J. Nonlinear Funct. Anal. 2022 (2022), Article ID 28 https://doi.org/10.23952/jnfa.2022.28

MATHRES

Journal of Nonlinear Functional Analysis

Available online at http://jnfa.mathres.org

THE EXPONENTIAL STABILITY OF A TWO-UNIT SYSTEM WITH NON-PREEMPTIVE PRIORITY

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Abstract. This study analyzes a repairable system with two dissimilar units, one of which has non-preemptive priority. We transform the model to an abstract Cauchy problem and study the well-posedness via the C_0- semigroup theory of linear operators. By using spectral properties of the corresponding operators, we derive the exponential convergence of the time-dependent solution to its steady-state solution. We also present the asymptotic behavior of several time-dependent reliability indices and numerical examples.

Keywords. Availability; C_0-semigroup; Exponential Stability; Priority system; Reliability.

1. INTRODUCTION

We consider the system consisting of two dissimilar unit with non-preemptive priority with two types of components, Type I and Type II, in the system. The Type I unit has priority with non-preemptive repair disciplines, that is, the Type II unit's repair is continued and the repair of the Type I unit is entertained only when the Type II unit's repair is done. The system runs in a degraded state but does not fail if the Type II unit fails, while the system can collapse fully if the Type I component fails. The system, According to Govil [5], can be characterized by the partial differential equations below

$$
\begin{aligned}
&\frac{\mathrm{d}P_0(t)}{\mathrm{d}t} = -(\lambda_1+\lambda_2)P_0(t) + \int_0^\infty \eta_1(x)P_f(x,t)\mathrm{d}x + \int_0^\infty \eta_2(x)P_D(x,t)\mathrm{d}x,\\
&\frac{\partial P_D(x,t)}{\partial t} + \frac{\partial P_D(x,t)}{\partial x} = -(\lambda_1+\eta_2(x))P_D(x,t), \frac{\partial P_f(x,t)}{\partial t} + \frac{\partial P_f(x,t)}{\partial x} = -\eta_1(x)P_f(x,t), \quad (1.1)\\
&\frac{\partial P_{sf}(x,t)}{\partial t} + \frac{\partial P_{sf}(x,t)}{\partial x} = -\eta_2(x)P_{sf}(x,t) + \lambda_1 P_D(x,t)
\end{aligned}
$$

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Received February 14, 2022; Accepted June 7, 2022.

with the boundary conditions

$$P_D(0,t)=\lambda_2 P_0(t), P_f(0,t)=\lambda_1 P_0(t)+\int_0^\infty \eta_2(x)P_{sf}(x,t)\mathrm{d}x, P_{sf}(0,t)=0, \tag{1.2}$$

and initial conditions

$$P_0(0)=1,\ P_D(x,0)=0,\ P_f(x,0)=0,\ P_{sf}(x,0)=0, \tag{1.3}$$

where $(x,t)\in[0,\infty)\times[0,\infty)$, $P_0(t)$ is the probability that the system is working at normal efficiency at time t, $P_D(x,t)\,\mathrm{d}x$ is the probability that the system is degraded at time t owing to a Type II component failure, and the elapsed repair time is in $(x,x+\Delta x)$, $P_f(x,t)\,\mathrm{d}x$ is the probability that the system is failed at time t owing to a Type I component failure, and the elapsed repair time is in $(x,x+\Delta x)$, $P_{sf}(x,t)\,\mathrm{d}x$ is the probability that the system is still failed at time t owing to a Type I component failure, and the elapsed repair time of Type II lies in the interval $(x,x+\Delta x)$, whereas Type I is awaiting repair. The failure rates of Type I and Type II units are denoted by λ and λ_2, respectively. The repair rates of Type I and Type II units are denoted by $\eta_1(x)$ and $\eta_2(x)$, respectively, and satisfies $\eta_i(x)\geq 0$, $\int_0^\infty \eta_i(x)\mathrm{d}x=\infty$, $i=1,2$. A system composed of two dissimilar units and a repairman is a common system in real life. Therefore, many researchers studied different types of such systems; see, e.g. [1, 6, 7, 8, 10, 11, 12, 14, 15, 16, 17] and the references therein. In 1972, Govil [5] used the supplementary variable method, which was first applied in a reliability model by Gaver [4], to establish a repairable system consisting of two different components with three different priority repair disciplines, i.e., Non preemptive, Preemptive Resume, and Preemptive Repeat, and the Laplace transforms of state probabilities were derived. The effects of varying repair priorities on the system's pointwise availability was explored using numerical examples for a special case. Kasim and Gupur [10] used the C_0-semigroup theory to perform dynamic analysis on a system composed of two-dissimilar units under preemptive repeat repair discipline and proved that the time-dependent solution converges exponentially to its steady-state solution. Furthermore, the time-dependent reliability indices were theoretically and numerically investigated.

In this paper, we investigate the system consisting of two-different units under non preemptive repair discipline. The remainder of the paper is organized as follows. Section 2 begins by converting the preceding system into an abstract Cauchy problem and proving the well-posedness. In Section 3, we obtain that the time-dependent solution converges exponentially to its steady-state solution. Section 4 discusses the asymptotic behavior of instantaneous reliability indices. In Section 5, we use numerical examples to demonstrate the effect of changes in system parameters on reliability indices such as instantaneous availability, instantaneous failure frequency, and instantaneous renewal frequency, in a specific case.

2. Well-posedness of the System

Based on the C_0-semigroup theory, we demonstrate the well-posedness in this section. To this end, we first transform the model into an abstract Cauchy problem.

For simplicity, we use the notation:

$$\Gamma_0=\begin{pmatrix} e^{-x} & 0 & 0 & 0\\ \lambda_2 e^{-x} & 0 & 0 & 0\\ \lambda_1 e^{-x} & 0 & 0 & \eta_2(x)\\ 0 & 0 & 0 & 0\end{pmatrix}.$$

Choose the following Banach space

$$\mathscr{X} = \left\{ \mathbf{P} \left| \begin{array}{l} \mathbf{P} = (P_0, P_D, P_f, P_{sf}) \in \mathbb{R} \times \left(L^1[0,\infty)\right)^3 \\ \|\mathbf{P}\| = |P_0| + \sum\limits_{j=D,f,sf} \|P_j\|_{L^1[0,\infty)} < \infty. \end{array} \right. \right\}$$

as the state space. Define $\mathscr{A}\mathbf{P} = (A+U+E)\mathbf{P}$, and

$$D(\mathscr{A}) = \left\{ \mathbf{P} \in \mathscr{X} \left| \begin{array}{l} \frac{dP_j(x)}{\mathrm{d}x} \in L^1[0,\infty), P_j(x)(j = D, f, sf) \text{ are absolutely} \\ \text{continuous and } \mathbf{P}(0) = \int_0^\infty \Gamma_0 \, \mathbf{P}(x)\mathrm{d}x \end{array} \right. \right\},$$

where

$$A := \begin{pmatrix} (\lambda_1+\lambda_2) & 0 & 0 & 0 \\ 0 & -\frac{d}{\mathrm{d}x} - (\lambda_1 + \eta_2(x)) & 0 & \\ 0 & 0 & -\frac{d}{\mathrm{d}x} - \eta_1(x) & 0 \\ 0 & 0 & 0 & -\frac{d}{\mathrm{d}x} - \eta_2(x) \end{pmatrix},$$

$$U := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \end{pmatrix}, \quad E := \begin{pmatrix} \int_0^\infty \eta_2(x) P_D(x)\mathrm{d}x + \int_0^\infty \eta_1(x) P_f(x)\mathrm{d}x \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Then the original equations (1.1)-(1.3) can be written as the following abstract Cauchy problem on $\mathscr{X}$

$$\begin{cases} \frac{d\mathbf{P}(t)}{dt} = \mathscr{A}\mathbf{P}(t), \quad t \in (0,\infty), \\ \mathbf{P}(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \end{cases} \tag{2.1}$$

Now we proceed to show that operator $\mathscr{A}$ generates a positive contraction C_0- semigroup $\mathscr{T}(t)$ on $\mathscr{X}$.

Theorem 2.1. *If $\eta_1(x)$ and $\eta_2(x)$ satisfy $\overline{\eta_1} = \sup\limits_{x\in[0,\infty)} \eta_1(x) < \infty$ and $\overline{\eta_2} = \sup\limits_{x\in[0,\infty)} \eta_2(x) < \infty$, then $\mathscr{A}$ generates a positive contraction C_0- semigroup $\mathscr{T}(t)$.*

Proof. First of all, we estimate $\|(\gamma I - A)^{-1}\|$. Taking $(\gamma I - A)\mathbf{P} = Y$, $\forall Y \in \mathscr{X}$, we have

$$(\gamma + \lambda_1 + \lambda_2)P_0 = Y_0, \tag{2.2}$$

$$\frac{dP_D(x)}{dx} + (\gamma + \lambda_1 + \eta_2(x))P_D(x) = Y_D(x), \tag{2.3}$$

$$\frac{dP_f(x)}{dx} + (\gamma + \eta_1(x))P_f(x) = Y_f(x), \tag{2.4}$$

$$\frac{dP_{sf}(x)}{dx} + (\gamma + \eta_2(x))P_{sf}(x) = Y_{sf}(x), \tag{2.5}$$

$$P_D(0) = \lambda_2 P_0, P_f(0) = \lambda_1 P_0 + \int_0^\infty P_{sf}(x)\eta_2(x)\mathrm{d}x, P_{sf}(0) = 0. \tag{2.6}$$

Solving (2.2)-(2.5) yields

$$P_0 = \frac{1}{\gamma+\lambda_1+\lambda_2} Y_0, \tag{2.7}$$

$$P_D(x) = a_D e^{-(\gamma+\lambda_1)x - \int_0^x \eta_2(\tau)d\tau} + e^{-(\gamma+\lambda_1)x - \int_0^x \eta_2(\tau)d\tau} \int_0^x Y_D(\tau) e^{(\gamma+\lambda_1)\tau + \int_0^\tau \eta_2(\sigma)d\sigma} d\tau, \tag{2.8}$$

$$P_f(x) = a_f e^{-\gamma x - \int_0^x \eta_1(\tau)d\tau} + e^{-\gamma x - \int_0^x \eta_1(\tau)d\tau} \int_0^x Y_f(\tau) e^{\gamma\tau + \int_0^\tau \eta_1(\sigma)d\sigma} d\tau, \tag{2.9}$$

$$P_{sf}(x) = e^{-\gamma x - \int_0^x \eta_2(\tau)d\tau} \int_0^x Y_{sf}(\tau) e^{\gamma\tau + \int_0^\tau \eta_2(\sigma)d\sigma} d\tau. \tag{2.10}$$

Applying (2.6) in (2.7)-(2.10), we have

$$a_D = P_D(0) = \lambda_2 P_0 = \frac{\lambda_2}{\gamma+\lambda_1+\lambda_2} Y_0, \tag{2.11}$$

$$\begin{aligned} a_f =& P_f(0) = \lambda_1 P_0 + \int_0^\infty P_{sf}(x)\eta_2(x)\mathrm{d}x \\ =& \frac{\lambda_1}{\gamma+\lambda_1+\lambda_2} Y_0 + \int_0^\infty \eta_2(x) e^{-\gamma x - \int_0^x \eta_2(\tau)d\tau} \mathrm{d}x \int_0^x Y_{sf}(\tau) e^{\gamma\tau + \int_0^\tau \eta_2(\sigma)d\sigma} d\tau \mathrm{d}x. \end{aligned} \tag{2.12}$$

Using the Fubini theorem and combining (2.11)-(2.12) with (2.7)-(2.10), we can now estimate (assume $\gamma > \overline{\eta_2}$)

$$\begin{aligned} \|\mathbf{P}\| =& |P_0| + \|P_D\|_{L^1[0,\infty)} + \|P_f\|_{L^1[0,\infty)} + \|P_{sf}\|_{L^1[0,\infty)} \\ \le& \frac{1}{\gamma+\lambda_1+\lambda_2}|Y_0| + \frac{1}{\gamma+\lambda_1}|a_D| + \int_0^\infty e^{-(\gamma+\lambda_1)x} \int_0^x |Y_D(\tau)| e^{(\gamma+\lambda_1)\tau} d\tau \mathrm{d}x \\ &+ \frac{1}{\gamma}|a_f| + \int_0^\infty e^{-\gamma x} \int_0^x |Y_f(\tau)| e^{\gamma\tau} d\tau \mathrm{d}x + \int_0^\infty e^{-\gamma x} \int_0^x |Y_{sf}(\tau)| e^{\gamma\tau} d\tau \mathrm{d}x \\ \le& \frac{1}{\gamma+\lambda_1+\lambda_2}|Y_0| + \frac{\lambda_2}{(\gamma+\lambda_1)(\gamma+\lambda_1+\lambda_2)}|Y_0| + \frac{1}{\gamma+\lambda_1}\|Y_D\|_{L^1[0,\infty)} \\ &+ \frac{\lambda_1}{\gamma(\gamma+\lambda_1+\lambda_2)}|Y_0| + \frac{\overline{\varphi}}{\gamma^2}\|Y_{sf}\|_{L^1[0,\infty)} + \frac{1}{\gamma}\|Y_f\|_{L^1[0,\infty)} + \frac{1}{\gamma}\|Y_{sf}\|_{L^1[0,\infty)} \\ \le& \frac{1}{\gamma}\|Y\|, \end{aligned} \tag{2.13}$$

which yields that, for $\gamma > \overline{\eta_2}$, $(\gamma I - A)^{-1}$ exist, and

$$(\gamma I - A)^{-1} : \mathscr{X} \to D(A), \quad \|(\gamma I - A)^{-1}\| \le \frac{1}{\gamma}.$$

Following that, we should demonstrate that $D(A)$ is dense in $\mathscr{X}$. This is done in the same way as Gupur [9, P.76]. Therefore, we omit the detailed proof. As a result, we deduce that operator A generates a C_0--semigroup according to the Hille-Yosida Theorem (see Nagel citeNagel). An easy computation shows that $\|U\mathbf{P}\| \le \lambda\|\mathbf{P}\|$, $\|E\mathbf{P}\| \le \max\{\overline{\varphi}, \overline{\eta}\}\|\mathbf{P}\|$, which implies that both U and E are bounded linear operators. Hence, $\mathscr{A}$ generates a C_0- semigroup $\mathscr{T}(t)$ [9, Theorem 1.80]. The left is to show that $\mathscr{A}$ is a dispersive operator.

Choosing ψ, for $\mathbf{P}\in D(\mathscr{A})$,

$$\psi(x)=\left(\frac{[P_0]^+}{P_0},\frac{[P_D(x)]^+}{P_D(x)},\frac{[P_f(x)]^+}{P_f(x)},\frac{[P_{sf}(x)]^+}{P_{sf}(x)}\right),$$

where

$$[P_0]^+=\begin{cases}0 & \text{if} \quad P_0\le 0\\ P_0 & \text{if} \quad P_0>0\end{cases},\qquad [P_l(x)]^+=\begin{cases}0 & \text{if} \quad P_l(x)\le 0\\ P_l(x) & \text{if} \quad P_l(x)>0\end{cases},\quad l=R,f,sf.$$

From the boundary condition, we have

$$[P_D(0)]^+\le\lambda_2[P_0]^+,[P_f(0)]^+\le\lambda_1[P_0]^++\int_0^\infty\eta_2(x)[P_{sf}(x)]^+\mathrm{d}x,[P_{sf}(0)]^+=0. \tag{2.14}$$

Now, we define $\mathscr{W}_0=\{x\in[0,\infty)\big|y(x)\le 0\}$ and $\mathscr{W}_1=\{x\in[0,\infty)\big|y(x)>0\}$. By replacing $y(x)$ in $\mathscr{W}_0$, $\mathscr{W}_0$ with $P_D(x),P_f(x),P_{sf}(x)$, we obtain

$$\begin{aligned}\int_0^\infty\frac{dP_D(x)}{\mathrm{d}x}\frac{[P_D(x)]^+}{P_D(x)}\mathrm{d}x&=\sum_{i=1}^2\int_{\mathscr{W}_i}\frac{dP_D(x)}{\mathrm{d}x}\frac{[P_D(x)]^+}{P_D(x)}\mathrm{d}x=\int_{\mathscr{W}_1}\frac{dP_D(x)}{\mathrm{d}x}\frac{[P_D(x)]^+}{P_D(x)}\mathrm{d}x=\int_{\mathscr{W}_1}\frac{dP_D(x)}{\mathrm{d}x}\mathrm{d}x\\&=\int_{\mathscr{W}_1}\frac{d[P_D(x)]^+}{\mathrm{d}x}\mathrm{d}x=-[P_D(0)]^+,\end{aligned} \tag{2.15}$$

$$\int_0^\infty\frac{dP_f(x)}{\mathrm{d}x}\frac{[P_f(x)]^+}{P_f(x)}\mathrm{d}x=-[P_f(0)]^+,\int_0^\infty\frac{dP_{sf}(x)}{\mathrm{d}x}\frac{[P_{sf}(x)]^+}{P_{sf}(x)}\mathrm{d}x=-[P_{sf}(0)]^+. \tag{2.16}$$

Using (2.14)-(2.16), we deduce

$$\begin{aligned}\langle\mathscr{A}\mathbf{P},\psi\rangle=&\left\{-(\lambda_1+\lambda_2)P_0+\int_0^\infty\eta_2(x)P_D(x)dy+\int_0^\infty\eta_1(x)P_f(x)\mathrm{d}x\right\}\frac{[P_0]^+}{P_0}\\&+\int_0^\infty\left\{-\frac{dP_D(x)}{dx}-(\lambda_1+\eta_2(x))P_D(x)\right\}\frac{[P_D(x)]^+}{P_D(x)}\mathrm{d}x\\&+\int_0^\infty\left\{-\frac{dP_f(x)}{dx}-\eta_1(x)P_f(x)\right\}\frac{[P_f(x)]^+}{P_f(x)}\mathrm{d}x\\&+\int_0^\infty\left\{-\frac{dP_{sf}(x)}{dx}-\eta_2(x)P_{sf}(x)+\lambda_1P_{sf}(x)\right\}\frac{[P_{sf}(x)]^+}{P_{sf}(x)}\mathrm{d}x\\\le&-(\lambda_1+\lambda_2)[P_0]^++\left\{\int_0^\infty\eta_2(x)[P_D(x)]^+\mathrm{d}x+\int_0^\infty\eta_1(x)[P_f(x)]^+\mathrm{d}x\right\}\frac{[P_0]^+}{P_0}\\&+\lambda_2[P_0]^+-\int_0^\infty(\lambda_1+\eta_2(x))[P_D(x)]^+\mathrm{d}x+\lambda_1[P_0]^++\int_0^\infty\eta_2(x)[P_{sf}(x)]^+\mathrm{d}x\\&-\int_0^\infty\eta_1(x)[P_f(x)]^+\mathrm{d}x-\int_0^\infty\eta_2(x)[P_{sf}(x)]^+\mathrm{d}x+\lambda_1\int_0^\infty[P_D(x)]^+\mathrm{d}x\\=&\left(\frac{[P_{0,0}]^+}{P_{0,0}}-1\right)\int_0^\infty\eta_2(x)[P_R(x)]^+\mathrm{d}x+\left(\frac{[P_{0,0}]^+}{P_{0,0}}-1\right)\int_0^\infty\eta_1(x)\mathrm{d}x[P_f(x)]^+\Big\}\\\le&\,0,\end{aligned}$$

which shows the dispersivity of the operator $\mathscr{A}$.

Finally, we can deduce from the preceding results and the Fillips theorem (see Nagel [13]) that $\mathscr{A}$ generates a positive contraction C_0-semigroup $\mathscr{T}(t)$. □

It is clear that $\mathscr{X}^*$, $\mathscr{X}$'s dual space, is as follows

$$\mathscr{X}^* = \left\{ \mathscr{P}^* \left| \begin{array}{l} \mathscr{P}^* = \left(Q_0^*, Q_D^*(x), Q_f^*(x), Q_{sf}^*(x)\right) \in \mathbb{R}\times (L^\infty[0,\infty))^3 \\ |||\mathscr{P}^*||| = \sup\left\{|Q_0^*|, \|Q_D^*\|_{L^\infty[0,\infty)}, \|Q_f^*\|_{L^\infty[0,\infty)}, \|Q_{sf}^*\|_{L^\infty[0,\infty)}\right\} < \infty \end{array} \right. \right\}.$$

Clearly, $\mathscr{X}^*$ is a Banach space. Set

$$\mathscr{Y} = \left\{\mathbf{P}\in\mathscr{X} \,\middle|\, \mathbf{P}(x) = (P_0, P_D(x), P_f(x), P_{sf}(x)),\ P_0\geq 0,\ P_D(x), P_f(x), P_{sf}(x)\geq 0, \quad \forall x\in[0,\infty)\right\} \subset \mathscr{X}.$$

Then Theorem 2.1 guarantees that $\mathscr{T}(t)\mathscr{Y}\subset\mathscr{Y}$. For $\mathbf{P}\in D(A)\cap\mathscr{Y}$, choose $\mathscr{P}^*(x) = \|\mathbf{P}\|(1,1,1,1)^T$. Thus $\mathscr{P}^*\in\mathscr{X}^*$ and

$$\begin{aligned}
\langle \mathscr{A}\mathbf{P}, \mathscr{P}^*\rangle =& \|\mathbf{P}\|\left\{-(\lambda_1+\lambda_2)P_0 + \int_0^\infty \eta_2(x)P_D(x)\mathrm{d}x + \int_0^\infty \eta_1(x)P_f(x)\mathrm{d}x\right\}\\
&+\|\mathbf{P}\|\int_0^\infty\left\{-\frac{dP_D(x)}{dx} - \lambda_1 P_D(x) - \eta_2(x)P_D(x)\right\}\mathrm{d}x\\
&+\|\mathbf{P}\|\int_0^\infty\left\{-\frac{dP_f(x)}{dx} - \eta_1(x)P_f(x)\right\}\mathrm{d}x\\
&+\|\mathbf{P}\|\int_0^\infty\left\{-\frac{dP_{sf}(x)}{dx} - \eta_2(x)P_{sf}(x) + \lambda_1 P_D(x)\right\}\mathrm{d}x\\
=&\|\mathbf{P}\|\left\{-(\lambda_1+\lambda_2)P_0 + \int_0^\infty \eta_2(x)P_D(x)\mathrm{d}x + \int_0^\infty \eta_1(x)P_f(x)\mathrm{d}x\right\}\\
&+\|\mathbf{P}\|\left\{\lambda_2 P_0 - \lambda_1\int_0^\infty P_D(x)\mathrm{d}x - \int_0^\infty \eta_2(x)P_D(x)\mathrm{d}x\right\}\\
&+\|\mathbf{P}\|\left\{\lambda_1 P_0 + \int_0^\infty \eta_2(x)P_{sf}(x)\mathrm{d}x - \int_0^\infty \eta_1(x)P_f(x)\mathrm{d}x\right\}\\
&+\|\mathbf{P}\|\left\{-\int_0^\infty \eta_2(x)P_{sf}(x)\mathrm{d}x + \lambda_1\int_0^\infty P_D(x)\mathrm{d}x\right\} = 0.
\end{aligned}$$

This implies that $\mathscr{A}$ is conservative with respect to the set

$$\Theta(\mathbf{P}) = \left\{\mathscr{P}^*\in\mathscr{X}^* \mid \langle\mathbf{P},\mathscr{P}^*\rangle = \|P\|^2 = |||\mathscr{P}^*|||^2\right\}.$$

Because $\mathbf{P}(0)\in D(A^2)\cap\mathscr{Y}$, the Fattorini theorem [3, P.155] leads to the following.

Theorem 2.2. *$\mathscr{T}(t)$ is isometric for $\mathbf{P}(0)$, that is,* $\|\mathscr{T}(t)\mathbf{P}(0)\| = \|\mathbf{P}(0)\|, \quad \forall t\in[0,\infty)$.

We obtain the system's well-posedness by combining Theorem 2.1 and 2.2.

Theorem 2.3. *Under the condition of Theorem 2.1, system (2.1) has a unique positive time-dependent solution $\mathbf{P}(x,t)$ satisfying* $\|\mathbf{P}(\cdot,t)\| = 1$, $\forall t\in[0,\infty)$.

Proof. Since $\mathbf{P}(0)\in D(A^2)\cap\mathscr{Y}$, then, by Theorem 2.1 and [9, Theorem 1.81], the unique positive time-dependent solution $\mathbf{P}(x,t)$ to system (2.1) is obtained by $\mathbf{P}(x,t) = \mathscr{T}(t)\mathbf{P}(0)$, $t\in[0,\infty)$. It follows that $\|\mathbf{P}(\cdot,t)\| = \|\mathscr{T}(t)\mathbf{P}(0)\| = 1$, $\forall t\in[0,\infty)$. □

3. Exponential Stability of the Time-dependent Solution to System (2.1)

In this section, we study the asymptotic behavior. First, we provide the following useful lemma.

Lemma 3.1. *Under the same condition of Theorem 2.1, operator $A+U$ generates a positive contraction C_0-semigroup $\mathscr{V}(t)$.*

Lemma 3.2. *If $\mathbf{P}(x,t)=(\mathscr{V}(t)\phi)(x)$ is a solution of to*

$$\begin{aligned} &\frac{d\mathbf{P}}{dt}=(A+U)\mathbf{P}(t), \quad t\in[0,\infty),\\ &\mathbf{P}(0)=\phi\in D(A), \end{aligned} \tag{3.1}$$

then

$$\mathbf{P}(x,t)=(\mathscr{V}(t)\phi)(x)=\begin{cases} \begin{pmatrix} \phi_0 e^{-(\lambda_1+\lambda_2)t}\\ P_D(0,t-x)e^{-\lambda x-\int_0^x \eta_2(\tau)\,d\tau}\\ P_f(0,t-x)e^{-\int_0^x \eta_1(\tau)\,d\tau}\\ P_D(0,t-x)e^{-\int_0^x \eta_2(\tau)\,d\tau}(1-e^{-\lambda x}) \end{pmatrix}, & x<t,\\[2ex] \begin{pmatrix} \phi_0 e^{-(\lambda_1+\lambda_2)t}\\ \phi_D(x-t)e^{-\lambda_1 t-\int_{x-t}^x \eta_2(\tau)\,d\tau}\\ \phi_f(x-t)e^{-\int_{x-t}^x \eta_1(\tau)\,d\tau}\\ \phi_{sf}(x-t)e^{-\int_{x-t}^x \eta_2(\tau)\,d\tau}+e^{-\int_{x-t}^x \eta_2(\tau)\,d\tau}\phi_D(x-t)\left(1-e^{-\lambda_1 t}\right) \end{pmatrix}, & x>t. \end{cases}$$

where $P_j(0,t-x)$ $(j=R,f)$ is given by (1.2).

Proof. Obviously, $\mathbf{P}(x,t)$ satisfies

$$\frac{dP_0(t)}{dt}=-(\lambda_1+\lambda_2)P_0(t), \tag{3.2}$$

$$\frac{\partial P_R(x,t)}{\partial t}+\frac{\partial P_D(x,t)}{\partial x}=-(\lambda_1+\eta_2(x))P_D(x,t), \tag{3.3}$$

$$\frac{\partial P_f(x,t)}{\partial t}+\frac{\partial P_f(x,t)}{\partial x}=-\eta_1(x)P_f(x,t), \tag{3.4}$$

$$\frac{\partial P_{sf}(x,t)}{\partial t}+\frac{\partial P_{sf}(x,t)}{\partial x}=-\eta_2(x)P_{sf}(x,t)+\lambda_1 P_D(x,t), \tag{3.5}$$

$$P_D(0)=\lambda_2 P_0(t), P_f(0,t)=\lambda_1 P_0(t)+\int_0^\infty P_{sf}(x,t)\eta_2(x)\mathrm{d}x, P_{sf}(0,t)=0, \tag{3.6}$$

$$P_0(0)=\phi_0, \quad P_l(x,0)=\phi_l(x), \quad l=D,f,sf.$$

Let $\upsilon=x-t$ and $F_l(t)=P_l(\upsilon+t,t)$, $l=R,f,sf$, From (3.3)-(3.5), we have

$$\frac{dF_D(t)}{dt}=-(\lambda_1+\varphi(\upsilon+t))F_D(t), \tag{3.7}$$

$$\frac{dF_f(t)}{dt}=-\eta(\upsilon+t)F_f(t), \tag{3.8}$$

$$\frac{dF_{sf}(t)}{dt}=-\varphi(\upsilon+t)F_{sf}(t)+\lambda_1 F_D(t). \tag{3.9}$$

If $\upsilon<0$ (i.e., $x<t$), then by integrating (3.7)-(3.9) from $-\upsilon$ to t separately and using $F_l(-\upsilon)=P_l(0,-\upsilon)=P_l(0,t-x)$ $(l=D,f,sf)$, we have

$$P_D(x,t)=F_D(t)=F_D(-\upsilon)e^{-\lambda_1(\upsilon+t)-\int_{-\upsilon}^{t}\eta_2(\upsilon+\tau)\,d\tau}\overset{s=\upsilon+\tau}{=\!=\!=}P_D(0,t-x)e^{-\lambda_1 x-\int_0^x\eta_2(s)\,ds},\tag{3.10}$$

$$P_f(x,t)=F_f(t)=F_f(-\upsilon)e^{-\int_{-\upsilon}^{t}\eta_1(\upsilon+\tau)\,d\tau}\overset{s=\upsilon+\tau}{=\!=\!=}P_f(0,t-x)e^{-\int_0^x\eta_1(s)\,ds},\tag{3.11}$$

$$\begin{aligned}P_{sf}(x,t)&=F_{sf}(t)=F_{sf}(-\upsilon)e^{-\int_{-\upsilon}^{t}\eta_2(\upsilon+\tau)\,d\tau}+\lambda e^{-\int_{-\upsilon}^{t}\eta_2(\upsilon+\tau)\,d\tau}\int_{-\upsilon}^{t}F_D(\sigma)e^{\int_{-\upsilon}^{\sigma}\eta_2(\upsilon+\tau)\,d\tau}d\sigma\\&\overset{s=\upsilon+\tau}{=\!=\!=}P_{sf}(0,t-x)e^{-\int_0^x\varphi(s)\,ds}+\lambda_1 e^{-\int_0^x\eta_2(s)\,ds}\int_{-\upsilon}^{t}F_D(\sigma)e^{\int_0^{\sigma+\upsilon}\eta_2(s)\,ds}d\sigma\\&\overset{z=\sigma+\upsilon}{=\!=\!=}\lambda_1 e^{-\int_0^x\eta_2(s)\,ds}\int_0^x F_D(z-\upsilon)e^{\int_0^z\eta_2(s)\,ds}dz\\&=P_R(0,t-x)e^{-\int_0^x\eta_2(s)\,ds}\left(1-e^{-\lambda_1 x}\right).\end{aligned}\tag{3.12}$$

Combining (3.2) with (3.6) gives

$$P_0(t)=\phi_0 e^{-(\lambda_1+\lambda_2)t}.\tag{3.13}$$

If $\upsilon>0$ (i.e., $x>t$), by integrating (3.7)-(3.9) from 0 to t, we deduce

$$\begin{aligned}P_D(x,t)&=F_D(t)=F_D(0)e^{-\lambda_1 t-\int_0^t\eta_2(\upsilon+\tau)\,d\tau}\\&\overset{\sigma=\upsilon+\tau}{=\!=\!=}\phi_D(x-t)e^{-\lambda_1 t-\int_{\upsilon}^{\upsilon+t}\eta_2(\sigma)\,d\sigma}=\phi_D(x-t)e^{-\lambda_1 t-\int_{x-t}^{x}\eta_2(\tau)\,d\tau},\end{aligned}\tag{3.14}$$

$$\begin{aligned}P_f(x,t)&=F_f(t)=F_f(0)e^{-\int_0^t\eta_1(\upsilon+\tau)\,d\tau}\\&\overset{\sigma=\upsilon+\tau}{=\!=\!=}\phi_f(x-t)e^{-\int_{\upsilon}^{\upsilon+t}\eta_1(\sigma)\,d\sigma}=\phi_f(x-t)e^{-\int_{x-t}^{x}\eta_1(\tau)\,d\tau},\end{aligned}\tag{3.15}$$

$$\begin{aligned}P_{sf}(x,t)&=F_{sf}(t)=F_{sf}(0)e^{-\int_0^t\eta_2(\upsilon+\tau)\,d\tau}+\lambda_1 e^{-\int_0^t\eta_2(\upsilon+\tau)\,d\tau}\int_0^t F_D(\sigma)e^{\int_0^{\sigma}\eta_2(\upsilon+\tau)\,d\tau}d\sigma\\&\overset{s=\upsilon+\tau}{=\!=\!=}\phi_{sf}(x-t)e^{-\int_{\upsilon}^{\upsilon+t}\eta_2(s)\,ds}+\lambda e^{-\int_{\upsilon}^{\upsilon+t}\eta_2(s)\,ds}\int_0^t F_D(\sigma)e^{\int_{\upsilon}^{\upsilon+\sigma}\eta_2(s)\,ds}d\sigma\\&\overset{z=\sigma+\upsilon}{=\!=\!=}\phi_{sf}(x-t)e^{-\int_{x-t}^{x}\eta_2(s)\,ds}+\lambda_1 e^{-\int_{x-t}^{x}\eta_2(s)\,ds}\int_{\upsilon}^{\upsilon+t}F_D(z-\upsilon)e^{\int_{\upsilon}^{z}\eta_2(s)\,ds}dz\\&=\phi_{sf}(x-t)e^{-\int_{x-t}^{x}\eta_2(s)\,ds}+\lambda e^{-\int_{x-t}^{x}\eta_2(s)\,ds}\int_{x-t}^{x}P_D(z,z-\upsilon)e^{\int_{\upsilon}^{z}\eta_2(s)\,ds}dz\\&=\phi_{sf}(x-t)e^{-\int_{x-t}^{x}\eta_2(s)\,ds}+\lambda_1 e^{-\int_{x-t}^{x}\eta_2(s)\,ds}\phi_D(x-t)e^{\lambda_1(x-t)}\int_{x-t}^{x}e^{-\lambda_1 z}dz\\&=\phi_{sf}(x-t)e^{-\int_{x-t}^{x}\eta_2(s)\,ds}+e^{-\int_{x-t}^{x}\eta_2(s)\,ds}\phi_D(x-t)\left(1-e^{-\lambda_1 t}\right).\end{aligned}\tag{3.16}$$

By using (3.10)-(3.16), we conclude the desired conclusion immediately. □

The following result is based on [9, Th. 1.35].

Corollary 3.3. *A bounded subset $G\subset\mathscr{X}$ is relatively compact if and only if the following conditions hold* $\lim_{h\to 0}\sum_{n=1}^{3}\int_0^\infty|g_n(x+h)-g_n(x)|\,\mathrm{d}x=0$, *uniformly for* $g=(g_0,g_1,g_2,g_3)\in G$ *and* $\lim_{h\to\infty}\sum_{n=1}^{3}\int_h^\infty|g_n(x)|\,\mathrm{d}x=0$, *uniformly for* $g=(g_0,g_1,g_2,g_3)\in G$.

Now, we need to prove that $\mathscr{V}(t)$ is a quasi compact on $\mathscr{X}$. Define, for $\mathbf{P} \in \mathscr{X}$,

$$(\mathscr{V}_1(t)\mathbf{P})(x) = \begin{cases} (\mathscr{V}(t)\mathbf{P})(x) & x \in [0,t), \\ 0 & x \in [t,\infty), \end{cases} \quad (\mathscr{V}_2(t)\mathbf{P})(x) = \begin{cases} 0 & x \in [0,t), \\ (\mathscr{V}(t)\mathbf{P})(x) & x \in [t,\infty), \end{cases}$$

Obviously, $(\mathscr{V}(t)\mathbf{P})(x) = (\mathscr{V}_1(t)\mathbf{P})(x) + (\mathscr{V}_2(t)\mathbf{P})(x)$, $\forall \mathbf{P} \in \mathscr{X}$.

Theorem 3.4. *Assume that $\eta_1(x)$ and $\eta_2(x)$ are Lipschitz continuous and satisfy $0 < \underline{\eta_1} \leq \eta_1(x) \leq \overline{\eta_1} < \infty$ and $0 < \underline{\eta_2} \leq \eta_2(x) \leq \overline{\eta_2} < \infty$. Then $\mathscr{V}_1(t)$ is a compact on $\mathscr{X}$.*

Proof. We only need to show that the condition (1) hold in Corollary 3.3. Set $\mathbf{P}(x,t) = (\mathbb{S}(t)\phi)(x)$, $x \in [0,t)$, for $\phi \in \mathscr{X}$, then $\mathbf{P}(x,t)$ is a solution of the system (4.1). So, by Lemma 3.2 we get, for $x \in [0,t), \sigma \in (0,t], x+\sigma \in [0,t)$

$$\begin{aligned}
&\sum_{l=D,f,sf} \int_0^t |P_l(x+\sigma,t) - P_l(x,t)| \mathrm{d}x \\
&\leq \int_0^t |P_D(0,t-x-\sigma)| \left| e^{-\lambda_1(x+\sigma) - \int_0^{x+\sigma} \eta_2(s)\,ds} - e^{-\lambda x - \int_0^x \eta_2(s)\,ds} \right| \mathrm{d}x \\
&\quad + \int_0^t |P_D(0,t-x-\sigma) - P_D(0,t-x)| \left| e^{-\lambda x - \int_0^x \eta_2(s)\,ds} \right| \mathrm{d}x \\
&\quad + \int_0^t |P_f(0,t-x-\sigma)| \left| e^{-\int_0^{x+\sigma} \eta_1(s)\,ds} - e^{-\int_0^x \eta_2(s)\,ds} \right| \mathrm{d}x \\
&\quad + \int_0^t |P_f(0,t-x-\sigma) - P_f(0,t-x)| \left| e^{-\int_0^x \eta_1(s)\,ds} \right| \mathrm{d}x \\
&\quad + \int_0^t |P_D(0,t-x-\sigma)| \left| e^{-\int_0^{x+\sigma} \eta_2(s)\,ds} \left(1 - e^{-\lambda_1(x+\sigma)}\right) - e^{-\int_0^x \eta_2(s)\,ds} \left(1 - e^{-\lambda_1 x}\right) \right| \mathrm{d}x \\
&\quad + \int_0^t |P_D(0,t-x-\sigma) - P_D(0,t-x)| \left| e^{-\int_0^x \eta_2(s)\,ds} \left(1 - e^{-\lambda_1 x}\right) \right| \mathrm{d}x. \qquad (3.17)
\end{aligned}$$

Now, we proceed to estimate each term in (3.17). Observe that

$$|P_D(0,t-x-\sigma)| \leq \lambda_2 \left| \phi_0 e^{-(\lambda_1+\lambda_2)(t-x-\sigma)} \right| \leq \lambda_1 \|\phi\|, \qquad (3.18)$$

$$\begin{aligned}
|P_f(0,t-y-\sigma)| &\leq \max\{\lambda_1, \overline{\eta_2}\} \left\{ |P_0(t-y-\sigma)| + \int_0^\infty |P_{sf}(s,t-x-\sigma)|\,ds \right\} \\
&\leq \max\{\lambda_1, \overline{\eta_2}\} \|P(\cdot,t-x-\sigma)\|_{\mathscr{X}} = \max\{\lambda_1, \overline{\eta_2}\} \|V(t-x-\sigma)\phi(\cdot)\|_{\mathscr{X}} \\
&\leq \max\{\lambda_1, \overline{\eta_2}\} \|\phi\|, \qquad (3.19)
\end{aligned}$$

$$|P_{sf}(0,t-x-\sigma)| = 0. \qquad (3.20)$$

Hence, the first, third and fifth term of (3.17) vanish as $|\sigma|$ approaches 0,

$$\begin{aligned}
&\int_0^t |P_D(0,t-x-\sigma)| \left| e^{-\lambda_1(x+\sigma) - \int_0^{x+\sigma} \eta_2(s)\,ds} - e^{-\lambda_1 x - \int_0^x \eta_2(s)\,ds} \right| \mathrm{d}x \\
&\leq \lambda_2 \|\phi\| \int_0^t \left| e^{-\lambda_1(x+\sigma) - \int_0^{x+\sigma} \eta_2(s)\,ds} - e^{-\lambda_1 x - \int_0^x \eta_2(s)\,ds} \right| \mathrm{d}x \to 0 \text{ as } |\sigma| \to 0 \text{ uniformly for } \phi,
\end{aligned}$$

$$\begin{aligned}
&\int_0^t |P_f(0,t-x-\sigma)| \left| e^{-\int_0^{x+\sigma} \eta_1(s)\,ds} - e^{-\int_0^x \eta_1(s)\,ds} \right| \mathrm{d}x \\
&\leq \max\{\lambda_1, \overline{\eta_1}\} \|\phi\| \int_0^t \left| e^{-\int_0^{x+\sigma} \eta_1(s)\,ds} - e^{-\int_0^x \eta_1(s)\,ds} \right| \mathrm{d}x \to 0 \text{ as } |\sigma| \to 0 \text{ uniformly for } \phi
\end{aligned}$$

and

$$\int_0^t |P_D(0,t-x-\sigma)| \left| e^{-\int_0^{x+\sigma} \eta_2(s)\, ds}\left(1-e^{-\lambda_1(x+\sigma)}\right) - e^{-\int_0^{x} \eta_2(s)\, ds}\left(1-e^{-\lambda_1 x}\right)\right| \mathrm{d}x$$
$$\leq \lambda_2 \|\phi\| \int_0^t \left| e^{-\int_0^{x+\sigma} \eta_2(s)\, ds}\left(1-e^{-\lambda_1(x+\sigma)}\right) - e^{-\int_0^{x} \eta_2(s)\, ds}\left(1-e^{-\lambda_1 x}\right)\right| \mathrm{d}x \to 0$$

as $|\sigma| \to 0$ uniformly for ϕ. By the condtion of this theorem, we know that $\eta_2(x)$ are Lipschitz continuous (Assume that the Lipschitz constant is equal to one without losing generality.). Using the boundary condition, we obtain

$$|P_D(0,t-x-\sigma) - P_D(0,t-x)| \leq \lambda_2 \left| \phi_0 e^{-(\lambda_1+\lambda_2)(t-x-\sigma)} - \phi_0 e^{-(\lambda_1+\lambda_2)(t-x)} \right| \to 0$$

as $|\sigma| \to 0$ uniformly for ϕ,

$$\begin{aligned}
&\left| P_f(0,t-x-\sigma) - P_f(0,t-x) \right| \\
&\leq \lambda_1 |\phi_0| \left| e^{-(\lambda_1+\lambda_2)(t-x-\sigma)} - e^{-(\lambda_1+\lambda_2)(t-x)} \right| \\
&\quad + \left| \int_0^{t-x-\sigma} P_{sf}(t-x-\sigma-z, t-x-\sigma)\eta_2(t-x-\sigma-z) dz \right. \\
&\quad \left. - \int_0^{t-x} P_{sf}(t-x-z, t-x)\eta_2(t-x-z) dz \right| \\
&\quad + \left| \int_{t-x-\sigma}^{\infty} \phi_{sf}(\xi - t + x + \sigma)\eta_2(\xi) e^{-\int_{\xi-t+x+\sigma}^{\xi} \eta_2(s) ds} d\xi \right. \\
&\quad \left. - \int_{t-x}^{\infty} \phi_{sf}(\xi - t + x)\eta_2(\xi) e^{-\int_{\xi-t+x}^{\xi} \eta_2(s) ds} d\xi \right| \\
&\quad + \left| \int_{t-x-\sigma}^{\infty} \phi_R(\xi - t + x + \sigma)\eta_2(\xi) e^{-\int_{\xi-t+x+\sigma}^{\xi} \eta_2(s) ds} \left(1 - e^{-\lambda_1(t-x-\sigma)}\right) d\xi \right. \\
&\quad \left. - \int_{t-x}^{\infty} \phi_R(\xi - t + x)\eta_2(\xi) e^{-\int_{\xi-t+x}^{\xi} \eta_2(s) ds} \left(1 - e^{-\lambda_1(t-x)}\right) d\xi \right|.
\end{aligned}$$

Letting $\xi - t + x + \sigma = z$ and $\xi - t + x = z$ yields that

$$
\begin{aligned}
&\left|P_f(0,t-x-\sigma) - P_f(0,t-x)\right| \\
&\le \lambda_1 |\phi_0| \left| e^{-(\lambda_1+\lambda_2)(t-x-\sigma)} - e^{-(\lambda_1+\lambda_2)(t-x)} \right| \\
&\quad + \left| \int_0^{t-x-\sigma} \eta_2(t-x-\sigma-z) P_R(0,z) e^{-\int_0^{t-x-\sigma-z}\eta_2(s)ds}\left(1-e^{-\lambda_1(t-x-\sigma)}\right)dz \right. \\
&\quad \left. - \int_0^{t-x} \eta_2(t-x-z) P_D(0,z) e^{-\int_0^{t-x-z}\eta_2(s)ds}\left(1-e^{-\lambda_1(t-x)}\right)dz \right| \\
&\quad + \left| \int_0^{\infty} \phi_{sf}(z)\eta_2(z+t-x-\sigma) e^{-\int_z^{z+t-x-\sigma}\eta_2(s)ds} dz \right. \\
&\quad \left. - \int_0^{\infty} \phi_{sf}(z)\eta_2(z+t-x) e^{-\int_z^{z+t-x}\eta_2(s)ds} dz \right| \\
&\quad + \left| \int_0^{\infty} \phi_R(z)\eta_2(z+t-x-\sigma) e^{-\int_z^{z+t-x-\sigma}\eta_2(s)ds}\left(1-e^{-\lambda_1(t-x-\sigma)}\right) dz \right. \\
&\quad \left. - \int_0^{\infty} \phi_R(z)\eta_2(z+t-x) e^{-\int_z^{z+t-x}\eta_2(s)ds}\left(1-e^{-\lambda_1(t-x)}\right) dz \right| \\
&\le \lambda_1 |\phi_0| \left| e^{-(\lambda_1+\lambda_2)(t-x-\sigma)} - e^{-(\lambda_1+\lambda_2)(t-x)} \right| \\
&\quad + \int_{t-x-\sigma}^{t-x} \eta_2(t-x-\sigma-z)\,|P_R(0,z)|\, e^{-\int_0^{t-x-\sigma-z}\eta_2(s)ds}\left(1-e^{-\lambda_1(t-x-\sigma)}\right)dz \\
&\quad + \int_0^{t-x} |P_R(0,z)|\,|\eta_2(t-x-\sigma-z)-\eta_2(t-x-z)|\, e^{-\int_0^{t-x-z}\eta_2(s)ds}\left(1-e^{-\lambda_1(t-x)}\right)dz \\
&\quad + \int_0^{t-x} \eta_2(t-x-z)\,|P_R(0,z)|\, \left| e^{-\int_0^{t-x-\sigma-z}\eta_2(s)ds}\left(1-e^{-\lambda_1(t-x-\sigma)}\right) \right. \\
&\quad \left. - e^{-\int_0^{t-x-z}\eta_2(s)ds}\left(1-e^{-\lambda_1(t-x)}\right) \right| dz \\
&\quad + \left| \int_0^{\infty} \left|\phi_{sf}(z)\right| |\eta_2(z+t-x-\sigma)-\eta_2(z+t-x)|\, e^{-\int_z^{z+t-x-\sigma}\eta_2(s)ds} dz \right. \\
&\quad \left. - \int_0^{\infty} \eta_2(z+t-x)\left|\phi_{sf}(z)\right| \left| e^{-\int_z^{z+t-x-\sigma}\eta_2(s)ds} - e^{-\int_z^{z+t-x}\eta_2(s)ds}\right| dz \right. \\
&\quad + \left| \int_0^{\infty} |\phi_R(z)|\,|\eta_2(z+t-x-\sigma)-\eta_2(z+t-x)|\, e^{-\int_z^{z+t-x-\sigma}\eta_2(s)ds}\left(1-e^{-\lambda_1(t-x-\sigma)}\right) dz \right. \\
&\quad - \int_0^{\infty} \eta_2(z+t-x)\,|\phi_R(z)|\, \left| e^{-\int_z^{z+t-x-\sigma}\eta_2(\tau)d\tau}\left(1-e^{-\lambda_1(t-x-\sigma)}\right) \right. \\
&\quad \left. - e^{-\int_z^{z+t-x}\eta_2(\tau)d\tau}\left(1-e^{-\lambda_1(t-x)}\right) \right| dz \\
&\le \lambda_1 |\phi_0| \left| e^{-(\lambda_1+\lambda_2)(t-x-\sigma)} - e^{-(\lambda_1+\lambda_2)(t-x)} \right| \\
&\quad + \lambda_1 \|\phi\| \left\{ \overline{\eta_2}|\sigma| \sup_{z\in[0,\infty)} e^{-\int_0^{t-x-\sigma-z}\eta_2(s)ds} + |\sigma| \int_0^{t-x} e^{-\int_0^{t-x-\sigma-z}\eta_2(s)ds}\left(1-e^{-\lambda_1(t-x-\sigma)}\right)dz \right. \\
&\quad \left. + \overline{\eta_2} \int_0^{t-x} \left| e^{-\int_0^{t-x-\sigma-z}\eta_2(s)ds}\left(1-e^{-\lambda_1(t-x-\sigma)}\right) - e^{-\int_0^{t-x-z}\eta_2(s)ds}\left(1-e^{-\lambda_1(t-x)}\right) \right| dz \right\} \\
&\quad + \|\phi\| \left\{ |\sigma| \sup_{z\in[0,\infty)} e^{-\int_z^{z+t-x-\sigma}\eta_2(s)ds} + \overline{\eta_2} \sup_{z\in[0,\infty)} \left| e^{-\int_z^{z+t-x-\sigma}\eta_2(s)ds} - e^{-\int_z^{z+t-x}\eta_2(s)ds} \right| \right.
\end{aligned}
$$

$$+|\sigma| \sup_{z\in[0,\infty)} e^{-\int_z^{z+t-x-\sigma}\eta_2(s)ds}$$

$$+\overline{\eta_2} \sup_{z\in[0,\infty)} \left| e^{-\int_z^{z+t-x-\sigma}\eta_2(s)ds}\left(1-e^{-\lambda_1(t-x-\sigma)}\right) - e^{-\int_z^{z+t-x}\eta_2(s)ds}\left(1-e^{-\lambda_1(t-x)}\right)\right| \Bigg\} \to 0$$

as $|\sigma| \to 0$ uniformly for ϕ. Thus we derive the remaining term of (3.17) vanish as $|\sigma| \to 0$. For $x \in [0,t), \sigma \in (0,t], x+\sigma \in [0,t)$, we have

$$\sum_{l=D,f,sf} \int_0^t |P_l(x+\sigma,t) - P_l(x,t)| \mathrm{d}x \quad \text{as } |\sigma| \to 0 \text{ uniformly for } \phi. \tag{3.21}$$

The same conclusion as (3.21) can be drawn for $x+\sigma \in [0,t)$,$\sigma \in [-t,0)$,, and the proof is complete. ☐

Theorem 3.5. *If the conditions of Theorem 3.4 hold, then $\mathscr{V}_2(t)$ satisfies*

$$\|\mathscr{V}_2(t)\phi\|_{\mathscr{X}} \le e^{-\min\{\lambda_1+\lambda_2,\underline{\eta_1},\underline{\eta_2}\}t}\|\phi\|_{\mathscr{X}}, \ \forall \phi \in \mathscr{X}.$$

From Theorems 3.4 and 3.5, we have that

$$\|\mathscr{V}(t) - \mathscr{V}V_1(t)\| = \|\mathscr{V}_2(t)\| \leqslant e^{-\min\{\lambda_1+\lambda_2,\underline{\eta_1},\underline{\eta_2}\}t} \to 0, \quad t \to \infty.$$

Theorem 3.6. *$\mathscr{V}(t)$ is a quasi-compact operator on $\mathscr{X}$.*

Because E is a compact operator on $\mathscr{X}$, we can derive the following result from Theorem 3.6 and Proposition 2.9 in Nagel [13, P.215].

Corollary 3.7. *$\mathscr{T}(t)$ is a quasi-compact operator on $\mathscr{X}$.*

Lemma 3.8. $0 \in \sigma_p(\mathscr{A})$, *and geometric multiplicity of* 0 *is one.*

Proof. Take $\mathscr{A}\mathbf{P} = 0$, i.e.,

$$(\lambda_1+\lambda_2)P_0 = \int_0^\infty \eta_2(x)P_D(x)\mathrm{d}x + \int_0^\infty \eta_1(x)P_f(x)\mathrm{d}x, \tag{3.22}$$

$$\frac{dP_D(x)}{dx} = -(\lambda_1+\eta_2(x))P_D(x), \tag{3.23}$$

$$\frac{dP_f(x)}{dx} = -\eta_1(x)P_f(x), \tag{3.24}$$

$$\frac{dP_{sf}(x)}{dx} = -\eta_2(x)P_{sf}(x) + \lambda_1 P_D(x), \tag{3.25}$$

$$P_D(0) = \lambda_2 P_0, \tag{3.26}$$

$$P_f(0) = \lambda_1 P_0 + \int_0^\infty P_{sf}(x)\eta_2(x)\mathrm{d}x, \tag{3.27}$$

$$P_3(0) = 0. \tag{3.28}$$

Solving (3.23)-(3.25), we have

$$P_D(x) = \alpha_D e^{-\lambda_1 x - \int_0^x \eta_2(\tau)d\tau}, \tag{3.29}$$

$$P_f(x) = \alpha_f e^{-\int_0^x \eta_1(\tau)d\tau}, \tag{3.30}$$

$$P_{sf}(x) = \lambda_1 e^{-\int_0^x \eta_2(\tau)d\tau} \int_0^x P_D(\xi) e^{\int_0^\xi \eta_2(\tau)d\tau} d\xi. \tag{3.31}$$

From (3.26) and (3.27), we deduce

$$\alpha_{D} = \lambda_2 P_0, \tag{3.32}$$

$$\begin{aligned}\alpha_{f} &= P_{f}(0) = \lambda_1 P_0 + \lambda_1 \int_0^\infty \eta_2(x) e^{-\int_0^x \eta_2(\tau)d\tau} \int_0^x P_{D}(\xi) e^{\int_0^\xi \eta_2(\tau)d\tau} d\xi \mathrm{d}x \\ &= \lambda_1 \left(1 + \lambda_2 \int_0^\infty e^{-\lambda_1 x - \int_0^x \eta_2(\tau)d\tau} \mathrm{d}x\right) P_0.\end{aligned} \tag{3.33}$$

Combining (3.29)-(3.31) with (3.32)-(3.33) yields

$$\begin{aligned}\|P\| &= |P_0| + \|P_{D}\|_{L^1[0,\infty)} + \|P_{f}\|_{L^1[0,\infty)} + \|P_{sf}\|_{L^1[0,\infty)} \\ &\leq |P_0| + |\alpha_{D}| \int_0^\infty e^{-\lambda_1 x - \int_0^x \eta_2(\tau)d\tau} \mathrm{d}x + |\alpha_{f}| \int_0^\infty e^{-\int_0^x \eta_1(\tau)d\tau} \mathrm{d}x \\ &\quad + \int_0^\infty \lambda_1 e^{-\int_0^x \eta_2(\tau)d\tau} \int_0^x |P_{D}(\xi)| e^{\int_0^\xi \eta_2(\tau)d\tau} d\xi \mathrm{d}x \\ &= |P_0| + \lambda_2 \int_0^\infty e^{-\int_0^x \eta_2(\tau)d\tau} \mathrm{d}x \, |P_0| \\ &\quad + \lambda_1 \left(1 + \lambda_2 \int_0^\infty e^{-\lambda_1 x - \int_0^x \eta_2(\tau)d\tau} \mathrm{d}x\right) \int_0^\infty e^{-\int_0^x \eta_1(\tau)d\tau} \mathrm{d}x \, |P_0| \\ &\quad + \lambda_2 \left(1 - \int_0^\infty e^{-\lambda_1 x - \int_0^x \eta_2(\tau)d\tau} \mathrm{d}x\right) |P_0| < \infty.\end{aligned}$$

This demonstrates that 0 is an eigenvalue of $\mathscr{A}$. Furthermore, it is obvious that the geometric multiplicity of 0 is one by (3.32) and (3.33). □

Lemma 3.9. *$\mathscr{A}^*$ is given by*

$$\mathscr{A}^* \mathbf{Q}^* = (\mathscr{G} + \mathscr{F}) \mathbf{Q}^*, \quad \mathbf{Q}^* \in D(\mathscr{A}^*) = D(\mathscr{G}),$$

where

$$\mathscr{G}\mathbf{Q}^* = \left(\begin{pmatrix} -(\lambda_1+\lambda_2) & 0 & 0 & 0 \\ 0 & \frac{d}{\mathrm{d}x} - (\lambda_1 + \eta_2(x)) & 0 & 0 \\ 0 & 0 & \frac{d}{\mathrm{d}x} - \eta_1(x) & 0 \\ 0 & 0 & 0 & \frac{d}{\mathrm{d}x} - \eta_2(x) \end{pmatrix} \begin{pmatrix} Q_0^* \\ Q_{D}^*(x) \\ Q_{f}^*(x) \\ Q_{sf}^*(x) \end{pmatrix} \right),$$

$$\mathscr{F}\mathbf{Q}^* = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} Q_0^* \\ Q_{D}^*(x) \\ Q_{f}^*(x) \\ Q_{sf}^*(x) \end{pmatrix} + \begin{pmatrix} 0 & \lambda_2 & \lambda_1 & 0 \\ \eta_2(x) & 0 & 0 & 0 \\ \eta_1(x) & 0 & 0 & 0 \\ 0 & 0 & \eta_2(x) & 0 \end{pmatrix} \begin{pmatrix} Q_0^* \\ Q_{D}^*(0) \\ Q_{f}^*(0) \\ Q_{sf}^*(0) \end{pmatrix}$$

$$D(\mathscr{G}) = \left\{ \mathbf{Q}^* \in \mathscr{X}^* \,\middle|\, \frac{dQ_l^*(x)}{dx} \text{ exists and } Q_l^*(\infty) = \zeta \ (l = D, f, sf) \right\},$$

in $D(\mathscr{G})$, and ζ is a constant.

Lemma 3.10. $0 \in \sigma_p(\mathscr{A}^*)$, *and geometric multiplicity of* 0 *is one.*

Proof. Consider $\mathscr{A}^* \mathbf{Q}^* = 0$, that is,

$$-(\lambda_1 + \lambda_2) Q_0^* + \lambda_2 Q_{D}^*(0) + \lambda Q_{f}^*(0) = 0, \tag{3.34}$$

$$\frac{dQ_D^*(x)}{dx} - (\lambda_1 + \eta_2(x))Q_D^*(x) + \eta_2(x)Q_0^* + \lambda_1 Q_{sf}^*(x) = 0, \tag{3.35}$$
$$\frac{dQ_f^*(x)}{dx} - \eta_1(x)Q_f^*(x) + \eta_1(x)Q_0^* = 0, \tag{3.36}$$
$$\frac{dQ_{sf}^*(x)}{dx} - \eta_1(x)Q_{sf}^*(x) + \eta_2(x)Q_f^*(0) = 0, \tag{3.37}$$
$$Q_D^*(\infty) = Q_f^*(\infty) = Q_{sf}^*(\infty) = \zeta. \tag{3.38}$$

Solving (3.35)-(3.37), we have

$$Q_D^*(x) = \beta_D e^{\int_0^x(\lambda+\varphi(\tau))d\tau} - e^{\int_0^x(\lambda_1+\eta_2(\tau))d\tau}\int_0^x \left[\eta_2(\xi)Q_0^* + \lambda_1 Q_{sf}^*(\xi)\right] e^{-\int_0^\xi(\lambda_1+\eta_2(\tau))d\tau}d\xi, \tag{3.39}$$

$$Q_f^*(x) = \beta_f e^{\int_0^x \eta_1(\tau)d\tau} - e^{\int_0^x \eta_1(\tau)d\tau}\int_0^x \eta_1(\xi)Q_0^* e^{-\int_0^\xi \eta_1(\tau)d\tau}d\xi, \tag{3.40}$$

$$Q_{sf}^*(x) = \beta_{sf} e^{\int_0^x \eta_2(\tau)d\tau} - e^{\int_0^x \eta_2(\tau)d\tau}\int_0^x \eta_2(\xi)Q_f^*(0)e^{-\int_0^\xi \eta_2(\tau)d\tau}d\xi. \tag{3.41}$$

Multiplying $e^{-\int_0^y(\lambda_1+\eta_2(\tau))d\tau}$, $e^{-\int_0^x \eta_1(\tau)d\tau}$ and $e^{-\int_0^x \eta_2(\tau)d\tau}$ to the both side of (3.39), (3.40) and (3.41) separately, we have

$$\beta_D = \int_0^\infty \left[\varphi(\xi)Q_0^* + \lambda_1 Q_{sf}^*(\xi)\right] e^{-\int_0^\xi(\lambda_1+\eta_2(\tau))d\tau}d\xi, \tag{3.42}$$
$$\beta_f = \int_0^\infty \eta_1(\xi)Q_0^* e^{-\int_0^\xi \eta_1(\tau)d\tau}d\xi, \tag{3.43}$$
$$\beta_{sf} = \int_0^\infty \eta_2(\xi)Q_f^*(0)e^{-\int_0^\xi \eta_2(\tau)d\tau}d\xi. \tag{3.44}$$

Substituting (3.43) into (3.40), we have

$$\begin{aligned} Q_f^*(x) &= Q_0^* e^{\int_0^x \eta_1(\tau)d\tau}\int_x^\infty \eta_1(\xi)e^{-\int_0^\xi \eta_1(\tau)d\tau}d\xi \\ &= Q_0^* e^{\int_0^x \eta_1(\tau)d\tau}\left(-e^{-\int_0^\xi \eta_1(\tau)d\tau}\Big|_x^\infty\right) = Q_0^*. \end{aligned} \tag{3.45}$$

Combining (3.45), (3.44), and (3.41) yields

$$Q_{sf}^*(x) = Q_f^*(0)\, e^{\int_0^x \eta_2(\tau)d\tau}\int_x^\infty \eta_2(\xi)e^{-\int_0^\xi \varphi(\tau)d\tau}d\xi = Q_f^*(0) = Q_0^*. \tag{3.46}$$

Substituting (3.46) and (3.42) into (3.39), we deduce

$$\begin{aligned} Q_D^*(x) &= e^{\int_0^x(\lambda_1+\eta_2(\tau))d\tau}\int_x^\infty \left[\eta_2(\xi)Q_0^* + \lambda_1 Q_{sf}^*(\xi)\right] e^{-\int_0^\xi(\lambda_1+\eta_2(\tau))d\tau}d\xi \\ &= Q_0^* e^{\int_0^x(\lambda_1+\eta_2(\tau))d\tau}\int_x^\infty [\lambda_1+\eta_2(\xi)]\, e^{-\int_0^\xi(\lambda_1+\eta_2(\tau))d\tau}d\xi \\ &= Q_0^* e^{\int_0^x(\lambda_1+\eta_2(\tau))d\tau}\left(-e^{-\int_0^\xi(\lambda_1+\eta_2(\tau))d\tau}\Big|_x^\infty\right) = Q_0^*, \end{aligned} \tag{3.47}$$

Thus we have the following estimation

$$|||\,\mathbf{Q}^*||| = \sup\left\{|Q_0^*|, \|Q_D^*\|_{L^\infty[0,\infty)}, \|Q_f^*\|_{L^\infty[0,\infty)}, \|Q_{sf}^*\|_{L^\infty[0,\infty)}\right\} = |Q_0^*| < \infty.$$

Hence, $0 \in \sigma_p(\mathscr{A}^*)$, and the geometric multiplicity of 0 is one from (3.45)-(3.47). □

We conclude from Lemmas 3.8 and 3.10 with Theorem 2.3 that the algebraic multiplicity of 0 is one and $s(\mathscr{A})=0$. Therefore, According to [9, Theorem 1.90], we obtain the following theorem.

Theorem 3.11. *If $\eta_1(x)$ and $\eta_2(x)$ are Lipschitz continuous and satisfy $\underline{\eta_1}\leq\eta_1(x)\leq\overline{\eta_1}<\infty$ and $0<\underline{\eta_2}\leq\eta_2(x)\leq\overline{\eta_2}<\infty$, then there exist $\delta>0$, $M\geq 0$, and a positive projection $\mathbb{P}\mathbb{r}$ of rank one such that $\|\ \mathscr{T}(t)-\mathbb{P}\mathbb{r}\ \|\leq Me^{-\delta t}$, where $\mathbb{P}\mathbb{r}=\frac{1}{2\pi i}\int_{\overline{\Gamma}}(zI-\mathscr{A})^{-1}\mathrm{d}z$, and $\overline{\Gamma}$ is a circle with a zero-centered and a sufficiently small radius.*

In the following, we give the expression of the $\mathbb{P}\mathbb{r}$.

Lemma 3.12. *For $\gamma\in\rho(\mathscr{A})$, $(\gamma I-\mathscr{A})^{-1}\begin{pmatrix}\mathbb{z}_0\\ \mathbb{z}_D\\ \mathbb{z}_f\\ \mathbb{z}_{sf}\end{pmatrix}=\begin{pmatrix}\mathbb{y}_0\\ \mathbb{y}_D\\ \mathbb{y}_f\\ \mathbb{y}_{sf}\end{pmatrix},\quad\forall\mathbb{z}\in\mathscr{X},$*

where

$$
\begin{aligned}
\mathbb{y}_0=&\Bigg[\int_0^\infty\eta_2(x)e^{-(\gamma+\lambda_1)x-\int_0^x\eta_2(\tau)d\tau}\int_0^x\mathbb{z}_D(\tau)e^{(\gamma+\lambda_1)\tau+\int_0^\tau\eta_2(\xi)d\xi}d\tau\mathrm{d}x\\
&+\int_0^\infty\eta_2(x)e^{-\gamma x-\int_0^x\eta_2(\tau)d\tau}\int_0^x\mathbb{z}_D(s)e^{(\gamma+\lambda_1)s+\int_0^s\eta_2(\xi)d\xi}\big(e^{-\lambda_1 s}\\
&-e^{-\lambda_1 x}\big)ds\,\mathrm{d}x\int_0^\infty\eta_1(x)\,e^{-\gamma x-\int_0^x\eta_1(\tau)d\tau}\mathrm{d}x\\
&+\int_0^\infty\eta_2(x)e^{-\gamma x-\int_0^x\eta_2(\tau)d\tau}\int_0^x\mathbb{z}_{sf}(\tau)e^{\gamma\tau+\int_0^\tau\eta_2(\xi)d\xi}d\tau\mathrm{d}x\int_0^\infty\eta_1(x)\,e^{-\gamma x-\int_0^x\eta_1(\tau)d\tau}\mathrm{d}x\\
&+\int_0^\infty\eta_1(x)e^{-\gamma x-\int_0^x\eta_1(\tau)d\tau}\int_0^x\mathbb{z}_f(\tau)e^{\gamma\tau+\int_0^\tau\eta_1(\xi)d\xi}d\tau\mathrm{d}x+\mathbb{z}_0\Bigg]\\
&\Bigg/\Bigg[\gamma+\lambda_1+\lambda_2-\lambda_2\int_0^\infty\eta_2(x)\,e^{-(\gamma+\lambda_1)x-\int_0^x\eta_2(\tau)d\tau}\mathrm{d}x-\lambda_1\int_0^\infty\eta_1(x)\,e^{-\gamma x-\int_0^x\eta_1(\tau)d\tau}\mathrm{d}x\\
&-\lambda_2\int_0^\infty\eta_2(x)e^{-\gamma x-\int_0^x\eta_2(\tau)d\tau}\big(1-e^{-\lambda_1 x}\big)\mathrm{d}x\int_0^\infty\eta_1(x)\,e^{-\gamma x-\int_0^x\eta_1(\tau)d\tau}\mathrm{d}x\Bigg],
\end{aligned}
$$

$$
\mathbb{y}_D(x)=\lambda_2 e^{-(\gamma+\lambda_1)x-\int_0^x\eta_2(\tau)d\tau}\mathbb{y}_0+e^{-(\gamma+\lambda_1)x-\int_0^x\eta_2(\tau)d\tau}\int_0^x\mathbb{z}_D(\tau)e^{(\gamma+\lambda_1)\tau+\int_0^\tau\eta_2(\xi)d\xi}d\tau,
$$

$$
\begin{aligned}
\mathbb{y}_f(x)=&\left(\lambda_1+\lambda_2\int_0^\infty\eta_2(x)e^{-\gamma x-\int_0^x\eta_2(\tau)d\tau}\big(1-e^{-\lambda_1 x}\big)\mathrm{d}x\right)e^{-\gamma x-\int_0^x\eta_1(\tau)d\tau}\mathbb{y}_0\\
&+e^{-\gamma x-\int_0^x\eta_1(\tau)d\tau}\int_0^\infty\eta_2(x)e^{-\gamma x-\int_0^x\eta_2(\tau)d\tau}\int_0^x\mathbb{z}_D(s)e^{(\gamma+\lambda_1)s+\int_0^s\eta_2(\xi)d\xi}\big(e^{-\lambda_1 s}-e^{-\lambda_1 x}\big)ds\,\mathrm{d}x\\
&+e^{-\gamma x-\int_0^x\eta_1(\tau)d\tau}\int_0^\infty\eta_2(x)e^{-\gamma x-\int_0^x\eta_2(\tau)d\tau}\int_0^x\mathbb{z}_{sf}(\tau)e^{\gamma\tau+\int_0^\tau\eta_2(\xi)d\xi}d\tau\mathrm{d}x\\
&+e^{-\gamma x-\int_0^x\eta_1(\tau)d\tau}\int_0^x\mathbb{z}_f(\tau)e^{\gamma\tau+\int_0^\tau\eta_1(\xi)d\xi}d\tau,
\end{aligned}
$$

and

$$\mathbb{y}_{sf}(x) = \lambda_2 e^{-\gamma x-\int_0^x \eta_2(\tau)d\tau}\left(1-e^{-\lambda_1 x}\right)\mathbb{y}_0 + e^{-\gamma x-\int_0^x \eta_2(\tau)d\tau}\int_0^x \mathbb{z}_{sf}(\tau)e^{\gamma\tau+\int_0^\tau \eta_2(\xi)d\xi}d\tau$$
$$+e^{-\gamma x-\int_0^x \eta_2(\tau)d\tau}\int_0^x \mathbb{z}_D(s)e^{(\gamma+\lambda_1)s+\int_0^s \eta_2(\xi)d\xi}\left(e^{-\lambda_1 s}-e^{-\lambda_1 x}\right)ds.$$

Proof. Let $(\gamma I-\mathscr{A})\mathbb{y}=\mathbb{z}$, for $\mathbb{z}\in\mathscr{X}$, then,

$$(\gamma+\lambda_1+\lambda_2)\mathbb{y}_0 = \int_0^\infty \eta_2(x)\mathbb{y}_D(x)\mathrm{d}x + \int_0^\infty \eta_1(x)\mathbb{y}_f(x)\mathrm{d}x + \mathbb{z}_0, \tag{3.48}$$

$$\frac{d\mathbb{y}_D(x)}{dx} = -(\gamma+\lambda_1+\eta_2(x))\mathbb{y}_D(x)+\mathbb{z}_D(x), \tag{3.49}$$

$$\frac{d\mathbb{y}_f(x)}{dx} = -(\gamma+\eta_1(x))\mathbb{y}_f(x)+\mathbb{z}_f(x), \tag{3.50}$$

$$\frac{d\mathbb{y}_{sf}(x)}{dx} = -(\gamma+\eta_2(x))\mathbb{y}_{sf}(x)+\lambda_1\mathbb{y}_D(x)+\mathbb{z}_{sf}(x), \tag{3.51}$$

$$\mathbb{y}_D(0) = \lambda_2\mathbb{y}_0, \tag{3.52}$$

$$\mathbb{y}_f(0) = \lambda_1\mathbb{y}_0 + \int_0^\infty \eta_2(x)\mathbb{y}_{sf}(x)\mathrm{d}x, \tag{3.53}$$

$$\mathbb{y}_{sf}(0) = 0. \tag{3.54}$$

Solving (3.49)-(3.51), we have

$$\mathbb{y}_D(x) = \mathbb{y}_D(0)\, e^{-(\gamma+\lambda_1)x-\int_0^x \eta_2(\tau)d\tau} + e^{-(\gamma+\lambda_1)x-\int_0^x \eta_2(\tau)d\tau}\int_0^x \mathbb{z}_D(\tau)e^{(\gamma+\lambda_1)\tau+\int_0^\tau \eta_2(\xi)d\xi}d\tau, \tag{3.55}$$

$$\mathbb{y}_f(x) = \mathbb{y}_f(0)\, e^{-\gamma x-\int_0^x \eta_1(\tau)d\tau} + e^{-\gamma x-\int_0^x \eta_1(\tau)d\tau}\int_0^x \mathbb{z}_f(\tau)e^{\gamma\tau+\int_0^\tau \eta_1(\xi)d\xi}d\tau, \tag{3.56}$$

$$\mathbb{y}_{sf}(x) = \lambda_1 e^{-\gamma x-\int_0^x \eta_2(\tau)d\tau}\int_0^x \mathbb{y}_D(\tau)e^{\gamma\tau+\int_0^\tau \eta_2(\xi)d\xi}d\tau + e^{-\gamma x-\int_0^x \eta_2(\tau)d\tau}\int_0^x \mathbb{z}_{sf}(\tau)e^{\gamma\tau+\int_0^\tau \eta_2(\xi)d\xi}d\tau, \tag{3.57}$$

Using (3.55)-(3.57) in (3.52)-(3.54), we calculate

$$\mathbb{y}_D(0) = \lambda_2\mathbb{y}_0, \tag{3.58}$$

$$\begin{aligned}\mathbb{y}_f(0) =& \lambda_1\mathbb{y}_0 + \lambda_1\int_0^\infty \eta_2(x)e^{-\gamma x-\int_0^x \eta_2(\tau)d\tau}\int_0^x \Big[\mathbb{y}_D(0)\, e^{-(\gamma+\lambda_1)\tau-\int_0^\tau \eta_2(\xi)d\xi} \\ &+ e^{-(\gamma+\lambda_1)\tau-\int_0^\tau \eta_2(\xi)d\xi}\int_0^\tau \mathbb{z}_D(s)e^{(\gamma+\lambda_1)s+\int_0^s \eta_2(\xi)d\xi}ds\Big]e^{\gamma\tau+\int_0^\tau \eta_2(\xi)d\xi}d\tau\mathrm{d}x \\ &+ \int_0^\infty \eta_2(x)e^{-\gamma x-\int_0^x \eta_2(\tau)d\tau}\int_0^x \mathbb{z}_{sf}(\tau)e^{\gamma\tau+\int_0^\tau \eta_2(\xi)d\xi}d\tau\mathrm{d}x \\ =& \lambda_1\mathbb{y}_0 + \lambda_2\int_0^\infty \eta_2(x)e^{-\gamma x-\int_0^x \eta_2(\tau)d\tau}\left(1-e^{-\lambda_1 x}\right)\mathrm{d}x\,\mathbb{y}_0 \\ &+ \int_0^\infty \eta_2(x)e^{-\gamma x-\int_0^x \eta_2(\tau)d\tau}\int_0^x \mathbb{z}_D(s)e^{(\gamma+\lambda_1)s+\int_0^s \eta_2(\xi)d\xi}\left(e^{-\lambda_1 s}-e^{-\lambda_1 x}\right)ds\,\mathrm{d}x \\ &+ \int_0^\infty \eta_2(x)e^{-\gamma x-\int_0^x \eta_2(\tau)d\tau}\int_0^x \mathbb{z}_{sf}(\tau)e^{\gamma\tau+\int_0^\tau \eta_2(\xi)d\xi}d\tau\mathrm{d}x.\end{aligned} \tag{3.59}$$

Combining (3.58), (3.59), (3.55)-(3.57), and (3.48), we derive

$$
\begin{aligned}
&(\gamma+\lambda_1+\lambda_2)\mathbb{y}_0=\int_0^\infty \eta_2(x)\mathbb{y}_D(x)\mathrm{d}x+\int_0^\infty \eta_1(x)\mathbb{y}_f(x)\mathrm{d}x+\mathbb{z}_0\\
&=\lambda_2\int_0^\infty \eta_2(x)\, e^{-(\gamma+\lambda_1)x-\int_0^x\eta_2(\tau)d\tau}\mathrm{d}x\, \mathbb{y}_0\\
&\quad+\int_0^\infty \eta_2(x)e^{-(\gamma+\lambda_1)x-\int_0^x\eta_2(\tau)d\tau}\int_0^x \mathbb{z}_D(\tau)e^{(\gamma+\lambda_1)\tau+\int_0^\tau\eta_2(\xi)d\xi}d\tau\mathrm{d}x\\
&\quad+\lambda_1\int_0^\infty \eta_1(x)\, e^{-\gamma x-\int_0^x\eta_1(\tau)d\tau}\mathrm{d}x\, \mathbb{y}_0\\
&\quad+\lambda_2\int_0^\infty \eta_2(x)e^{-\gamma x-\int_0^x\eta_2(\tau)d\tau}\left(1-e^{-\lambda_1 x}\right)\mathrm{d}x\int_0^\infty \eta_1(x)\, e^{-\gamma x-\int_0^x\eta_1(\tau)d\tau}\mathrm{d}x\, \mathbb{y}_0\\
&\quad+\int_0^\infty \eta_2(x)e^{-\gamma x-\int_0^x\eta_2(\tau)d\tau}\int_0^x \mathbb{z}_D(s)e^{(\gamma+\lambda_1)s+\int_0^s\eta_2(\xi)d\xi}\left(e^{-\lambda_1 s}-e^{-\lambda_1 x}\right)ds\,\mathrm{d}x\\
&\quad\times\int_0^\infty \eta_1(x)\, e^{-\gamma x-\int_0^x\eta_1(\tau)d\tau}\mathrm{d}x\\
&\quad+\int_0^\infty \eta_2(x)e^{-\gamma x-\int_0^x\eta_2(\tau)d\tau}\int_0^x \mathbb{z}_{sf}(\tau)e^{\gamma\tau+\int_0^\tau\eta_2(\xi)d\xi}d\tau\mathrm{d}x\int_0^\infty \eta_1(x)\, e^{-\gamma x-\int_0^x\eta_1(\tau)d\tau}\mathrm{d}x\\
&\quad+\int_0^\infty \eta_1(x)e^{-\gamma x-\int_0^x\eta_1(\tau)d\tau}\int_0^x \mathbb{z}_f(\tau)e^{\gamma\tau+\int_0^\tau\eta_1(\xi)d\xi}d\tau\mathrm{d}x+\mathbb{z}_0\\
&\Longrightarrow
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{y}_0=&\Bigg[\int_0^\infty \eta_2(x)e^{-(\gamma+\lambda_1)x-\int_0^x\eta_2(\tau)d\tau}\int_0^x \mathbb{z}_D(\tau)e^{(\gamma+\lambda_1)\tau+\int_0^\tau\eta_2(\xi)d\xi}d\tau\mathrm{d}x\\
&+\int_0^\infty \eta_2(x)e^{-\gamma x-\int_0^x\eta_2(\tau)d\tau}\int_0^x \mathbb{z}_D(s)e^{(\gamma+\lambda_1)s+\int_0^s\eta_2(\xi)d\xi}\left(e^{-\lambda_1 s}-e^{-\lambda_1 x}\right)ds\,\mathrm{d}x\\
&\times\int_0^\infty \eta_1(x)\, e^{-\gamma x-\int_0^x\eta_1(\tau)d\tau}\mathrm{d}x\\
&+\int_0^\infty \eta_2(x)e^{-\gamma x-\int_0^x\eta_2(\tau)d\tau}\int_0^x \mathbb{z}_{sf}(\tau)e^{\gamma\tau+\int_0^\tau\eta_2(\xi)d\xi}d\tau\mathrm{d}x\int_0^\infty \eta_1(x)\, e^{-\gamma x-\int_0^x\eta_1(\tau)d\tau}\mathrm{d}x\\
&+\int_0^\infty \eta_1(x)e^{-\gamma x-\int_0^x\eta_1(\tau)d\tau}\int_0^x \mathbb{z}_f(\tau)e^{\gamma\tau+\int_0^\tau\eta_1(\xi)d\xi}d\tau\mathrm{d}x+\mathbb{z}_0\Bigg]\\
&\Bigg/\Bigg[\gamma+\lambda_1+\lambda_2-\lambda_2\int_0^\infty \eta_2(x)\, e^{-(\gamma+\lambda_1)x-\int_0^x\eta_2(\tau)d\tau}\mathrm{d}x-\lambda_1\int_0^\infty \eta_1(x)\, e^{-\gamma x-\int_0^x\eta_1(\tau)d\tau}\mathrm{d}x\\
&-\lambda_2\int_0^\infty \eta_2(x)e^{-\gamma x-\int_0^x\eta_2(\tau)d\tau}\left(1-e^{-\lambda_1 x}\right)\mathrm{d}x\int_0^\infty \eta_1(x)\, e^{-\gamma x-\int_0^x\eta_1(\tau)d\tau}\mathrm{d}x\Bigg].
\end{aligned}\tag{3.60}
$$

We can easily find the remaining results of this Lemma by substituting (3.58) - (3.60) into (3.55)-(3.57), separately. □

Theorem 3.13. *If $\eta_1(x)$ and $\eta_2(x)$ are Lipschitz continuous and satisfy*

$$0<\underline{\eta_1}\le\eta_1(x)\le\overline{\eta_1}<\infty,\ \underline{\eta_2}\le\eta_2(x)\le\overline{\eta_2}<\infty,$$

then $\|\mathbf{P}(\cdot,t)-\mathbf{P}(\cdot)\|\le Me^{-\delta t},\ t>0.$

Proof. From Theorem 3.5 and 3.7, we have

$$\|\mathscr{V}(t)-\mathscr{V}_1(t)\| = \|\mathscr{V}_2(t)\| \le e^{-\min\{\lambda_1+\lambda_2,\lambda_1+\underline{\eta_2},\underline{\eta_1}\}t}$$
$$\Longrightarrow$$
$$\ln\|\mathscr{V}(t)-\mathscr{V}_1(t)\| \le -\min\{\lambda_1+\lambda_2,\lambda_1+\underline{\eta_2},\underline{\eta_1}\}t$$
$$\Longrightarrow$$
$$\lim_{t\to\infty}\frac{\ln\|\mathscr{V}(t)-\mathscr{V}_1(t)\|}{t} \le -\min\{\lambda_1+\lambda_2,\lambda_1+\underline{\eta_2},\underline{\eta_1}\}t.$$

We can deduce from this and Proposition 2.10 in Engel and Nagel [2, P.258] that $\omega_{ess}(\mathscr{V}(t))$ satisfies $\omega_{ess}(\mathscr{V}(t)) \le -\min\{\lambda_1+\lambda_2,\lambda_1+\underline{\eta_2},\underline{\eta_1}\}$. By Proposition 2.12 in [2, P.258] and compactness of E, we have $\omega_{ess}(\mathscr{A}) = \omega_{ess}(\mathscr{T}(t)) = \omega_{ess}(\mathscr{V}(t)) \le -\min\{\lambda_1+\lambda_2,\lambda_1+\underline{\eta_2},\underline{\eta_1}\}$. Hence, 0 is a pole of $(\gamma I-\mathscr{A})^{-1}$ of order 1 by Corollary 2.11 in Engel and Nagel [2, P.258] and Theorem 3.7. As a result of the residue theorem, Theorem 3.13, and Lemma 3.12, it follows that

$$\mathbb{Pr}\begin{pmatrix}\mathbb{z}_0\\ \mathbb{z}_D\\ \mathbb{z}_f\\ \mathbb{z}_{sf}\end{pmatrix} = \lim_{\gamma\to0}\gamma(\gamma I-\mathscr{A})^{-1}\begin{pmatrix}\mathbb{z}_0\\ \mathbb{z}_D(x)\\ \mathbb{z}_f(x)\\ \mathbb{z}_{sf}(x)\end{pmatrix} = \begin{pmatrix}\lim_{\gamma\to0}\gamma\mathbb{y}_0\\ \lim_{\gamma\to0}\gamma\mathbb{y}_D(x)\\ \lim_{\gamma\to0}\gamma\mathbb{y}_f(x)\\ \lim_{\gamma\to0}\gamma\mathbb{y}_{sf}(x)\end{pmatrix}.$$

Using L'Hospital rule we, calculate

$$\begin{aligned}
&\lim_{\gamma\to0}\gamma\Bigg/\Bigg[\gamma+\lambda_1+\lambda_2-\lambda_2\int_0^\infty\eta_2(x)\,e^{-(\gamma+\lambda_1)x-\int_0^x\eta_2(\tau)d\tau}\mathrm{d}x-\lambda_1\int_0^\infty\eta_1(x)\,e^{-\gamma x-\int_0^x\eta_1(\tau)d\tau}\mathrm{d}x\\
&\quad-\lambda_2\int_0^\infty\eta_2(x)e^{-\gamma x-\int_0^x\eta_2(\tau)d\tau}\left(1-e^{-\lambda_1 x}\right)\mathrm{d}x\int_0^\infty\eta_1(x)\,e^{-\gamma x-\int_0^x\eta_1(\tau)d\tau}\mathrm{d}x\Bigg]\\
&=\lim_{\gamma\to0}1\Bigg/\Bigg[1+\lambda_2\int_0^\infty x\eta_2(x)\,e^{-(\gamma+\lambda_1)x-\int_0^x\eta_2(\tau)d\tau}\mathrm{d}x+\lambda_1\int_0^\infty x\eta_1(x)\,e^{-\gamma x-\int_0^x\eta_1(\tau)d\tau}\mathrm{d}x\\
&\quad-\Bigg\{\lambda_2\int_0^\infty x\,\eta_2(x)e^{-\gamma x-\int_0^x\eta_2(\tau)d\tau}\left(1-e^{-\lambda_1 x}\right)\mathrm{d}x\int_0^\infty\eta_1(x)\,e^{-\gamma x-\int_0^x\eta_1(\tau)d\tau}\mathrm{d}x\\
&\quad+\lambda_2\int_0^\infty\eta_2(x)e^{-\gamma x-\int_0^x\eta_2(\tau)d\tau}\left(1-e^{-\lambda_1 x}\right)\mathrm{d}x\int_0^\infty x\,\eta_1(x)\,e^{-\gamma x-\int_0^x\eta_1(\tau)d\tau}\mathrm{d}x\Bigg\}\Bigg]\\
&=1\Bigg/\Bigg[1+\lambda_1\int_0^\infty x\eta_1(x)\,e^{-\int_0^x\eta_1(\tau)d\tau}\mathrm{d}x+\lambda_2\int_0^\infty x\,\eta_2(x)e^{-\int_0^x\eta_2(\tau)d\tau}\mathrm{d}x\\
&\quad+\lambda_2\lambda_1\int_0^\infty e^{-\lambda_1 x-\int_0^x\eta_2(\tau)d\tau}\mathrm{d}x\int_0^\infty x\,\eta_1(x)\,e^{-\int_0^x\eta_1(\tau)d\tau}\mathrm{d}x\Bigg]\\
&\doteq\frac{1}{M}.
\end{aligned}$$

Applying Fubini theorem, we have

$$\begin{aligned}
&\int_0^\infty \eta_2(x)e^{-\int_0^x \eta_2(\tau)d\tau}\int_0^x \mathbb{z}_D(\tau)e^{\int_0^\tau \eta_2(\xi)d\xi}d\tau \mathrm{d}x \\
&= \int_0^\infty \mathbb{z}_D(\tau)e^{-\int_0^\tau \eta_2(\xi)d\xi}\int_\tau^\infty d\{-e^{-\int_0^x \eta_2(\tau)d\tau}\}d\tau \\
&= \int_0^\infty \mathbb{z}_D(\tau)e^{-\int_0^\tau \eta_2(\xi)d\xi}\left(-e^{-\int_0^x \eta_2(\xi)d\xi}\Big|_{x=\tau}^{x=\infty}\right)d\tau = \int_0^\infty \mathbb{z}_D(x)\mathrm{d}x, \\
&\int_0^\infty \eta_1(x)e^{-\int_0^x \eta_1(\tau)d\tau}\int_0^x \mathbb{z}_f(\tau)e^{-\int_0^\tau \eta_1(\xi)d\xi}d\tau \mathrm{d}x = \int_0^\infty \mathbb{z}_f(x)\mathrm{d}x, \\
&\int_0^\infty \eta_2(x)e^{-\int_0^x \eta_2(\tau)d\tau}\int_0^x \mathbb{z}_{sf}(\tau)e^{\int_0^\tau \eta_2(\xi)d\xi}d\tau \mathrm{d}x = \int_0^\infty \mathbb{z}_{sf}(x)\mathrm{d}x.
\end{aligned}$$

Hence

$$\begin{aligned}
\lim_{\gamma\to 0}\gamma \mathbb{y}_0 = \Bigg[&\int_0^\infty \eta_2(x)e^{-\lambda_1 x-\int_0^x \eta_2(\tau)d\tau}\int_0^x \mathbb{z}_D(\tau)e^{\lambda_1\tau+\int_0^\tau \eta_2(\xi)d\xi}d\tau \mathrm{d}x \\
&+\int_0^\infty \eta_2(x)e^{-\int_0^x \eta_2(\tau)d\tau}\int_0^x \mathbb{z}_D(s)e^{\lambda_1 s+\int_0^s \eta_2(\xi)d\xi}\left(e^{-\lambda_1 s}-e^{-\lambda_1 x}\right)ds\,\mathrm{d}x \\
&+\int_0^\infty \eta_2(x)e^{-\int_0^x \eta_2(\tau)d\tau}\int_0^x \mathbb{z}_{sf}(\tau)e^{\int_0^\tau \eta_2(\xi)d\xi}d\tau \mathrm{d}x \\
&+\int_0^\infty \eta_1(x)e^{-\int_0^x \eta_1(\tau)d\tau}\int_0^x \mathbb{z}_f(\tau)e^{\int_0^\tau \eta_1(\xi)d\xi}d\tau \mathrm{d}x+\mathbb{z}_0\Bigg]\Bigg/M \\
=\Bigg[&\int_0^\infty \eta_2(x)e^{-\int_0^x \eta_2(\tau)d\tau}\int_0^x \mathbb{z}_D(s)e^{\int_0^s \eta_2(\xi)d\xi}ds\,\mathrm{d}x \\
&+\int_0^\infty \eta_2(x)e^{-\int_0^x \eta_2(\tau)d\tau}\int_0^x \mathbb{z}_{sf}(\tau)e^{\int_0^\tau \eta_2(\xi)d\xi}d\tau \mathrm{d}x \\
&+\int_0^\infty \eta_1(x)e^{-\int_0^x \eta_1(\tau)d\tau}\int_0^x \mathbb{z}_f(\tau)e^{\int_0^\tau \eta_1(\xi)d\xi}d\tau \mathrm{d}x+\mathbb{z}_0\Bigg]\Bigg/M \\
=&\frac{1}{M} \doteq P_0, \qquad (3.61)
\end{aligned}$$

$$\lim_{\gamma\to 0}\gamma \mathbb{y}_D(x) = \lambda_2 e^{-\lambda_1 x-\int_0^x \eta_2(\tau)d\tau}\lim_{\gamma\to 0}\gamma \mathbb{y}_0 = \frac{\lambda_2 e^{-\lambda_1 x-\int_0^x \eta_2(\tau)d\tau}}{M} \doteq P_D(x), \qquad (3.62)$$

$$\begin{aligned}
\lim_{\gamma\to 0}\gamma \mathbb{y}_f(x) =& \left(\lambda_1+\lambda_2\int_0^\infty \eta_2(x)e^{-\int_0^x \eta_2(\tau)d\tau}\left(1-e^{-\lambda_1 x}\right)\mathrm{d}x\right)e^{-\int_0^x \eta_1(\tau)d\tau}\lim_{\gamma\to 0}\gamma \mathbb{y}_0 \\
=& \frac{\lambda_1\left(1+\lambda_2\int_0^\infty e^{-\lambda_1 x-\int_0^x \eta_2(\tau)d\tau}\mathrm{d}x\right)e^{-\int_0^x \eta_1(\tau)d\tau}}{M} \doteq P_f(x), \qquad (3.63)
\end{aligned}$$

$$\lim_{\gamma\to 0}\gamma\, \mathbb{y}_{sf}(x) = \frac{\lambda_2 e^{-\int_0^x \eta_2(\tau)d\tau}\left(1-e^{-\lambda_1 x}\right)}{M} \doteq P_{sf}(x). \qquad (3.64)$$

Combining (3.61)-(3.64) with Theorem 3.13, we obtain $\mathbb{P}\mathrm{r}\mathbf{P}(0) = \mathbf{P}(x)$. Furthermore, by Theorem 2.3 and 3.13, we find that

$$\|\mathbf{P}(\cdot,t) - \mathbf{P}(\cdot)\| \leq \|T(t) - \mathbb{P}\mathrm{r}\|\|\mathbf{P}(0)\| \leq Me^{-\delta t}\|\mathbf{P}(0)\| = Me^{-\delta t}, \quad t \geq 0,$$

which shows that the time-dependent solution exponentially converges to its steady-state solution. □

4. Asymptotic Behavior of Some Reliability Indices

Based on the results in Section 3, we conclude that

$$\lim_{t\to\infty} P_0(t) = P_0, \quad \lim_{t\to\infty}\int_0^\infty |P_l(x,t) - P_l(x)|\,\mathrm{d}x = 0, \quad l = D, f, sf, \tag{4.1}$$

which implies, for $i = 1, 2$, $\lim_{t\to\infty}\int_0^\infty |\eta_i(x)P_l(x,t) - \eta_i(x)P_l(x)|\,\mathrm{d}x = 0$, $l = D, f, sf$. The time-dependent availability, failure frequency, and renewal frequency are given by

$$\begin{aligned} A(t) =& P_0(t) + \int_0^\infty P_D(x,t)\mathrm{d}x, \\ m_f(t) =& \lambda_1 P_0(t) + \lambda_1 \int_0^\infty P_D(x,t)\mathrm{d}x, \\ m_r(t) =& \int_0^\infty \eta_2(x)P_D(x,t)\mathrm{d}x + \int_0^\infty \eta_1(x)P_f(x,t)\mathrm{d}x. \end{aligned}$$

which, together with (3.29)-(3.33) and (4.1), yields

$$\begin{aligned} A &= \lim_{t\to\infty} A(t) = P_0 + \int_0^\infty P_D(x)\mathrm{d}x = \left\{1 + \lambda_2\int_0^\infty e^{-\lambda_1 x - \int_0^x \eta_2(\tau)d\tau}\mathrm{d}x\right\} P_0, \\ m_f &= \lim_{t\to\infty} m_f(t) = \lambda_1 P_0 + \lambda_1\int_0^\infty P_D(x)\mathrm{d}x = \lambda_1 A, \\ m_r &= \lim_{t\to\infty} m_r(t) = \int_0^\infty \eta_2(x)P_D(x)\mathrm{d}x + \int_0^\infty \eta_1(x)P_f(x)\mathrm{d}x = (\lambda_1 + \lambda_2)P_0. \end{aligned}$$

Since the system's time-dependent reliability is similar to that of [10], it is omitted.

5. Numerical Results

In this section, we present some numerical examples to investigate how each system parameters affects the time-dependent reliability indices. First of all, we assume that the system's repair time is Gamma distributed: taking the constant repair rates as $\eta_1(x) = \eta_1,\ \eta_2(x) = \eta_2$, and choose $\eta_1 = 0.004,\ \eta_2 = 0.006,\ \lambda_1 = 0.0004,\ \lambda_2 = 0.0006$. In Fig.1, we show how β (β is another parameter) affects the system reliability indices. According to Fig.1, $A(t)$ (Fig.1a) and $m_f(t)$ (Fig.1b) decrease rapidly as time increases in each case and approach constant value after a long run. $m_r(t)$ (Fig.1c) increase rapidly in the early stage and approach constant value after a long run. Furthermore, we see that the parameter β has considerable effects on the reliability indices such that they decrease as β increases.

We assume $\beta = 1$ in the following, (i.e., the system's repair time is exponentially distributed) and the effect of system reliability indices with different values of parameters $\eta_1, \eta_2, \lambda_1, \lambda_2$ is depicted in Figs.2-4. Fig.2 demonstrates the behavior of the system's time-dependent availability $A(t)$. As expected, $A(t)$ decrease as λ_1, λ_2 increase and $A(t)$ increases as η_1, η_2 increase. The change of time-dependent failure frequency are plotted in Fig.3. $m_f(t)$ decreases initially

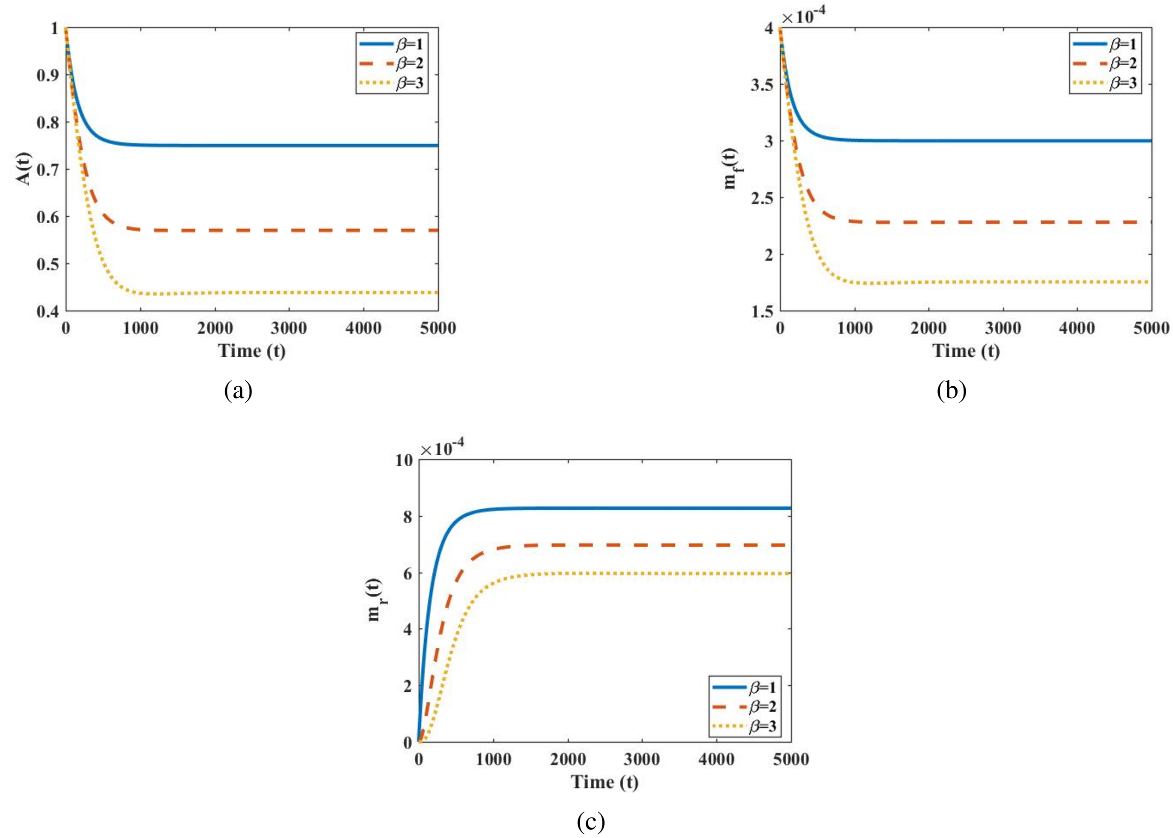

FIGURE 1. Time-dependent reliability indices for Gamma distributed repair time

and reaches constant value as time increase. Fig.3 also demonstrates that $m_f(t)$ increases with increase of the parameters. Fig.4 demonstrates that time-dependent renewal frequency increase inially and the attain constant number for large number of t. In addition, $m_r(t)$ increases when $\eta_1, \eta_2, \lambda_1$, and λ_2 increase.

6. CONCLUSION

We performed a dynamic analysis for the two-unit priority system with non preemptive repair disciplines in this papaer. The model is described by a partial differential equations with integral boundary conditions, which are then transformed into an abstract Cauchy problem. We obtained that the system is well-posed, and that the model's time-dependent solution converges exponentially to its steady-state solution. In addition, we gave the concrete expression of the corresponding projection operator, in which the repair time follows the general distribution. Furthermore, we investigated the asymptotic behavior of the system's time-dependent reliability indices and discussed the influence of each parameter on the reliability indices by numerical examples.

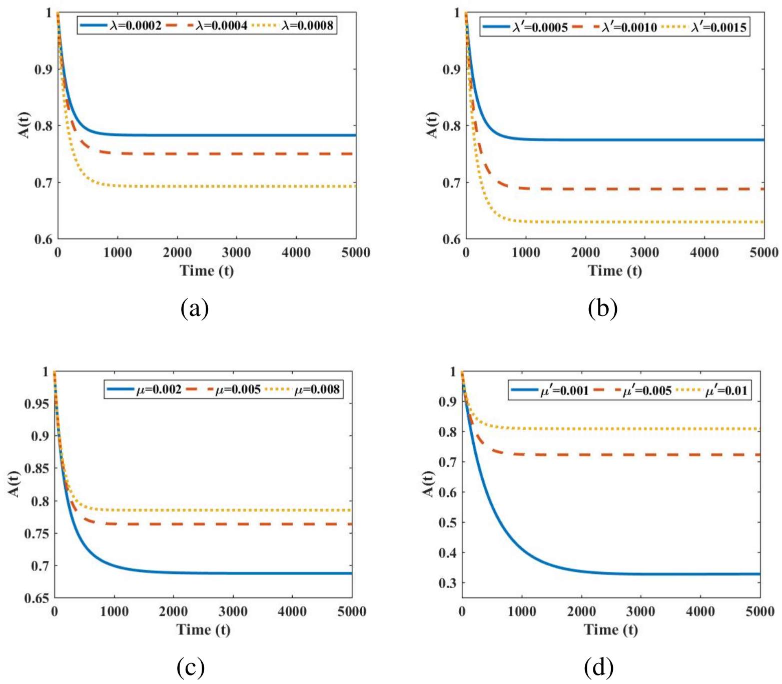

FIGURE 2. Effect of parameters on the time-dependent availability for Exponential distributed repair time

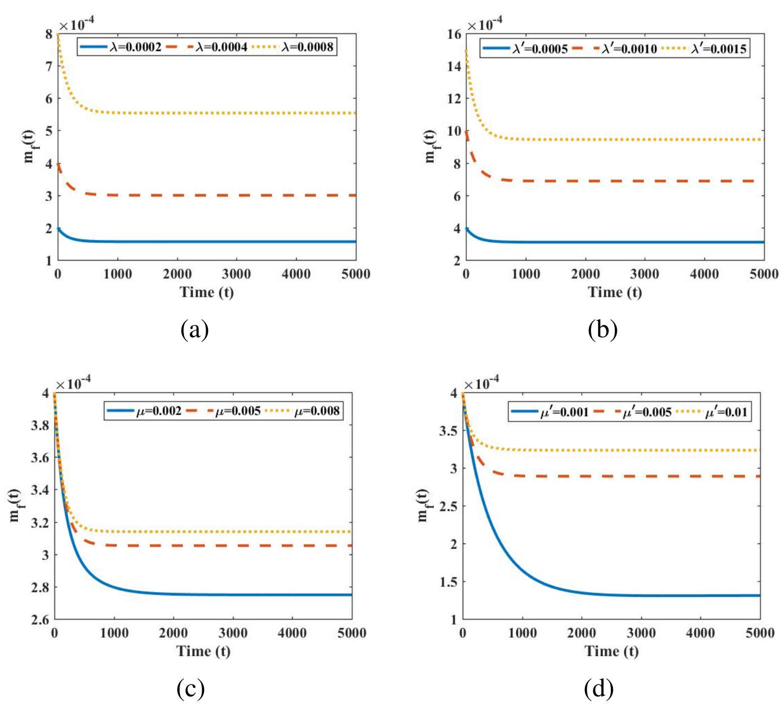

FIGURE 3. Effect of parameters on the time-dependent failure frequency for Exponential distributed repair time

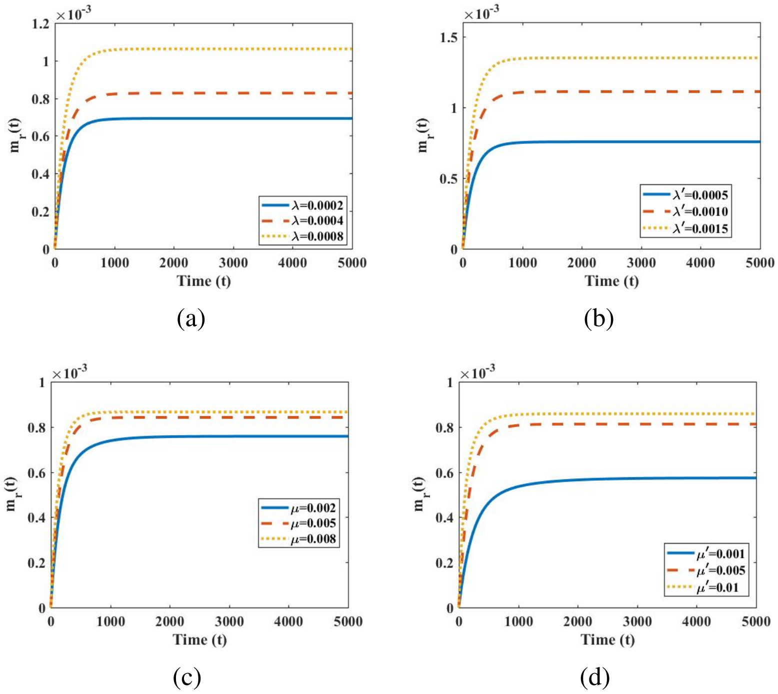

FIGURE 4. Effect of parameters on the time-dependent failure frequency for Exponential distributed repair time

Funding

This research work was supported by the National Natural Science Foundation of China (No. 11801485) and by the Natural Science Foundation of Xinjiang Uygur Autonomous Region (No. 2022D01C46).

Acknowledgements

The author would like to express their sincere thanks to the anonymous referees and associated editor for his/her careful reading of the manuscript.

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