\alpha_n\overline{\gamma}. \end{aligned}$$

□

Theorem 3.5. *Let H be an infinite dimensional real Hilbert space and let C be a nonempty closed convex subset of H. Let $\{T_i\}_{i=1}^{\infty}: C\to C$ be an infinite family of nonexpansive mappings and $f: H\to H$ be a Meir-Keeler-type contraction. Let A be a strongly positive bounded linear self-adjoint operator on H with coefficient $\overline{\gamma}>0$ and F be an α-inverse-strongly monotone mapping of C into H. Let B and G be maximal monotone mappings on H such that the domains of B and G are included in C. Let $J_\rho=(I+\rho B)^{-1}$ and $T_\eta=(I+\eta G)^{-1}$ for $\rho>0$ and $\eta>0$. Assume that $\Gamma=\bigcap_{i=1}^{\infty}F(T_i)\bigcap(F+B)^{-1}0\bigcap G^{-1}0\neq\emptyset$. Let $\{x_n\}$ be a sequence generated from an arbitrary $x_1\in H$ by the following algorithm:*

$$\begin{cases} y_n = J_{\rho_n}(I-\rho_n F)T_{\eta_n}x_n, \\ x_{n+1} = \alpha_n\gamma f(x_n)+\beta_n x_n+((1-\beta_n)I-\alpha_n A)W_n y_n. \end{cases} \tag{3.9}$$

where W_n is the W-mapping generated by $T_n, T_{n-1},\dots,T_1$ and $\lambda_n,\lambda_{n-1},\dots,\lambda_1$, constant $\gamma\le\bar{\gamma}$ and $\|A\|\le 1$. If $\{\alpha_n\},\{\beta_n\}\subset(0,1)$, $\{\rho_n\}$, and $\{\eta_n\}$ satisfying the following conditions:

(i) $\lim_{n\to\infty}\alpha_n=0$, $\sum_{n=0}^{\infty}\alpha_n=\infty$;
(ii) $0<\liminf_{n\to\infty}\beta_n\le\limsup_{n\to\infty}\beta_n<1$;
(iii) $0<a\le\rho_n\le b<2\alpha$, $\lim_{n\to\infty}|\rho_{n+1}-\rho_n|=0$;
(iv) $\liminf_{n\to\infty}\eta_n>0$, $\lim_{n\to\infty}|\eta_{n+1}-\eta_n|=0$,

then $\{x_n\}$ converges strongly to a point $\omega\in\Gamma$, which solves variational inequality (3.2).

Proof. First, we show that $\{x_n\}$ is bounded. From the condition $\lim_{n\to\infty}\alpha_n=0$, we may assume, without loss of generality, that $\alpha_n\le(1-\beta_n)\|A\|^{-1}$. By Lemma 3.4, we know that $\|(1-\beta_n)I-\alpha_n A\|\le 1-\beta_n-\alpha_n\overline{\gamma}$, $\forall n\ge1$. Suppose that $\forall p\in\Gamma$. By the same argument as in the proof of Theorem 3.3, we have that $\|y_n-p\|\le\|x_n-p\|$. Fixed ε_0, for $\forall n\ge 1$.

Case 1. $\|x_n-p\|<\varepsilon_0$. It is obvious that $\{x_n\}$ is bounded.

Case 2. $\|x_n-p\|\ge\varepsilon_0$. By Proposition 2.6, there exists $k_{\varepsilon_0}\in(0,1)$ such that $\|f(x_n)-f(p)\|\le k_{\varepsilon_0}\|x_n-p\|$. Then

$$\begin{aligned} \|x_{n+1}-p\| &= \|\alpha_n\gamma f(x_n)-\alpha_n\gamma f(p)+\beta_n x_n-\beta_n p+[(1-\beta_n)I-\alpha_n A](W_n y_n-p) \\ &\quad +\alpha_n\gamma f(p)+\beta_n p+[(1-\beta_n)I-\alpha_n A]p-p\| \\ &\le \alpha_n\gamma k_{\varepsilon_0}\|x_n-p\|+\beta_n\|x_n-p\|+\|(1-\beta_n)I-\alpha_n A\|\|W_n y_n-p\|+\alpha_n\|\gamma f(p)-Ap\| \\ &\le (1-\alpha_n(\overline{\gamma}-\gamma k_{\varepsilon_0}))\|x_n-p\|+\alpha_n(\overline{\gamma}-\gamma k_{\varepsilon_0})\tfrac{1}{\overline{\gamma}-\gamma k_{\varepsilon_0}}\|\gamma f(p)-Ap\|. \end{aligned}$$

Set $M=\max\{\|x_1-p\|,\frac{1}{\overline{\gamma}-\gamma k_{\varepsilon_0}}\|\gamma f(p)-Ap\|\}$. Assume that $\|x_n-p\|\le M$, By induction, we have $\|x_{n+1}-p\|\le M$. Hence $\{x_n\}$ is bounded and $\{y_n\}$ is also bounded.

Second, we show that $\|x_{n+1}-x_n\|\to 0$. Set $x_{n+1}=\beta_n x_n+(1-\beta_n)z_n$. Then $z_n=\frac{x_{n+1}-\beta_n x_n}{1-\beta_n}$ and

$$\begin{aligned}
&\|z_{n+1}-z_n\|\\
&=\|\frac{\alpha_{n+1}\gamma f(x_{n+1})+((1-\beta_{n+1})I-\alpha_{n+1}A)W_{n+1}y_{n+1}}{1-\beta_{n+1}}\\
&\quad-\frac{\alpha_n\gamma f(x_n)+((1-\beta_n)I-\alpha_nA)W_ny_n}{1-\beta_n}\|\\
&=\|\frac{\alpha_{n+1}\gamma}{1-\beta_{n+1}}(f(x_{n+1})-f(x_n))+(\frac{\alpha_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n}{1-\beta_n})\\
&\quad\times(\gamma f(x_n)-AW_ny_n)+W_{n+1}y_{n+1}-W_ny_n+\frac{\alpha_{n+1}}{1-\beta_{n+1}}(AW_ny_n-AW_{n+1}y_{n+1})\|\\
&\le\frac{\alpha_{n+1}\gamma}{1-\beta_{n+1}}\|f(x_{n+1})-f(x_n)\|+|\frac{\alpha_{n+1}}{1-\beta_{n+1}}\\
&\quad-\frac{\alpha_n}{1-\beta_n}|\|\gamma f(x_n)-AW_ny_n\|+\|W_{n+1}y_{n+1}-W_ny_n\|+\frac{\alpha_{n+1}}{1-\beta_{n+1}}\|AW_ny_n-AW_{n+1}y_{n+1}\|.
\end{aligned}\tag{3.10}$$

Observe that

$$\|W_{n+1}y_{n+1}-W_ny_n\|\le\|y_{n+1}-y_n\|+\|W_{n+1}y_n-W_ny_n\|\tag{3.11}$$

and

$$\begin{aligned}
\|W_{n+1}y_n-W_ny_n\|&=\|\lambda_1T_1U_{n+1,2}x_n-\lambda_1T_1U_{n,2}x_n\|\\
&\le\lambda_1\|U_{n+1,2}x_n-U_{n,2}x_n\|\\
&=\lambda_1\|\lambda_2T_2U_{n+1,3}x_n-\lambda_2T_2U_{n,3}x_n\|\\
&\le\lambda_1\lambda_2\|U_{n+1,3}x_n-U_{n,3}x_n\|\\
&\le\cdots\\
&\le\lambda_1\lambda_2\cdots\lambda_n\|U_{n+1,n+1}x_n-U_{n,n+1}x_n\|\\
&\le M\prod_{i=1}^{n}\lambda_i,
\end{aligned}\tag{3.12}$$

for some appropriate constant $M>0$ such that $M\ge\|U_{n+1,n+1}x_n-U_{n,n+1}x_n\|$. From Lemma 2.12, Lemma 2.13, and the nonexpansivity of $I-\rho_nF$, we have that

$$\begin{aligned}
\|y_{n+1}-y_n\|&=\|J_{\rho_{n+1}}(I-\rho_{n+1}F)T_{\eta_{n+1}}x_{n+1}-J_{\rho_{n+1}}(I-\rho_{n+1}F)T_{\eta_{n+1}}x_n\\
&\quad+J_{\rho_{n+1}}(I-\rho_{n+1}F)T_{\eta_{n+1}}x_n-J_{\rho_{n+1}}(I-\rho_{n+1}F)T_{\eta_n}x_n\\
&\quad+J_{\rho_{n+1}}(I-\rho_{n+1}F)T_{\eta_n}x_n-J_{\rho_{n+1}}(I-\rho_nF)T_{\eta_n}x_n\\
&\quad+J_{\rho_{n+1}}(I-\rho_nF)T_{\eta_n}x_n-J_{\rho_n}(I-\rho_nF)T_{\eta_n}x_n\|\\
&\le\|x_{n+1}-x_n\|+\|T_{\eta_{n+1}}x_n-T_{\eta_n}x_n\|+|\rho_{n+1}-\rho_n|\|FT_{\eta_n}x_n\|\\
&\quad+\|J_{\rho_{n+1}}(I-\rho_nF)T_{\eta_n}x_n-J_{\rho_n}(I-\rho_nF)T_{\eta_n}x_n\|\\
&\le\|x_{n+1}-x_n\|+\frac{|\eta_{n+1}-\eta_n|}{\eta_{n+1}}\|x_n-T_{\eta_{n+1}}x_n\|\\
&\quad+|\rho_{n+1}-\rho_n|\|FT_{\eta_n}x_n\|+\frac{|\rho_{n+1}-\rho_n|}{\rho_{n+1}}\|(I-\rho_nF)T_{\eta_n}x_n-J_{\rho_{n+1}}(I-\rho_n)FT_{\eta_n}x_n\|.
\end{aligned}\tag{3.13}$$

Substituting (3.11), (3.12), and (3.13) into (3.10), we obtain that

$$\begin{aligned}\|z_{n+1}-z_n\| &\le \tfrac{\alpha_{n+1}\gamma}{1-\beta_{n+1}}\|f(x_{n+1})-f(x_n)\|+|\tfrac{\alpha_{n+1}}{1-\beta_{n+1}}-\tfrac{\alpha_n}{1-\beta_n}|\\ &\quad\times\|\gamma f(x_n)-AW_ny_n\|+\tfrac{\alpha_{n+1}}{1-\beta_{n+1}}\|AW_ny_n-AW_{n+1}y_{n+1}\|\\ &\quad+\|x_{n+1}-x_n\|+\tfrac{|\eta_{n+1}-\eta_n|}{\eta_{n+1}}\|x_n-T_{\eta_{n+1}}x_n\|\\ &\quad+|\rho_{n+1}-\rho_n|\|FT_{\eta_n}x_n\|+\tfrac{|\rho_{n+1}-\rho_n|}{\rho_{n+1}}\|(I-\rho_nF)T_{\eta_n}x_n\\ &\quad-J_{\rho_{n+1}}(I-\rho_n)FT_{\eta_n}x_n\|+M\textstyle\prod_{i=1}^n\lambda_i.\end{aligned}$$

Then

$$\begin{aligned}\|z_{n+1}-z_n\|-\|x_{n+1}-x_n\| &\le \tfrac{\alpha_{n+1}\gamma}{1-\beta_{n+1}}\|f(x_{n+1})-f(x_n)\|+|\tfrac{\alpha_{n+1}}{1-\beta_{n+1}}-\tfrac{\alpha_n}{1-\beta_n}|\\ &\quad\times\|\gamma f(x_n)-AW_ny_n\|+\tfrac{\alpha_{n+1}}{1-\beta_{n+1}}\|AW_ny_n\\ &\quad-AW_{n+1}y_{n+1}\|+\tfrac{|\eta_{n+1}-\eta_n|}{\eta_{n+1}}\|x_n-T_{\eta_{n+1}}x_n\|\\ &\quad+|\rho_{n+1}-\rho_n|\|FT_{\eta_n}x_n\|+\tfrac{|\rho_{n+1}-\rho_n|}{\rho_{n+1}}\\ &\quad\times\|(I-\rho_nF)T_{\eta_n}x_n-J_{\rho_{n+1}}(I-\rho_n)FT_{\eta_n}x_n\|+M\textstyle\prod_{i=1}^n\lambda_i.\end{aligned}$$

In view of $0<\liminf_{n\to\infty}\beta_n\le\limsup_{n\to\infty}\beta_n<1$, $\lim_{n\to\infty}\alpha_n=0$, $0<a\le\rho_n\le b<2\alpha$, $\lim_{n\to\infty}|\rho_{n+1}-\rho_n|=0$, $\lim_{n\to\infty}|\eta_{n+1}-\eta_n|=0$, and $\liminf_{n\to\infty}\eta_n>0$, we have

$$\limsup_{n\to\infty}(\|z_{n+1}-z_n\|-\|x_{n+1}-x_n\|)\le 0.$$

By Lemma 2.16, we obtain that $\lim_{n\to\infty}\|z_n-x_n\|=0$, and hence $\|x_{n+1}-x_n\|=(1-\beta_n)\|z_n-x_n\|\to 0$.

Third, we show that $\omega_w(x_n)\sqsubseteq\Gamma$. For $\forall q\in\omega_w(x_n)$, then there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i}\rightharpoonup q$.

By the same argument as in the proof of Theorem 3.3, $\forall p\in\Gamma$, we have that

$$\|y_n-p\|^2\le\|x_n-p\|^2-\frac{\frac14-\frac{\rho_n}{8\alpha}}{\frac34+\frac{\rho_n}{8\alpha}}\|y_n-x_n\|^2.$$

Therefore

$$\begin{aligned}\|x_{n+1}-p\|^2 &= \|\alpha_n(\gamma f(x_n)-Ap)+\beta_n(x_n-p)+((1-\beta_n)I-\alpha_nA)(W_ny_n-p)\|^2\\ &= \alpha_n^2\|\gamma f(x_n)-Ap\|^2+\beta_n^2\|x_n-p\|^2+\|((1-\beta_n)I-\alpha_nA)(W_ny_n-p)\|^2\\ &\quad+2\langle\alpha_n(\gamma f(x_n)-Ap),\beta_n(x_n-p)\rangle\\ &\quad+2\langle\alpha_n(\gamma f(x_n)-Ap),((1-\beta_n)I-\alpha_nA)(W_ny_n-p)\rangle\\ &\quad+2\langle\beta_n(x_n-p),((1-\beta_n)I-\alpha_nA)(W_ny_n-p)\rangle\\ &\le \alpha_n^2\|\gamma f(x_n)-Ap\|^2+\beta_n^2\|x_n-p\|^2+(1-\beta_n-\alpha_n\overline{\gamma})^2\|y_n-p\|^2\\ &\quad+2\alpha_n\beta_n\|\gamma f(x_n)-Ap\|\|x_n-p\|+2\alpha_n(1-\beta_n-\alpha_n\overline{\gamma})\|\gamma f(x_n)-Ap\|\|x_n-p\|\\ &\quad+2\beta_n(1-\beta_n-\alpha_n\overline{\gamma})\|x_n-p\|^2\\ &\le(\beta_n^2+2\beta_n(1-\beta_n-\alpha_n\overline{\gamma}))\|x_n-p\|^2+\alpha_n^2\|\gamma f(x_n)-Ap\|^2\\ &\quad+2\alpha_n(1-\alpha_n\overline{\gamma})\|\gamma f(x_n)-Ap\|\|x_n-p\|\\ &\quad+(1-\beta_n-\alpha_n\overline{\gamma})^2(\|x_n-p\|^2-\frac{\frac14-\frac{\rho_n}{8\alpha}}{\frac34+\frac{\rho_n}{8\alpha}}\|y_n-x_n\|^2).\end{aligned}\tag{3.14}$$

Then

$$(1-\beta_n-\alpha_n\overline{\gamma})^2\frac{\frac{1}{4}-\frac{\rho_n}{8\alpha}}{\frac{3}{4}+\frac{\rho_n}{8\alpha}}\|y_n-x_n\|^2 \le (1-\alpha_n\overline{\gamma})^2\|x_n-p\|^2+\alpha_n^2\|\gamma f(x_n)-Ap\|^2 +2\alpha_n(1-\alpha_n\overline{\gamma})\|\gamma f(x_n)-Ap\|\|x_n-p\|-\|x_{n+1}-p\|^2.$$

Since $\lim_{n\to\infty}\alpha_n=0$, $0<\limsup_{n\to\infty}\beta_n<1$, and $0<a\le\rho_n\le b<2\alpha$, we have

$$\lim_{n\to\infty}\|y_n-x_n\|=0. \tag{3.15}$$

Since

$$\begin{aligned}
&\|x_{n+1}-W_nx_n\|\\
&=\|\alpha_n(\gamma f(x_n)-AW_nx_n)+\beta_n(x_n-W_nx_n)+((1-\beta_n)I-\alpha_nA)(W_ny_n-W_nx_n)\|\\
&\le\alpha_n\|\gamma f(x_n)-AW_nx_n\|+\beta_n\|x_n-W_nx_n\|+(1-\beta_n-\alpha_n\overline{\gamma})\|W_ny_n-W_nx_n\|\\
&\le\alpha_n\|\gamma f(x_n)-AW_nx_n\|+\beta_n\|x_n-W_nx_n\|+(1-\beta_n-\alpha_n\overline{\gamma})\|y_n-x_n\|,
\end{aligned}$$

we have

$$\begin{aligned}
&\|x_n-W_nx_n\|\\
&\le\|x_n-x_{n+1}\|+\|x_{n+1}-W_nx_n\|\\
&\le\|x_n-x_{n+1}\|+\alpha_n\|\gamma f(x_n)-AW_nx_n\|+\beta_n\|x_n-W_nx_n\|+(1-\beta_n-\alpha_n\overline{\gamma})\|y_n-x_n\|.
\end{aligned}$$

Obviously

$$(1-\beta_n)\|x_n-W_nx_n\|\le\|x_n-x_{n+1}\|+\alpha_n\|\gamma f(x_n)-AW_nx_n\|+(1-\beta_n-\alpha_n\overline{\gamma})\|y_n-x_n\|.$$

From $\lim_{n\to\infty}\alpha_n=0$, $0<\limsup_{n\to\infty}\beta_n<1$, $\|y_n-x_n\|\to0$, and $\|x_{n+1}-x_n\|\to0$, we have that $\|x_n-W_nx_n\|\to0$ as $n\to\infty$. By the same argument as in the proof of Theorem 3.3, we can obtain $q\in\bigcap_{i=1}^{\infty}F(T_i)$.

Next, we show that $q\in(F+B)^{-1}(0)\bigcap G^{-1}(0)$. Put $u_n=T_{\eta_n}x_n$. Since $\{x_n\}$ is bounded, then $\{y_n\}$ and $\{u_n\}$ are also bounded. By the same argument as in the proof of Theorem 3.3, we have that $\|y_n-p\|\le\|u_n-p\|$ and $\|u_n-p\|^2\le\|x_n-p\|^2-\|u_n-x_n\|^2$. From (3.14), we obtain that

$$\begin{aligned}
&\|x_{n+1}-p\|^2\\
&\le\alpha_n^2\|\gamma f(x_n)-Ap\|^2+\beta_n^2\|x_n-p\|^2+(1-\beta_n-\alpha_n\overline{\gamma})^2\|y_n-p\|^2\\
&\quad+2\alpha_n\beta_n\|\gamma f(x_n)-Ap\|\|x_n-p\|+2\alpha_n(1-\beta_n-\alpha_n\overline{\gamma})\\
&\quad\times\|\gamma f(x_n)-Ap\|\|x_n-p\|+2\beta_n(1-\beta_n-\alpha_n\overline{\gamma})\|x_n-p\|^2\\
&\le\alpha_n^2\|\gamma f(x_n)-Ap\|^2+\beta_n^2\|x_n-p\|^2+(1-\beta_n-\alpha_n\overline{\gamma})^2\|u_n-p\|^2\\
&\quad+2\alpha_n\beta_n\|\gamma f(x_n)-Ap\|\|x_n-p\|+2\alpha_n(1-\beta_n-\alpha_n\overline{\gamma})\\
&\quad\times\|\gamma f(x_n)-Ap\|\|x_n-p\|+2\beta_n(1-\beta_n-\alpha_n\overline{\gamma})\|x_n-p\|^2.
\end{aligned}$$

Thus

$$\begin{aligned}
&(1-\beta_n-\alpha_n\overline{\gamma})^2\|u_n-x_n\|^2\\
&\le(1-\alpha_n\overline{\gamma})^2\|x_n-p\|^2+\alpha_n^2\|\gamma f(x_n)-Ap\|^2\\
&\quad+2\alpha_n(1-\alpha_n\overline{\gamma})\|\gamma f(x_n)-Ap\|\|x_n-p\|-\|x_{n+1}-p\|^2.
\end{aligned}$$

Since $\lim_{n\to\infty}\alpha_n=0$ and $0<\limsup_{n\to\infty}\beta_n<1$, we have that $\lim_{n\to\infty}\|u_n-x_n\|=0$. It follows from (3.15) that $\lim_{n\to\infty}\|u_n-y_n\|=0$. By the same argument as in the proof of Theorem 3.3, we have that $q\in(F+B)^{-1}(0)\bigcap G^{-1}(0)$. So, $q\in\Gamma$. That is, $\omega_w(x_n)\sqsubseteq\Gamma$.

Fourth, we show that

$$\limsup_{n\to\infty}\langle(A-rf)\omega,x_n-\omega\rangle\ge0, \tag{3.16}$$

where $\omega = P_{U\bigcap(F+B)^{-1}(0)\bigcap G^{-1}(0)}f(\omega)$. Indeed, take a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle (A-rf)\omega, x_n-\omega\rangle = \lim_{i\to\infty}\langle (A-rf)\omega, x_{n_i}-\omega\rangle.$$

Since $\{x_n\}$ is bounded, we may assume, without loss of generality, that $x_{n_i} \rightharpoonup q \in U\bigcap(F+B)^{-1}(0)\bigcap G^{-1}(0)$. Then $\limsup_{n\to\infty}\langle (A-rf)\omega, x_n-\omega\rangle = \langle (A-rf)\omega, q-\omega\rangle \geq 0$. In view of $\|x_n - W_n x_n\| \to 0$ as $n\to\infty$, we have that

$$\limsup_{n\to\infty}\langle (A-rf)\omega, W_n x_n-\omega\rangle = \langle (A-rf)\omega, q-\omega\rangle \geq 0. \tag{3.17}$$

Finally, we show that $x_n \to \omega$ as $n\to\infty$. Assume that the sequence $\{x_n\}$ does not converge strongly to $\omega\in\Gamma$. Thus there exists $\varepsilon > 0$ and a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\|x_{n_i}-\omega\| \geq \varepsilon$ for all $i\geq 0$. From Proposition 2.6, there exists $k_\varepsilon \in (0,1)$ for this ε such that $\|f(x_{n_i}) - f(\omega)\| \leq k_\varepsilon\|x_{n_i}-\omega\|$. Then

$$\begin{aligned}
&\|x_{n_i+1}-\omega\|^2\\
&= \alpha_{n_i}^2\|\gamma f(x_{n_i}) - A\omega\|^2 + \|\beta_{n_i}(x_{n_i}-\omega) + ((1-\beta_{n_i})I - \alpha_{n_i}A)(W_{n_i}y_{n_i}-\omega)\|^2\\
&\quad + 2\langle \alpha_{n_i}(\gamma f(x_{n_i}) - A\omega), \beta_{n_i}(x_{n_i}-\omega) + ((1-\beta_{n_i})I - \alpha_{n_i}A)(W_{n_i}y_{n_i}-\omega)\rangle\\
&\leq \alpha_{n_i}^2\|\gamma f(x_{n_i}) - A\omega\|^2 + [\beta_{n_i}\|x_{n_i}-\omega\| + (1-\beta_{n_i}-\alpha_{n_i}\overline{\gamma})\|W_{n_i}y_{n_i}-\omega\|]^2\\
&\quad + 2\alpha_{n_i}\beta_{n_i}\langle \gamma f(x_{n_i}) - A\omega, x_{n_i}-\omega\rangle + 2\alpha_{n_i}\langle \gamma f(x_{n_i}) - A\omega, ((1-\beta_{n_i})I - \alpha_{n_i}A)(W_{n_i}y_{n_i}-\omega)\rangle\\
&\leq \alpha_{n_i}^2\|\gamma f(x_{n_i}) - A\omega\|^2 + (1-\alpha_{n_i}\overline{\gamma})^2\|x_{n_i}-\omega\|^2\\
&\quad + 2\alpha_{n_i}\beta_{n_i}\langle \gamma f(x_{n_i}) - \gamma f(\omega), x_{n_i}-\omega\rangle + 2\alpha_{n_i}\beta_{n_i}\langle \gamma f(\omega) - A\omega, x_{n_i}-\omega\rangle\\
&\quad + 2\alpha_{n_i}\langle \gamma f(x_{n_i}) - \gamma f(\omega), ((1-\beta_{n_i})I - \alpha_{n_i}A)(W_{n_i}y_{n_i}-\omega)\rangle\\
&\quad + 2\alpha_{n_i}\langle \gamma f(\omega) - A\omega, ((1-\beta_{n_i})I - \alpha_{n_i}A)(W_{n_i}y_{n_i}-\omega)\rangle\\
&\leq [(1-\alpha_{n_i}\overline{\gamma})^2 + 2\alpha_{n_i}\beta_{n_i}\gamma k_\varepsilon + 2\alpha_{n_i}\gamma k_\varepsilon(1-\beta_{n_i}-\alpha_{n_i}\overline{\gamma})]\|x_{n_i}-\omega\|^2\\
&\quad + \alpha_{n_i}^2\|\gamma f(x_{n_i}) - A\omega\|^2 + 2\alpha_{n_i}\beta_{n_i}\langle \gamma f(\omega) - A\omega, x_{n_i}-\omega\rangle\\
&\quad + 2\alpha_{n_i}(1-\beta_{n_i})\langle \gamma f(\omega) - A\omega, W_{n_i}y_{n_i} - W_{n_i}x_{n_i}\rangle\\
&\quad + 2\alpha_{n_i}(1-\beta_{n_i})\langle \gamma f(\omega) - A\omega, W_{n_i}x_{n_i}-\omega\rangle - 2\alpha_{n_i}^2\langle \gamma f(\omega) - A\omega, A(W_{n_i}y_{n_i}-\omega)\rangle\\
&\leq [1 - 2\alpha_{n_i}(\overline{\gamma}-\gamma k_\varepsilon) + \alpha_{n_i}^2\overline{\gamma}^2]\|x_{n_i}-\omega\|^2 + \alpha_{n_i}^2\|\gamma f(x_{n_i}) - A\omega\|^2 + 2\alpha_{n_i}\beta_{n_i}\langle \gamma f(\omega) - A\omega, x_{n_i}-\omega\rangle\\
&\quad + 2\alpha_{n_i}(1-\beta_{n_i})\|\gamma f(\omega) - A\omega\|\|y_{n_i}-x_{n_i}\| + 2\alpha_{n_i}(1-\beta_{n_i})\langle \gamma f(\omega) - A\omega, W_{n_i}x_{n_i}-\omega\rangle\\
&\quad + 2\alpha_{n_i}^2\|\gamma f(\omega) - A\omega\|\|A(W_{n_i}y_{n_i}-\omega)\|\\
&\leq [1 - 2\alpha_{n_i}(\overline{\gamma}-\gamma k_\varepsilon)]\|x_{n_i}-\omega\|^2 + \alpha_{n_i}\xi_{n_i},
\end{aligned} \tag{3.18}$$

where

$$\begin{aligned}
\xi_{n_i} &= \alpha_{n_i}\overline{\gamma}^2\|x_{n_i}-\omega\|^2 + \alpha_{n_i}\|\gamma f(x_{n_i}) - A\omega\|^2 + 2\beta_{n_i}\langle \gamma f(\omega) - A\omega, x_{n_i}-\omega\rangle\\
&\quad + 2(1-\beta_{n_i})\|\gamma f(\omega) - A\omega\|\|y_{n_i}-x_{n_i}\| + 2\alpha_{n_i}\|\gamma f(\omega) - A\omega\|\\
&\quad \times \|A(W_{n_i}y_{n_i}-\omega)\| + 2(1-\beta_{n_i})\langle \gamma f(\omega) - A\omega, W_{n_i}x_{n_i}-\omega\rangle.
\end{aligned}$$

Setting $b_{n_i} = 2\alpha_{n_i}(\overline{\gamma}-\gamma k_\varepsilon)$, $c_{n_i} = \alpha_{n_i}\xi_{n_i}$, we see that (3.18) is reduced to

$$\|x_{n_i+1}-\omega\|^2 \leq (1-b_{n_i})\|x_{n_i}-\omega\|^2 + c_{n_i}.$$

From condition $\sum_{n=0}^{\infty}\alpha_n=\infty$, we know that $\sum_{i=0}^{\infty}b_{n_i}=\infty$. Also from condition $\lim_{n\to}\alpha_n=0$, (3.15), (3.16), and (3.17), we have that $\limsup_{i\to\infty}\frac{c_{n_i}}{b_{n_i}}=\limsup_{i\to\infty}\frac{\xi_{n_i}}{2(\overline{\gamma}-\gamma k_\varepsilon)}\le 0$. It follows form Lemma 2.17 that $x_{n_i}\to\omega$ as $i\to\infty$. The contradiction permits us to conclude that $\{x_n\}$ converges strongly to $\omega\in\Gamma$. □

Remark 3.6. (i) In Theorem 3.3 and Theorem 3.5, set $\gamma=1$ and $A=I$. Then the sequence $\{x_n\}$ generated by algorithm (3.1) and (3.9), respectively, converges strongly to a point $\omega\in\Gamma$, which is also the unique solution to the following variational inequality $\langle(I-f)x,z-x\rangle\ge 0$, $\forall z\in\Gamma$, where f is a Meir-Keeler-type contraction.

(ii) In Theorem 3.3 and Theorem 3.5, set $T_i=I,\forall i\ge 1$. Thus the sequence $\{x_n\}$ generated by algorithm (3.1) and (3.9), respectively, converges strongly to a point $\omega\in\bigcap(F+B)^{-1}0\bigcap G^{-1}0$, which is also the unique solution to the following variational inequality $\langle(A-\gamma f)x,z-x\rangle\ge 0$, $\forall z\in\bigcap(F+B)^{-1}0\bigcap G^{-1}0$, where f is a Meir-Keeler-type contraction.

(iii) Observe that Meir-Keeler-type contraction is a generalization of contraction. In Theorem 3.3 and Theorem 3.5, if f is a contraction, then the corresponding conclusions can also be obtained.

4. Applications

In this section, we apply our results (taking the explicit iterative algorithm as an example) to variational inequality problem, equilibrium problem, constrained convex minimization problem and generalized split feasibility problem. Similarly, the results of implicit iterative algorithm also have the corresponding applications.

Let H be a Hilbert space and let $g:H\to(-\infty,\infty]$ be a proper lower semicontinuous convex function. The subdifferential ∂g of g is defined by

$$\partial g(x):=\{z\in H: g(x)+\langle z,y-x\rangle\le g(y),\forall y\in H\},x\in H.$$

From Rockafellar [14], we know that ∂g is a maximal monotone operator. In particular, if C is a nonempty, closed, and convex subset of H, and i_C is the *indicator function* of C defined by

$$i_C(x)=\begin{cases}0, & if\ x\in C,\\ \infty, & else.\end{cases}$$

then ∂i_C is a maximal monotone operator and we can define the resolvent J_β of ∂i_C for each $\beta>0$ as $J_\beta x=(I+\lambda\partial i_C)^{-1}x$, $\forall x\in H$. Recall that the normal cone to C at u is defined by

$$N_Cu:=\{z\in H:\langle z,v-u\rangle\le 0,\forall v\in C\}.$$

It is easy to verify that $\partial i_C(u)=N_Cu$ for $u\in C$ and $J_\beta x=P_Cx$ for $\forall x\in H, beta>0$.

Let A be a mapping of C into H. The variational inequality problem (VIP) is to find an element $x^*\in C$ satisfying the inequality

$$\langle Ax^*,x-x^*\rangle\ge 0,\forall x\in C. \tag{4.1}$$

The set of solutions of VIP (4.1) is denoted by $VI(C,A)$. We can easily show that $x^*\in VI(C,A)$ is equivalent to $x^*=P_C(I-\lambda A)x^*$ A simple iterative method algorithm for solving VIP (4.1) is the following projection-gradient method $x_{n+1}=P_C(I-\lambda A)x_n$ for each $n\in N$, where λ is a positive real number. Assume that A is α-inverse strongly monotone, $VI(C,A)$ is nonempty,

$0 < \lambda < 2\alpha$, then $VI(C,A)$ is a closed and convex subset of H. The sequence $\{x_n\}$ generated by the projection-gradient method above converges weakly to a point in $VI(C,A)$.

Let $\phi : C \times C \to R$ is a bifunction, the equilibrium problem (EP) is to: find $z \in C$ such that $\phi(z,y) \geq 0$ for each $y \in C$. We denote the set of solutions of equilibrium problem by $EP(\phi)$. In order to solve the equilibrium problem for bifunction ϕ, we assume that ϕ satisfies the following conditions:

(A1) $\phi(x,x) = 0, \ \forall x \in C$;
(A2) ϕ is monotone, i.e.,$\phi(x,y) + \phi(y,x) \leq 0, \ \forall x,y \in C$;
(A3) $\lim_{t\downarrow 0} \phi(tz + (1-t)x, y) \leq \phi(x,y), \ \forall x,y,z \in C$;
(A4) for fixed $x \in C$, the function $y \mapsto \phi(x,y)$ is convex and lower semicontinuous.

Lemma 4.1. [2] *Let C be a nonempty, closed, and convex subset of H. let ϕ be a bifunction from $C \times C \to R$ satisfies (A1)-(A4). Then, for $r > 0$ and each $x \in H$, there exists $z \in C$ such that $\phi(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0$ for all $y \in C$.*

Lemma 4.2. [4] *Let C be a nonempty, closed, and convex subset of H. let ϕ be a bifunction from $C \times C \to R$ satisfies (A1)-(A4). Then, for $r > 0$, define the resolvent $Q_r^\phi : H \to C$ as follows: $Q_r^\phi x := \{z \in C : \phi(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C\}$ for all $x \in H$. Then,*

(i) for each $x \in H$, $Q_r^\phi(x) \neq \emptyset$;
(ii) Q_r^ϕ is single-valued;
(iii) Q_r^ϕ is firmly nonexpansive, that is, $\|Q_r^\phi x - Q_r^\phi y\|^2 \leq \langle Q_r^\phi x - Q_r^\phi y, x - y\rangle, \ \forall x,y \in H$;
(iv) $F(Q_r^\phi) = EP(\phi)$;
(v) $EP(\phi)$ is closed and convex.

We call such Q_r^ϕ the resolvent of ϕ for $r > 0$. Using Lemmas 4.1 and 4.2, Takahashi et al. [23] showed the following lemma; see [1] for a more general result.

Lemma 4.3. [23] *Let C be a nonempty, closed, and convex subset of a Hilbert space H and let $\phi : C \times C \to R$ be a bifunction satisfying conditions (A1)-(A4). Define A_ϕ by*

$$A_\phi x := \begin{cases} \{z \in H : \phi(x,y) \geq \langle y - x, z\rangle, \forall y \in C\}, \forall x \in C. \\ \emptyset, else. \end{cases}$$

Then, $EP(\phi) = A_\phi^{-1}0$ and A_ϕ is a maximal monotone operator with the domain of $A_\phi \subset C$. Furthermore, for any $x \in H$ and $r > 0$, the resolvent Q_r^ϕ of ϕ coincides with the resolvent of A_ϕ, i.e., $Q_r^\phi x = (I + rA_\phi)^{-1}x$.

Next, we apply the result of Theorem 3.5 to the variational inequality problem and the equilibrium problem.

Theorem 4.4. *Let H be an infinite dimensional real Hilbert space and C be a nonempty closed convex subset of H. Let $\{T_i\}_{i=1}^\infty : C \to C$ be an infinite family of nonexpansive mappings and $f : H \to H$ be a Meir-Keeler-type contraction. Let A be a strongly positive bounded linear self-adjoint operator on H with coefficient $\overline{\gamma} > 0$ and F be an α-inverse-strongly monotone mapping of C into H. Let $\phi : C \times C \to R$ is a bifunction satisfying (A1)-(A4). Assume that $\Gamma = \bigcap_{i=1}^\infty F(T_i) \bigcap VI(C,F) \bigcap EP(\phi) \neq \emptyset$. For an arbitrary $x_1 \in H$, Let $\{x_n\}$ be a sequence* *generated by the following algorithm:*

$$\begin{cases} y_n = P_C(I-\rho_n F)Q^{\phi}_{\eta_n}x_n, \\ x_{n+1} = \alpha_n\gamma f(x_n)+\beta_n x_n+((1-\beta_n)I-\alpha_n A)W_n y_n. \end{cases}$$

where W_n is the W-mapping generated by $T_n, T_{n-1}, \ldots, T_1$ and $\lambda_n, \lambda_{n-1}, \ldots, \lambda_1$, $Q^{\phi}_{\eta_n}$ is a resolvent of ϕ for $\eta_n > 0$, constant $\gamma \le \bar{\gamma}$ and $\|A\| \le 1$. If $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$, $\{\rho_n\}$, and $\{\eta_n\}$ satisfy the following conditions:

(i) $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=0}^{\infty}\alpha_n = \infty$;
(ii) $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$;
(iii) $0 < a \le \rho_n \le b < 2\alpha$, $\lim_{n\to\infty}|\rho_{n+1}-\rho_n| = 0$;
(iv) $\liminf_{n\to\infty}\eta_n > 0$, $\lim_{n\to\infty}|\eta_{n+1}-\eta_n| = 0$,

then $\{x_n\}$ converges strongly to a point $\omega \in \Gamma$, which is also the unique solution to variational inequality (3.2).

Proof. In Theorem 3.5, we set $G = A_\phi$ and $B = \partial i_C$. Then, for $\rho_n > 0$ and $\eta_n > 0$, we have $J_{\rho_n} = P_C$ and $T_{\eta_n} = (I+\eta_n A_\phi)^{-1} = Q^{\phi}_{\eta_n}$. Furthermore, $A_\phi^{-1}(0) = F((I+\eta_n A_\phi)^{-1}) = F(Q^{\phi}_{\eta_n}) = EP(\phi)$. Also, we have $(F+\partial i_C)^{-1}0 = VI(C,F)$. Indeed, for $x \in C$,

$$\begin{aligned} x \in (F+\partial i_C)^{-1}0 \quad &\Leftrightarrow 0 \in Fx+\partial i_C x \\ &\Leftrightarrow 0 \in Fx+N_C x \\ &\Leftrightarrow \langle -Fx, y-x\rangle \le 0, \forall y \in C \\ &\Leftrightarrow \langle Fx, y-x\rangle \ge 0, \forall y \in C \\ &\Leftrightarrow x \in VI(C,F). \end{aligned}$$

Thus we can obtain the desired result by Theorem 3.5. ☐

Let $\varphi : C \to R$ is a real-valued convex function, the constrained convex minimization problem is to find $\hat{x} \in C$ such that

$$\varphi(\hat{x}) = \min_{x\in C}\varphi(x). \tag{4.2}$$

The set of solutions of the constrained convex minimization problem is denoted by $\operatorname{argmin}_{x\in C}\varphi(x)$.

Lemma 4.5. [25] *Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. Let φ be a convex function of H into R. If φ is differentiable, then z is a solution to* (4.2) *if and only if $z \in VI(C,\nabla\varphi)$.*

From Lemma 4.5, set $F = \nabla\varphi$ in Theorem 4.4. Thus we can obtain a strong convergence theorem for the equilibrium problem and the constrained convex minimization problem in a Hilbert space.

Theorem 4.6. *Let H be an infinite dimensional real Hilbert space and C be a nonempty closed convex subset of H. Let $\{T_i\}_{i=1}^{\infty} : C \to C$ be an infinite family of nonexpansive mappings and $f : H \to H$ be a Meir-Keeler-type contraction. Let A be a strongly positive bounded linear self-adjoint operator on H with coefficient $\overline{\gamma} > 0$. Let $\varphi : H \to R$ be a differentiable convex function and $\nabla\varphi$ be an α-inverse-strongly monotone mapping of C into H. Let $\phi : C \times C \to R$ is a bifunction satisfying (A1)-(A4). Assume that $\Gamma = \{z \in C : z \in \bigcap_{i=1}^{\infty}F(T_i), z \in \arg\min_{x\in C}\varphi(x)$ and $z \in$*

$EP(\phi)\} \neq \emptyset$. For an arbitrary $x_1 \in H$, Let $\{x_n\}$ be a sequence generated by the following algorithm:

$$\begin{cases} y_n = P_C(I-\rho_n\nabla\varphi)Q^{\phi}_{\eta_n}x_n, \\ x_{n+1} = \alpha_n\gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A)W_n y_n. \end{cases}$$

where W_n is the W-mapping generated by $T_n, T_{n-1}, \ldots, T_1$ and $\lambda_n, \lambda_{n-1}, \ldots, \lambda_1$, $Q^{\phi}_{\eta_n}$ is a resolvent of ϕ for $\eta_n > 0$, constant $\gamma \leq \bar{\gamma}$ and $\|A\| \leq 1$. If $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$, $\{\rho_n\}$, and $\{\eta_n\}$ satisfy the following conditions:

- *(i)* $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=0}^{\infty}\alpha_n = \infty$;
- *(ii)* $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$;
- *(iii)* $0 < a \leq \rho_n \leq b < 2\alpha$, $\lim_{n\to\infty}|\rho_{n+1} - \rho_n| = 0$;
- *(iv)* $\liminf_{n\to\infty}\eta_n > 0$, $\lim_{n\to\infty}|\eta_{n+1} - \eta_n| = 0$,

then $\{x_n\}$ converges strongly to a point $\omega \in \Gamma$ which is also the unique solution to variational inequality (3.2).

Lemma 4.7. [24] *Let H_1 and H_2 be two Hilbert spaces, $A : H_1 \to H_2$ be a bounded linear operator such that $A \neq \emptyset$, and A^* be the adjoint of A. let $T : H_2 \to H_2$ be a nonexpansive mapping. Then*

- *(i)* $A^*(I-T)A$ *is* $\frac{1}{2\|A\|^2}$*-ism.*
- *(ii) For* $0 < r < \frac{1}{\|A\|^2}$,
  - *(iia)* $I - rA^*(I-T)A$ *is* $r\|A\|^2$*-averaged;*
  - *(iib)* $J_\lambda(I - rA^*(I-T)A)$ *is* $\frac{1+r\|A\|^2}{2}$*-averaged.*
- *(iii) If* $r = \|A\|^{-2}$, *then* $I - rA^*(I-T)A$ *is nonexpansive.*

Lemma 4.8. [24] *Let H_1 and H_2 be two Hilbert spaces, $B : H_1 \to 2^{H_1}$ be be a maximal monotone mapping, and $J_\lambda = (I+\lambda B)^{-1}$ be the resolvent of B for $\lambda > 0$. Let $T : H_2 \to H_2$ be a nonexpansive mapping and let $A : H_1 \to H_2$ be a bounded linear operator. Suppose that $B^{-1}0 \bigcap A^{-1}F(T) \neq \emptyset$. Let $\lambda, r > 0$ and $z \in H_1$. Then the following are equivalent:*

- *(i)* $z = J_\lambda(I - rA^*(I-T)A)z$;
- *(ii)* $0 \in A^*(I-T)Az + Bz$;
- *(iii)* $z \in B^{-1}0 \bigcap A^{-1}F(T)$.

Consequently, $F(J_\lambda(I - rA^*(I-T)A)) = (A^*(I-T)A) + B)^{-1}0 = B^{-1}0 \bigcap A^{-1}F(T)$. Moreover, $B^{-1}0 \bigcap A^{-1}F(T)$ is closed and convex.

Lemma 4.9. *Let C_1 and C_2 be nonempty closed convex subsets of Hilbert spaces H_1 and H_2, respectively. Let S be a nonexpansive mapping of H_2 into H_2. Let $U : H_1 \to H_2$ be a bounded linear operator and U^* be the adjoint of U. Let B and G be a maximal monotone mapping on H_1 such that the domains of B and G are included in C_1. Let $J_\rho = (I+\rho B)^{-1}$ and $T_\eta = (I+\eta G)^{-1}$ for $\rho > 0$ and $\eta > 0$. If $0 < r < \frac{1}{\|U\|^2}$, then $F(J_\rho(I - rU^*(I-S)U)T_\eta) = B^{-1}0 \bigcap U^{-1}F(S) \bigcap G^{-1}0$ for any $\rho > 0$, $r > 0$ and $\eta > 0$.*

Proof. From Lemma 4.7 (iib), we know that $J_\rho(I - rU^*(I-S)U)$ is $\frac{1+r\|U\|^2}{2}$-averaged. From Lemma 2.12, we know that T_η is $\frac{1}{2}$-averaged. It follows from Proposition 2.1(iv) and (v) that $J_\rho(I - rU^*(I-S)U)T_\eta$ is also averaged and $F(J_\rho(I - rU^*(I-S)U)T_\eta) = F(J_\rho(I - rU^*(I -$

$S)U))\bigcap F(T_\eta)$. From Lemma 2.12 and Lemma 4.8, we have $F(J_\rho(I-rU^*(I-S)U)T_\eta) = B^{-1}0\bigcap U^{-1}F(S)\bigcap G^{-1}0$. □

Theorem 4.10. *Let C_1 and C_2 be nonempty closed convex subsets of Hilbert spaces H_1 and H_2, respectively. Let S be a nonexpansive mapping of H_2 into H_2. Let $U: H_1 \to H_2$ be a bounded linear operator and U^* be the adjoint of U. Let $\{T_i\}_{i=1}^{\infty}: C_1 \to C_1$ be an infinite family of nonexpansive mappings and $f: H_1 \to H_1$ be a Meir-Keeler-type contraction. Let A be a strongly positive bounded linear self-adjoint operator on H_1 with coefficient $\overline{\gamma} > 0$. Let B and G be a maximal monotone mapping on H_1 such that the domains of B and G are included in C_1. Let $J_\rho = (I+\rho B)^{-1}$ and $T_\eta = (I+\eta G)^{-1}$ for $\rho > 0$ and $\eta > 0$. Assume that $\Gamma = \{z \in C_1 : z \in \bigcap_{i=1}^{\infty} F(T_i)\bigcap B^{-1}0\bigcap G^{-1}0$ such that $Uz \in F(S)\} \neq \emptyset$. For an arbitrary $x_1 \in H$, Let $\{x_n\}$ be a sequence generated by the following algorithm:*

$$\begin{cases} y_n = J_{\rho_n}(I-\rho_n U^*(I-S)U)T_{\eta_n}x_n, \\ x_{n+1} = \alpha_n\gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A)W_n y_n. \end{cases} \tag{4.3}$$

where W_n is the W-mapping generated by $T_n, T_{n-1}, \ldots, T_1$ and $\lambda_n, \lambda_{n-1}, \ldots, \lambda_1$, constant $\gamma \leq \bar{\gamma}$ and $\|A\| \leq 1$. If $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$, $\{\rho_n\}$, and $\{\eta_n\}$ satisfy the following conditions:

- *(i)* $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=0}^{\infty}\alpha_n = \infty$;
- *(ii)* $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$;
- *(iii)* $0 < a \leq \rho_n \leq b < \frac{1}{\|A\|^2}$, $\lim_{n\to\infty}|\rho_{n+1}-\rho_n| = 0$;
- *(iv)* $\liminf_{n\to\infty}\eta_n > 0$, $\lim_{n\to\infty}|\eta_{n+1}-\eta_n| = 0$,

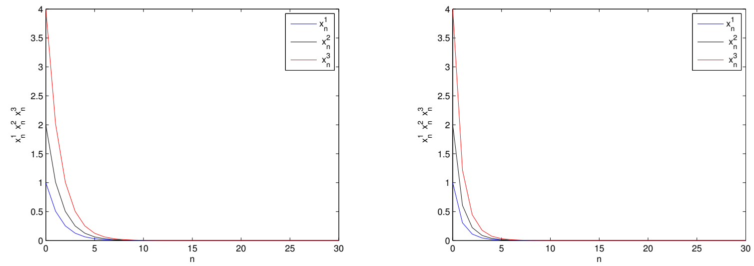
then $\{x_n\}$ converges strongly to a point $\omega \in \Gamma$ which is also the unique solution of to variational inequality (3.2).

Proof. Since S is a nonexpansive mapping of H_2 into H_2, it follows from Lemma 4.7 that $U^*(I-S)U$ is a $\frac{1}{2\|A\|^2}$-ism. Setting $F = A^*(I-S)A$ in Theorem 4.10, we see that algorithm (4.3) is just algorithm (3.9) in Theorem 3.5. Since Γ is nonempty, there exist $z \in C_1$ such that $z \in \bigcap_{i=1}^{\infty} F(T_i)\bigcap B^{-1}0\bigcap G^{-1}0$ and $Uz \in F(S)$. By Lemma 4.9, we have that $z = J_\rho(I-\rho U^*(I-S)U)T_\eta z = J_\rho(I-\rho F)T_\eta z$, that is, $z \in (F+B)^{-1}(0)\bigcap G^{-1}(0)$. Hence $z \in (F+B)^{-1}(0)\bigcap G^{-1}(0)\bigcap(\bigcap_{i=1}^{\infty} F(T_i)) \neq \emptyset$. By Theorem 3.5, we know that $x_n \to \omega$ and $\omega = P_\Gamma f(\omega)$. That is, ω is also the unique solution to variational inequality (3.2). □

5. Numerical Experiments

In this section, We give two numerical examples of Theorem 3.5 to illustrate the implementation of algorithm (3.9). All codes were written in Matlab 2010b and run on Dell i - 5 Dual-Core laptop.

Example 5.1. Let $H := R^3$ with the inner product $\langle\cdot,\cdot\rangle : R^3 \times R^3 \to R$ defined by $\langle \mathbf{x}, \mathbf{y}\rangle = x_1\cdot y_1 + x_2\cdot y_2 + x_3\cdot y_3$ and the usual norm $\|\cdot\| : R^3 \to R$ defined by $\|\mathbf{x}\| = \sqrt{x_1^2+x_2^2+x_3^2}$ for all $\mathbf{x} = (x_1,x_2,x_3)$ and $\mathbf{y} = (y_1,y_2,y_3)$. For $\forall \mathbf{x} \in R^3$, let $F, f, A, B, G : R^3 \to R^3$ be defined by $F(\mathbf{x}) = 2\mathbf{x}$, $f(\mathbf{x}) = \frac{1}{2}\mathbf{x}$, $A\mathbf{x} = \mathbf{x}$, $B\mathbf{x} = \begin{pmatrix} 12 & 0 & 0 \\ 0 & 18 & 0 \\ 0 & 0 & 27 \end{pmatrix}\mathbf{x}$, $G\mathbf{x} = \begin{pmatrix} 15 & 0 & 0 \\ 0 & 19 & 0 \\ 0 & 0 & 14 \end{pmatrix}\mathbf{x}$, and $T_i : R^3 \to R^3$ be defined by $T_i\mathbf{x} = \frac{\mathbf{x}}{i}$, $\forall i \geq 1$. It is clear that $(F+B)^{-1}(0)\bigcap G^{-1}(0)\bigcap(\bigcap_{i=1}^{\infty} F(T_i)) = \{0\}$.



(a) $\gamma = 1$ (b) $\gamma = \frac{1}{5}$

FIGURE 1. Numerical example of Theorem 3.5 with taking $\gamma = 1$ and $\gamma = \frac{1}{5}$.

TABLE 1. Numerical example of Theorem 3.5 with $\lambda_n = \frac{4}{5}$

n	0	1	2	3	4	5	6	7	$\cdots$
x_1	1	0.3184	0.1198	0.0487	0.0207	0.0091	0.0041	0.0018	$\cdots$
x_2	1	0.3161	0.1187	0.0483	0.0205	0.0090	0.0040	0.0018	$\cdots$
x_3	1	0.3157	0.1186	0.0482	0.0205	0.0090	0.0040	0.0018	$\cdots$

TABLE 2. Numerical example of Theorem 3.5 with $\lambda_n = \frac{1}{2}$

n	0	1	2	3	4	5	6	7	$\cdots$
x_1	1	0.3199	0.1205	0.0490	0.0209	0.0091	0.0041	0.0019	$\cdots$
x_2	1	0.3169	0.1191	0.0485	0.0206	0.0090	0.0040	0.0018	$\cdots$
x_3	1	0.3165	0.1190	0.0484	0.0206	0.0090	0.0040	0.0018	$\cdots$

TABLE 3. Numerical example of Theorem 3.5 with $\lambda_n = \frac{1}{3}$

n	0	1	2	3	4	5	6	7	$\cdots$
x_1	1	0.3207	0.1209	0.0492	0.0210	0.0092	0.0041	0.0019	$\cdots$
x_2	1	0.3174	0.1194	0.0486	0.0207	0.0090	0.0040	0.0018	$\cdots$
x_3	1	0.3170	0.1192	0.0485	0.0206	0.0090	0.0040	0.0018	$\cdots$

Let us choose $\alpha_n = \frac{1}{n+1}$, $\beta_n = \frac{n}{2(n+1)}$, $\rho_n = \frac{1}{2} - \frac{1}{4n}$, $\eta_n = \frac{3}{4} - \frac{1}{2n}$, and $\lambda_n = \frac{1}{2}$ $\forall n \geq 1$. Obviously, T_i, F, f, A, B, G, α_n, β_n, ρ_n, η_n, and λ_n satisfy all the conditions of Theorem 3.5. Below we take the same initial value $\mathbf{x}_0 = (1,2,3)$ and different γ values to observe the convergence of algorithm 3.9. Taking $\gamma = 1$ and $\gamma = \frac{1}{5}$, respectively, we have the following numerical results in Figure 1.

Example 5.2. Let H, F, f, B, G, α_n, β_n, ρ_n, and η_n satisfy the same conditions of Example 5.1, and let T_i, A be defined by $T_i\mathbf{x} = \frac{i}{i+1}\mathbf{x}$, $\forall i \geq 1$, and $A\mathbf{x} = \frac{3}{4}\mathbf{x}$. Put $\gamma = \frac{1}{4}$. Obviously, T_i, F, f, A, B, G, α_n, β_n, ρ_n, η_n, and γ satisfy all the conditions of Theorem 3.5.

Below we take the same initial value $\mathbf{x}_0 = (1,1,1)$ and different λ_n values to observe the convergence of the algorithm 3.9. Taking $\lambda_n = \frac{4}{5}$, $\lambda_n = \frac{1}{2}$, and $\lambda_n = \frac{1}{3}$, respectively, we have the following numerical results in Table 1, 2, and 3.

Observing the results of the above Tables, we know that, the greater the value of λ_n is, the faster algorithm 3.9 converges.

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