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A NEW SUPERCONVERGENCE OF FINITE ELEMENTS FOR BILINEAR PARABOLIC OPTIMAL CONTROL PROBLEMS

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Abstract. In this paper, we consider a fully discrete finite element approximation of bilinear parabolic optimal control problems with an integral constraint. First, we give an approximation scheme of the model problem, where triangular finite element and backward Euler methods are used. Second, we introduce some useful intermediate variables, interpolation operators and related error estimates. Third, we derive a new superconvergence between the numerical solutions and projection functions of exact solutions. Last, a numerical example is provided to verify our results.

Keywords. Bilinear parabolic optimal control problems; Finite element method; Superconvergence.

1. Introduction

There are a variety of research on finite element methods (FEMs) for optimal control problems (OCPs), and most of them focused on elliptic problems [15]. The superconvergence properties of FEMs for linear and semi-linear elliptic OCPs were obtained in [17] and [2], respectively, and for FEMs of bilinear elliptic OCPs [25]. Some superconvergence results of mixed FEMs for elliptic OCPs can be found in [3, 4]. In recent years, there has been considerable related research for FEMs of parabolic OCPs; see, e.g., [5, 21]. For linear and semi-linear parabolic OCPs, a priori error estimate of space-time finite element discretization was derived in [18, 19], and the superconvergence properties of semi-discrete and fully discrete FEMs were obtained in [8, 10], and [22, 23], respectively. For bilinear parabolic OCPs, some convergence and superconvergence results of FEMs and mixed FEMs can be found in [6, 16, 20, 24].

To the best of our knowledge if the control u is approximated by piecewise constant functions while the state y and co-state p are approximated by piecewise linear functions in [2, 22], then the superconvergence result between the numerical solution u_h and projection function $Q_h u$ of exact solution u is $\|Q_h u - u_h\| = \mathscr{O}(h^{\frac{3}{2}})$. It have been improved to $\|u_I - u_h\| = \mathscr{O}(h^2)$ by introducing the element centroid interpolation function u_I in [17] or $\|u - u_h\| = \mathscr{O}(h^2)$ by using

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variational discretization conception in [9]. In this paper, we investigate a fully discrete FEMs for bilinear parabolic OCPs and improve the superconvergence to $|||Q_h u - u_h||| = \mathscr{O}(h^2 + k)$.

We focus on the following bilinear parabolic OCPs:

$$\min_{u\in K} \frac{1}{2}\int_0^T \left(\|y(t,x)-y_d(t,x)\|^2 + \|u(t,x)\|^2\right) dt, \tag{1.1}$$
$$y_t(t,x) - \mathrm{div}(A(x)\nabla y(t,x)) + u(t,x)y(t,x) = f(t,x), \qquad t\in J, x\in\Omega, \tag{1.2}$$
$$y(t,x) = 0, \qquad t\in J, x\in\partial\Omega, \tag{1.3}$$
$$y(0,x) = y_0(x), \qquad x\in\Omega, \tag{1.4}$$

where $\Omega\in\mathbb{R}^2$ is a bounded convex polygon domain with the boundary $\partial\Omega$, $J=[0,T]\,(T>0)$. The coefficient $A(x) = (a_{ij}(x))_{2\times 2} \in (W^{1,\infty}(\bar{\Omega}))^{2\times 2}$ is a symmetric positive definite matrix. Moreover, we suppose that $f(t,x), y_d(t,x)\in C(J;L^2(\Omega))$, $y_0(x)\in H_0^1(\Omega)$, and the set K is defined by

$$K = \left\{ v(t,x)\in L^\infty(J;L^2(\Omega)) : \int_\Omega v(t,x)\,dx \geq 0, \forall t\in J \right\}.$$

In this paper, we adopt the standard notation $W^{m,q}(\Omega)$ for Sobolev spaces on Ω with norm $\|\cdot\|_{W^{m,q}(\Omega)}$ and semi-norm $|\cdot|_{W^{m,q}(\Omega)}$. We set $H_0^1(\Omega)\equiv\left\{v\in H^1(\Omega) : v|_{\partial\Omega}=0\right\}$ and denote $W^{m,2}(\Omega)$ by $H^m(\Omega)$. We denote by $L^s(J;W^{m,q}(\Omega))$ the Banach space of all L^s integrable functions from J into $W^{m,q}(\Omega)$ with norm $\|v\|_{L^s(J;W^{m,q}(\Omega))} = (\int_0^T \|v\|^s_{W^{m,q}(\Omega)} dt)^{\frac{1}{s}}$ for $s\in[1,\infty)$ and the standard modification for $s=\infty$. Similarly, we can define $H^l(J;W^{m,q}(\Omega))$ and $C^k(J;W^{m,q}(\Omega))$ (see e.g., [13]). In addition, c or C denotes a generic positive constant.

The organization of this article is as follows. In Section 2, we construct a fully discrete finite element approximation for (1.1)-(1.4), and introduce some useful intermediate variables and related error estimates in Section 3. In Section 4, we derive a new superconvergence of the fully discrete finite element approximation for bilinear parabolic OCPs, and a numerical example is provided in Section 5, which is also the last section.

2. Fully Discrete Finite Element Approximation

A fully discrete finite element approximation for problem (1.1)-(1.4) is presented in this section. First of all, we denote $L^p(J;W^{m,q}(\Omega))$ and $\|\cdot\|_{L^p(J;W^{m,q}(\Omega))}$ by $L^p(W^{m,q})$ and $\|\cdot\|_{L^p(W^{m,q})}$, respectively. Let $W = H_0^1(\Omega)$ and $U = L^2(\Omega)$. Moreover, we denote $\|\cdot\|_{H^m(\Omega)}$ and $\|\cdot\|_{L^2(\Omega)}$ by $\|\cdot\|_m$ and $\|\cdot\|$, respectively. Let

$$a(v,w) = \int_\Omega (A\nabla v)\cdot\nabla w, \qquad \forall v,\, w\in W,$$
$$(f_1,f_2) = \int_\Omega f_1\cdot f_2, \qquad \forall f_1,\, f_2\in U.$$

Since matrix A is symmetric and positive definite,

$$a(v,v)\geq c\|v\|_1^2, \quad |a(v,w)|\leq C\|v\|_1\|w\|_1, \qquad \forall v,w\in W.$$

We recast (1.1)-(1.4) as the following weak formulation:

$$J(u) = \min_{u\in K} \frac{1}{2}\int_0^T \left(\|y-y_d\|^2 + \|u\|^2\right) dt, \tag{2.1}$$

$$(y_t, w) + a(y,w) + (uy, w) = (f, w), \qquad \forall w \in W, t \in J, \tag{2.2}$$

$$y(x,0) = y_0(x), \qquad \forall x \in \Omega. \tag{2.3}$$

It follows from (see e.g., [14]) that problem (2.1)-(2.3) has at least one solution (y,u), and that if the pair $(y,u) \in \left(H^2(L^2)\cap L^2(H^1)\right)\times K$ is a solution to (2.1)-(2.3), then there is a co-state $p \in H^2(L^2)\cap L^2(H^1)$ such that the triplet (y,p,u) satisfies the following conditions:

$$(y_t, w) + a(y,w) + (uy, w) = (f, w), \qquad \forall w \in W, t \in J, \tag{2.4}$$

$$y(0,x) = y_0(x), \qquad \forall x \in \Omega, \tag{2.5}$$

$$-(p_t, q) + a(q,p) + (up, q) = (y-y_d, q), \qquad \forall q \in W, t \in J, \tag{2.6}$$

$$p(T,x) = 0, \qquad \forall x \in \Omega, \tag{2.7}$$

$$(u-yp, v-u) \geq 0, \qquad \forall v \in K, t \in J. \tag{2.8}$$

As in [16], it is easy to prove the following result.

Lemma 2.1. *Let (y,p,u) be the solution to (2.4)-(2.8). Then*

$$u = yp - \min(0, \overline{yp}), \qquad \forall t \in J, \tag{2.9}$$

where $\overline{yp} = \int_\Omega yp dx / \int_\Omega 1 dx$ denotes the integral average on Ω of yp.

Remark 2.2. It follows from the standard regularity argument of second order parabolic equations that $y, p \in L^2(H^2)\cap L^\infty(H_0^1)$. Thus, according to (2.9), we have $u \in L^2(H^2)\cap L^\infty(L^2)$.

Let $\mathscr{T}^h$ be regular triangulations of Ω such that $\bar{\Omega} = \bigcup_{\tau\in\mathscr{T}^h} \bar{\tau}$ and $h = \max_{\tau\in\mathscr{T}^h}\{h_\tau\}$, where h_τ is the diameter of the element τ. Furthermore, we set

$$U_h = \left\{ v_h \in L^2(\Omega) : v_h|_\tau = \text{constant}, \ \forall \tau \in \mathscr{T}^h \right\},$$

$$W_h = \left\{ v_h \in C(\bar{\Omega}) : v_h|_\tau \in \mathbb{P}_1, \ \forall \tau \in \mathscr{T}^h, w_h|_{\partial\Omega} = 0 \right\},$$

where $\mathbb{P}_1$ denotes the space of polynomials up to order 1, and

$$K_h = \left\{ v_h \in U_h : \int_\Omega v_h dx \geq 0 \right\}.$$

Let $N \in \mathbb{Z}^+$, $k = T/N$, and $t_n = nk$, $n = 0,1,\cdots,N$. Set $\varphi^n = \varphi(x,t_n)$ and

$$d_t\varphi^n = \frac{\varphi^n - \varphi^{n-1}}{k}, n = 1,2,\cdots,N.$$

Moreover, we define the discrete time-dependent norms when $1 \leq p < \infty$,

$$|||\varphi|||_{l^p(J;W^{m,q}(\Omega))} := \left(k\sum_{n=1-l}^{N-l} \|\varphi^n\|^p_{W^{m,q}(\Omega)}\right)^{\frac{1}{p}},$$

where $l=0$ for the control u and the state y and $l=1$ for the adjoint state p, with the standard modification for $p=\infty$. For convenience, we denote $|||\cdot|||_{l^s(J;W^{m,q}(\Omega))}$ by $|||\cdot|||_{l^s(W^{m,q})}$, and let

$$l_D^p(J;H^s(\Omega)) := \left\{ f : |||f|||_{l^p(H^s)} < \infty \right\}, \qquad 1 \le p \le \infty.$$

Then a possible fully discrete finite element approximation of (1.1)-(1.4) is as follows:

$$J_{hk}(u_h) = \min_{u_h^n \in K_h} \frac{1}{2} \sum_{n=1}^{N} k \left(\|y_h^n - y_d^n\|^2 + \|u_h^n\|^2 \right), \tag{2.10}$$

$$(d_t y_h^n, w_h) + a(y_h^n, w_h) + (u_h^n y_h^n, w_h) = (f^n, w_h), \qquad \forall w_h \in W_h, n = 1,2,\cdots,N, \tag{2.11}$$

$$y_h^0(x) = y_0^h(x), \qquad \forall x \in \Omega, \tag{2.12}$$

where $y_0^h(x) = R_h(y_0(x))$ and R_h will be specified later.

Bilinear parabolic OCPs (2.10)-(2.12) again has a solution $\left(y_h^n, u_h^n\right)$, $n=1,2,\cdots,N$, and that if $\left(y_h^n, u_h^n\right) \in W_h \times K_h$, $n=1,2,\cdots,N$, is a solution of (2.10)-(2.12), then there is a co-state $p_h^{n-1} \in W_h$, $n=1,2,\cdots,N$, such that the triplet $\left(y_h^n, p_h^{n-1}, u_h^n\right) \in W_h \times W_h \times K_h$, $n=1,2,\cdots,N$, satisfies the following optimality conditions:

$$(d_t y_h^n, w_h) + a(y_h^n, w_h) + (u_h^n y_h^n, w_h) = (f^n, w_h), \qquad \forall w_h \in W_h, \tag{2.13}$$

$$y_h^0(x) = y_0^h(x), \qquad \forall x \in \Omega, \tag{2.14}$$

$$-(d_t p_h^n, q_h) + a\left(q_h, p_h^{n-1}\right) + (u_h^n p_h^{n-1}, q_h) = (y_h^n - y_d^n, q_h), \qquad \forall q_h \in W_h, \tag{2.15}$$

$$p_h^N(x) = 0, \qquad \forall x \in \Omega, \tag{2.16}$$

$$\left(u_h^n - y_h^n p_h^{n-1}, v_h - u_h^n\right) \ge 0, \qquad \forall v_h \in K_h. \tag{2.17}$$

We introduce an element integral averaging operator π_h^c from U onto U_h such that

$$(\pi_h^c v)|_\tau = \frac{1}{|\tau|} \int_\tau v dx, \qquad \forall \tau \in \mathscr{T}^h, \tag{2.18}$$

where $|\tau|$ is the measure of τ. Thus inequality (2.17) is equivalent to

$$u_h^n = \pi_h^c\left(y_h^n p_h^{n-1}\right) - \min\left(0, \overline{y_h^n p_h^{n-1}}\right), \qquad n = 1,2,\cdots,N, \tag{2.19}$$

where $\overline{y_h^n p_h^{n-1}} = \frac{1}{|\Omega|} \int_\Omega y_h^n p_h^{n-1} dx$. The proof of (2.19) is just as in [7].

3. Error Estimates of Intermediate Variables

In this section, we introduce some useful intermediate variables and related error estimates. For any $v \in K$, let $y(v), p(v) \in H^1(L^2) \cap L^2(H^2)$ be the solution of the following equations:

$$(y_t(v), w) + a(y(v), w) + (vy(v), w) = (f, w), \qquad \forall w \in W, t \in J, \tag{3.1}$$

$$y(v)(0,x) = y_0(x), \qquad \forall x \in \Omega, \tag{3.2}$$

$$-(p_t(v), q) + a(q, p(v)) + (vp(v), q) = (y(v) - y_d, q), \qquad \forall q \in W, t \in J, \tag{3.3}$$

$$p(v)(T,x) = 0, \qquad \forall x \in \Omega, \tag{3.4}$$

and $y_h^n(v), p_h^n(v) \in W_h$ for $n = 1, 2 \cdots, N$ satisfy the following system:

$$(d_t y_h^n(v), w_h) + a(y_h^n(v), w_h) + (v^n y_h^n(v), w_h) = (f^n, w_h), \qquad \forall w_h \in W_h, \tag{3.5}$$

$$y_h^0(v) = y_0^h(x), \qquad \forall x \in \Omega, \tag{3.6}$$

$$-(d_t p_h^n(v), q_h) + a\left(q_h, p_h^{n-1}(v)\right) + (v^n p_h^{n-1}(v), q_h) = (y_h^n(v) - y_d^n, q_h), \qquad \forall q_h \in W_h, \tag{3.7}$$

$$p_h^N(v) = 0, \qquad \forall x \in \Omega. \tag{3.8}$$

Let u and u_h be the solutions of (2.4)-(2.8) and (2.13)-(2.17), respectively. It is clear that $(y, p) = (y(u), p(u))$ and $(y_h, p_h) = (y_h(u_h), p_h(u_h))$.

We introduce the standard L^2-orthogonal projection operator $Q_h : U \to U_h$, which satisfies, for any $\psi \in U$, $(Q_h\psi - \psi, v_h) = 0$, $\forall v_h \in U_h$, and the elliptic projection operator $R_h : W \to W_h$, which satisfies, for any $\phi \in W$, $a(R_h\phi - \phi, w_h) = 0$, $\forall w_h \in W_h$. Q_h and R_h have the following properties (see e.g., [2]):

$$\|Q_h\psi - \psi\|_{-s} \le Ch^{1+s}|\psi|_1, \qquad \forall \psi \in H^1(\Omega), s = 0, 1, \tag{3.9}$$

$$\|R_h\phi - \phi\|_s \le Ch^{2-s}\|\phi\|_2, \qquad \forall \phi \in H^2(\Omega), s = 0, 1. \tag{3.10}$$

As in [17], we define the interpolation function $u_I(t, x) \in U_h$ for any $t \in J$ such that $u_I(t, x) = u(t, S_i)$, $\forall x, S_i \in \tau_i, \tau_i \in \mathscr{T}^h$, where S_i is the centroid of the triangle τ_i. The following lemmas are very important for our superconvergence analysis.

Lemma 3.1. [17] *If $f \in H^2(\Omega)$, then*

$$\left|\int_{\tau_i} (f(x) - f(S_i))\, dx\right| \le Ch^2\sqrt{|\tau_i|}|f|_{H^2(\tau_i)},$$

and

$$\sum_{\tau_i \in \mathscr{T}^h} \left|\int_{\tau_i} (f(x) - f(S_i))\, dx\right| \le Ch^2\left(\sum_{\tau_i \in \mathscr{T}^h} |f|^2_{H^2(\tau_i)}\right)^{\frac{1}{2}}.$$

Lemma 3.2. *Let $(y_h(u), p_h(u))$ and $(y_h(Q_hu), p_h(Q_hu))$ be the discrete solutions of (3.5)-(3.8) with $v = u$ and $v = Q_hu$, respectively. If the exact control $u \in l_D^2(H^1)$, then*

$$|||y_h(Q_hu) - y_h(u)|||_{l^2(H^1)} + |||p_h(Q_hu) - p_h(u)|||_{l^2(H^1)} \le Ch^2|||u|||_{l^2(H^1)}.$$

Proof. Set $v = Q_hu$ and $v = u$ in (3.5), respectively. For $n = 1, 2, \cdots, N$, we obtain

$$\begin{aligned}&(d_t y_h^n(Q_hu) - d_t y_h^n(u), w_h) + a(y_h^n(Q_hu) - y_h^n(u), w_h) + (u^n(y_h^n(Q_hu) - y_h^n(u)), w_h)\\&= (y_h^n(Q_hu)(u^n - Q_hu^n), w_h), \qquad \forall w_h \in W_h.\end{aligned}$$

Observe that

$$\begin{aligned}&(d_t y_h^n(Q_hu) - d_t y_h^n(u), y_h^n(Q_hu) - y_h^n(u))\\&\ge \frac{1}{2k}\left(\|y_h^n(Q_hu) - y_h^n(u)\|^2 - \|y_h^{n-1}(Q_hu) - y_h^{n-1}(u)\|^2\right)\end{aligned} \tag{3.11}$$

and

$$\sum_{n=1}^N k\,(u^n(y_h^n(Q_hu) - y_h^n(u)), y_h^n(Q_hu) - y_h^n(u)) \ge \sum_{n=1}^N k\int_\Omega (y_h^n(Q_hu) - y_h^n(u))^2\, u^n dx \ge 0. \tag{3.12}$$

By choosing $w_h = y_h^n(Q_h u) - y_h^n(u)$ in (3.12), (3.9) and Young's inequality with ε, then multiplying both sides of (3.11) by $2k$ and summing n from 1 to N, we obtain

$$\|y_h^N(Q_h u) - y_h^N(u)\|^2 + c\sum_{n=1}^{N} k\|y_h^n(Q_h u) - y_h^n(u)\|_1^2$$
$$\leq C(\varepsilon)h^4 \sum_{n=1}^{N} k\|u^n\|_1^2 + \varepsilon \sum_{n=1}^{N} k\|y_h^n(Q_h u) - y_h^n(u)\|_1^2.$$

Letting ε be small enough, we derive

$$|||y_h(Q_h u) - y_h(u)|||_{l^2(H^1)} \leq Ch^2 |||u|||_{l^2(H^2)}. \tag{3.13}$$

On setting $v = Q_h u$ and $v = u$ in (3.7) respectively, we obtain

$$\begin{aligned}&\left(p_h^{n-1}(Q_h u)(u^n - Q_h u^n), q_h\right) + (y_h^n(Q_h u) - y_h^n(u), q_h)\\ &= -\left(d_t p_h^n(Q_h u) - d_t p_h^n(u), q_h\right) + a\left(q_h, p_h^{n-1}(Q_h u) - p_h^{n-1}(u)\right)\\ &\quad + \left(u^n(p_h^{n-1}(Q_h u) - p_h^{n-1}(u)), q_h\right), \qquad \forall q_h \in W_h, n = 1, 2\cdots, N.\end{aligned}$$

Similarly, we derive

$$|||p_h(Q_h u) - p_h(u)|||_{l^2(H^1)} \leq C|||y_h(Q_h u) - y_h(u)|||_{l^2(H^1)} + Ch^2|||u|||_{l^2(H^1)}. \tag{3.14}$$

From (3.13) and (3.14), we conclude the desired conclusion immediately. □

Lemma 3.3. *For any $v \in K \cap H^1(L^2)$, let $(y(v), p(v))$ and $(y_h(v), p_h(v))$ be the solutions to (3.1)-(3.4) and (3.5)-(3.8), respectively. Assume that $y(v), p(v) \in l_D^2(H^2) \cap H^1(H^2) \cap H^2(L^2)$ and $y_d \in H^1(L^2)$. Then*

$$|||R_h y(v) - y_h(v)|||_{l^2(H^1)} + |||R_h p(v) - p_h(v)|||_{l^2(H^1)} \leq C\left(h^2 + k\right). \tag{3.15}$$

Proof. From (3.1) and (3.5), for any $w_h \in W_h$ and $n = 1, 2\cdots, N$, we obtain

$$(y_t^n(v) - d_t y_h^n(v), w_h) + a\left(y^n(v) - y_h^n(v), w_h\right) + \left(v^n(y^n(v) - y_h^n(v)), w_h\right) = 0.$$

According to the definition of R_h, we have

$$\begin{aligned}&(dtR_h y^n(v) - d_t y_h^n(v), w_h) + a\left(R_h y^n(v) - y_h^n(v), w_h\right) + \left(v^n(R_h y^n(v) - y_h^n(v)), w_h\right)\\ &= \left(d_t R_h y^n(v) - d_t y^n(v) + d_t y^n(v) - y_t^n(v), w_h\right) + \left(v^n(R_h y^n(v) - y^n(v)), w_h\right).\end{aligned} \tag{3.16}$$

Note that

$$\begin{aligned}&(d_t R_h y^n(v) - d_t y^n(v), R_h y^n(v) - y_h^n(v))\\ &\leq \|d_t R_h y^n(v) - d_t y^n(v)\|\,\|R_h y^n(v) - y_h^n(v)\|\\ &\leq Ch^2\|d_t y^n(v)\|_2\,\|R_h y^n(v) - y_h^n(v)\|\\ &\leq Ch^2 k^{-1}\int_{t_{n-1}}^{t_n}\|y_t(v)\|_2\,dt\,\|R_h y^n(v) - y_h^n(v)\|\\ &\leq Ch^2 k^{-\frac{1}{2}}||y_t(v)||_{L^2(t_{n-1},t_n;H^2(\Omega))}\,\|R_h y^n(v) - y_h^n(v)\|,\end{aligned}$$

and

$$\left(v^n(R_h y^n(v) - y_h^n(v)), R_h y^n(v) - y_h^n(v)\right) = \int_\Omega \left(R_h y^n(v) - y_h^n(v)\right)^2 v^n dx \geq 0. \tag{3.17}$$

We also have

$$
\begin{aligned}
&(d_t y^n(v) - y^n_t(v), R_h y^n(v) - y^n_h(v)) \\
&= k^{-1}\left(y^n(v) - y^{n-1}(v) - k y^n_t(v), R_h y^n(v) - y^n_h(v)\right) \\
&\leq k^{-1}\left\|y^n(v) - y^{n-1}(v) - k y^n_t(v)\right\| \|R_h y^n(v) - y^n_h(v)\| \\
&= k^{-1}\left\|\int_{t_{n-1}}^{t_n} (t_{n-1} - s)(y_{tt}(v))(s)ds\right\| \|R_h y^n(v) - y^n_h(v)\| \\
&\leq C k^{\frac{1}{2}} ||y_{tt}(v)||_{L^2(t_{n-1},t_n;L^2(\Omega))} \|R_h y^n(v) - y^n_h(v)\|.
\end{aligned}
$$

Similar to Lemma 3.2, from (3.16), (3.17) and Young's inequality, we have

$$
\begin{aligned}
&\left\|R_h y^N(v) - y^N_h(v)\right\|^2 + c\sum_{n=1}^{N} k \|R_h y^n(v) - y^n_h(v)\|_1^2 \\
&\leq C(\varepsilon)\left(h^4||y_t(v)||^2_{L^2(H^2)} + k^2||y_{tt}(v)||^2_{L^2(L^2)} + h^4|||y(v)|||^2_{l^2(H^2)}\right) \qquad (3.18)\\
&\quad + \varepsilon\sum_{n=1}^{N} k \|R_h y^n(v) - y^n_h(v)\|^2.
\end{aligned}
$$

Let ε be small enough. Then

$$
|||R_h y(v) - y_h(v)|||_{l^2(H^1)} \leq C\left(h^2||y_t(v)||_{L^2(H^2)} + k||y_{tt}(v)||_{L^2(L^2)} + h^2|||y(v)|||_{l^2(H^2)}\right).
$$

From (3.3) and (3.7), we obtain, for any $q_h \in W^h$, $n = N, \cdots, 2, 1$,

$$
\begin{aligned}
&-\left(p^{n-1}_t(v) - d_t p^n_h(v), q_h\right) + a\left(q_h, p^{n-1}(v) - p^{n-1}_h(v)\right) + \left(v^{n-1}\left(p^{n-1}(v) - p^{n-1}_h(v)\right), q_h\right) \\
&= \left(y^{n-1}(v) - y^n_h(v) - y^{n-1}_d + y^n_d, q_h\right) + \left(p^{n-1}_h(v)(v^n - v^{n-1}), q_h\right).
\end{aligned}
$$

By using the definition of R_h, we derive

$$
\begin{aligned}
&-\left(d_t R_h p^n(v) - d_t p^n_h(v), q_h\right) + a\left(q_h, R_h p^{n-1}(v) - p^{n-1}_h(v)\right) + \left(v^{n-1}\left(p^{n-1}(v) - p^{n-1}_h(v)\right), q_h\right) \\
&= \left(-d_t R_h p^n(v) + p^{n-1}_t(v) + y^{n-1}(v) - y^n_h(v) + y^n_d - y^{n-1}_d, q_h\right) + \left(p^{n-1}_h(v)(v^n - v^{n-1}), q_h\right).
\end{aligned} \qquad (3.19)
$$

Similarly, we can prove that

$$
\begin{aligned}
|||R_h p(v) - p_h(v)|||_{l^2(H^1)} &\leq C h^2\left(||p_t(v)||_{L^2(H^2)} + |||y(v)|||_{l^2(H^2)}\right) + C|||R_h y(v) - y_h(v)|||_{l^2(H^1)} \\
&\quad + Ck\left(||p_{tt}(v)||_{L^2(L^2)} + ||y_t(v)||_{L^2(L^2)} + ||(y_d)_t||_{L^2(L^2)} + ||v_t||_{L^2(L^2)}\right).
\end{aligned} \qquad (3.20)
$$

From (3.18) and (3.19), we obtain (3.15) immediately. □

4. SUPERCONVERGENCE PROPERTIES

In this section, we derive the superconvergence properties between the approximation solutions and the projections of the exact solutions. We assume that there exist a neighborhood of u in K and a constant $c > 0$ such that, for any v in this neighborhood, the objective functional $J(\cdot)$ satisfies the convexity condition:

$$
c|||u - v|||^2_{l^2(L^2)} \leq \left(J'_{hk}(u) - J'_{hk}(v), u - v\right). \qquad (4.1)
$$

It was assumed in many studies on numerical methods of the problem; see, e.g., [1].

Theorem 4.1. *Let (y,p,u) and (y_h,p_h,u_h) be the solutions to (2.4)-(2.8) and (2.13)-(2.17), respectively. Assume that all the conditions in lemmas 3.2-3.3 are valid. If the exact control $u\in l_D^2(H^2)$, then*

$$|||Q_hu-u_h|||_{l^2(L^2)}\le C\left(h^2+k\right). \tag{4.2}$$

Proof. It follows from (2.8) that

$$(u^n-y^np^n,u_h^n-u^n)\ge 0,\quad n=1,2,\cdots,N. \tag{4.3}$$

By choosing $v_h=Q_hu$ in (2.17), we derive $(u_h^n-y_h^np_h^{n-1},Q_hu^n-u_h^n)\ge 0$, $n=1,2,\cdots,N$. Thus

$$\begin{aligned}(Q_hu^n-u_h^n,Q_hu^n-u_h^n)=&(u^n-u_h^n,Q_hu^n-u_h^n)\\ \le&(u^n-y_h^n(u_h)p_h^{n-1}(u_h),Q_hu^n-u_h^n).\end{aligned} \tag{4.4}$$

Moreover, from (4.1) and (4.3)-(4.4), we obtain

$$\begin{aligned}&c|||Q_hu-u_h|||_{l^2(L^2)}^2\\ &\le\left(J_{hk}'(Q_hu^n)-J_{hk}'(u_h^n),Q_hu^n-u_h^n\right)\\ &=k\sum_{n=1}^N\left(Q_hu^n-y_h^n(Q_hu)p_h^{n-1}(Q_hu)-u_h^n+y_h^n(u_h)p_h^{n-1}(u_h),Q_hu^n-u_h^n\right)\\ &\le k\sum_{n=1}^N\left(u^n-y_h^n(Q_hu)p_h^{n-1}(Q_hu),Q_hu^n-u_h^n\right)\\ &\le k\sum_{n=1}^N(u^n-y^np^n,Q_hu^n-u^n)+k\sum_{n=1}^N\left(y^np^n-y_h^n(u)p_h^{n-1}(u),Q_hu^n-u_h^n\right)\\ &\quad+k\sum_{n=1}^N\left(y_h^n(u)p_h^{n-1}(u)-y_h^n(Q_hu)p_h^{n-1}(Q_hu),Q_hu^n-u_h^n\right)\\ &:=I_1+I_2+I_3.\end{aligned} \tag{4.5}$$

Note that $u_I^n\in U_h, n=1,2,\cdots,N$. By using embedding theorem $\|v\|_{L^4(\Omega)}\le C\|v\|_{H^1(\Omega)}$ and Young's inequality with ε, we have

$$\begin{aligned}I_1=&k\sum_{n=1}^N(u^n-y^np^n,Q_hu^n-u^n)\\ =&k\sum_{n=1}^N(u^n-u_I^n,Q_hu^n-u^n)+k\sum_{n=1}^N(y_I^np_I^n-y^np^n,Q_hu^n-u^n)\\ \le&C(\varepsilon)h^4\left(|||u|||_{l^2(H^2)}^2+|||y|||_{l^2(H^2)}^2+|||p|||_{l^2(H^2)}^2\right)+3\varepsilon|||Q_hu-u_h|||_{l^2(L^2)}^2.\end{aligned} \tag{4.6}$$

Similarly, we have

$$
\begin{aligned}
I_2 =& k\sum_{n=1}^{N}\left(y^n p^n - y_h^n(u)p_h^{n-1}(u), Q_h u^n - u_h^n\right)\\
=& k\sum_{n=1}^{N}\left(y^n p^n - R_h y^n R_h p^{n-1} + R_h y^n R_h p^{n-1} - y_h^n(u)p_h^{n-1}(u), Q_h u^n - u_h^n\right)\\
\leq& C(\varepsilon)h^4\left(|||y|||_{l^2(H^1)}^2 + |||p|||_{l^2(H^1)}^2 + |||R_h y - y_h(u)|||_{l^2(H^1)}^2 + |||R_h p - p_h(u)|||_{l^2(H^1)}^2\right)\\
&+ 4\varepsilon|||Q_h u - u_h|||_{l^2(L^2)}^2,
\end{aligned}
\tag{4.7}
$$

and

$$
\begin{aligned}
I_3 =& k\sum_{n=1}^{N}\left(y_h^n(u)p_h^{n-1}(u) - y_h^n(Q_h u)p_h^{n-1}(Q_h u), Q_h u^n - u_h^n\right)\\
\leq& C(\varepsilon)\left(|||y_h(u) - y_h(Q_h u)|||_{l^2(H^1)}^2 + |||p_h(u) - p_h(Q_h u)|||_{l^2(H^1)}^2\right) + 2\varepsilon|||Q_h u - u_h|||_{l^2(L^2)}^2.
\end{aligned}
\tag{4.8}
$$

From Lemmas 3.1-3.3 and (4.5)-(4.8), we obtain (4.2) immediately. □

Theorem 4.2. *Let* (y,p,u) *and* (y_h,p_h,u_h) *be the solutions to (2.4)-(2.8) and (2.13)-(2.17), respectively. Assume that all the conditions in Theorem 4.1 are valid. Then*

$$
|||R_h y - y_h|||_{l^2(H^1)} + |||R_h p - p_h|||_{l^2(H^1)} \leq C\left(h^2 + k\right). \tag{4.9}
$$

Proof. From (2.4) and (2.13), for any $w_h \in W^h$ and $n = 1,2,\cdots,N$, we have

$$
(y_t^n - d_t y_h^n, w_h) + a(y^n - y_h^n, w_h) + (u^n(y^n - y_h^n), w_h) = (y_h^n(u_h^n - u^n), w_h). \tag{4.10}
$$

According to the definition of R_h, we have

$$
\begin{aligned}
&(d_t R_h y^n - d_t y_h^n, w_h) + a(R_h y^n - y_h^n, w_h) + (u^n(R_h y^n - y_h^n), w_h)\\
&= (d_t R_h y^n - d_t y^n + d_t y^n - y_t^n + u^n(R_h y^n - y^n) + y_h^n(u_h^n - Q_h u^n + Q_h u^n - u^n), w_h).
\end{aligned}
\tag{4.11}
$$

Note that

$$
\begin{aligned}
(u_h^n - Q_h u^n, R_h y^n - y_h^n) \leq& C\,\|Q_h u^n - u_h^n\|\,\|R_h y^n - y_h^n\|\\
\leq& C(\varepsilon)\,\|Q_h u^n - u_h^n\|^2 + \varepsilon\,\|R_h y^n - y_h^n\|^2,
\end{aligned}
\tag{4.12}
$$

and

$$
\begin{aligned}
(Q_h u^n - u^n, R_h y^n - y_h^n) \leq& C\,\|Q_h u^n - u^n\|_{-1}\,\|R_h y^n - y_h^n\|_1\\
\leq& C(\varepsilon)\,\|u^n - Q_h u^n\|_{-1}^2 + \varepsilon\,\|R_h y^n - y_h^n\|_1^2.
\end{aligned}
\tag{4.13}
$$

Similar to Lemma 3.3, according to (4.10)-(4.13), Lemma 3.1 and Theorem 4.1, we derive

$$
|||R_h y - y_h|||_{l^2(H^1)} \leq C\left(h^2\left(|||y|||_{l^2(H^2)} + |||u|||_{l^2(H^1)} + ||y_t||_{L^2(H^2)}\right) + k||y_{tt}||_{L^2(L^2)}\right). \tag{4.14}
$$

From (2.6) and (2.15), for any $q_h \in W^h$ and $n = N,\cdots,2,1$, we obtain

$$
\begin{aligned}
&-\left(p_t^{n-1} - d_t p_h^n, q_h\right) + a\left(q_h, p^{n-1} - p_h^{n-1}\right) + \left(u^{n-1}p^{n-1} - u_h^n p_h^{n-1}, q_h\right)\\
&= \left(y^{n-1} - y_h^n - y_d^{n-1} + y_d^n, q_h\right).
\end{aligned}
$$

By using the definition of R_h, we have

$$
\begin{aligned}
& -\left(d_t R_h p^n - d_t p_h^n, q_h\right) + a\left(q_h, R_h p^{n-1} - p_h^{n-1}\right) + \left(u^{n-1}\left(p^{n-1} - p_h^{n-1}\right), q_h\right) \\
& = \left(-d_t R_h p^n + p_t^n + p_h^{n-1}\left(u_h^n - u^{n-1}\right) + y^{n-1} - y_h^n - y_d^{n-1} + y_d^n, q_h\right).
\end{aligned}
$$

Similarly, we can prove that

$$
\begin{aligned}
& |||R_h p - p_h|||_{l^2(H^1)}^2 \\
& \leq C(\varepsilon)\left(|||R_h y - y_h|||_{l^2(H^1)}^2 + h^4|||y|||_{l^2(H^2)}^2 + h^4||p_t||_{L^2(J;H^2(\Omega))}^2\right) \\
& \quad + C(\varepsilon)k^2\left(|||p|||_{l^2(H^1)}^2 + ||p_{tt}||_{L^2(L^2)}^2 + ||y_t||_{L^2(L^2)}^2 + ||(y_d)_t||_{L^2(L^2)}^2\right).
\end{aligned}
\tag{4.15}
$$

From (4.14) and (4.15), we derive (4.9) immediately. □

5. Numerical Experiments

For a constrained optimization problem: $\min_{u\in K} J(u)$, where $J(u)$ is a convex functional on U, and K is a close convex subset of U, the iterative scheme reads ($n = 0,1,2,\cdots$):

$$
\begin{cases}
b(u_{n+\frac{1}{2}}, v) = b(u_n, v) - \rho_n\left(J'(u_n), v\right), \quad \forall v \in U, \\
u_{n+1} = P_K^b(u_{n+\frac{1}{2}}),
\end{cases}
\tag{5.1}
$$

where ρ_n is a step size of iteration, $b(\cdot,\cdot) = \int_0^T(\cdot,\cdot)$ is a symmetric positive definite bilinear form, and the projection operator P_K^b can be obtained like in [12].

By applying (5.1) to fully discretized problem (2.10)-(2.12), for an acceptable error Tol, we present the following algorithm in which we omitted the subscript h just for ease of exposition.

Algorithm 5.1. Projection Gradient Algorithm

Step 1. Initialize u_0.

Step 2. Solve the following equations:

$$
\begin{cases}
b(u_{n+\frac{1}{2}}, v) = b(u_n, v) - \rho_n \int_0^T\left(u_n - y_n p_n, v\right), \qquad u_{n+\frac{1}{2}}, u_n \in U_h, \forall v \in U_h, \\
\left(\frac{y_n^i - y_n^{i-1}}{k}, w\right) + a\left(y_n^i, w\right) + \left(u_n^i y_n^i, w\right) = \left(f^i, w\right), \qquad y_n^i, y_n^{i-1} \in W_h, \forall w \in W_h, \\
\left(\frac{p_n^{i-1} - p_n^i}{k}, q\right) + a\left(q, p_n^{i-1}\right) + \left(u_n^i p_n^{i-1}\right) = \left(y_n^i - y_d^i, q\right), \quad up_n^i, p_n^{i-1} \in W_h, \forall q \in W_h, \\
u_{n+1} = P_K^b(u_{n+\frac{1}{2}}).
\end{cases}
\tag{5.2}
$$

Step 3. Calculate the iterative error: $E_{n+1} = |||u_{n+1} - u_n|||_{l^2(L^2)}$.

Step 4. If $E_{n+1} \leq Tol$, stop; else go to Step 2.

The following numerical example was dealt numerically with AFEPack. It is freely available. The details can be found at [11]. Let $\Omega = [0,1] \times [0,1]$, $T = 1$, and $A(x)$ be the identity matrix. We solve the following bilinear parabolic OCPs:

$$
\begin{cases}
\min\limits_{u\in K} \frac{1}{2}\int_0^T\left(\|y(t,x) - y_d(t,x)\|^2 + \|u(t,x)\|^2\right)dt, \\
y_t(t,x) - \operatorname{div}(A(x)\nabla y(t,x)) + u(t,x)y(t,x) = f(t,x), \quad \text{in}\, \Omega \times (0,T], \\
y(t,x) = 0, \quad \text{on}\, \partial\Omega \times (0,T], \\
y(0,x) = y_0(x), \quad \text{in}\, \Omega.
\end{cases}
\tag{5.3}
$$

The discretization of control variable $u(t,x)$, state variable $y(t,x)$, and co-state variable $p(t,x)$ was described in Section 2. We denote $|||\cdot|||_{l^2(H^1)}$ and $|||\cdot|||_{l^2(L^2)}$ by $|||\cdot|||_1$ and $|||\cdot|||$, respectively. The convergence order rate:

$$Rate = \frac{log(e_{i+1}) - log(e_i)}{log(h_{i+1}) - log(h_i)},$$

where e_i and e_{i+1} denote errors when mesh size $h = h_i$ and $h = h_{i+1}$, respectively.

Example 5.1. The data are as follows:

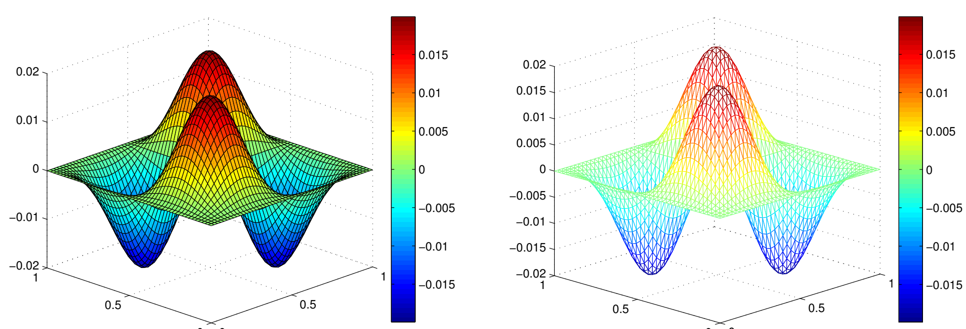
$$\begin{aligned}
y(t,x) &= \sin(\pi t)\sin(2\pi x_1)\sin(2\pi x_2),\\
p(t,x) &= (1-t)x_1(x_1-1)x_2(x_2-1),\\
u(t,x) &= y(t,x)p(t,x) - \min(0, \overline{y(t,x)p(t,x)}),\\
f(t,x) &= y_t(t,x) - \mathrm{div}(A(x)\nabla y(t,x)) + u(t,x)y(t,x),\\
y_d(t,x) &= y(t,x) + p_t(t,x) + \mathrm{div}(A^*(x)\nabla p(t,x)) - u(t,x)p(t,x).
\end{aligned}$$

The errors $|||Q_h u - u_h|||$, $|||R_h y - y_h|||_1$, and $|||R_h p - p_h|||_1$ on a sequence of uniformly meshes are shown in Table 1. When $t = 0.5$, we plot the profile of exact solution u and numerical solution u_h with $h = \frac{1}{40}$ and $k = \frac{1}{160}$ in Figure 1.

TABLE 1. Numerical results, Example 1.

h	k	$\vert\vert\vert Q_h u - u_h\vert\vert\vert$	Rate	$\vert\vert\vert R_h y - y_h\vert\vert\vert_1$	Rate	$\vert\vert\vert R_h p - p_h\vert\vert\vert_1$	Rate
$\frac{1}{10}$	$\frac{1}{10}$	6.20655e-03	—	4.07151e-02	—	4.07573e-03	—
$\frac{1}{20}$	$\frac{1}{40}$	1.61526e-03	1.94	1.05228e-02	1.95	1.031750e-03	1.98
$\frac{1}{40}$	$\frac{1}{160}$	4.10932e-04	1.97	2.64138e-03	1.99	2.581838e-04	2.00
$\frac{1}{80}$	$\frac{1}{640}$	1.04025e-04	1.98	6.56049e-04	2.01	6.431860e-05	2.01

FIGURE 1. Exact solution u (left) and numerical solution u_h (right).



From the numerical results in Example 5.1, we see that $|||Q_h u - u_h|||$, $|||R_h y - y_h|||_1$, and $|||R_h p - p_h|||_1$ are the second order convergent. Our numerical results and theoretical results are consistent.

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