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# BIFURCATION ANALYSIS FOR A DISCRETE SPACE-TIME MODEL OF LOGISTIC TYPE

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**Abstract.** In this paper, a discrete space-time logistic model with Neumann boundry conditions is considered. It is shown that the system undergoes a flip bifurcation at the unique positive equilibrium by means of the theory of the normal form and center manifolds.

**Keywords.** Bifurcation; Center manifold; Discrete space-time; Logistic equations.

## 1. Introduction

It is known that bifurcation phenomena can occur in parameter dependent dynamical systems. When the value of one or more parameters is smoothly varied, changes in the qualitative structure of the solutions for certain parameter values may occur. There have been numerous publications concerning the bifurcation analysis of the solutions of dynamic models which contain continuous dynamics models in physical systems, and discrete dynamics ones in biological, social and economical systems; see, e.g., [1, 2, 4, 5, 6, 7, 8, 10, 17, 22] and the references therein.

Within the latter class of systems, particularly in population dynamics, a model is the set of logistic equations, henceforth, which can be written as $x_{t+1} = \dfrac{px_t}{1+qx_t}$, which is the discrete-time version of the following widely used logistic differential equation [15] $\frac{dx(t)}{dt} = x(t)(a - bx(t))$, where $a$ and $b$ are constants.

In almost all the real ecosystems, the population dynamics include both temporal reproduction processes and spatial diffusion processes; see [16]. Then the following discrete logistic model with diffusion can be obtained

$$u_i^{t+1} = d\nabla^2 u_i^t + \frac{pu_i^t}{1+u_i^t}, i \in [1,m], \tag{1.1}$$

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with Neumann boundry conditions

$$u_0^t = u_1^t, u_m^t = u_{m+1}^t, \tag{1.2}$$

where $m$ is a positive integer, $d > 0$ is a diffusion parameter, and

$$\nabla^2 u_i^t = u_{i+1}^t - 2u_i^t + u_{i-1}^t.$$

The bifurcation of continuous diffusive models were studied extensively recently; see, e.g., [9, 18, 19, 20, 21, 26, 27, 28]. For example, in [9], the existence of the Hopf bifurcation of a diffusive single species model with stage structure and strong Allee effect subject to homogeneous Neumann boundary was investigated by analyzing the corresponding characteristic equation, and the stability and direction of bifurcating periodic solutions were determined by means of the theory of the normal form and center manifold. We also know that numerical solutions or approximate solutions of the discrete-time models can be obtained more easily, and many results were shown that simple difference equations may exhibit more richer properties, such as period-doubling bifurcation and chaos than continuous-time models [11]. Then the bifurcation analysis on discrete systems with diffusion may be interesting and be more challenging in the study of mathematical theory. Turing bifurcation, named diffusion driven instability, on discrete space-time systems attracted much attention recently; see, e.g., [12, 13, 14, 23, 24] and the references therein. Some authors focus on other bifurcation, such as flip bifurcation; see, e.g., [25, 29, 30]. In [30], Zhang et al. discussed the following model which can generate flip bifurcation

$$\begin{cases} u_{t+1} = \frac{pu_t}{1+u_t} + d(-u_t + v_t), \\ v_{t+1} = \frac{pv_t}{1+v_t} + d(u_t - v_t), \end{cases}$$

which is the spatial form of system (1.1)-(1.2) when $m = 2$. The same analysis can be found in [29] and [25]. To the best of the authors' knowledge, there is no result on the bifurcations of the general discrete space-time logistic model, such as model (1.1)-(1.2).

Based on the above, this paper focuses on the bifurcation analysis (not including Turing bifurcation) of discrete diffusion system (1.1)-(1.2). The organization of the work is as follows. In Section 2, we study the stability of equilibria and existence of flip bifurcation of system (1.1)-(1.2). We end up our investigation by drawing some conclusions in Section 3, the last section.

## 2. Stability and Bifurcation Analysis

The dynamics of system (1.1)-(1.2) is qualitatively the same as that of the following system

$$\begin{cases} u_1^{t+1} = \frac{pu_1^t}{1+u_1^t} + d(-u_1^t + u_2^t), \\ u_2^{t+1} = \frac{pu_2^t}{1+u_2^t} + d(u_1^t - 2u_2^t + u_3^t), \\ u_3^{t+1} = \frac{pu_3^t}{1+u_3^t} + d(u_2^t - 2u_3^t + u_4^t), \\ \qquad\vdots \\ u_{m-1}^{t+1} = \frac{pu_{m-1}^t}{1+u_{m-1}^t} + d(u_{m-2}^t - 2u_{m-1}^t + u_m^t), \\ u_m^{t+1} = \frac{pu_m^t}{1+u_m^t} + d(u_{m-1}^t - u_m^t). \end{cases} \tag{2.1}$$

Then we only need to analyze system (2.1) qualitatively.

For eigenvalue problem:

$$\begin{cases} -\nabla^2 x_{i-1} = \lambda x_i \\ x_0 = x_1, x_m = x_{m+1} \end{cases} i \in [1,m] = \{1,2,\ldots,m\},$$

there exist [30]

$$\lambda_k = 4\sin^2\frac{(k-1)\pi}{2m},$$

and the corresponding eigenvector

$$\varphi_i^{(k)} = \cos\frac{(k-1)(2i-1)\pi}{2m}, i \in [1,m],$$

for $k \in [1,m]$.

Clearly, the point $E^* = (u_1^*, u_2^*, \cdots, u_m^*) = (u^*, u^*, \cdots, u^*) = (p-1, p-1, \cdots, p-1)$ is the unique positive equilibrium of (2.1). The linearization equation of (2.1) about $E^*$ is

$$u_i^{t+1} = \frac{1}{p}u_i^t + d\nabla^2 u_i^t = Lu_i^t.$$

By means of $-\nabla^2 \varphi_{i-1}^{(k)} = \lambda_k \varphi_i^{(k)}$, we can obtain

$$L_k \varphi_{i-1}^{(k)} = (\frac{1}{p} - d\lambda_k)\varphi_i^{(k)} = \beta_k \varphi_i^{(k)}, k = 1,2,\cdots,m,$$

where $L_k$ is confined to the eigen-subspace for $L$.

It is well known that the positive steady state $E^*$ of (2.1) is locally asymptotically stable if and only if $|\beta_k| < 1$, namely,

$$\frac{1-p}{p\lambda_k} < d < \frac{1+p}{p\lambda_k},$$

for $k = 1,2,\cdots,m$ holds. And the bifurcation may occur when it is not satisfied.

**Theorem 2.1.** *System (2.1) does not undergo a Neimark-Sacker, Saddle-Node, Transcritical and Pitchfork bifurcation.*

*Proof.* Since all $\beta_k$ are real, system (2.1) does not undergo a Neimark-Sacker or Hopf bifurcation. For $p > 1, d > 0$, the condition $\frac{1}{p} - d\lambda_l = 1$ has not been satisfied, and then system (2.1) does not undergo a Saddle-Node, Transcritical or Pitchfork bifurcation. The proof is complete. □

To make $\frac{1}{p} - d\lambda_l = -1$ hold for some a $l$, we can obtain $d^* = \frac{p+1}{p\lambda_l}$. Furthermore, to make $\left|\frac{1}{p} - d\lambda_j\right| < 1$ hold for $j \neq l$, we can obtain $\lambda_j < \lambda_l$, which means that

$$\lambda_l = \max_{j\in[1,m]} \lambda_j = 4\cos^2\frac{\pi}{2m},$$

and we can determine the bifurcation equation.

Next, choose $d$ as a bifurcation parameter. By using the central manifold method and the theory of normal form, we obtain the following flip bifurcation theorem.

**Theorem 2.2.** *If $p > 1$, then system (2.1) undergoes a flip bifurcation at $d = d^*$.*

*Proof.* Let $x_i^t = u_i^t - u_i^*, \delta^t = d - d^*$, then

$$\begin{pmatrix} x_1^{t+1} \\ x_2^{t+1} \\ x_3^{t+1} \\ \vdots \\ x_{m-1}^{t+1} \\ x_m^{t+1} \\ \delta^{t+1} \end{pmatrix} = \begin{pmatrix} \frac{1}{p}-d^* & d^* & 0 & \cdots & 0 & 0 & 0 \\ d^* & \frac{1}{p}-2d^* & d^* & \cdots & 0 & 0 & 0 \\ 0 & d^* & \frac{1}{p}-2d^* & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & 0 \\ 0 & 0 & 0 & \cdots & \frac{1}{p}-2d^* & d^* & 0 \\ 0 & 0 & 0 & \cdots & d^* & \frac{1}{p}-d^* & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1^t \\ x_2^t \\ x_3^t \\ \vdots \\ x_{m-1}^t \\ x_m^t \\ \delta^t \end{pmatrix}$$
$$+ \begin{pmatrix} f_1(x_1^t, x_2^t, x_3^t, \cdots, x_{m-1}^t, x_m^t, \delta^t) \\ f_2(x_1^t, x_2^t, x_3^t, \cdots, x_{m-1}^t, x_m^t, \delta^t) \\ f_3(x_1^t, x_2^t, x_3^t, \cdots, x_{m-1}^t, x_m^t, \delta^t) \\ \vdots \\ f_{m-1}(x_1^t, x_2^t, x_3^t, \cdots, x_{m-1}^t, x_m^t, \delta^t) \\ f_m(x_1^t, x_2^t, x_3^t, \cdots, x_{m-1}^t, x_m^t, \delta^t) \\ 0 \end{pmatrix}, \tag{2.2}$$

where

$$\begin{aligned} & f_i(x_1^t, x_2^t, x_3^t, \cdots, x_{m-1}^t, x_m^t, \delta^t) \\ = & \begin{cases} -\frac{2}{p^2}(x_i^t)^2 - x_i^t\delta^t + x_{i+1}^t\delta^t + \frac{6}{p^3}(x_i^t)^3 & \\ +O\big(\big(|x_i^t| + |x_{i+1}^t| + |\delta^t|\big)^4\big), & i = 1, \\ -\frac{2}{p^2}(x_i^t)^2 + x_{i-1}^t\delta^t - 2x_i^t\delta^t + x_{i+1}^t\delta^t + \frac{6}{p^3}(x_i^t)^3 & \\ +O\big(\big(|x_{i-1}^t| + |x_i^t| + |x_{i+1}^t| + |\delta^t|\big)^4\big), & 2 \le i \le m-1, \\ -\frac{2}{p^2}(x_i^t)^2 + x_{i-1}^t\delta^t - x_i^t\delta^t + \frac{6}{p^3}(x_i^t)^3 & \\ +O\big(\big(|x_{i-1}^t| + |x_i^t| + |\delta^t|\big)^4\big), & i = m. \end{cases} \end{aligned}$$

We construct an invertible matrix

$$T = \begin{pmatrix} e_{11} & e_{21} & e_{31} & \cdots & e_{m-1,1} & e_{m1} \\ e_{12} & e_{22} & e_{32} & \cdots & e_{m-1,2} & e_{m2} \\ e_{13} & e_{23} & e_{33} & \cdots & e_{m-1,3} & e_{m3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ e_{1,m-1} & e_{2,m-1} & e_{3,m-1} & \cdots & e_{m-1,m-1} & e_{m,m-1} \\ e_{1m} & e_{2m} & e_{3m} & \cdots & e_{m-1,m} & e_{mm} \end{pmatrix}$$

$$
= \begin{pmatrix}
\frac{1}{\sqrt{m}}\varphi_1^{(1)} & \sqrt{\frac{2}{m}}\varphi_1^{(2)} & \sqrt{\frac{2}{m}}\varphi_1^{(3)} & \cdots & \sqrt{\frac{2}{m}}\varphi_1^{(m-1)} & \sqrt{\frac{2}{m}}\varphi_1^{(m)} \\
\frac{1}{\sqrt{m}}\varphi_2^{(1)} & \sqrt{\frac{2}{m}}\varphi_2^{(2)} & \sqrt{\frac{2}{m}}\varphi_2^{(3)} & \cdots & \sqrt{\frac{2}{m}}\varphi_2^{(m-1)} & \sqrt{\frac{2}{m}}\varphi_2^{(m)} \\
\frac{1}{\sqrt{m}}\varphi_3^{(1)} & \sqrt{\frac{2}{m}}\varphi_2^{(2)} & \sqrt{\frac{2}{m}}\varphi_2^{(3)} & \cdots & \sqrt{\frac{2}{m}}\varphi_3^{(m-1)} & \sqrt{\frac{2}{m}}\varphi_3^{(m)} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\frac{1}{\sqrt{m}}\varphi_{m-1}^{(1)} & \sqrt{\frac{2}{m}}\varphi_{m-1}^{(2)} & \sqrt{\frac{2}{m}}\varphi_{m-1}^{(3)} & \cdots & \sqrt{\frac{2}{m}}\varphi_{m-1}^{(m-1)} & \sqrt{\frac{2}{m}}\varphi_{m-1}^{(m)} \\
\frac{1}{\sqrt{m}}\varphi_m^{(1)} & \sqrt{\frac{2}{m}}\varphi_m^{(2)} & \sqrt{\frac{2}{m}}\varphi_m^{(3)} & \cdots & \sqrt{\frac{2}{m}}\varphi_m^{(m-1)} & \sqrt{\frac{2}{m}}\varphi_m^{(m)}
\end{pmatrix},
$$

which satisfies $T^{-1} = T'$. By using the translation

$$
\begin{pmatrix} x_1^t \\ x_2^t \\ x_3^t \\ \vdots \\ x_{m-1}^t \\ x_m^t \end{pmatrix} = T \begin{pmatrix} w_1^t \\ w_2^t \\ w_3^t \\ \vdots \\ w_{m-1}^t \\ w_m^t \end{pmatrix},
$$

system (2.2) can be transformed into

$$
\begin{pmatrix} w_1^{t+1} \\ w_2^{t+1} \\ w_3^{t+1} \\ \vdots \\ w_{m-1}^{t+1} \\ w_m^{t+1} \\ \delta^{t+1} \end{pmatrix}
= \begin{pmatrix}
\frac{1}{p} - d^*\lambda_1 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & \frac{1}{p} - d^*\lambda_2 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & \frac{1}{p} - d^*\lambda_3 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & \frac{1}{p} - d^*\lambda_{m-1} & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & -1 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix} w_1^t \\ w_2^t \\ w_3^t \\ \vdots \\ w_{m-1}^t \\ w_m^t \\ \delta^t \end{pmatrix}
+ \begin{pmatrix}
g_1(w_1^t, w_2^t, x_3^t, \cdots, w_{m-1}^t, w_m^t, \delta^t) \\
g_2(w_1^t, w_2^t, w_3^t, \cdots, w_{m-1}^t, w_m^t, \delta^t) \\
g_3(w_1^t, w_2^t, w_3^t, \cdots, w_{m-1}^t, w_m^t, \delta^t) \\
\vdots \\
g_{m-1}(w_1^t, w_2^t, w_3^t, \cdots, w_{m-1}^t, w_m^t, \delta^t) \\
g_m(w_1^t, w_2^t, w_3^t, \cdots, w_{m-1}^t, w_m^t, \delta^t) \\
0
\end{pmatrix},
$$

where

$$\begin{aligned} g_i(w_1^t, w_2^t, w_3^t, \cdots, w_{m-1}^t, w_m^t, \delta^t) &= -\frac{2}{p^2}\sum_{k=1}^{m} e_{ik}\Big(\sum_{j=1}^{m} e_{jk}w_j^t\Big)^2 \\ &+\Big(-\sum_{k=1}^{m} e_{ik}\sum_{j=1}^{m} e_{jk}w_j^t - \sum_{k=2}^{m-1} e_{ik}\sum_{j=1}^{m} e_{jk}w_j^t\Big)\delta^t \\ &+\Big(\sum_{k=1}^{m-1} e_{ik}\sum_{j=1}^{m} e_{j,k+1}w_j^t + \sum_{k=2}^{m} e_{ik}\sum_{j=1}^{m} e_{j,k-1}w_j^t\Big)\delta^t \\ &+\frac{6}{p^3}\sum_{k=1}^{m} e_{ik}\Big(\sum_{j=1}^{m} e_{jk}w_j^t\Big)^3 \\ &+O\Big((|w_1^t| + |w_2^t| + \cdots + |w_m^t| + |\delta^t|)^4\Big). \end{aligned}$$

From the center manifold theorem [3], we know that there exists a center manifold $W^c(0,0,\cdots,0)$, which can be approximately represented as follows:

$$W^c(\underbrace{0,0,\cdots,0}_{m+1}) = \left\{(w_1^t, w_2^t, \cdots, w_{m-1}^t, w_m^t, \delta^t)\,\middle|\, w_i = h_i(w_m^t, \delta^t), i = 1,2,\cdots,m-1\right\},$$

where

$$h_i(w_m^t, \delta^t) = a_{i1}(w_m^t)^2 + a_{i2}w_m^t\delta^t + a_{i3}(\delta^t)^2, i = 1,2,\cdots,m-1,$$

and we can obtain

$$\begin{aligned} w_i^{t+1} &= h_i\big(-w_m^t + g_i(w_1^t, w_2^t, w_3^t, \cdots, w_{m-1}^t, w_m^t, \delta^t), \delta^{t+1}\big) \\ &= (\frac{1}{p} - d^*\lambda_i)h_i - \frac{2}{p^2}\sum_{k=1}^{m} e_{ik}\Big(\sum_{j=1}^{m-1} e_{jk}h_j + e_{mk}w_m^t\Big)^2 \\ &+\Big[-\sum_{k=1}^{m} e_{ik}\Big(\sum_{j=1}^{m-1} e_{jk}h_j + e_{mk}w_m^t\Big) - \sum_{k=2}^{m-1} e_{ik}\Big(\sum_{j=1}^{m-1} e_{jk}h_j + e_{mk}w_m^t\Big)\Big]\delta^t \\ &+\Big[\sum_{k=1}^{m-1} e_{ik}\Big(\sum_{j=1}^{m-1} e_{j,k+1}h_j + e_{m,k+1}w_m^t\Big) + \sum_{k=2}^{m} e_{ik}\Big(\sum_{j=1}^{m-1} e_{j,k-1}h_j + e_{m,k-1}w_m^t\Big)\Big]\delta^t \\ &+\frac{6}{p^3}\sum_{k=1}^{m} e_{ik}\Big(\sum_{j=1}^{m-1} e_{jk}h_j + e_{mk}w_m^t\Big)^3 + O\Big((|w_m^t| + |\delta^t|)^4\Big), j = 1,2,\cdots,m-1. \end{aligned}$$

It follows that

$$\begin{aligned} a_{i1} &= -\frac{2}{p^2}\sum_{k=1}^{m} e_{ik}e_{mk}, \\ a_{i2} &= -\sum_{k=1}^{m} e_{ik}e_{mk} - \sum_{k=2}^{m} e_{ik}e_{mk} + \sum_{k=1}^{m-1} e_{ik}e_{m,k+1} + \sum_{k=2}^{m} e_{ik}e_{m,k-1}, \\ a_{i3} &= 0. \end{aligned}$$

Thus we consider the map which (2.2) is restricted to the center manifold $W^c(0,0,\cdots,0)$:

$$
\begin{aligned}
G \quad : \quad w_m^{t+1} &= -w_m^t - \frac{2}{p^2}\sum_{k=1}^{m} e_{mk}\Big(\sum_{j=1}^{m-1} e_{jk}w_j^t + e_{mk}w_m^t\Big)^2 \\
&+ \Big[-\sum_{k=1}^{m} e_{mk}\Big(\sum_{j=1}^{m-1} e_{jk}w_j^t + e_{mk}w_m^t\Big) - \sum_{k=2}^{m-1} e_{mk}\Big(\sum_{j=1}^{m-1} e_{jk}w_j^t + e_{mk}w_m^t\Big)\Big]\delta^t \\
&+ \Big[\sum_{k=1}^{m-1} e_{mk}\Big(\sum_{j=1}^{m-1} e_{j,k+1}w_j^t + e_{m,k+1}w_m^t\Big) + \sum_{k=2}^{m} e_{mk}\Big(\sum_{j=1}^{m-1} e_{j,k-1}w_j^t + e_{m,k-1}w_m^t\Big)\Big]\delta^t \\
&+ \frac{6}{p^3}\sum_{k=1}^{m} e_{mk}(\sum_{j=1}^{m-1} e_{jk}w_j^t + e_{mk}w_m^t\Big)^3 \\
&+ O\Big((|w_m^t| + |\delta^t|)^4\Big),
\end{aligned}
$$

where $w_j^t = a_{j1}(w_m^t)^2 + a_{j2}w_m^t\delta^t + a_{j3}(\delta^t)^2, j = 1,2,\cdots,m-1$. Observe that

$$
\begin{cases}
G(0,0) = 0, \\
\dfrac{\partial G}{\partial \delta}\,|_{(0,0)} = 0, \\
\dfrac{\partial G}{\partial w_m}\,|_{(0,0)} = -1, \\
\dfrac{\partial^2 G}{\partial w_m \partial \delta}\,|_{(0,0)} = -\sum_{k=1}^{m} e_{mk}^2 - \sum_{k=2}^{m} e_{mk}^2 + \sum_{k=1}^{m-1} e_{mk}e_{m,k+1} + \sum_{k=2}^{m} e_{mk}e_{m,k-1}, \\
\dfrac{\partial^2 G}{\partial w_m^2}\,|_{(0,0)} = -\dfrac{2}{p^2}\sum_{k=1}^{m} e_{mk}^2, \\
\dfrac{\partial^2 G}{\partial \delta^2}\,|_{(0,0)} = 0, \\
\dfrac{\partial^3 G}{\partial w_m^3}\,|_{(0,0)} = \dfrac{6}{p^3}\sum_{k=1}^{m} e_{mk}^4.
\end{cases}
$$

It follows that

$$
\begin{aligned}
(\frac{\partial G}{\partial \delta}\frac{\partial^2 G}{\partial w_m^2} + 2\frac{\partial^2 G}{\partial w_m \partial \delta}) \quad | \quad {}_{(0,0)} & \\
&= 2(-\sum_{k=1}^{m} e_{mk}^2 - \sum_{k=2}^{m} e_{mk}^2 + \sum_{k=1}^{m-1} e_{mk}e_{m,k+1} + \sum_{k=2}^{m} e_{mk}e_{m,k-1}) \\
&= -2(\sum_{k=1}^{m-1}(e_{m,k+1} - e_{m,k})^2 + e_{m,m}^2) \\
&\neq 0,
\end{aligned}
$$

and

$$
(\frac{1}{2}(\frac{\partial^2 G}{\partial w_m^2})^2 + \frac{1}{3}\frac{\partial^3 G}{\partial w_m^3})\,|_{(0,0)} = \frac{2}{p^4}(\sum_{k=1}^{m} e_{mk}^2)^2 + \frac{2}{p^3}\sum_{k=1}^{m} e_{mk}^4 \neq 0,
$$

hold. This completes the proof. □

**Remark 2.3.** If $m = 2$, then system (2.1) is reduced to

$$\begin{cases} u_{t+1} = \frac{pu_t}{1+u_t} + d(-u_t + v_t), \\ v_{t+1} = \frac{pv_t}{1+v_t} + d(u_t - v_t), \end{cases}$$

whose flip bifurcation was discussed in [30], and bifurcation diagrams were shown. So, we omit the numerical simulations in this work.

## 3. Conclusions

In this paper, we investigated the complex behaviors of a discrete-time logistic model with diffusion by means of the central manifold method and the theory of normal form, and proved that the unique positive fixed point of the system (1.1)-(1.2) can undergo flip bifurcation. Moreover, we obtained the following facts:

(1) the critical point on the bifurcation is completely determined by the maximum eigenvalue of the Laplace operator;

(2) the bifurcation at the critical point for the discrete diffusion system is completely determined by the dynamical behaviors on the corresponding eigen-subspace.

Are the above observations applicable to other discrete time and space systems? How to analyze the bifurcation of the coupled discrete time and space systems? We will investigate these questions in our future work.

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