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NEW MODIFIED HYBRID ALGORITHM FOR PSEUDO-CONTRACTIVE MAPPINGS IN HILBERT SPACES

KANNIKA KHOMPHURNGSON, NILOBOL KAMYUN, KAMONRAT NAMMANEE*

Department of Mathematics, School of Science, University of Phayao, Phayao, Thailand

Abstract. In this study, we construct a modified hybrid algorithm and prove a convergence theorem for pseudo-contractive mappings in Hilbert spaces. Our results improve and generalize some recent related results in the literature.

Keywords. Algorithm performance; Fixed point; Hybrid algorithm; Pseudo-contractive mapping; Strong convergence.

1. Introduction

Let C be a closed, convex, and nonempty subset of a Hilbert space H, and let $F : C \times C \to \mathbb{R}$ be bi-function. Recall that the equilibrium problem is to

$$\text{find } x^* \in C \text{ such that } F(x^*, y) \geq 0,\ \forall y \in C. \tag{1.1}$$

The set of solutions of (1.1) is denoted by $EP(F)$, that is,

$$EP(F) = \{x \in C : F(x,y) \geq 0, \forall y \in C\}.$$

The equilibrium problem, which includes variational inequality problems, saddle problems, and complementarity problems, is an important nonlinear problem and finds numerous applications in the real world. In the past decade, various solution methods were investigated for the solutions of the equilibrium problem; see, e.g., [5, 9, 10, 18, 19] and the references therein.

Let T be a self-mapping on C, and let $F(T)$ denote the fixed point set of T, that is, $F(T) := \{x \in C : Tx = x\}$. Recall that a mapping $T : C \to C$ is said to be a L-Lipschitzian mapping if there exists a positive constant L such that $\|Tx - Ty\| \leq L\|x - y\|$ for all $x, y \in C$. If $L = 1$, then $T : C \to C$ is said to be nonexpansive. Iterative methods for finding the fixed points of nonexpansive mappings are an important topic in the theory of nonexpansive mappings and have wide applications in a number of applied areas, such as the convex feasibility problem

*Corresponding author.

E-mail address: kamonrat.na@up.ac.th (K. Nammanee), nilobol.ka@up.ac.th (N. Kamyun), kannika.kh@up.ac.th (K. Khomphurngson).

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[1, 3, 14], the split feasibility problem [4, 6] and image recovery and signal processing [7, 8, 24] and so on.

In 2009, Lewicki and Marino [12] investigated a new hybrid method for the families of nonexpansive mappings in Hilbert spaces. Subsequently, various hybrid methods were introduced and studied; see, e.g., [15, 23, 25, 26] and the references therein.

Recall that a mapping $T : C \to C$ is said to be a strict pseudo-contraction [2] if there exists a constant $0 \le k < 1$ such that

$$\|Tx - Ty\|^2 \le \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in C.$$

If $k = 1$, then T is said to be a pseudo-contraction, i.e,

$$\|Tx - Ty\|^2 \le \|x - y\|^2 + \|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in C,$$

which is equivalent to $\langle Tx - Ty, x - y\rangle \le \|x - y\|^2$, $\forall x, y \in C$.

The class of strict pseudo-contractions extend the class of nonexpansive mappings. Note that, T is nonexpansive if and only if T is a 0-strict pseudo-contraction. The pseudo-contractive mapping includes the strict pseudo-contractive mapping. Recently, iterative methods for nonexpansive mappings have been extensively investigated; see, e.g., [16, 20, 21]. However iterative methods for the class of strictly pseudo-contractive mappings are far less developed than those for the class of nonexpansive mappings since Browder and Petryshyn introduced it in 1967. On the other hand, strictly pseudo-contractive mappings have more powerful applications than nonexpansive mappings do in solving inverse problems; see Scherzer [22]. Therefore it is interesting to develop iterative methods for strictly pseudo-contractive mappings.

In 2009, Yao, Liou and Marino [25] introduced the following hybrid iterative algorithm for pseudo-contractive mapping in Hilbert spaces as follows:

Let C be a nonempty, closed, and convex subset of a real Hilbert space H. Let $T : C \to C$ be a L-Lipschitz pseudo-contractive mapping. Assume that $\{\alpha_n\}$ is a sequence in $(0,1)$. Let $x_0 \in H$. For $C_1 = C$ and $x_1 = P_{C_1}x_0$, define a sequence $\{x_n\}$ of C as follow:

$$\langle 1\rangle \begin{cases} y_n = (1-\alpha_n)x_n + \alpha_n Tx_n, \\ C_{n+1} = \{z \in C_n : \|\alpha_n(I-T)y_n\|^2 \le 2\alpha_n\langle x_n - z, (I-T)y_n\rangle\}, \\ x_{n+1} = P_{C_{n+1}}x_0, n \ge 1. \end{cases} \tag{1.2}$$

They proved that the sequence generated above converges strongly to $P_{F(T)}x_0$ provided that $F(T) \ne \emptyset$ and $\{\alpha_n\}$ lies in $[a,b]$ for some $a, b \in (0, \frac{1}{1+L})$.

In 2011, Tang et al. [23] generalized the hybrid algorithm above to the Ishikawa iterative process as the following.

Let C be closed and convex subset of a real Hilbert space H. Let $T : C \to C$ be a L-Lipschitz pseudo-contractive mapping. Assume that $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$. Let $x_0 \in H$. For $C_1 = C$ and $x_1 = P_{C_1}x_0$, define a sequence $\{x_n\}$ of C as follow:

$$\langle 2\rangle \begin{cases} y_n = (1-\alpha_n)x_n + \alpha_n Tz_n, \\ z_n = (1-\beta_n)x_n + \beta_n Tx_n, \\ C_{n+1} = \{z \in C_n : \|\alpha_n(I-T)y_n\|^2 \le 2\alpha_n\langle x_n - z, (I-T)y_n\rangle \\ \qquad\qquad +2\alpha_n\beta_n L\|x_n - Tx_n\|\|y_n - x_n + \alpha_n(I-T)y_n\|\}, \\ x_{n+1} = P_{C_{n+1}}x_0, n \ge 1. \end{cases} \tag{1.3}$$

They proved that the sequence generated above converges strongly to $P_{F(T)}x_0$ provided that $F(T) \neq \emptyset$, $\{\alpha_n\}$ and $\{\beta_n\}$ are sequence in $(0,1)$ such that (i) $b \leq \alpha_n < \alpha_n(L+1)(1+\beta_n L) < a < 1$ for some $a,b \in (0,1)$ and (ii) $\lim_{n\to\infty} \beta_n = 0$.

In 2000, Noor [17] introduced the following three-step iteration (known as the Noor iteration)

$$\begin{cases} z_n = (1-\gamma_n)x_n + \gamma_n T x_n, \\ y_n = (1-\beta_n)x_n + \beta_n T z_n, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n, \quad n \geq 1, \end{cases}$$

where $\{\alpha_n\}, \{\beta_n\}$, and $\{\gamma_n\}$ are sequence in $[0,1]$. If $\gamma_n = 0$ for all n, then the Noor iteration reduce to the Ishikawa Iteration [11]

$$\begin{cases} y_n = (1-\beta_n)x_n + \beta_n T x_n \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n, \quad n \geq 1, \end{cases}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequence in $[0,1]$. If $\gamma_n = \beta_n = 0$ for all n, then the Noor iteration reduce to Mann Iteration [13]

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n, \quad n \geq 1,$$

where $\{\alpha_n\}$ is a sequence in $[0,1]$.

In this paper, we generalize the hybrid algorithm (1.3) to the modified Noor iterative process, which is defined as following.

Let C be a nonempty, close, and d convex subset of a real Hilbert space H. Let $T : C \to C$ be a pseudo-contractive mapping. Assume that $\{\alpha_n\}, \{\beta_n\}$, and $\{\gamma_n\}$ are sequences in $[0,1]$. Let $x_0 \in H$. For $C_1 = C$ and $x_1 = P_{C_1}x_0$ define a sequence $\{x_n\}$ of C by

$$\begin{cases} z_n = (1-\gamma_n)x_n + \gamma_n T x_n, \\ y_n = (1-\beta_n)x_n + \beta_n T z_n, \\ w_n = (1-\alpha_n)x_n + \alpha_n T y_n, \\ C_{n+1} = \{z \in C_n : \|\alpha_n (I-T)w_n\|^2 \leq 2\alpha_n \langle x_n - z, (I-T)w_n \rangle \\ \qquad\qquad + (2\alpha_n\beta_n L)(1+\gamma_n L)\|x_n - Tx_n\| \|w_n - x_n\| + \alpha_n (I-T)w_n\|, \\ x_{n+1} = P_{C_{n+1}}x_0, n \geq 1. \end{cases} \tag{1.4}$$

We prove the strong convergence of the hybrid algorithm in Hilbert spaces.

2. Preliminaries

In this section, we collect some useful results, which are used in the following section. Let H be a real Hilbert space with inner product $\langle .,. \rangle$ and norm $||.||$, and let C be a closed and convex subset of H. Recall that the nearest point projection P_C from H onto C assigns each $x \in H$ to its nearest point in C, denoted by $P_C x$, that is, $\|x - P_C x\| \leq \|x - y\|$ for all $y \in C$.

The following notations are used in this paper.

(i) $\rightharpoonup$ for weak convergence and $\to$ for strong convergence;

(ii) $\omega_\omega(x_n) = \{x : \exists x_{n_j} \rightharpoonup x\}$ denotes the weak $\omega - limit$ set of $\{x_n\}$.

The following Lemmas are well known.

Lemma 2.1. *Let C be a closed and convex subset of a real Hilbert space H. Given $x \in H$ and $z \in C$, $z = P_C x$ if and only if there hold the relation $\langle x - z, y - z \rangle \leq 0$ for all $y \in C$.*

Lemma 2.2. *Let H be a real Hilbert space. Then, for all $x,y \in H$, $\|x-y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x-y, y\rangle$.*

Lemma 2.3. [26] *Let H be a real Hilbert space, C a closed and convex subset of H, and $T : C \to C$ a continuous pseudo-contractive mapping. Then (i) $F(T)$ closed convex subset of C, and (ii) $I-T$ is demiclosed at zero, i.e., if $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup z$ and $(I-T)x_n \to 0$, then $(I-T)z = 0$.*

Lemma 2.4. [15] *Let C be a closed and convex subset of a real Hilbert space H. Let $\{x_n\}$ be a sequence in H and $u \in H$. Let $q = P_C u$. If $\{x_n\}$ is such that $\omega_\omega(x_n) \subseteq C$ and satisfies the condition: $\|x_n - u\| \le \|u-q\|$ for all $n \ge 1$, then $\{x_n\}$ converges strongly to q.*

3. Main Results

We prove the following strong convergence theorem for modified hybrid algorithms (1.4) for pseudo-contractive mappings in Hilbert spaces.

Theorem 3.1. *Let C be a nonempty, closed, and convex subset of a real Hilbert space H. Let $T : C \to C$ be a L-Lipschitz pseudo-contractive mapping with $F(T) \neq \emptyset$. Assume that $\{\alpha_n\}, \{\beta_n\}$,and $\{\gamma_n\}$ are sequences in $(0,1)$ satisfying:*
(i) $b \le \alpha_n \le \alpha_n(1+L)(1+\beta_n L + \beta_n\gamma_n L^2) < a < 1$ for some $a,b \in (0,1)$;
(ii) $\lim_{n\to\infty} \beta_n = 0$.
Then the sequence $\{x_n\}$ generated by (1.4) converges strongly to $P_{F(T)}x_0$.

Proof. By Lemma 2.3 (i), we have that $F(T)$ is closed and convex. Then $P_{F(T)}$ is well defined. It is easy to see that C_n is closed and convex.

Next, we show that $F(T) \subseteq C_n$ for all n. Let $p \in F(T)$. From Lemma 2.2 and $\langle (I-T)x - (I-T)y, x-y\rangle \ge 0$ for all $x,y \in C$, we obtain that

$$\begin{aligned}
&\|x_n - p - \alpha_n(I-T)w_n\|^2 \\
&= \|x_n-p\|^2 - \|\alpha_n(I-T)w_n\|^2 - 2\alpha_n\langle (I-T)w_n, x_n - w_n - \alpha_n(I-T)w_n\rangle \\
&\quad - 2\alpha_n\langle (I-T)w_n - (I-T)p, w_n - p\rangle \\
&\le \|x_n-p\|^2 - \|\alpha_n(I-T)w_n\|^2 - 2\alpha_n\langle (I-T)w_n, x_n - w_n - \alpha_n(I-T)w_n\rangle \\
&= \|x_n-p\|^2 - \|x_n - w_n\|^2 - \|w_n - x_n + \alpha_n(I-T)w_n\|^2 \\
&\quad - 2\langle x_n - w_n, w_n - x_n + \alpha_n(I-T)w_n\rangle \\
&\quad + 2\alpha_n\langle (I-T)w_n, w_n - x_n + \alpha_n(I-T)w_n\rangle \\
&\le \|x_n-p\|^2 - \|x_n - w_n\|^2 - \|w_n - x_n + \alpha_n(I-T)w_n\|^2 \\
&\quad + 2|\langle x_n - w_n - \alpha_n(I-T)w_n, w_n - x_n + \alpha(I-T)w_n\rangle|.
\end{aligned} \tag{3.1}$$

Considering the last item of (3.1) yields that

$$
\begin{aligned}
&|\langle x_n - w_n - \alpha_n(I-T)w_n, w_n - x_n + \alpha_n(I-T)w_n\rangle| \\
&\le \alpha_n|\langle (I-T)x_n - (I-T)w_n, w_n - x_n + \alpha_n(I-T)w_n\rangle| \\
&\quad + \alpha_n|\langle Tx_n - Ty_n, w_n - x_n + \alpha_n(I-T)w_n\rangle| \\
&\le \alpha_n\|(I-T)x_n - (I-T)w_n\|\|w_n - x_n + \alpha_n(I-T)w_n\| \\
&\quad + \alpha_n\|Tx_n - Ty_n\|\|w_n - x_n + \alpha_n(I-T)w_n\| \\
&\le \alpha_n(L+1)\|x_n - w_n\|\|w_n - x_n + \alpha_n(I-T)w_n\| \\
&\quad + \alpha_n L\|x_n - y_n\|\|w_n - x_n + \alpha_n(I-T)w_n\| \\
&\le \alpha_n(L+1)\|x_n - w_n\|\|w_n - x_n + \alpha_n(I-T)w_n\| \\
&\quad + \alpha_n\beta_n L[\|x_n - Tx_n\| + \|Tx_n - Tz_n\|]\|w_n - x_n + \alpha_n(I-T)w_n\| \\
&\le \alpha_n(L+1)\|x_n - w_n\|\|w_n - x_n + \alpha_n(I-T)w_n\| \\
&\quad + \alpha_n\beta_n L\|x_n - Tx_n\|\|w_n - x_n + \alpha_n(I-T)w_n\| \\
&\quad + \alpha_n\beta_n L^2\|x_n - z_n\|\|w_n - x_n + \alpha_n(I-T)w_n\| \\
&\le \frac{\alpha_n(L+1)}{2}(\|x_n - w_n\|^2 + \|w_n - x_n + \alpha_n(I-T)w_n\|^2) \\
&\quad + (\alpha_n\beta_n L)(1+\gamma_n L)\|x_n - Tx_n\|\|w_n - x_n + \alpha_n(I-T)w_n\|.
\end{aligned} \tag{3.2}
$$

Substituting (3.2) into (3.1), we obtain

$$
\begin{aligned}
\|x_n - p - \alpha_n(I-T)w_n\|^2 \le \|x_n - p\|^2 &- \|x_n - w_n\|^2 - \|w_n - x_n + \alpha_n(I-T)w_n\|^2 \\
&+ \alpha_n(L+1)(\|x_n - w_n\|^2 + \|w_n - x_n + \alpha_n(I-T)w_n\|^2) \\
&+ (2\alpha_n\beta_n L)(1+\gamma_n L)\|x_n - Tx_n\|\|w_n - x_n + \alpha_n(I-T)w_n\|.
\end{aligned} \tag{3.3}
$$

In view of $\|x_n - p - \alpha_n(I-T)w_n\|^2 = \|x_n - p\|^2 - 2\alpha_n\langle x_n - p, (I-T)w_n\rangle + \|\alpha_n(I-T)w_n\|^2$, we find from (3.3) that

$$
\begin{aligned}
\|\alpha_n(I-T)w_n\|^2 \le 2\alpha_n\langle x_n - p, (I-T)w_n\rangle &+ 2\alpha_n\beta_n L\|x_n - Tx_n\|\|w_n - x_n + \alpha_n(I-T)w_n\| \\
&+ 2\alpha_n\beta_n\gamma_n L^2\|x_n - Tx_n\|\|w_n - x_n + \alpha_n(I-T)w_n\|,
\end{aligned}
$$

i.e., $p \in C_{n+1}$ if $p \in C_n$. By induction, we have s that $F(T) \subseteq C_n$ for all n. From the definition of $\{x_n\}$, $x_n = P_{C_n}x_0$. This implies that $\|x_n - x_0\| \le \|z - x_0\|$ for all $z \in C_n$. Since $F(T) \subseteq C_n$, we have $\|x_n - x_0\| \le \|p - x_0\|$ for any $p \in F(T)$. In particular,

$$
\|x_n - x_0\| \le \|q - x_0\|, \tag{3.4}
$$

where $q = P_{F(T)}x_0$. Hence $\{x_n\}$ is bounded. Since T is L-Lipschitz continuous, then $\{w_n\}$, $\{Tx_n\}$, $\{Tw_n\}$, $\{z_n\}$, and $\{Tz_n\}$ are all bounded. From $x_n = P_{C_n}x_0$ and $x_{n+1} = P_{C_{n+1}}x_0 \in C_{n+1} \subseteq C_n$, we have $\langle x_n - x_0, x_{n+1} - x_n\rangle \ge 0$. Lemma 2.2 implies that

$$
\begin{aligned}
\|x_{n+1} - x_n\|^2 &= \|x_{n+1} - x_0 - (x_n - x_0)\|^2 \\
&= \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2 - 2\langle x_{n+1} - x_n, x_n - x_0\rangle \\
&\le \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2,
\end{aligned} \tag{3.5}
$$

which yields that $\|x_n - x_0\| \le \|x_{n+1} - x_0\|$ for all n. Then $\{\|x_n - x_0\|\}$ is a nondecreasing sequence, and hence $\{\|x_n - x_0\|\}$ is also bounded. Thus $\lim_{n\to\infty}\|x_n - x_0\|$ exists. At the same time, letting $n \to \infty$ in the right side of inequality (3.5), we have $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$. Since $x_{n+1} \in C_{n+1} \subseteq C_n$, we have

$$\begin{aligned}
\|\alpha_n(I-T)w_n\|^2 &\leq 2\alpha_n\langle x_n - x_{n+1}, (I-T)w_n\rangle \\
&\quad + (2\alpha_n\beta_n L)(1+\gamma_n L)\|x_n - Tx_n\|\|w_n - x_n + \alpha_n(I-T)w_n\| \\
&\leq 2\alpha_n\|x_n - x_{n+1}\|\|(I-T)w_n\| \\
&\quad + (2\alpha_n\beta_n L)(1+\gamma_n L)\|x_n - Tx_n\|\|w_n - x_n + \alpha_n(I-T)w_n\| \to 0
\end{aligned}$$

as $n \to \infty$, and

$$\begin{aligned}
&\|x_n - Tx_n\| \\
&\leq \|x_n - w_n\| + \|w_n - Tw_n\| + \|Tw_n - Tx_n\| \\
&= \alpha_n(L+1)\|x_n - Ty_n\| + \|w_n - Tw_n\| \\
&\leq \alpha_n(L+1)[\|x_n - Tx_n\| + \|Tx_n - Ty_n\|] + \|w_n - Tw_n\| \\
&\leq \alpha_n(L+1)\|x_n - Tx_n\| + \alpha_n(L+1)L\|x_n - y_n\| + \|w_n - Tw_n\| \\
&= \alpha_n(L+1)\|x_n - Tx_n\| + \alpha_n(L+1)L\beta_n\|x_n - Tz_n\| + \|w_n - Tw_n\| \\
&\leq \alpha_n(L+1)\|x_n - Tx_n\| + \alpha_n(L+1)L\beta_n[\|x_n - Tx_n\| + \|Tx_n - Tz_n\|] + \|w_n - Tw_n\| \\
&\leq \alpha_n(L+1)(1+\beta_n L + \beta_n\gamma_n L^2)\|x_n - Tx_n\| + \|w_n - Tw_n\|.
\end{aligned}$$

Since $0 < b \leq \alpha_n(1+L)(1+\beta_n L+\beta_n\gamma_n L^2) < a < 1$, we have $\|x_n - Tx_n\| \leq \frac{1}{1-a}\|w_n - Tw_n\| \to 0$ as $n \to \infty$. By Lemma 2.3 (ii), $I - T$ is demiclosed at zero. Since $\{x_n\}$ is bounded, we see that every weak limit point of $\{x_n\}$ is a fixed point of T. That is, $\omega_n(x_n) \subseteq F(T)$. Therefore, by inequality (3.4) and Lemma 2.4, we know that $\{x_n\}$ converges strongly to $q = P_{F(T)}x_0$. This completes the proof. □

If $\gamma_n = 0$ for all n in (1.4), then hybrid algorithm (1.4) reduces to hybrid algorithm (1.3). Moreover, if $\gamma_n = 0$ and $\beta_n = 0$ for all n in (1.4), then hybrid algorithm (1.4) reduces to hybrid algorithm (1.2). Thus [25, Theorem 3.1] and [23, Theorem 3.1] are a spacial case of Theorem 3.1.

As direct consequence of Theorem 3.1, we obtain the following.

Corollary 3.2. *Let C be a nonempty, closed, and convex subset of a real Hilbert space H. Let $T : C \to C$ be a nonexpansive mapping such that $F(T) \neq \emptyset$. Assume that $\{\alpha_n\}, \{\beta_n\}$, and $\{\gamma_n\}$ are sequences in $(0,1)$ satisfying:*
(i) $b \leq \alpha_n \leq 2\alpha_n(1+\beta_n+\beta_n\gamma_n) < a < 1$ for some $a, b \in (0,1)$;
(ii) $\lim_{n\to\infty}\beta_n = 0$.

Let $x_0 \in H$. For $C_1 = C$ and $x_1 = P_{C_1}x_0$. Then the sequence $\{x_n\}$ generated by

$$\begin{cases}
w_n = (1-\alpha_n)x_n + \alpha_n Ty_n, \\
y_n = (1-\beta_n)x_n + \beta_n Tz_n, \\
z_n = (1-\gamma_n)x_n + \gamma_n Tx_n, \\
C_{n+1} = \{z \in C_n : \|\alpha_n(I-T)w_n\|^2 \leq 2\alpha_n\langle x_n - z, (I-T)w_n\rangle \\
\qquad\qquad + (2\alpha_n\beta_n)(1+\gamma_n)\|x_n - Tx_n\|\|w_n - x_n + \alpha_n(I-T)w_n\|, \\
x_{n+1} = P_{C_{n+1}}x_0, n \geq 1,
\end{cases}$$

converges strongly to $P_{F(T)}x_0$.

Recall that a mapping A is said to be monotone if $\langle Ax - Ay, x - y\rangle \geq 0$ for all $x, y \in H$. The pseudo-contractive mapping is related to the monotone mapping. It is well known that A is monotone mapping if and only if $(I - A)$ is pseudo-contractive mapping. Hence, the fixed points of pseudo-contractive mapping actually is the zero of monotone e mapping. Due to Theorem 3.1, we have the following corollaries which generalize the corresponding results of Tang et al. [23].

Corollary 3.3. *Let $A : H \to H$ be a L-Lipschitz monotone mapping with $A^{-1}(0) \neq \emptyset$. Assume that $\{\alpha_n\}, \{\beta_n\}$, and $\{\gamma_n\}$ are sequences in $(0,1)$ satisfying:*
(i) $b \leq \alpha_n \leq \alpha_n(1+L)(1+\beta_n L+\beta_n\gamma_n L^2) < a < 1$ for some $a, b \in (0,1)$;
(ii) $\lim_{n\to\infty}\beta_n = 0$.

Let $x_0 \in H$. For $C_1 = C$ and $x_1 = P_{C_1}x_0$. Then the sequence $\{x_n\}$ generated by

$$\begin{cases} w_n = x_n + \alpha_n A y_n, \\ y_n = x_n + \beta_n A z_n, \\ z_n = x_n + \gamma_n A x_n, \\ C_{n+1} = \{z \in C_n : \|\alpha_n A w_n\|^2 \leq 2\alpha_n\langle x_n - z, A w_n\rangle \\ \qquad\qquad +(2\alpha_n\beta_n L)(1+\gamma_n L)\|Ax_n\|\|w_n - x_n + \alpha_n A w_n\| \\ x_{n+1} = P_{C_{n+1}}x_0, n \geq 1, \end{cases}$$

converges strongly to $P_{A^{-1}(0)}x_0$.

4. Numerical Examples

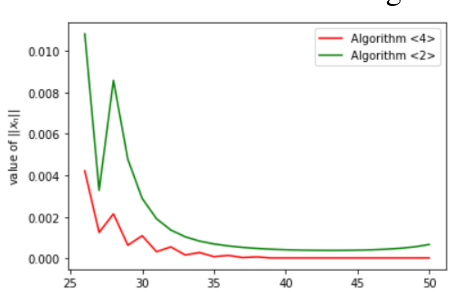
In this section, we give the following example which verifies theorem 3.1.

Example 4.1. Let $H = (\mathbb{R}^2, \langle .,.\rangle)$, where $\mathbb{R}$ is the set of real numbers and $\langle .,.\rangle$ is defined by the dot product on H. Let $x = (x_1, x_2) \in H$, and define $x^\perp = (x_2, -x_1)$. Let $C = \{x \in H : \|x\| = \sqrt{{x_1}^2 + {x_2}^2} \leq 5\}$, and let $T : C \to C$ be defined by $Tx = x + x^\perp$. Then C is a nonempty, closed, and convex subset of a real Hilbert H, and T is a Lipschitz (with $L = \sqrt{2}$) pseudo-contractive and $F(T) = \{(0,0)\}$.

Find some control condition on $\{\alpha_n\}, \{\beta_n\}$, and $\{\gamma_n\}$ such that $\{\alpha_n\} \subseteq (0, \frac{1}{1+L}) = (0, \frac{1}{1+\sqrt{2}})$ (see [25]), $\beta_n \to 0$ as $n \to \infty$ (see [23] and our results), $\gamma_n \in [0,1]$ and given an initial point $x_0 = (0,6)$. We numerically demonstrate the convergence of our algorithm and compare its behaviour with the hybrid algorithms in the sense of Ishikawa and Mann iterations. We present the numerical examples into two cases as following.

Case I: The comparative behaviour of our algorithm and algorithm (1.3) under some control condition as $\alpha_n = 0.25$, $\beta_n = \frac{1}{n+85}$, and $\gamma_n = 0.99$ for all n.

A Python program leads to the evaluation illustrated in Figure 1 and Table 1.



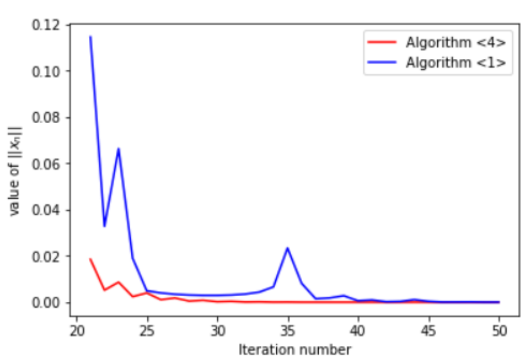
x_n	By Algorithm $< 4 >$	By Algorithm $< 2 >$
1	(0.0000,0.5000)	(0.0000,0.5000)
2	(1.1771,4.8595)	(1.1719,4.8607)
3	(1.7882,4.6693)	(1.7917,4.6680)
4	(2.3333,4.3504)	(2.3446,4.3445)
5	(2.5477,3.6648)	(2.5600,3.6476)
6	(2.6154,2.9800)	(2.6234,2.9515)
7	(2.5269,2.3140)	(2.5257,2.2760)
8	(2.3023,1.7040)	(2.2885,1.6601)
9	(1.9692,1.1782)	(1.9408,1.1328)
10	(1.5638,0.7555)	(1.5213,0.7134)
11	(1.1305,0.4440)	(1.0775,0.4093)
12	(0.7236,0.2386)	(0.6681,0.2138)
13	(0.4077,0.1204)	(0.3641,0.1052)
14	(0.2250,0.0598)	(0.1985,0.0513)
15	(0.0633,0.0287)	(0.0279,0.0243)
16	(0.1163,0.0293)	(0.0385,0.0243)
17	(0.0327,0.0143)	(0.0575,0.0244)
18	(0.0599,0.0143)	(0.1115,0.0246)
19	(0.0169,0.0071)	(0.0324,0.0124)
20	(0.0308,0.0069)	(0.0624,0.0115)
21	(0.0087,0.0036)	(0.0182,0.0063)
22	(0.0158,0.0034)	(0.0348,0.0052)
23	(0.0045,0.0018)	(0.0102,0.0031)
24	(0.0081,0.0016)	(0.0194,0.0022)
25	(0.0023,0.0009)	(0.0057,0.0015)
26	(0.0041,0.0008)	(0.0108,0.0009)
27	(0.0012,0.0004)	(0.0032,0.0007)
28	(0.0021,0.0004)	(-0.0086,0.0005)
29	(0.0006,0.0002)	(-0.0048,0.0005)
30	(0.0011,0.0002)	(-0.0029,0.0004)
31	(0.0003,0.0001)	(-0.0019,0.0004)
32	(0.0005,0.0001)	(-0.0013,0.0004)
33	(0.0002,0.0001)	(-0.0010,0.0004)
34	(0.0003,0.0000)	(-0.0007,0.0004)
35	(0.0001,0.0000)	(-0.0006,0.0004)
36	(0.0001,0.0000)	(-0.0005,0.0004)
37	(0.0000,0.0000)	(-0.0004,0.0004)
38	(0.0000,0.0000)	(-0.0003,0.0004)
⋮		
50		(0.0006,0.0004)

TABLE 1. Comparative results, Case I

We see that our algorithm performs better than the hybrid algorithm in the sense of Ishikawa.

Case II: The comparative behaviour of our algorithm and algorithm (1.2) under some control condition as $\alpha_n = 0.4$, $\beta_n = \frac{1}{n+14}$, and $\gamma_n = 0.01$ for all n.

A Python program leads to the evaluation illustrated in Figure 2 and Table 2.



x_n	By Algorithm $< 4 >$	By Algorithm $< 1 >$
1	(0.0000,0.5000)	(0.0000,0.5000)
2	(1.9151,4.6187)	(1.7843,4.6708)
3	(2.3054,3.6056)	(2.4173,3.7687)
4	(2.3084,2.5709)	(2.4315,2.5938)
5	(1.9973,1.6604)	(2.0666,1.5779)
6	(1.4881,0.9618)	(1.4419,0.8226)
7	(0.9364,0.5028)	(0.7815,0.3699)
8	(0.5128,0.2478)	(0.3601,0.1573)
9	(0.2783,0.1203)	(0.1860,0.0672)
10	(0.0366,0.0568)	(-0.0707,0.0333)
11	(0.0614,0.0569)	(-0.0469,0.0325)
12	(0.1128,0.0572)	(-0.0318,0.0319)
13	(0.4186,0.0527)	(-0.0211,0.0316)
14	(0.1612,0.0345)	(-0.0127,0.0313)
15	(0.0277,0.0251)	(-0.0054,0.0310)
16	(0.0428,0.0223)	(0.0015,0.0308)
17	(0.0796,0.0153)	(0.0087,0.0306)
18	(0.0209,0.0104)	(0.0171,0.0303)
19	(0.0384,0.0068)	(0.0286,0.0299)
20	(0.0100,0.0048)	(0.0485,0.0292)
21	(0.0183,0.0030)	(0.1113,0.0271)
22	(0.0047,0.0022)	(0.0298,0.0138)
23	(0.0086,0.0013)	(0.0658,0.0087)
24	(0.0022,0.0010)	(0.0180,0.0061)
25	(0.0040,0.0006)	(-0.0038,0.0032)
26	(0.0010,0.0005)	(-0.0026,0.0031)
27	(0.0018,0.0003)	(-0.0016,0.0031)
28	(0.0005,0.0002)	(-0.0009,0.0030)
29	(0.0008,0.0001)	(-0.0002,0.0030)
30	(0.0002,0.0001)	(0.0005,0.0029)
31	(0.0004,0.0001)	(0.0012,0.0029)
32	(0.0001,0.0000)	(0.0021,0.0029)
33	(0.0002,0.0000)	(0.0034,0.0028)
34	(0.0000,0.0000)	(0.0061,0.0026)
35	(0.0000,0.0000)	(0.0234,0.0017)
⋮		
49		(0.0000,0.0000)

TABLE 2. Comparative results, Case II

From the tables and figures, it can be seen that the sequence $\{x_n\}$ defined by our algorithm converges faster than the sequence generated by the hybrid algorithm in the sense of Ishikawa and Mann iterations.

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