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AN EXAMPLE OF A METRIC SPACE WITH FINITE DECOMPOSITION COMPLEXITY

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Abstract. We prove that a concrete metric space X_ξ whose transfinite asymptotic dimension is ξ has finite decomposition complexity.

Keywords. Asymptotic dimension; Asymptotic property C; Finite decomposition complexity; Transfinite asymptotic dimension.

1. Introduction

In coarse geometry, asymptotic dimension of a metric space is an important concept [5], which can be considered as an asymptotic analogue of the Lebesgue covering dimension. As a large scale analogue of Haver's property C, Dranishnikov introduced the notion of asymptotic property C in [3]. It is well known that every metric space with finite asymptotic dimension has asymptotic property C.

As another generalization of asymptotic dimension, Radul defined the transfinite asymptotic dimension (trasdim) and proved that, for a metric space X, X has asymptotic property C if and only if $\text{trasdim}(X) \leq \alpha$ for some ordinal number α in [8].

Guentner, Tessera, and Yu introduced the notion of finite decomposition complexity to study topological rigidity of manifolds in [6]. They proved that every metric space with finite asymptotic dimension has finite decomposition complexity in [7].

The relation between asymptotic property C and finite decomposition complexity was studied by Dranishnikov and Zarichnyi in [4]. Till now, there is no example of a metric space known which makes a difference between asymptotic property C and finite decomposition complexity. In [9], for every countable ordinal number ξ, we constructed a metric space X_ξ with $\text{trasdim}(X_\xi) = \xi$, which implies that X_ξ has asymptotic property C. In this paper, we prove that X_ξ has finite decomposition complexity.

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The paper is organized as follows: In Section 2, we recall some definitions and properties. In Section 3, which is also the last section, we prove the metric space X_ξ has finite decomposition complexity.

2. PRELIMINARIES

2.1. **Asymptotic dimension.** Let (X,d) be a metric space and $U,V \subseteq X$, let

$$\text{diam } U = \sup\{d(x,y) \mid x,y \in U\} \text{ and } d(U,V) = \inf\{d(x,y) \mid x \in U, y \in V\}.$$

Let $R > 0$ and $\mathscr{U}$ be a family of subsets of X. recall that $\mathscr{U}$ is said to be *R-bounded* if

$$\text{diam } \mathscr{U} = \sup\{\text{diam } U \mid U \in \mathscr{U}\} \leq R.$$

In this case, $\mathscr{U}$ is said to be *uniformly bounded*. Let $r > 0$. A family $\mathscr{U}$ is said to be *r-disjoint* if

$$d(U,V) > r \text{ for every } U,V \in \mathscr{U} \text{ with } U \neq V.$$

In this paper, we denote the union $\bigcup\{U \mid U \in \mathscr{U}\}$ by $\bigcup \mathscr{U}$, and denote the family $\{U \mid U \in \mathscr{U}_1 \text{ or } U \in \mathscr{U}_2\}$ by $\mathscr{U}_1 \cup \mathscr{U}_2$. Let A be a subset of X and $\varepsilon > 0$. We denote $\{x \in X \mid d(x,A) < \varepsilon\}$ by $N_\varepsilon(A)$. Let $\mathbb{N}$ be the set of all non-negative integers, and let $\mathbb{Z}^+$ be the set of all positive integers.

Definition 2.1. [5] A metric space X is said to have *finite asymptotic dimension* if there exists $n \in \mathbb{N}$ such that, for every $r > 0$, there exists a sequence of uniformly bounded families $\{\mathscr{U}_i\}_{i=0}^n$ of subsets of X such that the family $\bigcup_{i=0}^n \mathscr{U}_i$ covers X, and each $\mathscr{U}_i$ is r-disjoint for $i \in \{0,1,\cdots,n\}$. In this case, we say that the *asymptotic dimension* of X less than or equal to n, which is denoted by $\text{asdim}(X) \leq n$.

Lemma 2.2. [1] *Let X be a metric space with $X_1, X_2 \subseteq X$. Then*

$$asdim(X_1 \cup X_2) \leq max\{asdim(X_1), asdim(X_2)\}.$$

2.2. **Transfinite asymptotic dimension.** In 2010, Radul generalized asymptotic dimension of a metric space X to transfinite asymptotic dimension which is denoted by $\text{trasdim}(X)$; see [8].

Definition 2.3. [2] Let $Fin\mathbb{N}$ denote the collection of all finite, nonempty subsets of $\mathbb{N}$, and let $M \subseteq Fin\mathbb{N}$. For $\sigma \in \{\varnothing\} \cup Fin\mathbb{N}$, let

$$M^\sigma = \{\tau \in Fin\mathbb{N} \mid \tau \cup \sigma \in M \text{ and } \tau \cap \sigma = \varnothing\}.$$

Let M^a abbreviate $M^{\{a\}}$ for $a \in \mathbb{N}$. Define *the ordinal number* $\text{Ord}M$ inductively as follows:

$$\begin{aligned}
\text{Ord}M = 0 &\Leftrightarrow M = \varnothing, \\
\text{Ord}M \leq \alpha &\Leftrightarrow \forall\, a \in \mathbb{N},\ \text{Ord}M^a \leq \beta \text{ for some } \beta < \alpha, \\
\text{Ord}M = \alpha &\Leftrightarrow \text{Ord}M \leq \alpha \text{ and for any } \gamma < \alpha, \text{Ord}M \leq \gamma \text{ is not true}, \\
\text{Ord}M = \infty &\Leftrightarrow \text{for any ordinal number } \alpha, \text{Ord}M \leq \alpha \text{ is not true}.
\end{aligned}$$

Lemma 2.4. [2] *Let L and L' be sets. Let $FinL$ and $FinL'$ denote collections of all finite, nonempty subsets of L and L', respectively. Let $M \subseteq FinL$, $M' \subseteq FinL'$ and $\phi : L \to L'$ be a function such that, for every $\sigma \in M$, $\phi(\sigma) \in M'$ and $|\phi(\sigma)| = |\sigma|$. Then $OrdM \leq Ord\, M'$.*

Let $M \subseteq Fin\mathbb{N}$ and K be a infinite subset of $\mathbb{N}$. Then there is a standard bijection φ from $\mathbb{N}$ to K, which keeps the order. We define $M[K] = \left\{ \{\varphi(k_1), \varphi(k_2), ..., \varphi(k_m)\} \,\middle|\, \{k_1, k_2, ..., k_m\} \in M \right\}$.

By Lemma 2.4, we obtain the following result.

Corollary 2.5. *Let $M \subseteq Fin\mathbb{N}$ and K be a infinite subset of $\mathbb{N}$. Then $OrdM = OrdM[K]$.*

Definition 2.6. [8] Given a metric space X, define the following collection

$$A(X,d) = \big\{\sigma \in Fin\mathbb{N} \mid \text{ there are no uniformly bounded families } \mathscr{U}_i \text{ for } i \in \sigma \\ \text{such that each } \mathscr{U}_i \text{ is } i\text{-disjoint and } \bigcup_{i \in \sigma} \mathscr{U}_i \text{ covers } X\big\}.$$

The *transfinite asymptotic dimension* of X is defined as trasdim$X$ = Ord $A(X,d)$.

2.3. Finite decomposition complexity.

Definition 2.7. ([6, 7]) Let $\mathscr{X}$ and $\mathscr{Y}$ be metric families. A metric family $\mathscr{X}$ is *r-decomposable* over a metric family $\mathscr{Y}$ if every $X \in \mathscr{X}$ admits a decomposition

$$X = X_0 \cup X_1,\ X_i = \bigsqcup_{r\text{-disjoint}} X_{ij},$$

where each $X_{ij} \in \mathscr{Y}$. It is denoted by $\mathscr{X} \xrightarrow{r} \mathscr{Y}$.

Definition 2.8. ([6, 7])

(1) Let $\mathscr{D}_0$ be the collection of uniformly bounded families, i.e.,

$$\mathscr{D}_0 = \big\{ \mathscr{X} \mid \mathscr{X} \text{ is uniformly bounded } \big\}.$$

(2) Let α be an ordinal number greater than 0, and let $\mathscr{D}_\alpha$ be the collection of metric families decomposable over $\bigcup_{\beta<\alpha} \mathscr{D}_\beta$, i.e.,

$$\mathscr{D}_\alpha = \big\{ \mathscr{X} \mid \forall\, r > 0,\ \exists\, \beta < \alpha,\ \exists\, \mathscr{Y} \in \mathscr{D}_\beta,\ \text{ such that } \mathscr{X} \xrightarrow{r} \mathscr{Y} \big\}.$$

Remark 2.9.
- We view a metric space X as a singleton family $\{X\}$.
- $\mathscr{D}_\beta \subseteq \mathscr{D}_\alpha$ for every $\beta < \alpha$.
- It is known that X has finite asymptotic dimension if and only if X belongs to $\mathscr{D}_n$ for some $n \in \mathbb{N}$. Moreover, asdim$(X) \leq n$ implies $X \in \mathscr{D}_{n+1}$([6, 7]).

Lemma 2.10. *Let α and β be ordinal numbers and $\alpha \leq \beta$. Let X and Y be metric spaces such that $X \in \mathscr{D}_\alpha$ and $Y \in \mathscr{D}_\beta$. Then $X \cup Y \in \mathscr{D}_{\beta+1}$.*

Proof. Note that, for every $r > 0$, $X \cup Y \xrightarrow{r} \{X,Y\}$ and $\{X,Y\} \in \mathscr{D}_\beta$. Thus $X \cup Y \in \mathscr{D}_{\beta+1}$. This completes the proof. □

Definition 2.11. ([6, 7]) A metric family $\mathscr{X}$ has finite decomposition complexity if there exists a countable ordinal number α such that $\mathscr{X} \in \mathscr{D}_\alpha$.

A map of families $F : \mathscr{X} \to \mathscr{Y}$ from $\mathscr{X}$ to $\mathscr{Y}$ is a collection of functions $F = \{f\}$, each mapping some $X \in \mathscr{X}$ to some $Y \in \mathscr{Y}$ and such that every $X \in \mathscr{X}$ is the domain of at least one $f \in F$.

Definition 2.12. A map of metric space families $F:\mathscr{X}\to\mathscr{Y}$ is *uniformly expansive* if there exists a non-decreasing function $\theta:[0,\infty)\to[0,\infty)$ such that, for every $f\in F$ and every $x,y\in X_f$, $d(f(x),f(y))\leq\theta(d(x,y))$. A map of metric space families $F:\mathscr{X}\to\mathscr{Y}$ is *effectively proper* if there exists a proper non-decreasing function $\delta:[0,\infty)\to[0,\infty)$ such that, for every $f\in F$ and every $x,y\in X_f$, $d(f(x),f(y))\geq\delta(d(x,y))$. A map of metric space families $F:\mathscr{X}\to\mathscr{Y}$ is a *coarse embedding* if it is both uniformly expansive and effectively proper.

Lemma 2.13. *(Coarse invariance, [7]) Let $\mathscr{X}$ and $\mathscr{Y}$ be two metric families such that there is a coarse embedding $\phi:\mathscr{X}\to\mathscr{Y}$. If $\mathscr{Y}\in\mathscr{D}_\alpha$ for some countable ordinal number α, then $\mathscr{X}\in\mathscr{D}_\alpha$.*

2.4. **The metric space X_ξ.** For each $\tau=\{k_0,k_1,...,k_s\}\in Fin\mathbb{N}$, we choose an indexation such that $k_0<k_1<\cdots<k_s$. For every $n\in\mathbb{N}$, let

$$K(\tau,n)=\begin{cases}\{k_{n+1},k_{n+2}...,k_s\} & \text{if } n<s,\\ \varnothing & \text{if } n\geq s,\end{cases}$$

and let

$$i(\tau,n)=\begin{cases}\max\{k_0,k_1...,k_n\}=k_n & \text{if } n\leq s,\\ k_s & \text{if } n>s.\end{cases}$$

For each limit ordinal α, we fix an increasing sequence $\{\zeta_i(\alpha)+i\}$ of ordinals such that each $\zeta_i(\alpha)$ is a limit ordinal or 0, and

$$\alpha=\sup_i\{\zeta_i(\alpha)+i\}.$$

For each countable ordinal number ξ, we write $\xi=\gamma(\xi)+n(\xi)$, where $\gamma(\xi)$ is a limit ordinal or 0, and $n(\xi)\in\mathbb{N}$. We define a family $S_\xi\subseteq Fin\mathbb{N}$ by induction.

Definition 2.14. [9] Let $n\in\mathbb{N}$ and let ξ be a countable infinite ordinal number. Define

- $S_n=\left\{\sigma\in Fin\mathbb{N}\;\middle|\;|\sigma|\leq n\right\}$.
- $$\begin{aligned}S_\xi&=S_{\gamma(\xi)+n(\xi)}\\&=\left\{\sigma\in Fin\mathbb{N}\;\middle|\;K(\sigma,n(\xi))\in S_{\zeta_l(\gamma(\xi))+l}\cup\varnothing\ \text{ for some } l\in\{1,2,\cdots,i(\sigma,n(\xi))\}\right\}.\end{aligned}$$

Let $L=\{n+2\mid n\in\mathbb{N}\}$ and $S_\xi[L]=\left\{\{k_0+2,k_1+2,\cdots,k_s+2\}\;\middle|\;\{k_0,k_1,\cdots,k_s\}\in S_\xi\right\}$.

Definition 2.15. [9] For $\tau=\{k_0+2,k_1+2,...,k_m+2\}\in S_\xi[L]$, we define

$$X_\tau=\left\{(x_i)_{i=0}^m\in(2^{k_0}\mathbb{Z})^{m+1}\;\middle|\;|\{j\mid x_j\notin 2^{k_p}\mathbb{Z}\}|\leq p \text{ for every } p\in\{0,1,\cdots,m\}\right\}.$$

We consider X_τ with sup-metric.

Definition 2.16. [9] Let

$$\bigoplus\mathbb{Z}=\left\{(x_i)\mid x_i\in\mathbb{Z} \text{ and there exists } k\in\mathbb{N} \text{ such that } x_j=0 \text{ for each } j\geq k\right\}$$

with the sup-metric ρ. For every $\tau=\{k_0+2,k_1+2,...,k_m+2\}\in S_\xi[L]$, we define an isometric embedding

$$i_\tau:X_\tau\to\bigoplus\mathbb{Z}$$

by for every $x=(x_0,x_1,\cdots,x_m)\in X_\tau$,

$$i_\tau(x)_j=\begin{cases}x_j & \forall\, j\in\{0,1,\cdots,m\},\\ 0 & \forall\, j\notin\{0,1,\cdots,m\},\end{cases}$$

where $i_\tau(x)_j$ is the j-th coordinate of $i_\tau(x)$.

For every countable ordinal number ξ, we define X_ξ as the disjoint union of X_τ with $\tau\in S_\xi[L]$, i.e.,

$$X_\xi=\bigsqcup_{\tau\in S_\xi[L]}X_\tau$$

with the metric d_ξ which is defined as

$$d_\xi(x,y)=\begin{cases}\rho(i_\tau(x),i_\tau(y)) & \text{if } x,y\in X_\tau,\\ \max\{s(\tau_1),s(\tau_2),\rho(i_{\tau_1}(x),i_{\tau_2}(y))\} & \text{if } x\in X_{\tau_1},y\in X_{\tau_2}\text{ and }\tau_1\neq\tau_2,\end{cases}$$

where $s(\tau)=2^{\max\tau}$ for every $\tau\in S_\xi[L]$.

Lemma 2.17. [9] *Let $\xi=\gamma(\xi)+n(\xi)$ be a countable infinite ordinal number and $\tau=\{k_0,...,k_s\}\in S_\xi$. Then $\tau\in S_{\zeta_l(\gamma(\xi))+l+n(\xi)+1}$ for some $l\in\{1,2,\cdots,i(\tau,n(\xi))\}$.*

Lemma 2.18. [9] *$S_{\gamma+n}\subseteq S_{\gamma+m}$, where γ is a limit ordinal or 0 and $m,n\in\mathbb{N}$ such that $n<m$.*

Lemma 2.19. *$asdim(X_n)\le n$ for every $n\in\mathbb{N}$.*

Proof. For every $r\in\mathbb{N}$, let

$$Y_{0,r}=\bigsqcup_{\substack{\tau\in S_n[L]\\ s(\tau)\le r}}X_\tau,\ Y_{1,r}=\bigsqcup_{\substack{\tau\in S_n[L]\\ s(\tau)>r}}X_\tau.$$

Note that $X_\tau\subseteq\mathbb{R}^n$ for every $\tau\in S_n[L]$. Since $\operatorname{asdim}(\mathbb{R}^n)\le n$, there are r-disjoint, $B(r)$-bounded families $\mathscr{U}_0(\tau),\mathscr{U}_1(\tau),\cdots,\mathscr{U}_n(\tau)$ such that $\bigcup_{i=0}^n\mathscr{U}_i(\tau)$ covers X_τ. For $i\in\{0,1,2,\cdots,n\}$, let

$$\mathscr{U}_i=\Big\{U\mid U\in\mathscr{U}_i(\tau)\text{ for some }\tau\in S_n[L]\text{ and }s(\tau)>r\Big\}.$$

Then $\mathscr{U}_0,\mathscr{U}_1,\cdots,\mathscr{U}_n$ are r-disjoint, $B(r)$-bounded families such that $\bigcup_{i=0}^n\mathscr{U}_i$ covers $Y_{1,r}$.

Note that $\{\tau\mid\tau\in S_n[L],s(\tau)\le r\}$ is a finite set. By Lemma 2.2,

$$\operatorname{asdim}(Y_{0,r})\le\max\{\operatorname{asdim}(X_\tau)\mid\tau\in S_n[L],s(\tau)\le r\}\le\operatorname{asdim}(\mathbb{R}^n)\le n.$$

There are r-disjoint, uniformly bounded families $\mathscr{V}_0,\mathscr{V}_1,\cdots,\mathscr{V}_n$ such that $\bigcup_{i=0}^n\mathscr{V}_i$ covers $Y_{0,r}$. For $i\in\{0,1,2,\cdots,n\}$, let $\mathscr{W}_i=\mathscr{U}_i\cup\mathscr{V}_i$. Since $d(Y_{0,r},Y_{1,r})>r$, $\mathscr{W}_0,\mathscr{W}_1,\cdots,\mathscr{W}_n$ are r-disjoint, uniformly bounded families such that $\bigcup_{i=0}^n\mathscr{W}_i$ covers X_n, i.e., $\operatorname{asdim}(X_n)\le n$. □

3. Main Results

Now we prove that the metric space X_ξ has finite decomposition complexity. More specifically, $X_{\gamma(\xi)+n(\xi)}\in\mathscr{D}_{\gamma(\xi)+n(\xi)}$ for any countable infinite ordinal ξ with $\xi=\gamma(\xi)+n(\xi)$, where $\gamma(\xi)$ is a limit ordinal and $n(\xi)\in\mathbb{N}$.

Lemma 3.1.

$$X_n=\bigsqcup_{\tau\in S_n[L]}X_\tau\in\mathscr{D}_{n+1}.$$

Proof. By Lemma 2.19, we have asdim $(X_n) \leq n$, which implies that $X_n \in \mathscr{D}_{n+1}$. □

For every $n, i \in \mathbb{Z}^+$, let

$$Y_{i,n,1} = \left\{ (x_1, \ldots, x_i) \in \mathbb{Z}^i \;\middle|\; |\{ j \mid x_j \notin 2^n\mathbb{Z} \}| \leq 1 \right\},$$

which is considered as a subspace of the metric space $(\bigoplus \mathbb{Z}, \rho)$.

Lemma 3.2. *For every $r \in \mathbb{N}$ with $r \geq 4$, there exist $n = r \in \mathbb{N}$ and r-disjoint, 2^n-bounded families $\mathscr{U}_0, \mathscr{U}_1$ such that $\mathscr{U}_0 \cup \mathscr{U}_1$ covers $Y_{i,n,1}$.*

Proof. For every $r \in \mathbb{N}$ and $r \geq 4$, choose $n = r \in \mathbb{N}$. Let

$$\mathscr{U}_0 = \left\{ \Big(\prod_{t=1}^{i} (n_t 2^n - r, n_t 2^n + r) \Big) \cap Y_{i,n,1} \;\middle|\; n_t \in \mathbb{Z} \right\},$$

$$\mathscr{U}_1 = \Big\{ \Big(\prod_{t=1}^{j-1} (n_t 2^n - r, n_t 2^n + r) \times [n_j 2^n + r, (n_j+1)2^n - r] \times \prod_{t=j+1}^{i} (n_t 2^n - r, n_t 2^n + r) \Big) \cap Y_{i,n,1} \;\Big|\; n_t \in \mathbb{Z}, j \in \{1, 2, \cdots, i\} \Big\}$$

It is easy to see that $\mathscr{U}_0$ and $\mathscr{U}_1$ are r-disjoint and 2^n-bounded families.

Now for every $x = (x_1, \ldots, x_i) \in Y_{i,n,1} \setminus (\bigcup \mathscr{U}_0)$, there exists unique $j \in \{1, 2, \cdots, i\}$ such that

$$x_j \in [n_j 2^n + r, (n_j+1)2^n - r].$$

It follows that $x \in \bigcup \mathscr{U}_1$. Therefore, $\mathscr{U}_0 \cup \mathscr{U}_1$ covers $Y_{i,n,1}$. □

Lemma 3.3.

$$X_{\omega+1} = \bigsqcup_{\tau \in S_{\omega+1}[L]} X_\tau \in \mathscr{D}_\omega.$$

Moreover, $X_\omega \in \mathscr{D}_\omega$.

Proof. For every $r \in \mathbb{N}$, let

$$Y_{0,r} = \bigsqcup_{\substack{\tau \in S_{\omega+1}[L] \\ i(\tau,1) \leq r+2}} X_\tau, \; Y_{1,r} = \bigsqcup_{\substack{\tau \in S_{\omega+1}[L] \\ i(\tau,1) > r+2}} X_\tau.$$

For $\tau = \{k_0 + 2, k_1 + 2, \cdots, k_m + 2\} \in S_{\omega+1}[L]$ and $i(\tau, 1) \leq r + 2$, let $\widetilde{\tau} = \{k_0, k_1, \cdots, k_m\}$. Then

$$\widetilde{\tau} \in S_{\omega+1} \text{ and } i(\widetilde{\tau}, 1) \leq r.$$

Note that $\zeta_l(\omega) = 0$. By Lemma 2.17, one has

$$\widetilde{\tau} \in S_{l+2} \text{ for some } l \in \{1, 2, \cdots, i(\widetilde{\tau}, 1)\} \subseteq \{1, 2, \cdots, r\}.$$

Then

$$\widetilde{\tau} \in \bigcup_{l=1}^{r} S_{l+2},$$

which implies that

$$\tau \in \bigcup_{l=1}^{r} S_{l+2}[L].$$

It follows that

$$Y_{0,r} \subseteq \bigcup_{l=1}^{r} X_{l+2}.$$

By Lemma 2.10 and Lemma 3.1, $Y_{0,r} \in \mathscr{D}_{r+4}$. Note that, for every $\tau \in S_{\omega+1}[L]$ and $i(\tau,1) > r+2$, $X_\tau \subseteq Y_{j,r,1}$ for some $j \in \mathbb{N}$. By Lemma 3.2, there exist r-disjoint and 2^r-bounded families $\mathscr{U}_0(\tau), \mathscr{U}_1(\tau)$ such that $\mathscr{U}_0(\tau) \cup \mathscr{U}_1(\tau)$ covers X_τ. Let

$$\mathscr{U}_0 = \Big\{ U \mid U \in \mathscr{U}_0(\tau) \text{ for some } \tau \in S_{\omega+1}[L] \text{ and } i(\tau,1) > r+2 \Big\} \cup \{Y_{0,r}\}$$

and

$$\mathscr{U}_1 = \Big\{ U \mid U \in \mathscr{U}_1(\tau) \text{ for some } \tau \in S_{\omega+1}[L] \text{ and } i(\tau,1) > r+2 \Big\}.$$

Then $\mathscr{U}_0$, $\mathscr{U}_1$ are r-disjoint families such that $\mathscr{U}_0 \in \mathscr{D}_{r+4}$, $\mathscr{U}_1 \in \mathscr{D}_0$ and $\mathscr{U}_0 \cup \mathscr{U}_1$ covers $X_{\omega+1}$. Let $\mathscr{Y} = \mathscr{U}_0 \bigcup \mathscr{U}_1$. Then $X_{\omega+1} \xrightarrow{r} \mathscr{Y}$ and $\mathscr{Y} \in \mathscr{D}_{r+4}$. Thus $X_{\omega+1} \in \mathscr{D}_\omega$. By Lemma 2.18, we have

$$X_\omega = \bigsqcup_{\tau \in S_\omega[L]} X_\tau \subseteq \bigsqcup_{\tau \in S_{\omega+1}[L]} X_\tau = X_{\omega+1} \in \mathscr{D}_\omega.$$

□

Definition 3.4. Let $\xi = \gamma(\xi) + n(\xi)$ be a countable infinite ordinal number with $n(\xi) \geq 1$ and let $\tau = \{k_0+2, k_1+2, \cdots, k_m+2\} \in S_\xi[L]$ such that $m \geq n(\xi)$. For $i \in \{1,2,\cdots,n(\xi)\}$, let

$$X_{\tau,i} = \Big\{ (x_i)_{i=0}^m \in (2^{k_0}\mathbb{Z})^{m+1} \;\Big|\; |\{j \mid x_j \notin 2^{k_{n(\xi)}}\mathbb{Z}\}| \leq i \text{ and } |\{j \mid x_j \notin 2^{k_p}\mathbb{Z}\}| \leq p,\ \forall\, p \in \{1,\cdots,m\} \setminus \{n(\xi)\} \Big\},$$

and let

$$X_\xi^i = \bigsqcup_{\substack{\tau \in S_\xi[L] \\ |\tau| \geq n(\xi)+1}} X_{\tau,i}.$$

Remark 3.5. Note that

$$X_{\tau,1} \subseteq X_{\tau,2} \subseteq \cdots \subseteq X_{\tau,n(\xi)} = X_\tau, X_\xi^i \subseteq X_\xi \text{ and } X_\xi = X_\xi^{n(\xi)} \cup \Big(\bigsqcup_{\substack{\tau \in S_\xi[L] \\ |\tau| \leq n(\xi)}} X_\tau \Big).$$

Lemma 3.6. *For every countable infinite ordinal number $\xi = \gamma(\xi) + n(\xi)$ with $n(\xi) \geq 1$,*

$$X_\xi^1 = \bigsqcup_{\substack{\tau \in S_\xi[L] \\ |\tau| \geq n(\xi)+1}} X_{\tau,1} \in \mathscr{D}_{\gamma(\xi)}.$$

Proof. We prove it by induction on ξ.

- Let $\xi = \omega + 1$. By Lemma 3.3 and $X_{\omega+1}^1 \subseteq X_{\omega+1}$, $X_{\omega+1}^1 \in \mathscr{D}_\omega$. i.e., the result is true for $\xi = \omega + 1$.

- Assume that the result is true for every countable infinite ordinal number $\alpha = \gamma(\alpha) + n(\alpha)$ such that $\alpha < \xi$ and $n(\alpha) \geq 1$. For every $r \in \mathbb{N}$, let
$$Y_{0,r} = \bigsqcup_{\substack{\tau \in S_\xi[L] \\ |\tau| \geq n(\xi)+1 \\ i(\tau,n(\xi)) \leq r+2}} X_{\tau,1}, \; Y_{1,r} = \bigsqcup_{\substack{\tau \in S_\xi[L] \\ |\tau| \geq n(\xi)+1 \\ i(\tau,n(\xi)) > r+2}} X_{\tau,1},$$
then $X^1_\xi = Y_{0,r} \cup Y_{1,r}$.

For $\tau \in S_\xi[L]$ with $i(\tau, n(\xi)) \leq r+2$, by Lemma 2.17, $\tau \in \bigcup_{l=1}^{r+2} S_{\zeta_l(\gamma(\xi))+l+n(\xi)+1}[L]$. Then
$$Y_{0,r} \subseteq \bigcup_{l=1}^{r+2} X^1_{\zeta_l(\gamma(\xi))+l+n(\xi)+1}.$$
By inductive assumption,
$$X^1_{\zeta_l(\gamma(\xi))+l+n(\xi)+1} \in \mathscr{D}_{\zeta_l(\gamma(\xi))} \subseteq \mathscr{D}_{\zeta_l(\gamma(\xi))+l} \text{ for each } l \in \{1,2,\cdots,r+2\}.$$
Then by Lemma 2.10,
$$Y_{0,r} \in \mathscr{D}_{\zeta_{r+2}(\gamma(\xi))+r+3}.$$
Note that for every $\tau \in S_\xi[L]$ and $i(\tau,n(\xi)) > r+2$, $X_{\tau,1} \subseteq Y_{j,r,1}$ for some $j \in \mathbb{N}$. By Lemma 3.2, there exist r-disjoint, 2^r-bounded families $\mathscr{U}_0(\tau), \mathscr{U}_1(\tau)$ such that $\mathscr{U}_0(\tau) \cup \mathscr{U}_1(\tau)$ covers $X_{\tau,1}$. Let
$$\mathscr{U}_0 = \Big\{U \mid U \in \mathscr{U}_0(\tau) \text{ for some } \tau \in S_\xi[L], |\tau| \geq n(\xi)+1 \text{ and } i(\tau,n(\xi)) > r+2\Big\} \cup \{Y_{0,r}\}$$
and
$$\mathscr{U}_1 = \Big\{U \mid U \in \mathscr{U}_1(\tau) \text{ for some } \tau \in S_\xi[L], |\tau| \geq n(\xi)+1 \text{ and } i(\tau,n(\xi)) > r+2\Big\},$$
then $\mathscr{U}_0 \in \mathscr{D}_{\zeta_{r+2}(\gamma(\xi))+r+3}$ and $\mathscr{U}_1 \in \mathscr{D}_0$. Let $\mathscr{Y} = \mathscr{U}_0 \bigcup \mathscr{U}_1$, then
$$X^1_\xi \xrightarrow{r} \mathscr{Y} \text{ and } \mathscr{Y} \in \mathscr{D}_{\zeta_{r+2}(\gamma(\xi))+r+3}.$$
Therefore, $X^1_\xi \in \mathscr{D}_{\gamma(\xi)}$.

□

Lemma 3.7. *Let $\xi = \gamma(\xi) + n(\xi)$ be a countable infinite ordinal number with $n(\xi) \geq 1$ and let $\tau = \{k_0+2, k_1+2, \cdots, k_m+2\} \in S_\xi[L]$ with $m \geq n(\xi)$. Then, for each $i \in \{1,2,\cdots,n(\xi)\}$ and for every $r > 0$,*
$$X_{\tau,i} \cap N_r(X_{\tau,i-1}) = \Big\{(x_i)_{i=0}^m \in (2^{k_0}\mathbb{Z})^{m+1} \;\Big|\; |\{j \mid x_j \notin N_r(2^{k_{n(\xi)}}\mathbb{Z})\}| \leq i-1\Big\} \cap X_{\tau,i}.$$

Proof. • First we prove that
$$X_{\tau,i} \cap N_r(X_{\tau,i-1}) \subseteq \Big\{(x_i)_{i=0}^m \in (2^{k_0}\mathbb{Z})^{m+1} \;\Big|\; |\{j \mid x_j \notin N_r(2^{k_{n(\xi)}}\mathbb{Z})\}| \leq i-1\Big\} \cap X_{\tau,i}.$$
For every $x = (x_i)_{i=0}^m \in X_{\tau,i} \cap N_r(X_{\tau,i-1})$, there exists $y = (y_i)_{i=0}^m \in X_{\tau,i-1} \subseteq X_{\tau,i}$ such that $d(x,y) < r$, which implies that $d(x_i,y_i) < r$ for $i \in \{0,1,\cdots,m\}$. Note that
$$|\{j \mid y_j \notin 2^{k_{n(\xi)}}\mathbb{Z}\}| \leq i-1 \text{ and } y_j \in 2^{k_{n(\xi)}}\mathbb{Z} \text{ implies } x_j \in N_r(2^{k_{n(\xi)}}\mathbb{Z}).$$
Then
$$\left|\{j \mid x_j \notin N_r(2^{k_{n(\xi)}}\mathbb{Z})\}\right| \leq \left|\{j \mid y_j \notin 2^{k_{n(\xi)}}\mathbb{Z}\}\right| \leq i-1,$$

and then

$$x \in \left\{ (x_i)_{i=0}^m \in (2^{k_0}\mathbb{Z})^{m+1} \;\middle|\; |\{ j \mid x_j \notin N_r(2^{k_{n(\xi)}}\mathbb{Z})\}| \leq i-1 \right\} \cap X_{\tau,i}.$$

- Now we prove that

$$\left\{ (x_i)_{i=0}^m \in (2^{k_0}\mathbb{Z})^{m+1} \;\middle|\; |\{ j \mid x_j \notin N_r(2^{k_{n(\xi)}}\mathbb{Z})\}| \leq i-1 \right\} \cap X_{\tau,i} \subseteq X_{\tau,i} \cap N_r(X_{\tau,i-1}).$$

Assume that $x = (x_i)_{i=0}^m$ such that $x \in X_{\tau,i}$ and $|\{ j \mid x_j \notin N_r(2^{k_{n(\xi)}}\mathbb{Z})\}| \leq i-1$. Let

$$I_p = \{ j \mid x_j \in N_r(2^{k_p}\mathbb{Z})\} \text{ for } p \in \{0,\ldots,m\} \text{ and } I_{m+1} = \emptyset.$$

Since $k_0 < k_1 < \cdots < k_m$, $I_m \subseteq I_{m-1} \subseteq \cdots \subseteq I_1 \subseteq I_0$. $x \in X_{\tau,i}$ and $|\{ j \mid x_j \notin N_r(2^{k_{n(\xi)}}\mathbb{Z})\}| \leq i-1$ imply that $|I_p| \geq m+1-p, \forall p \in \{0,1,\cdots,m\} \setminus \{n(\xi)\}$ and $|I_{n(\xi)}| \geq m+1-(i-1)$. In particular, $I_m \neq \emptyset$ and $I_0 = \{0,1,\cdots,m\}$. Note that

$$I_0 = \bigsqcup_{p=0}^{m} \left(I_p \setminus I_{p+1}\right).$$

If $I_p \setminus I_{p+1} \neq \emptyset$, then

$$\forall j \in \left(I_p \setminus I_{p+1}\right), \text{ there exists } y_j \in 2^{k_i}\mathbb{Z} \text{ such that } d(x_j, y_j) < r.$$

Let $y = (y_i)_{i=0}^m$. Note that

$$\forall\, p \in \{0,1,\cdots,m\} \setminus \{n(\xi)\}, |\{ j \mid y_j \in 2^{k_p}\mathbb{Z}\}| = |I_p| \geq m+1-p$$

and

$$|\{ j \mid y_j \in 2^{k_{n(\xi)}}\mathbb{Z}\}| = |I_{n(\xi)}| \geq m+1-(i-1).$$

Then

$$\forall\, p \in \{0,1,\cdots,m\} \setminus \{n(\xi)\}, |\{ j \mid y_j \notin 2^{k_p}\mathbb{Z}\}| \leq p \text{ and } |\{ j \mid y_j \notin 2^{k_n(\xi)}\mathbb{Z}\}| \leq i-1.$$

It follows that $y \in X_{\tau,i-1}$. So $x \in N_r(X_{\tau,i-1})$.

□

Corollary 3.8. *Let $\xi = \gamma(\xi) + n(\xi)$ be a countable infinite ordinal number with $n(\xi) \geq 1$ and let $\tau = \{k_0+2, k_1+2, \cdots, k_m+2\} \in S_\xi[L]$ such that $m \geq n(\xi)$. Then, for each $i \in \{1,2,\cdots,n(\xi)\}$ and for every $r > 0$,*

$$X_{\tau,i} \subseteq N_r(X_{\tau,i-1}) \cup \left\{ (x_i)_{i=0}^m \in (2^{k_0}\mathbb{Z})^{m+1} \;\middle|\; |\{ j \mid x_j \notin N_r(2^{k_{n(\xi)}}\mathbb{Z})\}| = i \right\}.$$

Proof. For every $x = (x_i)_{i=0}^m \in X_{\tau,i} \setminus N_r(X_{\tau,i-1})$, $x \in X_{\tau,i}$ implies $x \in (2^{k_0}\mathbb{Z})^{m+1}$ and

$$|\{ j \mid x_j \notin N_r(2^{k_{n(\xi)}}\mathbb{Z})\}| \leq |\{ j \mid x_j \notin 2^{k_{n(\xi)}}\mathbb{Z}\}| \leq i.$$

Suppose that $\{ j \mid x_j \notin N_r(2^{k_{n(\xi)}}\mathbb{Z})\}| \leq i-1$. By Lemma 3.7, we have $x \in N_r(X_{\tau,i-1})$, which is a contradiction. Then $|\{ j \mid x_j \notin N_r(2^{k_{n(\xi)}}\mathbb{Z})\}| = i$. It follows that

$$X_{\tau,i} \setminus N_r(X_{\tau,i-1}) \subseteq \left\{ (x_i)_{i=0}^m \in (2^{k_0}\mathbb{Z})^{m+1} \;\middle|\; |\{ j \mid x_j \notin N_r(2^{k_{n(\xi)}}\mathbb{Z})\}| = i \right\}.$$

Thus

$$X_{\tau,i} \subseteq N_r(X_{\tau,i-1}) \cup \left\{ (x_i)_{i=0}^m \in (2^{k_0}\mathbb{Z})^{m+1} \;\middle|\; |\{ j \mid x_j \notin N_r(2^{k_{n(\xi)}}\mathbb{Z})\}| = i \right\}.$$

□

Proposition 3.9. *For every countable infinite ordinal number $\xi=\gamma(\xi)+n(\xi)$ such that $n(\xi)\geq 1$, and for each $i\in\{1,2,\cdots,n(\xi)\}$,*

$$X_{\xi}^{i}=\bigsqcup_{\substack{\tau\in S_{\xi}[L]\\|\tau|\geq n(\xi)+1}} X_{\tau,i}\in\mathscr{D}_{\gamma(\xi)+i-1}.$$

In particular, $X_{\xi}^{n(\xi)}\in\mathscr{D}_{\xi-1}$.

Proof. By Lemma 3.3, $X_{\omega+1}\in\mathscr{D}_{\omega}$. It follows that $X_{\omega+1}^{1}\in\mathscr{D}_{\omega}$. By Lemma 3.6, $X_{\xi}^{1}\in\mathscr{D}_{\gamma(\xi)}$. Assume that $X_{\tilde{\xi}}^{n(\tilde{\xi})}\in\mathscr{D}_{\tilde{\xi}-1}$ for every countable infinite ordinal number $\tilde{\xi}$ such that $\tilde{\xi}<\xi$, $n(\tilde{\xi})\geq 1$, and $X_{\xi}^{i-1}\in\mathscr{D}_{\gamma(\xi)+i-2}$. Now we show that $X_{\xi}^{i}\in\mathscr{D}_{\gamma(\xi)+i-1}$. For every $r\in\mathbb{N}$, let

$$Y_{0,r}=\bigsqcup_{\substack{\tau\in S_{\xi}[L]\\|\tau|\geq n(\xi)+1\\i(\tau,n(\xi))\leq r+2}} X_{\tau,i},\ Y_{1,r}=\bigsqcup_{\substack{\tau\in S_{\xi}[L]\\|\tau|\geq n(\xi)+1\\i(\tau,n(\xi))>r+2}} X_{\tau,i}.$$

Then $X_{\xi}^{i}=Y_{0,r}\cup Y_{1,r}$. For $\tau\in S_{\xi}[L]$ with $i(\tau,n(\xi))\leq r+2$, by Lemma 2.17,

$$\tau\in\bigcup_{l=1}^{r+2}S_{\zeta_l(\gamma(\xi))+l+n(\xi)+1}[L].$$

Then

$$Y_{0,r}\subseteq\bigcup_{l=1}^{r+2}X_{\zeta_l(\gamma(\xi))+l+n(\xi)+1}^{i}.$$

Note that, for each $l\in\{1,2,\cdots,r+2\}$,

$$X_{\zeta_l(\gamma(\xi))+l+n(\xi)+1}^{i}\subseteq X_{\zeta_l(\gamma(\xi))+l+n(\xi)+1}^{l+n(\xi)+1}\in\mathscr{D}_{\zeta_l(\gamma(\xi))+l+n(\xi)}$$

by inductive assumption. It follows from Lemma 2.10 that

$$Y_{0,r}\in\mathscr{D}_{\zeta_{r+2}(\gamma(\xi))+r+n(\xi)+3}.$$

Observe that $\tau=\{k_0+2,k_1+2,\cdots,k_m+2\}\in S_{\xi}[L]$ such that $m\geq n(\xi)$. By Corollary 3.8, for each $i\in\{1,2,\cdots,n(\xi)\}$,

$$X_{\tau,i}\subseteq N_r(X_{\tau,i-1})\cup\left\{(x_j)_{j=0}^{m}\in(2^{k_0}\mathbb{Z})^{m+1}\;\middle|\;|\{j\mid x_j\notin N_r(2^{k_{n(\xi)}}\mathbb{Z})\}|=i\right\}.$$

$\left\{(x_j)_{j=0}^{m}\in(2^{k_0}\mathbb{Z})^{m+1}\;\middle|\;|\{j\mid x_j\notin N_r(2^{k_{n(\xi)}}\mathbb{Z})\}|=i\right\}$ implies that $2^{k_{n(\xi)}}\geq 2r$. Let

$$\begin{aligned}\mathscr{U}(\tau)=\Big\{\Big(&\{x_t\}_{t=0}^{j_1-1}\times[n_{j_1}2^{k_{n(\xi)}}+r,(n_{j_1}+1)2^{k_{n(\xi)}}-r]\times(x_t)_{t=j_1+1}^{j_2-1}\\&\times[n_{j_2}2^{k_{n(\xi)}}+r,(n_{j_2}+1)2^{k_{n(\xi)}}-r]\times\{x_t\}_{t=j_2+1}^{j_3-1}\\&\times\cdots\times\{x_t\}_{t=j_{i-1}+1}^{j_i-1}\times[n_{j_i}2^{k_{n(\xi)}}+r,(n_{j_i}+1)2^{k_{n(\xi)}}-r]\\&\times\{x_t\}_{t=j_i+1}^{m}\Big)\cap(2^{k_0}\mathbb{Z})^{m+1}\\&\Big|\ x_t\in 2^{k_{n(\xi)}}\mathbb{Z},n_{j_k}\in\mathbb{Z},j_k\in\mathbb{Z}^{+},k\in\{1,2,\cdots,i\}\Big\}.\end{aligned}$$

It is easy to see that $\mathscr{U}(\tau)$ is r-disjoint and

$$\left\{(x_i)_{i=0}^m \in (2^{k_0}\mathbb{Z})^{m+1} \;\middle|\; |\{j \mid x_j \notin N_r(2^{k_{n(\xi)}}\mathbb{Z})\}| = i\right\} \subseteq (\bigcup \mathscr{U}(\tau)).$$

Therefore,

$$\begin{aligned} X_{\tau,i} &\subseteq N_r(X_{\tau,i-1}) \cup \left\{(x_i)_{i=0}^m \in (2^{k_0}\mathbb{Z})^{m+1} \;\middle|\; |\{j \mid x_j \notin N_r(2^{k_{n(\xi)}}\mathbb{Z})\}| = i\right\} \\ &\subseteq N_r(X_{\tau,i-1}) \cup (\bigcup \mathscr{U}(\tau)). \end{aligned}$$

Note that

$$\bigcup_{\substack{\tau \in S_\xi[L] \\ |\tau| \geq n(\xi)+1 \\ i(\tau,n(\xi)) > r+2}} N_r(X_{\tau,i-1}) \subseteq N_r\Big(\bigcup_{\substack{\tau \in S_\xi[L] \\ |\tau| \geq n(\xi)+1 \\ i(\tau,n(\xi)) > r+2}} X_{\tau,i-1}\Big) \subseteq N_r\left(X_\xi^{i-1}\right) \in \mathscr{D}_{\gamma(\xi)+i-2}$$

and there is a natural coarse embedding

$$F : \left\{U \mid U \in \mathscr{U}(\tau) \text{ for some } \tau \in S_\xi[L], |\tau| \geq n(\xi)+1 \text{ and } i(\tau,n(\xi)) > r+2\right\} \to \{\mathbb{Z}^i\}$$

such that, for every $f \in F$ and every $x,y \in X_f$, $d(f(x),f(y)) = d(x,y)$. By Lemma 2.13 and $\{\mathbb{Z}^i\} \in \mathscr{D}_i$, $\left\{U \mid U \in \mathscr{U}(\tau) \text{ for some } \tau \in S_\xi[L], |\tau| \geq n(\xi)+1 \text{ and } i(\tau,n(\xi)) > r+2\right\} \in \mathscr{D}_i$. Let

$$\mathscr{U}_0 = \{\bigcup_{\substack{\tau \in S_\xi[L] \\ |\tau| \geq n(\xi)+1 \\ i(\tau,n(\xi)) > r+2}} N_r(X_{\tau,i-1})\} \text{ and}$$

$$\mathscr{U}_1 = \left\{U \mid U \in \mathscr{U}(\tau) \text{ for some } \tau \in S_\xi[L], |\tau| \geq n(\xi)+1 \text{ and } i(\tau,n(\xi)) > r+2\right\} \cup \{Y_{0,r}\}.$$

Let $\mathscr{Y} = \mathscr{U}_0 \bigcup \mathscr{U}_1$. Then $X_\xi^i \xrightarrow{r} \mathscr{Y}$ and $\mathscr{Y} \in \mathscr{D}_{\gamma(\xi)+i-2}$. Therefore, $X_\xi^i \in \mathscr{D}_{\gamma(\xi)+i-1}$. □

Corollary 3.10. *$X_\xi \in \mathscr{D}_\xi$ for every countable infinite ordinal number ξ.*

Proof.
- If $n(\xi) \geq 1$, then we let

$$Y_0 = \bigsqcup_{\substack{\tau \in S_\xi[L] \\ |\tau| \leq n(\xi)}} X_\tau,\; Y_1 = \bigsqcup_{\substack{\tau \in S_\xi[L] \\ |\tau| > n(\xi)}} X_\tau.$$

By Lemma 3.1, we have

$$Y_0 \subseteq \bigsqcup_{\tau \in S_{n(\xi)}[L]} X_\tau = X_{n(\xi)} \in \mathscr{D}_{n(\xi)+1}.$$

Note that

$$Y_1 = \bigsqcup_{\substack{\tau \in S_\xi[L] \\ |\tau| \geq n(\xi)+1}} X_\tau = \bigsqcup_{\substack{\tau \in S_\xi[L] \\ |\tau| \geq n(\xi)+1}} X_{\tau,n(\xi)} = X_\xi^{n(\xi)} \in \mathscr{D}_{\xi-1}$$

by Proposition 3.9. Thus $X_\xi = Y_0 \cup Y_1 \in \mathscr{D}_\xi$.

- If $n(\xi) = 0$, then, for every $r \in \mathbb{N}$, we let

$$Y_{0,r} = \bigsqcup_{\substack{\tau \in S_\xi[L] \\ |\tau| \geq n(\xi)+1 \\ i(\tau,0) \leq r+2}} X_{\tau,i},\ Y_{1,r} = \bigsqcup_{\substack{\tau \in S_\xi[L] \\ |\tau| \geq n(\xi)+1 \\ i(\tau,0) > r+2}} X_{\tau,i}.$$

Thus $X_\xi = Y_{0,r} \cup Y_{1,r}$. For $\tau \in S_\xi[L]$ with $i(\tau,0) \leq r+2$, by Lemma 2.17, we have $\tau \in \bigcup_{l=1}^{r+2} S_{\zeta_l(\gamma(\xi))+l+1}[L]$. Then $Y_{0,r} \subseteq \bigcup_{l=1}^{r+2} X_{\zeta_l(\gamma(\xi))+l+1}$. Note that

$$X_{\zeta_l(\gamma(\xi))+l+1} \in \mathscr{D}_{\zeta_l(\gamma(\xi))+l+1}$$

for each $l \in \{1, 2, \cdots, r+2\}$. It follows from Lemma 2.10 that $Y_{0,r} \in \mathscr{D}_{\zeta_{r+2}(\gamma(\xi))+r+4}$. Let $\mathscr{U} = \{\{x\} \mid x \in Y_{1,r}\}$. Then $\mathscr{U}$ is a r-disjoint and uniformly bounded family. Let $\mathscr{Y} = \mathscr{U} \cup \{Y_{0,r}\}$. Thus $X_\xi \xrightarrow{r} \mathscr{Y}$ and $\mathscr{Y} \in \mathscr{D}_{\zeta_{r+2}(\gamma(\xi))+r+4}$. Therefore, $X_\xi \in \mathscr{D}_{\gamma(\xi)} = \mathscr{D}_\xi$. □

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