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CERTAIN ERROR BOUNDS ON THE BULLEN TYPE INTEGRAL INEQUALITIES IN THE FRAMEWORK OF FRACTAL SPACES

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Abstract. By virtue of Yang's fractal sets $\mathbb{R}^{\varpi}(0<\varpi\leq 1)$, this paper is devoted to studying a number of integral inequalities, which are associate with the celebrated Bullen type inequality. To this end, an integral equality via local differentiable mappings is presented from which we obtain certain error estimations in connection with the Bullen type inequalities. Moreover, we provide three examples to identify the correctness of the outcomes. Considering the applications of the derived findings, we investigate several local fractional integral inequalities for ϖ-type special means, numerical integrations, and extended probability distribution mappings, respectively.

Keywords. Bullen type inequalities; Generalized h-convexity; Local fractional integrals.

1. Introduction and Preliminaries

The convexity of functions is a forceful tool, which was mainly employed to deal with a variety of pure and applied analysis and related engineering problems. Recently, numerous scholars devoted themselves to exploring the properties and inequalities closely related to convexity in diverse fields; see, e.g., [5, 12, 20] and the references therein.

Suppose that $\zeta:\mathscr{Q}\subseteq\mathbb{R}\to\mathbb{R}$ is a convex mapping defined on the interval $\mathscr{Q}$ of real numbers, and $m,n\in\mathscr{Q}$ along with $m<n$. The subsequent inequalities, to be named as Hermite–Hadamard's inequalities, were frequently used in pure and applied mathematics

$$\zeta\left(\frac{m+n}{2}\right)\leq\frac{1}{n-m}\int_m^n\zeta(\tau)\mathrm{d}\tau\leq\frac{\zeta(m+n)}{2}.$$

The integral inequalities, which attracted much attention in engineering mathematics, provide error bounds for the mean value of a continuous convex mapping $\zeta:\mathscr{Q}\subseteq\mathbb{R}\to\mathbb{R}$.

There are numerous studies on the Hermite-Hadamard type inequalities with various convex mappings, such as s-convex mappings [16], h-convex mappings [10], h-preinvex mappings [23],

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and N-quasiconvex mappings [1]. For more results, we refer to [15, 24, 28] and the references therein.

In 2010, Tseng, Hwang, and Dragomir [30] proposed an improved version of the Hermite-Hadamard's inequalities. They testified the integral inequalities

$$\begin{aligned}\zeta\left(\frac{m+n}{2}\right) &\leq \frac{1}{2}\left[\zeta\left(\frac{3m+n}{4}\right)+\zeta\left(\frac{m+3n}{4}\right)\right]\\ &\leq \frac{1}{n-m}\int_m^n \zeta(\tau)\mathrm{d}\tau\\ &\leq \frac{1}{2}\left[\frac{\zeta(m)+\zeta(n)}{2}+\zeta\left(\frac{m+n}{2}\right)\right] \leq \frac{\zeta(m)+\zeta(n)}{2}.\end{aligned}$$

The third inequality above, in general, is named as the Bullen's inequality in the literature.

Since the Bullen type inequalities have potential real applications, in particular, the signal processing and quantum communication, they have been extensively studied. In accordance with the second-order optical sliding mode, Acu and Gonska [2] provided the error bounds in connection with the Bullen type inequalities for continuous mappings. By using Riemann-Liouville fractional integral operators in relevance with Gauss hypergeometric functions, Çakmak [7] constructed the Bullen-type inequalities involving s-convexity in the second meaning. Çakmak [8] further considered some Bullen-type inequalities with differentiable h-convex functions. Tseng, Hwang and Hsu [31] purposed several Bullen-type inequalities involving Lipschitzian mappings. Mitroi-Symeonidis proposed [25] certain error bounds of the gap within Bullen-type inequalities. For more results on the Bullen's inequality, we refer to [19, 22] and the references therein. In 2007, Varošanec [32] considered the following definition based on the h-convex mappings. They generalized p-convexity, non-negative convexity, and Godunova–Levin convex mappings.

Definition 1.1. Let $\mathscr{Q}$, $\mathscr{J}$ be two real-valued intervals within $\mathbb{R}$, $(0,1)\subseteq\mathscr{J}$, and the mapping $h:\mathscr{J}\to\mathbb{R}$ be nonnegative with $h\not\equiv 0$. A nonnegative mapping $\zeta:\mathscr{Q}\to\mathbb{R}$ is said to be h-convex if $\zeta(\lambda\tau+(1-\lambda)\upsilon)\leq h(\lambda)\zeta(\tau)+h(1-\lambda)\zeta(\upsilon)$ for any $\tau,\upsilon\in\mathscr{Q}$ along with $\lambda\in(0,1)$.

Now, let us start with reviewing Yang's axiom on the local fractional calculus, which are used throughout this paper. These fractal theories were mainly introduced by Yang in [34]. Here, we state certain rules in the theory of fractal space as follows

(1) $\xi^{\varpi}+\eta^{\varpi}\in\mathbb{R}^{\varpi}$;
(2) $\xi^{\varpi}+\eta^{\varpi}=\eta^{\varpi}+\xi^{\varpi}$;
(3) $\xi^{\varpi}+(\eta^{\varpi}+\varphi^{\varpi})=(\xi^{\varpi}+\eta^{\varpi})+\varphi^{\varpi}$;
(4) 0^{ϖ} is the identity for each $\xi^{\varpi}\in\mathbb{R}^{\varpi}$, $\xi^{\varpi}+0^{\varpi}=0^{\varpi}+\xi^{\varpi}=\xi^{\varpi}$;
(5) for any $\xi^{\varpi}\in\mathbb{R}^{\varpi}$, $(-\xi)^{\varpi}$ is the inverse element of ξ^{ϖ}, $\xi^{\varpi}+(-\xi)^{\varpi}=\big(\xi+(-\xi)\big)^{\varpi}=0^{\varpi}$;
(6) $\xi^{\varpi}\eta^{\varpi}\in\mathbb{R}^{\varpi}$;
(7) $\xi^{\varpi}\eta^{\varpi}=\eta^{\varpi}\xi^{\varpi}$;
(8) $\xi^{\varpi}(\eta^{\varpi}\varphi^{\varpi})=(\xi^{\varpi}\eta^{\varpi})\varphi^{\varpi}$;
(9) 1^{ϖ} is the identity for $(\mathbb{R}^{\varpi},\cdot)$, for any $\xi^{\varpi}\in\mathbb{R}^{\varpi}$, $\xi^{\varpi}1^{\varpi}=1^{\varpi}\xi^{\varpi}=\xi^{\varpi}$;
(10) for each $\xi^{\varpi}\in\mathbb{R}^{\varpi}\setminus\{0^{\varpi}\}$, $\big(\frac{1}{\xi}\big)^{\varpi}$ is the inverse element of ξ^{ϖ}, $\xi^{\varpi}\big(\frac{1}{\xi}\big)^{\varpi}=\big((\frac{1}{\xi})\xi\big)^{\varpi}=1^{\varpi}$;

Distributive law holds: $\xi^{\varpi}(\eta^{\varpi}+\varphi^{\varpi})=\xi^{\varpi}\eta^{\varpi}+\xi^{\varpi}\varphi^{\varpi}$.

To recall the axiom of the local fractional calculus on $\mathbb{R}^{\varpi}$, we evoke the notion involving the local fractional continuity.

Definition 1.2. [34] Let $\zeta:\mathbb{R}\to\mathbb{R}^{\varpi}$ be a non-differentiable mapping give by $\nu\to\zeta(\nu)$. If, for each $\varepsilon>0$, there exists $\kappa>0$ satisfying $|\zeta(\nu)-\zeta(\nu_0)|<\varepsilon^{\varpi}$ for $|\nu-\nu_0|<\kappa$, then $\zeta(\nu)$ is named as local fractional continuous at ν_0. If the mapping $\zeta(\nu)$ is local continuous at every point in the define interval (m,n), we write it by $\zeta(\nu)\in C_{\varpi}(m,n)$.

Definition 1.3. [34] With regard to the local fractional derivative of the mapping $\zeta(\nu)$ of order ϖ at $\nu=\nu_0$, we define

$$\zeta^{(\varpi)}(\nu_0)={}_{\nu_0}D_{\nu}^{\varpi}\zeta(\nu)=\frac{\mathrm{d}^{\varpi}\zeta(\nu)}{\mathrm{d}\nu^{\varpi}}\bigg|_{\nu=\nu_0}=\lim_{\nu\to\nu_0}\frac{\Delta^{\varpi}\big(\zeta(\nu)-\zeta(\nu_0)\big)}{(\nu-\nu_0)^{\varpi}},$$

where $\Delta^{\varpi}\big(\zeta(\nu)-\zeta(\nu_0)\big)=\Gamma(\varpi+1)\big(\zeta(\nu)-\zeta(\nu_0)\big)$.

Let $\zeta^{(\varpi)}(\nu)=D_{\nu}^{\varpi}\zeta(\nu)$. If there exists $\zeta^{((k+1)\varpi)}(\nu)=\overbrace{D_{\nu}^{\varpi}\cdots D_{\nu}^{\varpi}}^{(k+1)\text{ times}}\zeta(\nu)$ for every $\nu\in\mathscr{Q}\subseteq\mathbb{R}$, then it is written $\zeta\in D_{(k+1)\varpi}(\mathscr{Q})$, in which $k=0,1,2,\cdots$.

Definition 1.4. [34] Let $\zeta(\nu)\in C_{\varpi}[m,n]$, and $\triangle=\{\kappa_0,\kappa_1,\cdots,\kappa_N\}$, where $N\in\mathbb{N}$, be a partition of $[m,n]$ satisfying $m=\kappa_0<\kappa_1<\cdots<\kappa_N=n$. Then the local fractional integration of the mapping ζ on $[m,n]$ of order ϖ is defined by

$${}_m\mathscr{I}_n^{(\varpi)}\zeta(\nu)=\frac{1}{\Gamma(1+\varpi)}\int_m^n\zeta(\kappa)(\mathrm{d}\kappa)^{\varpi}:=\frac{1}{\Gamma(\varpi+1)}\lim_{\triangle\kappa\to0}\sum_{j=0}^{N-1}\zeta(\kappa_j)(\Delta\kappa_j)^{\varpi},$$

in which $\Delta\kappa:=\max\{\Delta\kappa_1,\Delta\kappa_2,\cdots,\Delta\kappa_{N-1}\}$ along with $\Delta\kappa_j:=\kappa_{j+1}-\kappa_j$, $j=0,\cdots,N-1$.

Here, ${}_m\mathscr{I}_n^{(\varpi)}\zeta(\nu)=0$ if $m=n$, and ${}_m\mathscr{I}_n^{(\varpi)}\zeta(\nu)=-{}_n\mathscr{I}_m^{(\varpi)}\zeta(\nu)$ if $m<n$. For each $\nu\in[m,n]$, if there exists ${}_m\mathscr{I}_{\nu}^{(\varpi)}\zeta(\nu)$, then it is denoted by $\zeta(\nu)\in\mathscr{I}_{\nu}^{\varpi}[m,n]$.

Lemma 1.5. [34] *(a) If $\zeta(\nu)=w^{(\varpi)}(\nu)\in C_{\varpi}[m,n]$, then ${}_m\mathscr{I}_n^{(\varpi)}\zeta(\nu)=w(n)-w(m)$. (b) If $\zeta(\nu),w(\nu)\in D_{\varpi}[m,n]$ and $\zeta^{(\varpi)}(\nu),w^{(\varpi)}(\nu)\in C_{\varpi}[m,n]$, then*

$${}_m\mathscr{I}_n^{(\varpi)}\zeta(\nu)w^{(\varpi)}(\nu)=\zeta(\nu)w(\nu)\Big|_m^n-{}_m\mathscr{I}_n^{(\varpi)}\zeta^{(\varpi)}(\nu)w(\nu).$$

(c) The local fractional derivative of the mapping $\nu^{k\varpi}$ is as follows

$$\frac{\mathrm{d}^{\varpi}\nu^{k\varpi}}{\mathrm{d}\nu^{\varpi}}=\frac{\Gamma(1+k\varpi)}{\Gamma\big(1+(k-1)\varpi\big)}\nu^{(k-1)\varpi},\ k\in\mathbb{R}.$$

(d) The local fractional integration with regard to the mapping $\nu^{k\varpi}$ is as below

$$\frac{1}{\Gamma(1+\varpi)}\int_m^n\nu^{k\varpi}(\mathrm{d}\nu)^{\varpi}=\frac{\Gamma(1+k\varpi)}{\Gamma\big(1+(k+1)\varpi\big)}\Big(n^{(k+1)\varpi}-m^{(k+1)\varpi}\Big),\ k\in\mathbb{R}.$$

Lemma 1.6. [34] *(Holder-Yang's inequality). If the mappings $\zeta,w\in C_{\varpi}[m,n]$, and $\theta,\psi>1$ with $\frac{1}{\theta}+\frac{1}{\psi}=1$, then*

$$\frac{1}{\Gamma(1+\varpi)}\int_m^n\big|\zeta(\nu)w(\nu)\big|(\mathrm{d}\nu)^{\varpi}\le\left(\frac{1}{\Gamma(1+\varpi)}\int_m^n\big|\zeta(\nu)\big|^{\theta}(\mathrm{d}\nu)^{\varpi}\right)^{\frac{1}{\theta}}\left(\frac{1}{\Gamma(1+\varpi)}\int_m^n\big|w(\nu)\big|^{\psi}(\mathrm{d}\nu)^{\varpi}\right)^{\frac{1}{\psi}}.$$

In 2021, Luo, Yu and Du modified the Hölder's integral inequality in the frame of fractal spaces, and offered a new upper bound.

Theorem 1.7. [21] *Let $\psi > 1$ with $\theta^{-1}+\psi^{-1}=1$. If the mappings ζ as well as w are both local fractional continuous over the interval $[m,n]$, i.e., the mappings $\zeta, w \in C_{\varpi}[m,n]$ as well as $|\zeta|^{\theta}$, $|w|^{\psi}$ are both local fractional integral over the interval $[m,n]$, then*

$$\begin{aligned}
&\frac{1}{\Gamma(1+\varpi)}\int_m^n |\zeta(\tau)w(\tau)|\,(\mathrm{d}\tau)^{\varpi} \\
&\leq \left(\frac{1}{n-m}\right)^{\varpi}\Bigg\{\left(\frac{1}{\Gamma(1+\varpi)}\int_m^n (n-\tau)^{\varpi}\,|\zeta(\tau)|^{\theta}\,(\mathrm{d}\tau)^{\varpi}\right)^{\frac{1}{\theta}} \\
&\quad\times\left(\frac{1}{\Gamma(1+\varpi)}\int_m^n (n-\tau)^{\varpi}\,|w(\tau)|^{\psi}\,(\mathrm{d}\tau)^{\varpi}\right)^{\frac{1}{\psi}} \\
&\quad+\left(\frac{1}{\Gamma(1+\varpi)}\int_m^n (\tau-m)^{\varpi}\,|\zeta(\tau)|^{\theta}\,(\mathrm{d}\tau)^{\varpi}\right)^{\frac{1}{\theta}} \\
&\quad\times\left(\frac{1}{\Gamma(1+\varpi)}\int_m^n (\tau-m)^{\varpi}\,|w(\tau)|^{\psi}\,(\mathrm{d}\tau)^{\varpi}\right)^{\frac{1}{\psi}}\Bigg\} \\
&\leq \left(\frac{1}{\Gamma(1+\varpi)}\int_m^n |\zeta(\tau)|^{\theta}\,(\mathrm{d}\tau)^{\varpi}\right)^{\frac{1}{\theta}}\left(\frac{1}{\Gamma(1+\varpi)}\int_m^n |w(\tau)|^{\psi}\,(\mathrm{d}\tau)^{\varpi}\right)^{\frac{1}{\psi}}.
\end{aligned} \tag{1.1}$$

In [33], Vivas, Hernández and Merente presented the following generalized h-convexity defined on the Yang's fractal set $\mathbb{R}^{\varpi}$.

Definition 1.8. [33] Let $h:\mathscr{J}\subseteq\mathbb{R}\to\mathbb{R}^{\varpi}$ be a nonnegative mapping with $h\not\equiv 0^{\varpi}$. If the mapping $\zeta:\mathscr{Q}\subseteq\mathbb{R}\to\mathbb{R}^{\varpi}$ is nonnegative, that is, $\zeta(\nu)\geq 0^{\varpi}$ and $\zeta\big(\lambda\tau+(1-\lambda)\upsilon\big)\leq h(\lambda)\zeta(\tau)+h(1-\lambda)\zeta(\upsilon)$ for each $\tau,\upsilon\in\mathscr{Q}$ with $\lambda\in(0,1)$, then ζ is generalized h-convex.

Notice that from [33] that $h\not\equiv 0^{\varpi}$ rather than $h\not\equiv 0$. From Definition 1.8, let us state the coming exceptional circumstances

(a) If $h(\lambda)=\lambda^{\varpi}$, then the generalized h-convexity is reduced to the non-negative generalized convexity.

(b) If $h(\lambda)=1^{\varpi}$, then the generalized h-convexity is reduced to the generalized p-convexity.

(c) If $h(\lambda)=\lambda^{s\varpi}$ for some fixed $0<s<1$, then the generalized h-convexity is reduced to the generalized s-convexity in second meaning.

(d) If $h(\lambda)=[\lambda(1-\lambda)]^{\varpi}$, then the generalized h-convexity is reduced to the generalized tgs-convexity.

Local fractional calculus axiom is of significance in multitudinous domains, in particular, in engineering mathematical and applied analysis. In 2017, Sarikaya and Budak [26] studied a number of celebrated local fractional integral inequalities. In the same year, Choi, Set and Tomar [9] considered generalized Ostrowski type inequalities with generalized convexity. In accordance with the generalized s-convexity on fractal spaces, Kiliçman and Saleh [17] discussed certain generalized Hermite-Hadamard type integral inequalities. By means of the generalized m-convexity on fractal sets, Du et al. [11] studied the Hermite-Hadamard-type, Hermite-Hadamard-Fejér-type and Simpson-type inequalities. Almutairi and Kiliçman [3, 4] established certain Hermite-Hadamard type by means of generalized $(h-m)$-convexity as well as s-convexity, respectively. Iftikhar [14] presented several Newton's type inequalities by virtue of local fractional integrals. On the basis of generalized h-convexity, Sun [29] investigated the local fractional Ostrowski-type integral inequalities. With regard to the multidimensional Hilbert-type inequalities, Krnić and Vuković [18] studied them in terms of local fractional calculus axiom. For recent inequalities on the local fractional integrals, we refer to [6, 13, 27] and the references therein.

Inspired by the axiom of the local fractional calculus described in [34] and the results mentioned above, we study some integral inequalities on $\mathbb{R}^{\varpi}$ for the first-order local differentiable mappings with regard to the Bullen type inequality. We discuss the three cases: (i) the local fractional derivative of the mapping is bounded; (ii) the local fractional derivative of the mapping holds the generalized h-convexity; (iii) the local fractional derivative of the mapping satisfies the local Lipschitz condition. Furthermore, we give several applications of our main results.

2. Main Results

To obtain the error bounds with regard to the generalized Bullen type inequalities, we present the following lemma.

Lemma 2.1. *Let $\mathscr{Q} \subseteq \mathbb{R}$ be an interval and $\zeta : \mathscr{Q} \to \mathbb{R}^{\varpi}$ be differentiable mapping defined on $\mathscr{Q}^o$ ($\mathscr{Q}^o$ is the interior of $\mathscr{Q}$). If $\zeta \in D_{\varpi}(\mathscr{Q})$ and $\zeta^{(\varpi)} \in C_{\varpi}[m,n]$ for $m,n \in \mathscr{Q}$ together with $m<n$, then, for each $\tau \in [m,n]$,*

$$\begin{aligned}
&\frac{1^{\varpi}}{2^{\varpi}}\left[\frac{\zeta(m)+\zeta(n)}{2^{\varpi}}+\zeta\left(\frac{m+n}{2}\right)\right]-\frac{\Gamma(1+\varpi)}{(n-m)^{\varpi}}{}_m\mathscr{I}_n^{(\varpi)}\zeta(\tau)\\
&=\frac{(n-m)^{\varpi}}{8^{\varpi}}\frac{1}{\Gamma(1+\varpi)}\int_0^1\left[(1-2\lambda)^{\varpi}\zeta^{(\varpi)}\left(\frac{1+\lambda}{2}m+\frac{1-\lambda}{2}n\right)\right.\\
&\quad\left.+(2\lambda-1)^{\varpi}\zeta^{(\varpi)}\left(\frac{1+\lambda}{2}n+\frac{1-\lambda}{2}m\right)\right](\mathrm{d}\lambda)^{\varpi}.
\end{aligned} \tag{2.1}$$

Proof. Integrating by parts on the local fractional space for the right side of 2.1, we see that

$$\begin{aligned}
&\frac{1}{\Gamma(1+\varpi)}\int_0^1(1-2\lambda)^{\varpi}\zeta^{(\varpi)}\left(\frac{1+\lambda}{2}m+\frac{1-\lambda}{2}n\right)(\mathrm{d}\lambda)^{\varpi}\\
&=\frac{2^{\varpi}}{(m-n)^{\varpi}}\left[(1-2\lambda)^{\varpi}\zeta\left(\frac{1+\lambda}{2}m+\frac{1-\lambda}{2}n\right)\Big|_0^1\right.\\
&\quad\left.-\frac{(-2)^{\varpi}}{\Gamma(1+\varpi)}\int_0^1\Gamma(1+\varpi)\zeta\left(\frac{1+\lambda}{2}m+\frac{1-\lambda}{2}n\right)(\mathrm{d}\lambda)^{\varpi}\right]\\
&=\frac{2^{\varpi}}{(m-n)^{\varpi}}\left[-\zeta(m)-\zeta\left(\frac{m+n}{2}\right)+\frac{2^{\varpi}}{\Gamma(1+\varpi)}\int_0^1\Gamma(1+\varpi)\zeta\left(\frac{1+\lambda}{2}m+\frac{1-\lambda}{2}n\right)(\mathrm{d}\lambda)^{\varpi}\right].
\end{aligned}$$

Letting $\tau=\frac{1+\lambda}{2}m+\frac{1-\lambda}{2}n$, we have

$$\frac{1}{\Gamma(1+\varpi)}\int_0^1\Gamma(1+\varpi)\zeta\left(\frac{1+\lambda}{2}m+\frac{1-\lambda}{2}n\right)(\mathrm{d}\lambda)^{\varpi}=\frac{2^{\varpi}\Gamma(1+\varpi)}{(m-n)^{\varpi}}{}_{\frac{m+n}{2}}\mathscr{I}_m^{(\varpi)}\zeta(\tau).$$

This leads us to

$$\begin{aligned}&\frac{1}{\Gamma(1+\varpi)}\int_0^1(1-2\lambda)^{\varpi}\zeta^{(\varpi)}\left(\frac{1+\lambda}{2}m+\frac{1-\lambda}{2}n\right)(\mathrm{d}\lambda)^{\varpi}\\&=\frac{2^{\varpi}}{(n-m)^{\varpi}}\left[\zeta(m)+\zeta\left(\frac{m+n}{2}\right)-\frac{4^{\varpi}\Gamma(1+\varpi)}{(n-m)^{\varpi}}{}_m\mathscr{I}^{(\varpi)}_{\frac{m+n}{2}}\zeta(\tau)\right].\end{aligned}$$

Similarly,

$$\begin{aligned}&\frac{1}{\Gamma(1+\varpi)}\int_0^1(2\lambda-1)^{\varpi}\zeta^{(\varpi)}\left(\frac{1+\lambda}{2}n+\frac{1-\lambda}{2}m\right)(\mathrm{d}\lambda)^{\varpi}\\&=\frac{2^{\varpi}}{(n-m)^{\varpi}}\left[\zeta(n)+\zeta\left(\frac{m+n}{2}\right)-\frac{4^{\varpi}\Gamma(1+\varpi)}{(n-m)^{\varpi}}{}_{\frac{m+n}{2}}\mathscr{I}^{(\varpi)}_{n}\zeta(\tau)\right].\end{aligned}$$

Using the two equalities, we have (2.1). This ends the proof. □

Remark 2.2. If $\varpi=1$ in Lemma 2.1, then we obtain the Bullen-type integral identity given in [7].

With the aid of Lemma 2.1, we prove the following theorem.

Theorem 2.3. *Let all the assumptions mentioned in Lemma 2.1 hold. If there exist constants $z^{\varpi}<Z^{\varpi}$ satisfying $z^{\varpi}\le\zeta^{(\varpi)}\le Z^{\varpi}$ for all $\tau\in[m,n]$, then*

$$\left|\frac{1^{\varpi}}{2^{\varpi}}\left[\frac{\zeta(m)+\zeta(n)}{2^{\varpi}}+\zeta\left(\frac{m+n}{2}\right)\right]-\frac{\Gamma(1+\varpi)}{(n-m)^{\varpi}}{}_m\mathscr{I}^{(\varpi)}_n\zeta(\tau)\right|\le\frac{(n-m)^{\varpi}(Z-z)^{\varpi}}{8^{\varpi}}\frac{\Gamma(1+\varpi)}{\Gamma(1+2\varpi)}.$$

Proof. In accordance with Lemma 2.1, we know that

$$\begin{aligned}&\frac{1^{\varpi}}{2^{\varpi}}\left[\frac{\zeta(m)+\zeta(n)}{2^{\varpi}}+\zeta\left(\frac{m+n}{2}\right)\right]-\frac{\Gamma(1+\varpi)}{(n-m)^{\varpi}}{}_m\mathscr{I}^{(\varpi)}_n\zeta(\tau)\\&=\frac{(n-m)^{\varpi}}{8^{\varpi}}\left\{\frac{1}{\Gamma(1+\varpi)}\int_0^1(1-2\lambda)^{\varpi}\left[\zeta^{(\varpi)}\left(\frac{1+\lambda}{2}m+\frac{1-\lambda}{2}n\right)-\frac{z^{\varpi}+Z^{\varpi}}{2^{\varpi}}+\frac{z^{\varpi}+Z^{\varpi}}{2^{\varpi}}\right](\mathrm{d}\lambda)^{\varpi}\right.\\&\quad\left.+\frac{1}{\Gamma(1+\varpi)}\int_0^1(2\lambda-1)^{\varpi}\left[\zeta^{(\varpi)}\left(\frac{1+\lambda}{2}n+\frac{1-\lambda}{2}m\right)-\frac{z^{\varpi}+Z^{\varpi}}{2^{\varpi}}+\frac{z^{\varpi}+Z^{\varpi}}{2^{\varpi}}\right](\mathrm{d}\lambda)^{\varpi}\right\}\\&=\frac{(n-m)^{\varpi}}{8^{\varpi}}\left\{\frac{1}{\Gamma(1+\varpi)}\int_0^1(1-2\lambda)^{\varpi}\left[\zeta^{(\varpi)}\left(\frac{1+\lambda}{2}m+\frac{1-\lambda}{2}n\right)-\frac{z^{\varpi}+Z^{\varpi}}{2^{\varpi}}\right](\mathrm{d}\lambda)^{\varpi}\right.\\&\quad\left.+\frac{1}{\Gamma(1+\varpi)}\int_0^1(2\lambda-1)^{\varpi}\left[\zeta^{(\varpi)}\left(\frac{1+\lambda}{2}n+\frac{1-\lambda}{2}m\right)-\frac{z^{\varpi}+Z^{\varpi}}{2^{\varpi}}\right](\mathrm{d}\lambda)^{\varpi}\right\}.\end{aligned}\tag{2.2}$$

If we denote S by the left hand of the equality (2.2), then we find that

$$\begin{aligned}|S| \leq &\frac{(n-m)^{\varpi}}{8^{\varpi}}\left\{\frac{1}{\Gamma(1+\varpi)}\int_0^1 |1-2\lambda|^{\varpi}\left|\zeta^{(\varpi)}\left(\frac{1+\lambda}{2}m+\frac{1-\lambda}{2}n\right)-\frac{z^{\varpi}+Z^{\varpi}}{2^{\varpi}}\right|(\mathrm{d}\lambda)^{\varpi}\right.\\ &\left.+\frac{1}{\Gamma(1+\varpi)}\int_0^1 |2\lambda-1|^{\varpi}\left|\zeta^{(\varpi)}\left(\frac{1+\lambda}{2}n+\frac{1-\lambda}{2}m\right)-\frac{z^{\varpi}+Z^{\varpi}}{2^{\varpi}}\right|(\mathrm{d}\lambda)^{\varpi}\right\}\\ \leq &\frac{(n-m)^{\varpi}(Z-z)^{\varpi}}{8^{\varpi}}\frac{\Gamma(1+\varpi)}{\Gamma(1+2\varpi)}.\end{aligned}$$

Since $\zeta^{(\varpi)}$ satisfies $z^{\varpi} \leq \zeta^{(\varpi)} \leq Z^{\varpi}$, we deduce that

$$z^{\varpi}-\frac{z^{\varpi}+Z^{\varpi}}{2^{\varpi}} \leq \zeta^{(\varpi)}-\frac{z^{\varpi}+Z^{\varpi}}{2^{\varpi}} \leq Z^{\varpi}-\frac{z^{\varpi}+Z^{\varpi}}{2^{\varpi}},$$

which implies that

$$\left|\zeta^{(\varpi)}-\frac{z^{\varpi}+Z^{\varpi}}{2^{\varpi}}\right| \leq \frac{Z^{\varpi}-z^{\varpi}}{2^{\varpi}}.$$

This ends the proof.

Theorem 2.4. *Let all the assumptions mentioned in Lemma 2.1 hold. If $|\zeta^{(\varpi)}|$ satisfies the generalized h-convexity defined on the interval $[m,n]$, and $\zeta^{(\varpi)}$ is bounded, i.e., $||\zeta^{(\varpi)}||_{\infty} = \sup_{\tau\in(m,n)}|\zeta^{(\varpi)}| < \infty$, then*

$$\begin{aligned}&\left|\frac{1^{\varpi}}{2^{\varpi}}\left[\frac{\zeta(m)+\zeta(n)}{2^{\varpi}}+\zeta\left(\frac{m+n}{2}\right)\right]-\frac{\Gamma(1+\varpi)}{(n-m)^{\varpi}}{}_m\mathscr{I}_n^{(\varpi)}\zeta(\tau)\right|\\ \leq &\frac{(n-m)^{\varpi}\|\zeta^{(\varpi)}\|_{\infty}}{4^{\varpi}}\left[\frac{1}{\Gamma(1+\varpi)}\int_0^{\frac{1}{2}}(1-2\lambda)^{\varpi}\left(h\left(\frac{1+\lambda}{2}\right)+h\left(\frac{1-\lambda}{2}\right)\right)(\mathrm{d}\lambda)^{\varpi}\right.\\ &\left.+\frac{1}{\Gamma(1+\varpi)}\int_{\frac{1}{2}}^1(2\lambda-1)^{\varpi}\left(h\left(\frac{1+\lambda}{2}\right)+h\left(\frac{1-\lambda}{2}\right)\right)(\mathrm{d}\lambda)^{\varpi}\right].\end{aligned}$$

Proof. Considering the integral identity derived in lemma 2.1 and taking advantage of the characteristics of modulus, we find that

$$\begin{aligned}&\left|\frac{1^{\varpi}}{2^{\varpi}}\left[\frac{\zeta(m)+\zeta(n)}{2^{\varpi}}+\zeta\left(\frac{m+n}{2}\right)\right]-\frac{\Gamma(1+\varpi)}{(n-m)^{\varpi}}{}_m\mathscr{I}_n^{(\varpi)}\zeta(\tau)\right|\\ \leq &\frac{(n-m)^{\varpi}}{8^{\varpi}}\frac{1}{\Gamma(1+\varpi)}\int_0^1\left(\left|1-2\lambda\right|^{\varpi}\left|\zeta^{(\varpi)}\left(\frac{1+\lambda}{2}m+\frac{1-\lambda}{2}n\right)\right|\right.\\ &\left.+\left|2\lambda-1\right|^{\varpi}\left|\zeta^{(\varpi)}\left(\frac{1+\lambda}{2}n+\frac{1-\lambda}{2}m\right)\right|\right)(\mathrm{d}\lambda)^{\varpi}.\end{aligned}$$

If we employ the generalized h-convexity of the mapping $|\zeta^{(\varpi)}|$ defined on $[m,n]$, then

$$\begin{aligned}
&\left|\frac{1^{\varpi}}{2^{\varpi}}\left[\frac{\zeta(m)+\zeta(n)}{2^{\varpi}}+\zeta\left(\frac{m+n}{2}\right)\right]-\frac{\Gamma(1+\varpi)}{(n-m)^{\varpi}}{}_{m}\mathscr{I}_{n}^{(\varpi)}\zeta(\tau)\right|\\
&\leq \frac{(n-m)^{\varpi}}{8^{\varpi}}\left[\frac{1}{\Gamma(1+\varpi)}\int_{0}^{\frac{1}{2}}(1-2\lambda)^{\varpi}\left(h\left(\frac{1+\lambda}{2}\right)|\zeta^{(\varpi)}(m)|+h\left(\frac{1-\lambda}{2}\right)|\zeta^{(\varpi)}(n)|\right)(\mathrm{d}\lambda)^{\varpi}\right.\\
&\quad+\frac{1}{\Gamma(1+\varpi)}\int_{0}^{\frac{1}{2}}(1-2\lambda)^{\varpi}\left(h\left(\frac{1+\lambda}{2}\right)|\zeta^{(\varpi)}(n)|+h\left(\frac{1-\lambda}{2}\right)|\zeta^{(\varpi)}(m)|\right)(\mathrm{d}\lambda)^{\varpi}\\
&\quad+\frac{1}{\Gamma(1+\varpi)}\int_{\frac{1}{2}}^{1}(2\lambda-1)^{\varpi}\left(h\left(\frac{1+\lambda}{2}\right)|\zeta^{(\varpi)}(m)|+h\left(\frac{1-\lambda}{2}\right)|\zeta^{(\varpi)}(n)|\right)(\mathrm{d}\lambda)^{\varpi}\\
&\quad\left.+\frac{1}{\Gamma(1+\varpi)}\int_{\frac{1}{2}}^{1}(2\lambda-1)^{\varpi}\left(h\left(\frac{1+\lambda}{2}\right)|\zeta^{(\varpi)}(n)|+h\left(\frac{1-\lambda}{2}\right)|\zeta^{(\varpi)}(m)|\right)(\mathrm{d}\lambda)^{\varpi}\right]\\
&\leq \frac{(n-m)^{\varpi}\|\zeta^{(\varpi)}\|_{\infty}}{4^{\varpi}}\left[\frac{1}{\Gamma(1+\varpi)}\int_{0}^{\frac{1}{2}}(1-2\lambda)^{\varpi}\left(h\left(\frac{1+\lambda}{2}\right)+h\left(\frac{1-\lambda}{2}\right)\right)(\mathrm{d}\lambda)^{\varpi}\right.\\
&\quad\left.+\frac{1}{\Gamma(1+\varpi)}\int_{\frac{1}{2}}^{1}(2\lambda-1)^{\varpi}\left(h\left(\frac{1+\lambda}{2}\right)+h\left(\frac{1-\lambda}{2}\right)\right)(\mathrm{d}\lambda)^{\varpi}\right],
\end{aligned}$$

which is the desired result asserted in Theorem 2.4. This completes the proof. □

If $|\zeta^{(\varpi)}|^{\psi}$ is generalized h-convex, then we achieve the following result.

Theorem 2.5. *Let ζ be defined as Lemma 2.1. Assume that $|\zeta^{(\varpi)}|^{\psi}$ is generalized h-convex on the interval $[m,n]$ for $\theta,\psi>1$ with $\frac{1}{\theta}+\frac{1}{\psi}=1$. Then*

$$\begin{aligned}
&\left|\frac{1^{\varpi}}{2^{\varpi}}\left[\frac{\zeta(m)+\zeta(n)}{2^{\varpi}}+\zeta\left(\frac{m+n}{2}\right)\right]-\frac{\Gamma(1+\varpi)}{(n-m)^{\varpi}}{}_{m}\mathscr{I}_{n}^{(\varpi)}\zeta(\tau)\right|\\
&\leq \frac{(n-m)^{\varpi}}{8^{\varpi}}\left(\frac{\Gamma(1+\theta\varpi)}{\Gamma(1+(\theta+1)\varpi)}\right)^{\frac{1}{\theta}}\\
&\quad\times\left\{\left[\frac{1}{\Gamma(1+\varpi)}\int_{0}^{1}\left(h\left(\frac{1+\lambda}{2}\right)\left|\zeta^{(\varpi)}(m)\right|^{\psi}+h\left(\frac{1-\lambda}{2}\right)\left|\zeta^{(\varpi)}(n)\right|^{\psi}\right)(\mathrm{d}\lambda)^{\varpi}\right]^{\frac{1}{\psi}}\right.\\
&\quad\left.+\left[\frac{1}{\Gamma(1+\varpi)}\int_{0}^{1}\left(h\left(\frac{1+\lambda}{2}\right)\left|\zeta^{(\varpi)}(n)\right|^{\psi}+h\left(\frac{1-\lambda}{2}\right)\left|\zeta^{(\varpi)}(m)\right|^{\psi}\right)(\mathrm{d}\lambda)^{\varpi}\right]^{\frac{1}{\psi}}\right\}.
\end{aligned}$$

Proof. Employing Lemma 2.1 and the Hölder-Yang's inequality, we have that

$$\begin{aligned}
&\left|\frac{1^{\varpi}}{2^{\varpi}}\left[\frac{\zeta(m)+\zeta(n)}{2^{\varpi}}+\zeta\left(\frac{m+n}{2}\right)\right]-\frac{\Gamma(1+\varpi)}{(n-m)^{\varpi}}{}_{m}\mathscr{I}_{n}^{(\varpi)}\zeta(\tau)\right| \\
&\leq \frac{(n-m)^{\varpi}}{8^{\varpi}}\left[\frac{1}{\Gamma(1+\varpi)}\int_0^1 |1-2\lambda|^{\varpi}\left|\zeta^{(\varpi)}\left(\frac{1+\lambda}{2}m+\frac{1-\lambda}{2}n\right)\right|(\mathrm{d}\lambda)^{\varpi}\right. \\
&\quad\left.+\frac{1}{\Gamma(1+\varpi)}\int_0^1 |2\lambda-1|^{\varpi}\left|\zeta^{(\varpi)}\left(\frac{1+\lambda}{2}n+\frac{1-\lambda}{2}m\right)\right|(\mathrm{d}\lambda)^{\varpi}\right] \\
&\leq \frac{(n-m)^{\varpi}}{8^{\varpi}}\left(\frac{1}{\Gamma(1+\varpi)}\int_0^1 |1-2\lambda|^{\varpi\theta}(\mathrm{d}\lambda)^{\varpi}\right)^{\frac{1}{\theta}} \\
&\quad\times\left(\frac{1}{\Gamma(1+\varpi)}\int_0^1 \left|\zeta^{(\varpi)}\left(\frac{1+\lambda}{2}m+\frac{1-\lambda}{2}n\right)\right|^{\psi}(\mathrm{d}\lambda)^{\varpi}\right)^{\frac{1}{\psi}} \\
&\quad+\frac{(n-m)^{\varpi}}{8^{\varpi}}\left(\frac{1}{\Gamma(1+\varpi)}\int_0^1 |2\lambda-1|^{\varpi\theta}(\mathrm{d}\lambda)^{\varpi}\right)^{\frac{1}{\theta}} \\
&\quad\times\left(\frac{1}{\Gamma(1+\varpi)}\int_0^1 \left|\zeta^{(\varpi)}\left(\frac{1+\lambda}{2}n+\frac{1-\lambda}{2}m\right)\right|^{\psi}(\mathrm{d}\lambda)^{\varpi}\right)^{\frac{1}{\psi}}.
\end{aligned}$$

Owing to the generalized h-convexity of the mapping $|\zeta^{(\varpi)}|^{\psi}$, we obtain that

$$\begin{aligned}
&\frac{1}{\Gamma(1+\varpi)}\int_0^1 \left|\zeta^{(\varpi)}\left(\frac{1+\lambda}{2}m+\frac{1-\lambda}{2}n\right)\right|^{\psi}(\mathrm{d}\lambda)^{\varpi} \\
&\leq \frac{1}{\Gamma(1+\varpi)}\int_0^1 \left[h\left(\frac{1+\lambda}{2}\right)|\zeta^{(\varpi)}(m)|^{\psi}+h\left(\frac{1-\lambda}{2}\right)|\zeta^{(\varpi)}(n)|^{\psi}\right](\mathrm{d}\lambda)^{\varpi}.
\end{aligned} \tag{2.3}$$

Analogously,

$$\begin{aligned}
&\frac{1}{\Gamma(1+\varpi)}\int_0^1 \left|\zeta^{(\varpi)}\left(\frac{1+\lambda}{2}n+\frac{1-\lambda}{2}m\right)\right|^{\psi}(\mathrm{d}\lambda)^{\varpi} \\
&\leq \frac{1}{\Gamma(1+\varpi)}\int_0^1 \left[h\left(\frac{1+\lambda}{2}\right)|\zeta^{(\varpi)}(n)|^{\psi}+h\left(\frac{1-\lambda}{2}\right)|\zeta^{(\varpi)}(m)|^{\psi}\right](\mathrm{d}\lambda)^{\varpi}.
\end{aligned} \tag{2.4}$$

On the other hand, we have

$$\begin{aligned}
\frac{1}{\Gamma(1+\varpi)}\int_0^1 |1-2\lambda|^{\varpi\theta}(\mathrm{d}\lambda)^{\varpi} &= \frac{1}{\Gamma(1+\varpi)}\int_0^1 |2\lambda-1|^{\varpi\theta}(\mathrm{d}\lambda)^{\varpi} \\
&= \frac{\Gamma(1+\theta\varpi)}{\Gamma(1+(\theta+1)\varpi)}.
\end{aligned} \tag{2.5}$$

Combing (2.3), (2.4), and (2.5), we conclude the proof. □

By virtue of the similar argument of Theorem 2.5, we can prove the Theorem 2.6.

Theorem 2.6. *If all the hypotheses in Theorem 2.5 hold, then*

$$\begin{aligned}&\left|\frac{1^{\varpi}}{2^{\varpi}}\left[\frac{\zeta(m)+\zeta(n)}{2^{\varpi}}+\zeta\left(\frac{m+n}{2}\right)\right]-\frac{\Gamma(1+\varpi)}{(n-m)^{\varpi}}{}_{m}\mathscr{I}_{n}^{(\varpi)}\zeta(\tau)\right|\\&\leq\frac{(n-m)^{\varpi}}{8^{\varpi}}\left(\frac{\Gamma(1+\theta\varpi)}{\Gamma(1+(\theta+1)\varpi)}\right)^{\frac{1}{\theta}}\\&\quad\times\left\{\left[\frac{1}{\Gamma(1+\varpi)}\int_{0}^{1}\left(h(\lambda)\left|\zeta^{(\varpi)}(m)\right|^{\psi}+h(1-\lambda)\left|\zeta^{(\varpi)}\left(\frac{m+n}{2}\right)\right|^{\psi}\right)(\mathrm{d}\lambda)^{\varpi}\right]^{\frac{1}{\psi}}\right.\\&\quad\left.+\left[\frac{1}{\Gamma(1+\varpi)}\int_{0}^{1}\left(h(\lambda)\left|\zeta^{(\varpi)}(n)\right|^{\psi}+h(1-\lambda)\left|\zeta^{(\varpi)}\left(\frac{m+n}{2}\right)\right|^{\psi}\right)(\mathrm{d}\lambda)^{\varpi}\right]^{\frac{1}{\psi}}\right\}.\end{aligned}$$

Proof. Following the proof of Theorem 2.5 and consider the convex combinations of endpoints m and n, i.e., $\frac{1+\lambda}{2}m+\frac{1-\lambda}{2}n=\lambda m+(1-\lambda)\frac{m+n}{2}$ and $\frac{1-\lambda}{2}m+\frac{1+\lambda}{2}n=\lambda n+(1-\lambda)\frac{m+n}{2}$, we conclude the desired conclusion immediately. □

Theorem 2.7. *If all the assumptions in Theorem 2.5 hold, then*

$$\begin{aligned}&\left|\frac{1^{\varpi}}{2^{\varpi}}\left[\frac{\zeta(m)+\zeta(n)}{2^{\varpi}}+\zeta\left(\frac{m+n}{2}\right)\right]-\frac{\Gamma(1+\varpi)}{(n-m)^{\varpi}}{}_{m}\mathscr{I}_{n}^{(\varpi)}\zeta(\tau)\right|\\&\leq\frac{(n-m)^{\varpi}}{8^{\varpi}}\left(\frac{\Gamma(1+\theta\varpi)}{2^{\alpha}\Gamma(1+(\theta+1)\varpi)}\right)^{\frac{1}{\theta}}\\&\quad\times\left\{\left[\left|\zeta^{(\varpi)}(m)\right|^{\psi}\cdot\frac{1}{\Gamma(1+\varpi)}\int_{0}^{1}(1-\lambda)^{\varpi}h\left(\frac{1+\lambda}{2}\right)(\mathrm{d}\lambda)^{\varpi}\right.\right.\\&\quad\left.+\left|\zeta^{(\varpi)}(n)\right|^{\psi}\cdot\frac{1}{\Gamma(1+\varpi)}\int_{0}^{1}(1-\lambda)^{\varpi}h\left(\frac{1-\lambda}{2}\right)(\mathrm{d}\lambda)^{\varpi}\right]^{\frac{1}{\psi}}\\&\quad+\left[\left|\zeta^{(\varpi)}(m)\right|^{\psi}\cdot\frac{1}{\Gamma(1+\varpi)}\int_{0}^{1}\lambda^{\varpi}h\left(\frac{1+\lambda}{2}\right)(\mathrm{d}\lambda)^{\varpi}\right.\\&\quad\left.+\left|\zeta^{(\varpi)}(n)\right|^{\psi}\cdot\frac{1}{\Gamma(1+\varpi)}\int_{0}^{1}\lambda^{\varpi}h\left(\frac{1-\lambda}{2}\right)(\mathrm{d}\lambda)^{\varpi}\right]^{\frac{1}{\psi}}\\&\quad+\left[\left|\zeta^{(\varpi)}(n)\right|^{\psi}\cdot\frac{1}{\Gamma(1+\varpi)}\int_{0}^{1}(1-\lambda)^{\varpi}h\left(\frac{1+\lambda}{2}\right)(\mathrm{d}\lambda)^{\varpi}\right.\\&\quad\left.+\left|\zeta^{(\varpi)}(m)\right|^{\psi}\cdot\frac{1}{\Gamma(1+\varpi)}\int_{0}^{1}(1-\lambda)^{\varpi}h\left(\frac{1-\lambda}{2}\right)(\mathrm{d}\lambda)^{\varpi}\right]^{\frac{1}{\psi}}\\&\quad+\left[\left|\zeta^{(\varpi)}(n)\right|^{\psi}\cdot\frac{1}{\Gamma(1+\varpi)}\int_{0}^{1}\lambda^{\varpi}h\left(\frac{1+\lambda}{2}\right)(\mathrm{d}\lambda)^{\varpi}\right.\\&\quad\left.\left.+\left|\zeta^{(\varpi)}(m)\right|^{\psi}\cdot\frac{1}{\Gamma(1+\varpi)}\int_{0}^{1}\lambda^{\varpi}h\left(\frac{1-\lambda}{2}\right)(\mathrm{d}\lambda)^{\varpi}\right]^{\frac{1}{\psi}}\right\}.\end{aligned}\tag{2.6}$$

Proof. In view of Lemma 2.1 and the properties of modulus, we infer that

$$\begin{aligned}
&\left|\frac{1^{\varpi}}{2^{\varpi}}\left[\frac{\zeta(m)+\zeta(n)}{2^{\varpi}}+\zeta\left(\frac{m+n}{2}\right)\right]-\frac{\Gamma(1+\varpi)}{(n-m)^{\varpi}}{}_{m}\mathscr{I}_{n}^{(\varpi)}\zeta(\tau)\right| \\
&\leq \frac{(n-m)^{\varpi}}{8^{\varpi}}\left[\frac{1}{\Gamma(1+\varpi)}\int_0^1 |1-2\lambda|^{\varpi}\left|\zeta^{(\varpi)}\left(\frac{1+\lambda}{2}m+\frac{1-\lambda}{2}n\right)\right|(\mathrm{d}\lambda)^{\varpi}\right. \\
&\left.+\frac{1}{\Gamma(1+\varpi)}\int_0^1 |2\lambda-1|^{\varpi}\left|\zeta^{(\varpi)}\left(\frac{1+\lambda}{2}n+\frac{1-\lambda}{2}m\right)\right|(\mathrm{d}\lambda)^{\varpi}\right].
\end{aligned} \tag{2.7}$$

On account of the modified version of the generalized Hölder's integral inequality (1.1), we find that

$$\begin{aligned}
&\frac{1}{\Gamma(1+\varpi)}\int_0^1 |1-2\lambda|^{\varpi}\left|\zeta^{(\varpi)}\left(\frac{1+\lambda}{2}m+\frac{1-\lambda}{2}n\right)\right|(\mathrm{d}\lambda)^{\varpi} \\
&\leq \left[\left(\frac{1}{\Gamma(1+\varpi)}\int_0^1 (1-\lambda)^{\varpi}|1-2\lambda|^{\theta\varpi}(\mathrm{d}\lambda)^{\varpi}\right)^{\frac{1}{\theta}}\right. \\
&\quad\times\left(\frac{1}{\Gamma(1+\varpi)}\int_0^1 (1-\lambda)^{\varpi}\left|\zeta^{(\varpi)}\left(\frac{1+\lambda}{2}m+\frac{1-\lambda}{2}n\right)\right|^{\psi}(\mathrm{d}\lambda)^{\varpi}\right)^{\frac{1}{\psi}} \\
&\quad+\left(\frac{1}{\Gamma(1+\varpi)}\int_0^1 \lambda^{\varpi}|1-2\lambda|^{\theta\varpi}(\mathrm{d}\lambda)^{\varpi}\right)^{\frac{1}{\theta}} \\
&\quad\left.\times\left(\frac{1}{\Gamma(1+\varpi)}\int_0^1 \lambda^{\varpi}\left|\zeta^{(\varpi)}\left(\frac{1+\lambda}{2}m+\frac{1-\lambda}{2}n\right)\right|^{\psi}(\mathrm{d}\lambda)^{\varpi}\right)^{\frac{1}{\psi}}\right].
\end{aligned} \tag{2.8}$$

Direct calculations yield that

$$\begin{aligned}
&\frac{1}{\Gamma(1+\varpi)}\int_0^1 (1-\lambda)^{\varpi}|1-2\lambda|^{\theta\varpi}(\mathrm{d}\lambda)^{\varpi} \\
&=\frac{1}{\Gamma(1+\varpi)}\int_0^{\frac{1}{2}} (1-\lambda)^{\varpi}(1-2\lambda)^{\theta\varpi}(\mathrm{d}\lambda)^{\varpi}+\frac{1}{\Gamma(1+\varpi)}\int_{\frac{1}{2}}^1 (1-\lambda)^{\varpi}(2\lambda-1)^{\theta\varpi}(\mathrm{d}\lambda)^{\varpi} \\
&=\frac{\Gamma(1+\theta\varpi)}{2^{\varpi}\Gamma(1+(\theta+1)\varpi)}
\end{aligned} \tag{2.9}$$

and

$$\begin{aligned}
&\frac{1}{\Gamma(1+\varpi)}\int_0^1 \lambda^{\varpi}|1-2\lambda|^{\theta\varpi}(\mathrm{d}\lambda)^{\varpi} \\
&=\frac{1}{\Gamma(1+\varpi)}\int_0^{\frac{1}{2}} \lambda^{\varpi}(1-2\lambda)^{\theta\varpi}(\mathrm{d}\lambda)^{\varpi}+\frac{1}{\Gamma(1+\varpi)}\int_{\frac{1}{2}}^1 \lambda^{\varpi}(2\lambda-1)^{\theta\varpi}(\mathrm{d}\lambda)^{\varpi} \\
&=\frac{\Gamma(1+\theta\varpi)}{2^{\varpi}\Gamma(1+(\theta+1)\varpi)}.
\end{aligned} \tag{2.10}$$

Taking advantage of the generalized h-convexity of $|\zeta^{(\varpi)}|^{\psi}$, we deduce that

$$\begin{aligned}&\frac{1}{\Gamma(1+\varpi)}\int_0^1(1-\lambda)^{\varpi}\left|\zeta^{(\varpi)}\left(\frac{1+\lambda}{2}m+\frac{1-\lambda}{2}n\right)\right|^{\psi}(\mathrm{d}\lambda)^{\varpi}\\ &\leq\left|\zeta^{(\varpi)}(m)\right|^{\psi}\cdot\frac{1}{\Gamma(1+\varpi)}\int_0^1(1-\lambda)^{\varpi}h\left(\frac{1+\lambda}{2}\right)(\mathrm{d}\lambda)^{\varpi}\\ &\quad+\left|\zeta^{(\varpi)}(n)\right|^{\psi}\cdot\frac{1}{\Gamma(1+\varpi)}\int_0^1(1-\lambda)^{\varpi}h\left(\frac{1-\lambda}{2}\right)(\mathrm{d}\lambda)^{\varpi}\end{aligned}\tag{2.11}$$

and

$$\begin{aligned}&\frac{1}{\Gamma(1+\varpi)}\int_0^1\lambda^{\varpi}\left|\zeta^{(\varpi)}\left(\frac{1+\lambda}{2}m+\frac{1-\lambda}{2}n\right)\right|^{\psi}(\mathrm{d}\lambda)^{\varpi}\\ &\leq\left|\zeta^{(\varpi)}(m)\right|^{\psi}\cdot\frac{1}{\Gamma(1+\varpi)}\int_0^1\lambda^{\varpi}h\left(\frac{1+\lambda}{2}\right)(\mathrm{d}\lambda)^{\varpi}\\ &\quad+\left|\zeta^{(\varpi)}(n)\right|^{\psi}\cdot\frac{1}{\Gamma(1+\varpi)}\int_0^1\lambda^{\varpi}h\left(\frac{1-\lambda}{2}\right)(\mathrm{d}\lambda)^{\varpi}.\end{aligned}\tag{2.12}$$

Substituting (2.9)-(2.12) into (2.8), we obtain that

$$\begin{aligned}&\frac{1}{\Gamma(1+\varpi)}\int_0^1|1-2\lambda|^{\varpi}\left|\zeta^{(\varpi)}\left(\frac{1+\lambda}{2}m+\frac{1-\lambda}{2}n\right)\right|(\mathrm{d}\lambda)^{\varpi}\\ &\leq\left(\frac{\Gamma(1+\theta\varpi)}{2^{\varpi}\Gamma(1+(\theta+1)\varpi)}\right)^{\frac{1}{\theta}}\Bigg[\Bigg(\left|\zeta^{(\varpi)}(m)\right|^{\psi}\cdot\frac{1}{\Gamma(1+\varpi)}\int_0^1(1-\lambda)^{\varpi}h\left(\frac{1+\lambda}{2}\right)(\mathrm{d}\lambda)^{\varpi}\\ &\quad+\left|\zeta^{(\varpi)}(n)\right|^{\psi}\cdot\frac{1}{\Gamma(1+\varpi)}\int_0^1(1-\lambda)^{\varpi}h\left(\frac{1-\lambda}{2}\right)(\mathrm{d}\lambda)^{\varpi}\Bigg)^{\frac{1}{\psi}}\\ &\quad+\Bigg(\left|\zeta^{(\varpi)}(m)\right|^{\psi}\cdot\frac{1}{\Gamma(1+\varpi)}\int_0^1\lambda^{\varpi}h\left(\frac{1+\lambda}{2}\right)(\mathrm{d}\lambda)^{\varpi}\\ &\quad+\left|\zeta^{(\varpi)}(n)\right|^{\psi}\cdot\frac{1}{\Gamma(1+\varpi)}\int_0^1\lambda^{\varpi}h\left(\frac{1-\lambda}{2}\right)(\mathrm{d}\lambda)^{\varpi}\Bigg)^{\frac{1}{\psi}}\Bigg].\end{aligned}\tag{2.13}$$

Similarly, we receive that

$$\begin{aligned}&\frac{1}{\Gamma(1+\varpi)}\int_0^1|2\lambda-1|^{\varpi}\left|\zeta^{(\varpi)}\left(\frac{1+\lambda}{2}n+\frac{1-\lambda}{2}m\right)\right|(\mathrm{d}\lambda)^{\varpi}\\ &\leq\left(\frac{\Gamma(1+\theta\varpi)}{2^{\varpi}\Gamma(1+(\theta+1)\alpha)}\right)^{\frac{1}{\theta}}\Bigg[\Bigg(\left|\zeta^{(\varpi)}(n)\right|^{\psi}\cdot\frac{1}{\Gamma(1+\varpi)}\int_0^1(1-\lambda)^{\varpi}h\left(\frac{1+\lambda}{2}\right)(\mathrm{d}\lambda)^{\varpi}\\ &\quad+\left|\zeta^{(\varpi)}(m)\right|^{\psi}\cdot\frac{1}{\Gamma(1+\varpi)}\int_0^1(1-\lambda)^{\varpi}h\left(\frac{1-\lambda}{2}\right)(\mathrm{d}\lambda)^{\varpi}\Bigg)^{\frac{1}{\psi}}\\ &\quad+\Bigg(\left|\zeta^{(\varpi)}(n)\right|^{\psi}\cdot\frac{1}{\Gamma(1+\varpi)}\int_0^1\lambda^{\varpi}h\left(\frac{1+\lambda}{2}\right)(\mathrm{d}\lambda)^{\varpi}\\ &\quad+\left|\zeta^{(\varpi)}(m)\right|^{\psi}\cdot\frac{1}{\Gamma(1+\varpi)}\int_0^1\lambda^{\varpi}h\left(\frac{1-\lambda}{2}\right)(\mathrm{d}\lambda)^{\varpi}\Bigg)^{\frac{1}{\psi}}\Bigg].\end{aligned}\tag{2.14}$$

Employing (2.13) and (2.14) in (2.7), one achieves (2.6). Hence, the proof is completed. □

Theorem 2.8. *Let all the assumptions in Lemma 2.1 hold. If $|\zeta^{(\varpi)}|^{\psi}$ is generalized h-convex on the interval $[m,n]$ for $\psi \geq 1$, then*

$$\begin{aligned}
&\left|\frac{1^{\varpi}}{2^{\varpi}}\left[\frac{\zeta(m)+\zeta(n)}{2^{\varpi}}+\zeta\left(\frac{m+n}{2}\right)\right]-\frac{\Gamma(1+\varpi)}{(n-m)^{\varpi}}{}_{m}\mathscr{I}_{n}^{(\varpi)}\zeta(\tau)\right| \\
&\leq \frac{(n-m)^{\varpi}}{8^{\varpi}}\left(\frac{\Gamma(1+\varpi)}{\Gamma(1+2\varpi)}\right)^{1-\frac{1}{\psi}}\Bigg\{\Bigg[\frac{1}{\Gamma(1+\varpi)}\int_{0}^{\frac{1}{2}}\left((1-2\lambda)^{\varpi}h\left(\frac{1+\lambda}{2}\right)|\zeta^{(\varpi)}(m)|^{\psi}\right. \\
&\left.+(1-2\lambda)^{\varpi}h\left(\frac{1-\lambda}{2}\right)|\zeta^{(\varpi)}(n)|^{\psi}\right)(\mathrm{d}\lambda)^{\varpi}+\frac{1}{\Gamma(1+\varpi)}\int_{\frac{1}{2}}^{1}\left((2\lambda-1)^{\varpi}h\left(\frac{1+\lambda}{2}\right)|\zeta^{(\varpi)}(m)|^{\psi}\right. \\
&\left.+(2\lambda-1)^{\varpi}h\left(\frac{1-\lambda}{2}\right)|\zeta^{(\varpi)}(n)|^{\psi}\right)(\mathrm{d}\lambda)^{\varpi}\Bigg]^{\frac{1}{\psi}} \\
&+\Bigg[\frac{1}{\Gamma(1+\varpi)}\int_{0}^{\frac{1}{2}}\left((1-2\lambda)^{\varpi}h\left(\frac{1+\lambda}{2}\right)|\zeta^{(\varpi)}(n)|^{\psi}\right. \\
&\left.+(1-2\lambda)^{\varpi}h\left(\frac{1-\lambda}{2}\right)|\zeta^{(\varpi)}(m)|^{\psi}\right)(\mathrm{d}\lambda)^{\varpi} \\
&+\frac{1}{\Gamma(1+\varpi)}\int_{\frac{1}{2}}^{1}\left((2\lambda-1)^{\varpi}h\left(\frac{1+\lambda}{2}\right)|\zeta^{(\varpi)}(n)|^{\psi}\right. \\
&\left.+(2\lambda-1)^{\varpi}h\left(\frac{1-\lambda}{2}\right)|\zeta^{(\varpi)}(m)|^{\psi}\right)(\mathrm{d}\lambda)^{\varpi}\Bigg]^{\frac{1}{\psi}}\Bigg\}.
\end{aligned} \tag{2.15}$$

Proof. From (2.1) and the generalized power-mean inequality, we have

$$\begin{aligned}
&\left|\frac{1^{\varpi}}{2^{\varpi}}\left[\frac{\zeta(m)+\zeta(n)}{2^{\varpi}}+\zeta\left(\frac{m+n}{2}\right)\right]-\frac{\Gamma(1+\varpi)}{(n-m)^{\varpi}}{}_{m}\mathscr{I}_{n}^{(\varpi)}\zeta(\tau)\right| \\
&\leq \frac{(n-m)^{\varpi}}{8^{\varpi}}\left[\frac{1}{\Gamma(1+\varpi)}\int_{0}^{1}|1-2\lambda|^{\varpi}\left|\zeta^{(\varpi)}\left(\frac{1+\lambda}{2}m+\frac{1-\lambda}{2}n\right)\right|(\mathrm{d}\lambda)^{\varpi}\right. \\
&\quad\left.+\frac{1}{\Gamma(1+\varpi)}\int_{0}^{1}|2\lambda-1|^{\varpi}\left|\zeta^{(\varpi)}\left(\frac{1+\lambda}{2}n+\frac{1-\lambda}{2}m\right)\right|(\mathrm{d}\lambda)^{\varpi}\right] \\
&\leq \frac{(n-m)^{\varpi}}{8^{\varpi}}\Bigg\{\left(\frac{1}{\Gamma(1+\varpi)}\int_{0}^{1}|1-2\lambda|^{\varpi}(\mathrm{d}\lambda)^{\varpi}\right)^{1-\frac{1}{\psi}} \\
&\quad\times\left(\frac{1}{\Gamma(1+\varpi)}\int_{0}^{1}|1-2\lambda|^{\varpi}\left|\zeta^{(\varpi)}\left(\frac{1+\lambda}{2}m+\frac{1-\lambda}{2}n\right)\right|^{\psi}(\mathrm{d}\lambda)^{\varpi}\right)^{\frac{1}{\psi}} \\
&\quad+\left(\frac{1}{\Gamma(1+\varpi)}\int_{0}^{1}|2\lambda-1|^{\varpi}(\mathrm{d}\lambda)^{\varpi}\right)^{1-\frac{1}{\psi}} \\
&\quad\times\left(\frac{1}{\Gamma(1+\varpi)}\int_{0}^{1}|2\lambda-1|^{\varpi}\left|\zeta^{(\varpi)}\left(\frac{1+\lambda}{2}n+\frac{1-\lambda}{2}m\right)\right|^{\psi}(\mathrm{d}\lambda)^{\varpi}\right)^{\frac{1}{\psi}}\Bigg\}
\end{aligned}$$

$$\leq \frac{(n-m)^{\varpi}}{8^{\varpi}}\left(\frac{1}{\Gamma(1+\varpi)}\int_0^1 \left|2\lambda-1\right|^{\varpi}(\mathrm{d}\lambda)^{\varpi}\right)^{1-\frac{1}{\psi}}\left\{\left[\frac{1}{\Gamma(1+\varpi)}\int_0^{\frac{1}{2}}\left((1-2\lambda)^{\varpi}h\left(\frac{1+\lambda}{2}\right)\right.\right.\right.$$

$$\times\left|\zeta^{(\varpi)}(m)\right|^{\psi}+(1-2\lambda)^{\varpi}h\left(\frac{1-\lambda}{2}\right)\left|\zeta^{(\varpi)}(n)\right|^{\psi}\Big)(\mathrm{d}\lambda)^{\varpi}+\frac{1}{\Gamma(1+\varpi)}\int_{\frac{1}{2}}^1\left((2\lambda-1)^{\varpi}h\left(\frac{1+\lambda}{2}\right)\right.$$

$$\left.\times\left|\zeta^{(\varpi)}(m)\right|^{\psi}+(2\lambda-1)^{\varpi}h\left(\frac{1-\lambda}{2}\right)\left|\zeta^{(\varpi)}(n)\right|^{\psi}\Big)(\mathrm{d}\lambda)^{\varpi}\right]^{\frac{1}{\psi}}$$

$$+\left[\frac{1}{\Gamma(1+\varpi)}\int_0^{\frac{1}{2}}\left((1-2\lambda)^{\varpi}h\left(\frac{1+\lambda}{2}\right)\left|\zeta^{(\varpi)}(n)\right|^{\psi}\right.\right.$$

$$+(1-2\lambda)^{\varpi}h\left(\frac{1-\lambda}{2}\right)\left|\zeta^{(\varpi)}(m)\right|^{\psi}\Big)(\mathrm{d}\lambda)^{\varpi}+\frac{1}{\Gamma(1+\varpi)}\int_{\frac{1}{2}}^1\left((2\lambda-1)^{\varpi}h\left(\frac{1+\lambda}{2}\right)\left|\zeta^{(\varpi)}(n)\right|^{\psi}\right.$$

$$\left.\left.+(2\lambda-1)^{\varpi}h\left(\frac{1-\lambda}{2}\right)\left|\zeta^{(\varpi)}(m)\right|^{\psi}\Big)(\mathrm{d}\lambda)^{\varpi}\right]^{\frac{1}{\psi}}\right\}.$$

Observe that

$$\frac{1}{\Gamma(1+\varpi)}\int_0^1 |1-2\lambda|^{\varpi}(\mathrm{d}\lambda)^{\varpi}=\frac{\Gamma(1+\varpi)}{\Gamma(1+2\varpi)}.$$

This concludes the proof. □

Theorem 2.9. *Let $|\zeta^{(\varpi)}|$ satisfy the local Lipschitz condition for certain $L^{\varpi}>0^{\varpi}$. Then*

$$\begin{aligned}&\left|\frac{1^{\varpi}}{2^{\varpi}}\left[\frac{\zeta(m)+\zeta(n)}{2^{\varpi}}+\zeta\left(\frac{m+n}{2}\right)\right]-\frac{\Gamma(1+\varpi)}{(n-m)^{\varpi}}{}_m\mathscr{I}_n^{(\varpi)}\zeta(\tau)\right|\\&\quad\leq\frac{(n-m)^{\varpi}}{8^{\varpi}}\frac{\Gamma(1+\varpi)}{\Gamma(1+2\varpi)}\left[\frac{(n-m)^{\varpi}L^{\varpi}}{2^{\varpi}}+\zeta^{(\varpi)}(n)+\zeta^{(\varpi)}\left(\frac{m+n}{2}\right)\right],\end{aligned}\tag{2.16}$$

where ζ is as in Lemma 2.1.

Proof. By using Lemma 2.1, one finds that

$$\left|\frac{1^{\varpi}}{2^{\varpi}}\left[\frac{\zeta(m)+\zeta(n)}{2^{\varpi}}+\zeta\left(\frac{m+n}{2}\right)\right]-\frac{\Gamma(1+\varpi)}{(n-m)^{\varpi}}{}_m\mathscr{I}_n^{(\varpi)}\zeta(\tau)\right|$$

$$\leq\frac{(n-m)^{\varpi}}{8^{\varpi}}\left[\frac{1}{\Gamma(1+\varpi)}\int_0^1\left|1-2\lambda\right|^{\varpi}\left|\zeta^{(\varpi)}\left(\frac{1+\lambda}{2}m+\frac{1-\lambda}{2}n\right)\right|(\mathrm{d}\lambda)^{\varpi}\right.$$

$$\left.+\frac{1}{\Gamma(1+\varpi)}\int_0^1\left|2\lambda-1\right|^{\varpi}\left|\zeta^{(\varpi)}\left(\frac{1+\lambda}{2}n+\frac{1-\lambda}{2}m\right)\right|(\mathrm{d}\lambda)^{\varpi}\right]$$

$$=\frac{(n-m)^{\varpi}}{8^{\varpi}}\left\{\frac{1}{\Gamma(1+\varpi)}\int_0^1\left(\left|1-2\lambda\right|^{\varpi}\left|\zeta^{(\varpi)}\left(\frac{m+n}{2}+\frac{m-n}{2}\lambda\right)-\zeta^{(\varpi)}\left(\frac{m+n}{2}\right)\right|\right.\right.$$

$$\left.+\left|2\lambda-1\right|^{\varpi}\left|\zeta^{(\varpi)}\left(\frac{m+n}{2}+\frac{n-m}{2}\lambda\right)-\zeta^{(\varpi)}(n)\right|\right)(\mathrm{d}\lambda)^{\varpi}$$

$$\left.+\left(\zeta^{(\varpi)}\left(\frac{m+n}{2}\right)+\zeta^{(\varpi)}(n)\right)\frac{1}{\Gamma(1+\varpi)}\int_0^1|1-2\lambda|^{\varpi}(\mathrm{d}\lambda)^{\varpi}\right\}.$$

Since $|\zeta^{(\varpi)}|$ satisfies the local Lipschitz conditions for $L^{\varpi} > 0^{\varpi}$, we see that

$$\begin{aligned}&\left|\zeta^{(\varpi)}\left(\frac{m+n}{2}+\frac{m-n}{2}\lambda\right)-\zeta^{(\varpi)}\left(\frac{m+n}{2}\right)\right|\\&\le L^{\varpi}\left|\left(\frac{m+n}{2}+\frac{m-n}{2}\lambda\right)^{\varpi}-\left(\frac{m+n}{2}\right)^{\varpi}\right|\\&=\left(\frac{|m-n|L}{2}\right)^{\varpi}|\lambda|^{\varpi},\end{aligned}$$

and

$$\begin{aligned}&\left|\zeta^{(\varpi)}\left(\frac{m+n}{2}+\frac{n-m}{2}\lambda\right)-\zeta^{(\varpi)}(n)\right|\\&\le L^{\varpi}\left|\left(\frac{m+n}{2}+\frac{n-m}{2}\lambda\right)^{\varpi}-\left(\frac{m+n}{2}+\frac{n-m}{2}\right)^{\varpi}\right|\\&=\left(\frac{|m-n|L}{2}\right)^{\varpi}|1-\lambda|^{\varpi}.\end{aligned}$$

Hence,

$$\begin{aligned}&\left|\frac{1^{\varpi}}{2^{\varpi}}\left[\frac{\zeta(m)+\zeta(n)}{2^{\varpi}}+\zeta\left(\frac{m+n}{2}\right)\right]-\frac{\Gamma(1+\varpi)}{(n-m)^{\varpi}}{}_m\mathscr{I}_n^{(\varpi)}\zeta(\tau)\right|\\&\le\frac{(n-m)^{\varpi}}{8^{\varpi}}\left\{\frac{1}{\Gamma(1+\varpi)}\int_0^1\left(\left|1-2\lambda\right|^{\varpi}\left(\frac{|m-n|L}{2}\right)^{\varpi}|\lambda|^{\varpi}\right.\right.\\&\quad\left.+\left|2\lambda-1\right|^{\varpi}\left(\frac{|m-n|L}{2}\right)^{\varpi}|1-\lambda|^{\varpi}(\mathrm{d}\lambda)^{\varpi}\right)\\&\quad\left.+\left(\zeta^{(\varpi)}\left(\frac{m+n}{2}\right)+\zeta^{(\varpi)}(n)\right)\frac{1}{\Gamma(1+\varpi)}\int_0^1|1-2\lambda|^{\varpi}(\mathrm{d}\lambda)^{\varpi}\right\}\\&=\frac{(n-m)^{\varpi}}{8^{\varpi}}\frac{\Gamma(1+\varpi)}{\Gamma(1+2\varpi)}\left[\frac{(n-m)^{\varpi}L^{\varpi}}{2^{\varpi}}+\zeta^{(\varpi)}(n)+\zeta^{(\varpi)}\left(\frac{m+n}{2}\right)\right].\end{aligned}$$

This concludes the proof. □

3. EXAMPLES

We present three examples in this section to support our main theorems.

Example 3.1. If $|\zeta^{(\varpi)}(\tau)| = \tau^{4\varpi}$ for $\tau\in(0,\infty)$, $\zeta(\tau)=\frac{\Gamma(1+4\varpi)}{\Gamma(1+5\varpi)}\tau^{5\varpi}$. The left-hand side term of (2.1) is

$$\begin{aligned}&\frac{1^{\varpi}}{2^{\varpi}}\left[\frac{\zeta(m)+\zeta(n)}{2^{\varpi}}+\zeta\left(\frac{m+n}{2}\right)\right]-\frac{\Gamma(1+\varpi)}{(n-m)^{\varpi}}{}_m\mathscr{I}_n^{(\varpi)}\zeta(\tau)\\&=\frac{1^{\varpi}}{2^{\varpi}}\frac{\Gamma(1+4\varpi)}{\Gamma(1+5\varpi)}\left[\frac{m^{5\varpi}+n^{5\varpi}}{2^{\varpi}}+\left(\frac{m+n}{2}\right)^{5\varpi}\right]-\frac{\Gamma(1+\varpi)}{(n-m)^{5\varpi}}\frac{\Gamma(1+4\varpi)}{\Gamma(1+6\varpi)}(n^{6\varpi}-m^{6\varpi}).\end{aligned}$$

The right-hand side terms of (2.1) is

$$\begin{aligned}&\frac{(n-m)^{\varpi}}{8^{\varpi}}\frac{1}{\Gamma(1+\varpi)}\int_0^1\left[(1-2\lambda)^{\varpi}\zeta^{(\varpi)}\left(\frac{1+\lambda}{2}m+\frac{1-\lambda}{2}n\right)\right.\\&\left.+(2\lambda-1)^{\varpi}\zeta^{(\varpi)}\left(\frac{1+\lambda}{2}n+\frac{1-\lambda}{2}m\right)\right](\mathrm{d}\lambda)^{\varpi}\\&=\frac{(n-m)^{\varpi}}{8^{\varpi}}\frac{1}{\Gamma(1+\varpi)}\int_0^1\left[(1-2\lambda)^{\varpi}\left(\frac{1+\lambda}{2}m+\frac{1-\lambda}{2}n\right)^{4\varpi}\right.\\&\left.+(2\lambda-1)^{\varpi}\left(\frac{1+\lambda}{2}n+\frac{1-\lambda}{2}m\right)^{4\varpi}\right](\mathrm{d}\lambda)^{\varpi}\\&=\frac{1^{\varpi}}{2^{\varpi}}\frac{\Gamma(1+4\varpi)}{\Gamma(1+5\varpi)}\left[\frac{m^{5\varpi}+n^{5\varpi}}{2^{\varpi}}+\left(\frac{m+n}{2}\right)^{5\varpi}\right]-\frac{\Gamma(1+\varpi)}{(n-m)^{5\varpi}}\frac{\Gamma(1+4\varpi)}{\Gamma(1+6\varpi)}(n^{6\varpi}-m^{6\varpi}).\end{aligned}$$

This illustrates the (2.1) in Lemma 2.1.

Example 3.2. If $|\zeta^{(\varpi)}(\tau)|^{\psi}=\tau^{4\psi\varpi}$ for $\tau\in(0,\infty)$, then $\zeta(\tau)=\frac{\Gamma(1+4\varpi)}{\Gamma(1+5\varpi)}\tau^{5\varpi}$. If $\psi=2$, then the mapping $|\zeta^{(\varpi)}(\tau)|^{2}=\tau^{8\varpi}$ is generalized h-convex with $h(\lambda)=\lambda^{\varpi}$. If $\varpi=\frac{1}{2}$, $m=1$, and $n=3$, then all the hypotheses in Theorem 2.8 are satisfied. The left-hand side term of inequality (2.15) is

$$\begin{aligned}&\left|\frac{1^{\varpi}}{2^{\varpi}}\left[\frac{\zeta(m)+\zeta(n)}{2^{\varpi}}+\zeta\left(\frac{m+n}{2}\right)\right]-\frac{\Gamma(1+\varpi)}{(n-m)^{\varpi}}{}_m\mathscr{I}_n^{(\varpi)}\zeta(\tau)\right|\\&=\left|\frac{1^{\frac{1}{2}}}{2^{\frac{1}{2}}}\left[\frac{\zeta(1)+\zeta(3)}{2^{\frac{1}{2}}}+\zeta\left(\frac{1+3}{2}\right)\right]-\left(\frac{1}{3-1}\right)^{\frac{1}{2}}\int_1^3\zeta(\tau)(\mathrm{d}\tau)^{\frac{1}{2}}\right|\approx 1.9677.\end{aligned}$$

The right-hand side term of inequality (2.15) is

$$\begin{aligned}&\frac{(3-1)^{\frac{1}{2}}}{8^{\frac{1}{2}}}\left(\frac{\Gamma(1+\frac{1}{2})}{\Gamma(1+1)}\right)^{1-\frac{1}{2}}\left\{\left[\frac{1}{\Gamma(1+\frac{1}{2})}\int_0^{\frac{1}{2}}\left((1-2\lambda)^{\frac{1}{2}}\left(\frac{1+\lambda}{2}\right)^{\frac{1}{2}}|\zeta^{(\frac{1}{2})}(1)|^2\right.\right.\right.\\&\left.+(1-2\lambda)^{\frac{1}{2}}\left(\frac{1-\lambda}{2}\right)^{\frac{1}{2}}|\zeta^{(\frac{1}{2})}(3)|^2\right)(\mathrm{d}\lambda)^{\frac{1}{2}}+\frac{1}{\Gamma(1+\frac{1}{2})}\int_{\frac{1}{2}}^1\left((2\lambda-1)^{\frac{1}{2}}\left(\frac{1+\lambda}{2}\right)^{\frac{1}{2}}|\zeta^{(\frac{1}{2})}(1)|^2\right.\\&\left.\left.+(2\lambda-1)^{\frac{1}{2}}\left(\frac{1-\lambda}{2}\right)^{\frac{1}{2}}|\zeta^{(\frac{1}{2})}(3)|^2\right)(\mathrm{d}\lambda)^{\frac{1}{2}}\right]^{\frac{1}{2}}+\left[\frac{1}{\Gamma(1+\frac{1}{2})}\int_0^{\frac{1}{2}}\left((1-2\lambda)^{\frac{1}{2}}\left(\frac{1+\lambda}{2}\right)^{\frac{1}{2}}|\zeta^{(\frac{1}{2})}(3)|^2\right.\right.\\&\left.+(1-2\lambda)^{\frac{1}{2}}\left(\frac{1-\lambda}{2}\right)^{\frac{1}{2}}|\zeta^{(\frac{1}{2})}(1)|^2\right)(\mathrm{d}\lambda)^{\frac{1}{2}}+\frac{1}{\Gamma(1+\frac{1}{2})}\int_{\frac{1}{2}}^1\left((2\lambda-1)^{\frac{1}{2}}\left(\frac{1+\lambda}{2}\right)|\zeta^{(\frac{1}{2})}(3)|^2\right.\\&\left.\left.\left.+(2\lambda-1)^{\frac{1}{2}}\left(\frac{1-\lambda}{2}\right)|\zeta^{(\frac{1}{2})}(1)|^2\right)(\mathrm{d}\lambda)^{\frac{1}{2}}\right]^{\frac{1}{2}}\right\}\approx 5.9596,\end{aligned}$$

which illustrates the (2.15) in Theorem 2.8 .

Example 3.3. Let $|\zeta^{(\varpi)}(\tau)|=\tau^{4\varpi}$ for $\tau\in(0,\infty)$. Then $\zeta(\tau)=\frac{\Gamma(1+4\varpi)}{\Gamma(1+5\varpi)}\tau^{5\varpi}$ and $|\zeta^{(\varpi)}|$ satisfies a local Lipschitz condition for $L^{\varpi}>0^{\varpi}$. If $\varpi=\frac{1}{2}$, $L^{\varpi}=4^{\varpi}$, $m=1$, and $n=3$, then all the hypotheses in Theorem 2.9 are satisfied. The left-hand side term of inequality (2.16) is

$$\begin{aligned}
&\left|\frac{1^{\varpi}}{2^{\varpi}}\left[\frac{\zeta(m)+\zeta(n)}{2^{\varpi}}+\zeta\left(\frac{m+n}{2}\right)\right]-\frac{\Gamma(1+\varpi)}{(n-m)^{\varpi}}{}_{m}\mathscr{I}_{n}^{(\varpi)}\zeta(\tau)\right| \\
&=\left|\frac{1^{\frac{1}{2}}}{2^{\frac{1}{2}}}\left[\frac{\zeta(1)+\zeta(3)}{2^{\frac{1}{2}}}+\zeta\left(\frac{1+3}{2}\right)\right]-\left(\frac{1}{3-1}\right)^{\frac{1}{2}}\int_{1}^{3}\zeta(\tau)(\mathrm{d}\tau)^{\frac{1}{2}}\right| \\
&\approx 1.9677.
\end{aligned}$$

The right-hand side term of inequality (2.16) is

$$\begin{aligned}
&\frac{(n-m)^{\varpi}}{8^{\varpi}}\frac{\Gamma(1+\varpi)}{\Gamma(1+2\varpi)}\left[\frac{(n-m)^{\varpi}L^{\varpi}}{2^{\varpi}}+\zeta^{(\varpi)}(n)+\zeta^{(\varpi)}\left(\frac{m+n}{2}\right)\right] \\
&=\frac{(3-1)^{\frac{1}{2}}}{8^{\frac{1}{2}}}\frac{\Gamma(1+\frac{1}{2})}{\Gamma(1+1)}\left[\frac{(3-1)^{\frac{1}{2}}4^{\frac{1}{2}}}{2^{\frac{1}{2}}}+\zeta^{(\frac{1}{2})}(3)+\zeta^{(\frac{1}{2})}\left(\frac{1+3}{2}\right)\right] \\
&\approx 6.6467,
\end{aligned}$$

which illustrates the (2.16) in Theorem 2.9 .

4. Some Applications

4.1. **Special means.** For $m<n$, the ϖ-type special means are considered as below

$$A_{\varpi}(m,n)=\left(\frac{m+n}{2}\right)^{\varpi}=\frac{m^{\varpi}+n^{\varpi}}{2^{\varpi}}$$

and

$$L_{\sigma\varpi}(m,n)=\left[\frac{\Gamma(1+\sigma\varpi)}{\Gamma(1+(1+\sigma)\varpi)}\frac{n^{(\sigma+1)\varpi}-m^{(\sigma+1)\varpi}}{(n-m)^{\varpi}}\right]^{\frac{1}{\sigma}},\ \sigma\in R\setminus\{-1,0\}.$$

Proposition 4.1. *Let* $0<m<n$, $\theta,\psi>1$, $\frac{1}{\theta}+\frac{1}{\psi}=1$, *and* $0<\varpi\leq 1$. *Then*

$$\begin{aligned}
&\left|\frac{\Gamma(1+4\varpi)}{\Gamma(1+5\varpi)}\left[\frac{1}{2^{\varpi}}\left(A_{\varpi}(m^5,n^5)+A_{\varpi}^5(m,n)\right)-\Gamma(1+\varpi)L_{5\varpi}^5(m,n)\right]\right| \\
&\leq\frac{(n-m)^{\varpi}}{8^{\varpi}}\left(\frac{\Gamma(1+\theta\varpi)}{\Gamma(1+(\theta+1)\varpi)}\right)^{\frac{1}{\theta}}\left(\frac{\Gamma(1+\varpi)}{\Gamma(1+2\varpi)}\right)^{\frac{1}{\psi}} \\
&\quad\times\left[\left(m^{4\psi\varpi}+\left(\frac{m+n}{2}\right)^{4\psi\varpi}\right)^{\frac{1}{\psi}}+\left(n^{4\psi\varpi}+\left(\frac{m+n}{2}\right)^{4\psi\varpi}\right)^{\frac{1}{\psi}}\right].
\end{aligned}$$

Proof. Letting $\zeta(\tau)=\frac{\Gamma(1+4\varpi)}{\Gamma(1+5\varpi)}\tau^{5\varpi}$ for $\tau>0$, we have $|\zeta^{(\varpi)}|^{\psi}=\tau^{4\psi\varpi}$ satisfying the generalized h-convexity with $h(\lambda)=\lambda^{\varpi}$. Then, we can obtain the desired conclusion immediately. □

4.2. **Probability distribution function.** Let $p:[m,n]\to[0^{\varpi},1^{\varpi}]$ be a generalized probability density mapping of random variable X which satisfies the generalized h-convexity, and the corresponding cumulative distribution mapping be as below

$$\mathrm{Pr}_{\varpi}(X\leq\tau)=\frac{1}{\Gamma(1+\varpi)}\int_{m}^{\tau}p(\lambda)(\mathrm{d}\lambda)^{\varpi}.$$

The following result, which connects the cumulative distribution mapping and the generalized expectation $E^{\varpi}(X)$, presentes several significant inequalities.

Proposition 4.2. *(a) If $h(\lambda)=\lambda^{\varpi}$ in Theorem 2.5, then*

$$\begin{aligned}&\left|\frac{1^{\varpi}}{2^{\varpi}}\left[\frac{\Pr_{\varpi}(X\le m)+\Pr_{\varpi}(X\le n)}{2^{\varpi}}+\Pr_{\varpi}\left(X\le\frac{m+n}{2}\right)\right]-\frac{n^{\varpi}-E^{\varpi}(X)}{(n-m)^{\varpi}}\right|\\&\le\frac{(n-m)^{\varpi}}{8^{\varpi}}\left(\frac{\Gamma(1+\theta\varpi)}{\Gamma(1+(\theta+1)\varpi)}\right)^{\frac{1}{\theta}}\left\{\left[\frac{\Gamma(1+\varpi)}{\Gamma(1+2\varpi)}\left(\left(\frac{3}{2}\right)^{\varpi}|p(m)|^{\psi}+\left(\frac{1}{2}\right)^{\varpi}|p(n)|^{\psi}\right)\right]^{\frac{1}{\psi}}\right.\\&\quad\left.+\left[\frac{\Gamma(1+\varpi)}{\Gamma(1+2\varpi)}\left(\left(\frac{3}{2}\right)^{\varpi}|p(n)|^{\psi}+\left(\frac{1}{2}\right)^{\varpi}|p(m)|^{\psi}\right)\right]^{\frac{1}{\psi}}\right\}.\end{aligned}$$

(b) If $h(\lambda)=\lambda^{s\varpi}$ with some fixed $0<s<1$ in Theorem 2.5, then

$$\begin{aligned}&\left|\frac{1^{\varpi}}{2^{\varpi}}\left[\frac{\Pr_{\varpi}(X\le m)+\Pr_{\varpi}(X\le n)}{2^{\varpi}}+\Pr_{\varpi}\left(X\le\frac{m+n}{2}\right)\right]-\frac{n^{\varpi}-E^{\varpi}(X)}{(n-m)^{\varpi}}\right|\\&\le\frac{(n-m)^{\varpi}}{8^{\varpi}}\left(\frac{\Gamma(1+\theta\varpi)}{\Gamma(1+(\theta+1)\varpi)}\right)^{\frac{1}{\theta}}\\&\quad\times\left\{\left[\frac{\Gamma(1+s\varpi)}{\Gamma(1+(s+1)\varpi)}\left(\left(2^{\varpi}-\left(\frac{1}{2}\right)^{s\varpi}\right)|p(m)|^{\psi}+\left(\frac{1}{2}\right)^{s\varpi}|p(n)|^{\psi}\right)\right]^{\frac{1}{\psi}}\right.\\&\quad\left.+\frac{\Gamma(1+s\varpi)}{\Gamma(1+(s+1)\varpi)}\left(\left(2^{\varpi}-\left(\frac{1}{2}\right)^{s\varpi}\right)|p(n)|^{\psi}+\left(\frac{1}{2}\right)^{s\varpi}|p(m)|^{\psi}\right)\right]^{\frac{1}{\psi}}\right\}.\end{aligned}$$

(c) If $h(\lambda)=1^{\varpi}$ in Theorem 2.5, then

$$\begin{aligned}&\left|\frac{1^{\varpi}}{2^{\varpi}}\left[\frac{\Pr_{\varpi}(X\le m)+\Pr_{\varpi}(X\le n)}{2^{\varpi}}+\Pr_{\varpi}\left(X\le\frac{m+n}{2}\right)\right]-\frac{n^{\varpi}-E^{\varpi}(X)}{(n-m)^{\varpi}}\right|\\&\le\frac{(n-m)^{\varpi}}{4^{\varpi}}\left(\frac{\Gamma(1+\theta\varpi)}{\Gamma(1+(\theta+1)\varpi)}\right)^{\frac{1}{\theta}}\left[\frac{1}{\Gamma(1+\varpi)}(|p(m)|^{\psi}+|p(n)|^{\psi})\right]^{\frac{1}{\psi}}.\end{aligned}$$

(d) If $h(\lambda)=\lambda^{\varpi}(1-\lambda)^{\varpi}$ in Theorem 2.5, then

$$\begin{aligned}&\left|\frac{1^{\varpi}}{2^{\varpi}}\left[\frac{\Pr_{\varpi}(X\le m)+\Pr_{\varpi}(X\le n)}{2^{\varpi}}+\Pr_{\varpi}\left(X\le\frac{m+n}{2}\right)\right]-\frac{n^{\varpi}-E^{\varpi}(X)}{(n-m)^{\varpi}}\right|\\&\le\frac{(n-m)^{\varpi}}{8^{\varpi}}\left(\frac{\Gamma(1+\theta\varpi)}{\Gamma(1+(\theta+1)\varpi)}\right)^{\frac{1}{\theta}}\left[\left(\frac{\Gamma(1+\varpi)}{\Gamma(1+2\varpi)}-\frac{\Gamma(1+2\varpi)}{2^{\varpi}\Gamma(1+3\varpi)}\right)(|p(m)|^{\psi}+|p(n)|^{\psi})\right]^{\frac{1}{\psi}}.\end{aligned}$$

4.3. **Numerical integration.** Let $I_N: m=\kappa_0<\kappa_1<\cdots<\kappa_{N-1}=n$ be a partition with regard to the interval $[m,n]$. We take into account the numerical integration formula arise from local fraction integrals ${}_m\mathscr{I}_n^{(\varpi)}\zeta(\kappa)=K(\zeta,I_N)+E(\zeta,I_N)$, where $K(\zeta,I_N)$ is approximation formula of the integral term ${}_m\mathscr{I}_n^{(\varpi)}\zeta(\kappa)$ and $E(\zeta,I_N)$ is the approximation error. We have the following error bounds.

Proposition 4.3. *Let all the hypotheses in Theorem 2.3 hold. Then, for every division I_N of $[m,n]$, the approximation error $E(\zeta, I_N)$ satisfies that*

$$\begin{aligned}|E(\zeta,I_N)| &= |{}_m\mathscr{I}_n^{(\varpi)}\zeta(\kappa) - K(\zeta,I_N)| \\ &\leq \frac{(Z-z)^{\varpi}}{8^{\varpi}\Gamma(1+2\varpi)}\sum_{i=0}^{N-1}(\kappa_{i+1}-\kappa_i)^{2\varpi},\end{aligned}$$

where $K(\zeta,I_N) := \frac{1^{\varpi}}{2^{\varpi}\Gamma(1+\varpi)}\sum_{i=0}^{N-1}(\kappa_{i+1}-\kappa_i)^{\varpi}\left[\frac{\zeta(\kappa_i)+\zeta(\kappa_{i+1})}{2^{\varpi}} + \zeta\left(\frac{\kappa_i+\kappa_{i+1}}{2}\right)\right]$.

Proof. On $[\kappa_i, \kappa_{i+1}]$, we find from Theorem 2.3 that

$$\begin{aligned}&\left|{}_{\kappa_i}\mathscr{I}_{\kappa_{i+1}}^{(\varpi)}\zeta(\kappa) - \frac{(\kappa_{i+1}-\kappa_i)^{\varpi}}{2^{\varpi}\Gamma(1+\varpi)}\left[\frac{\zeta(\kappa_i)+\zeta(\kappa_{i+1})}{2^{\varpi}} + \zeta\left(\frac{\kappa_i+\kappa_{i+1}}{2}\right)\right]\right| \\ &\leq \frac{(\kappa_{i+1}-\kappa_i)^{2\varpi}(Z-z)^{\varpi}}{8^{\varpi}\Gamma(1+2\varpi)},\end{aligned}$$

for each $i = 0,\cdots,N-1$. Summing over i from 0 to $N-1$, we obtain the error bound. This completes the proof. □

5. Conclusion

This paper studied some Bullen-type inequalities on fractal spaces. The results presented in this paper extended and generalized some known Bullen-type inequalities. Furthermore, the derived results were utilized to investigate some special cases. Some applications, such as ϖ-type special means, numerical integration, and generalized probability distribution mappings, were provided to support our new inequalities.

References

[1] S. Abramovich, L.E. Persson, Fejér and Hermite-Hadamard type inequalities for N-quasiconvex functions, Math. Notes 102 (2017) 599-609.

[2] A.M. Acu, H. Gonska, On Bullen's and related inequalities, General Math. 22 (2014) 19-26.

[3] O. Almutairi, A. Kiliçman, Generalized Fejér–Hermite–Hadamard type via generalized $(h-m)$-convexity on fractal sets and applications, Chaos, Solitons and Fractals 147 (2021) 110938.

[4] O. Almutairi, A. Kılıçman, Integral inequalities for s-convexity via generalized fractional integrals on fractal sets, Mathematics, 8 (2020) 53.

[5] H. Budak, T. Tunç, M.Z. Sarikaya, Fractional Hermite–Hadamard-type inequalities for interval-valued functions, Proc. Amer. Math. Soc. 148 (2020) 705-718.

[6] H. Budak, M.Z. Sarikaya, H. Yildirim, New inequalities for local fractional integrals, Iran. J. Sci. Technol. Trans. Sci. 41 (2017) 1039-1046.

[7] M. Çakmak, Refinements of Bullen-type inequalities for s-convex functions via Riemann–Liouville fractional integrals involving Gauss hypergeometric function, J. Interdiscip. Math. 22 (2019) 975-989.

[8] M. Çakmak, M. Tunç, A. Acem, Some new inequalities for differentiable h-convex functions and applications, Miskolc Math. Notes 22 (2021) 107-121.

[9] J. Choi, E. Set, M. Tomar, Certain generalized Ostrowski type inequalities for local fractional integrals, Commun. Korean Math. Soc. 32 (2017) 601-617.

[10] M.R. Delavar, M.De La Sen, A mapping associated to h-convex version of the Hermite–Hadamard inequality with applications, J. Math. Inequal. 14 (2020) 329-335.

[11] T.S. Du, H. Wang, M.A. Khan, Y. Zhang, Certain integral inequalities considering generalized m-convexity on fractal sets and their applications, Fractals, 27 (2019) 1950117.

[12] T.S. Du, M.U. Awan, A. Kashuri, S.S. Zhao, Some k-fractional extensions of the trapezium inequalities through generalized relative semi-(m,h)-preinvexity, Appl. Anal. 100 (2021) 642-662.

[13] S. Erden, M.Z. Sarikaya, Generalized Pompeiu type inequalities for local fractional integrals and its applications, Appl. Math. Comput. 274 (2016) 282-291.

[14] S. Iftikhar, S. Erden, P. Kumam, M.U. Awan, Local fractional Newton's inequalities involving generalized harmonic convex functions, Adv. Difference Equ. 2020 (2020) 185.

[15] M. Iqbal, M.I. Bhatti, K. Nazeer, Generalization of inequalities analogous to Hermite–Hadamard inequality via fractional integrals, Bull. Korean Math. Soc. 52 (2015) 707-716.

[16] P. Kórus, An extension of the Hermite–Hadamard inequality for convex and s-convex functions, Aequat. Math. 93 (2019) 527-534.

[17] A. Kiliçman, W. Saleh, Some generalized Hermite–Hadamard type integral inequalities for generalized s-convex functions on fractal sets, Adv. Difference Equ. 2015 (2015) 301.

[18] M. Krnić, P. Vuković, Multidimensional Hilbert-type inequalities obtained via local fractional calculus, Acta Appl. Math. 169 (2020) 667-680.

[19] M.A. Latif, On some new inequalities of Hermite–Hadamard type for functions whose derivatives are s-convex in the second sense in the absolute value, Ukrainian Math. J. 67 (2016) 1552-1571.

[20] J.G. Liao, S.H. Wu, T.S. Du, The Sugeno integral with respect to α-preinvex functions, Fuzzy Sets and Systems, 379 (2020) 102-114.

[21] C.Y. Luo, Y.P. Yu, T.S. Du, An improvement of Hölder integral inequality on fractal sets and some related Simpson-like inequalites, Fractals, 29 (2021), 1-20.

[22] M. Matić, J. Pečarić, A. Vukelić, On generalization of Bullen-Simpson's inequality, Rocky Mountain J. Math. 35 (2005) 1727-1754.

[23] M. Matłoka, Inequalities for h-preinvex functions, Appl. Math. Comput. 234 (2014) 52-57.

[24] D. Ş. Marinescu, M. Monea, A very short proof of the Hermite–Hadamard inequalities, Amer. Math. Monthly 127 (2020) 850-851.

[25] F.C. Mitroi-Symeonidis, N. Minculete, M. Niezgoda, Estimates on the gap in Bullen's inequality, Math. Inequal. Appl 22 (2019) 1493-1503.

[26] M.Z. Sarikaya, H. Budak, Generalized Ostrowski type inequalities for local fractional integrals, Proc. Amer. Math. Soc. 145 (2017) 1527-1538.

[27] K. Sayevand, Mittag-Leffler string stability of singularly perturbed stochastic systems within local fractal space, Math. Model. Anal. 24 (2019) 311-334.

[28] E. Set, S.I. Butt, A.O. Akdemir, A. Karaoğlan, T. Abdeljawad, New integral inequalities for differentiable convex functions via Atangana–Baleanu fractional integral operators, Chaos, Solitons and Fractals 143 (2021) 110554.

[29] W.B. Sun, Local fractional Ostrowski-type inequalities involving generalized h-convex functions and some applications for generalized moments, Fractals 29 (2021) 2150006.

[30] K.L. Tseng, S.R. Hwang, S.S. Dragomir, Fejér-type inequalities (I), J. Inequal. Appl. 2010 (2010) 531976.

[31] K.L. Tseng, S.R. Hwang, K.C. Hsu, Hadamard-type and Bullen-type inequalities for Lipschitzian functions and their applications, Comput. Math. Appl. 64 (2012) 651-660.

[32] S. Varošanec, On h-convexity, J. Math. Anal. Appl. 326 (2007) 303-311.

[33] M. Vivas, J. Hernández, N. Merentes, New Hermite-Hadamard and jensen type inequalities for h-convex functions on fractal sets, Rev. Colombiana Mat. 50 (2016) 145-164.

[34] X.J. Yang, Advanced Local Fractional Calculus and its Applications, World Science Publisher, New York, 2012.