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ITERATIVE ALGORITHM FOR ZERO POINTS OF THE SUM OF COUNTABLE ACCRETIVE-TYPE MAPPINGS AND VARIATIONAL INEQUALITIES

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Abstract. In a real uniformly convex and q-uniformly smooth Banach space, a new iterative algorithm is devised for approximating a zero point of the sum of countable accretive-type mappings. The zero point also solves generalized variational inequalities. The highlight of this paper is the framework of the space. The application to capillarity systems is also considered.

Keywords. m-accretive mapping; θ-inversely-strongly accretive mapping; Fixed points; Variational inequalities; Zero points.

1. Introduction and Preliminaries

Let E be a real Banach space with E^* being its dual space. The value of $g \in E^*$ at $x \in E$ is denoted by $\langle x, g\rangle$, and "$\to$" denotes strong convergence in E. A Banach space E is said to be uniformly convex [3, 18] if any two sequences $\{x_n\}$ and $\{y_n\}$ in E with $\|x_n\| = \|y_n\| = 1$ and $\lim_{n\to\infty} \|x_n + y_n\| = 2$ imply $\lim_{n\to\infty} \|x_n - y_n\| = 0$. The function $\delta : [0,+\infty) \to [0,+\infty)$ is called the modulus of smoothness of E [3, 18] if it is defined as follows: $\delta(t) = \sup\{\frac{1}{2}(\|x+y\| + \|x-y\|) - 1 : x, y \in E,\ \|x\| = 1,\ \|y\| \le t\}$. A Banach space E is said to be uniformly smooth [3, 18] if $\lim_{t\to 0} \frac{\delta(t)}{t} = 0$. Let $q > 1$ be a real number. A Banach space E is said to be q-uniformly smooth with constant K_q if $K_q > 0$ and $\delta(t) \le K_q t^q$ for $t > 0$. It is well-known that every q-uniformly smooth Banach space is uniformly smooth. For $q > 1$, the generalized duality mapping $J_q : E \to 2^{E^*}$ is defined by (see, e.g., [3, 18]) $J_q(x) = \{g \in E^* : \langle x, g\rangle = \|x\|^q, \|x\|^{q-1} = \|g\|\}, \forall x \in E$. In particular, $J := J_2$ is called the normalized duality mapping. The single-valued generalized duality mapping and the single-valued normalized duality mapping are denoted by j_q or j, respectively. Let $T : D(T) \subset E \to E$ be a nonlinear mapping. We use $T^{-1}0$ to denote

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the set of zero points of T. That is, $T^{-1}0 = \{x \in D(T) : Tx = 0\}$. We use $F(T)$ to denote the set of fixed points of T. That is, $F(T) = \{x \in D(T) : Tx = x\}$. Recall that

(1) T is said to be accretive if, for all $x, y \in D(T)$, $\langle Tx - Ty, j(x-y)\rangle \geq 0$, where $j(x-y) \in J(x-y)$;

(2) T is said to be m-accretive if T is accretive and $R(I + \lambda T) = E$, for $\forall \lambda > 0$;

(3) T is said to be θ-inversely-strongly accretive if, for all $x, y \in D(T)$, $\langle Tx - Ty, j_q(x-y)\rangle \geq \theta\|Tx - Ty\|^q$, where $j_q(x-y) \in J_q(x-y)$, for some $\theta > 0$;

(4) T is said to be ϑ-strongly accretive if, for all $x, y \in D(T)$, $\langle Tx - Ty, j(x-y)\rangle \geq \vartheta\|x-y\|^2$, where $j(x-y) \in J(x-y)$, for some $\vartheta > 0$;

(5) T is said to be a contraction with coefficient k if $k \in (0,1)$ such that, for all $x, y \in D(T)$, $\|Tx - Ty\| \leq k\|x-y\|$;

(6) T is said to be nonexpansive if, for all $x, y \in D(T)$, $\|Tx - Ty\| \leq \|x-y\|$;

(7) T is said to be a strongly positive mapping with coefficient ξ if $\xi > 0$ such that $\langle Tx, j(x)\rangle \geq \xi\|x\|^2$ for $x \in D(T)$, where $j(x) \in J(x)$. In this case, $\|aI - bT\| = \sup_{\|x\|\leq 1} |\langle (aI - bT)x, j(x)\rangle|$, where $j(x) \in J(x)$, I is the identity mapping, $a \in [0,1]$, and $b \in [-1,1]$;

(8) T is said to be μ-strictly pseudo-contractive if, for all $x, y \in D(T)$, there exists $j(x-y) \in J(x-y)$ such that $\langle Tx - Ty, j(x-y)\rangle \leq \|x-y\|^2 - \mu\|x-y-(Tx-Ty)\|^2$, for some $\mu \in (0,1)$.

The Lyapunov functional $\omega : E \times E \to R^+$ is defined by $\omega(x,y) = \|x\|^2 - 2\langle x, j(y)\rangle + \|y\|^2$, $\forall x, y \in E$, $j(y) \in J(y)$.

If E is a real uniformly smooth and uniformly convex Banach space, and C is a nonempty, closed, and convex subset of E, then [4, 11, 16] (1) for each $x \in E$, there exists a unique element $v \in C$ such that $\|x - v\| = \inf\{\|x-y\| : y \in C\}$. Such an element v is denoted by $P_C x$, and it is called the metric projection of E onto C; (2) $\forall x \in E$, there exists a unique element $x_0 \in C$ satisfying $\omega(x_0, x) = \inf\{\omega(z,x) : z \in C\}$. In this case, for any $x \in E$, define $\Pi_C : E \to C$ by $\Pi_C x = x_0$, and then Π_C is called the generalized projection from E onto C. If E is a real uniformly smooth Banach space, and K is a nonempty closed subset of E, then $T : K \to K$ is said to be generalized nonexpansive [17] if $F(T) \neq \emptyset$ and $\omega(Tx, y) \leq \omega(x,y)$ for each $x \in D(T)$ and $y \in F(T)$. Let Q be a mapping of E onto K, where K is a nonempty closed subset of E. Then Q is said to be sunny [4, 11, 16] if $Q(Q(x) + t(x - Q(x))) = Q(x)$, for all $x \in E$ and $t \geq 0$. A mapping $Q : E \to K$ is said to be a retraction [4, 11, 16] if $Q^2 = Q$. If E is a uniformly smooth and uniformly convex Banach space, then the sunny generalized nonexpansive retraction of E onto K is uniquely decided, which is denoted by R_K.

Recently, a hot topic to find zero points of the sum of two accretive-type mappings, namely, a solution of the following inclusion problem: $0 \in Au + Bu$, where $A : E \to 2^E$ is m-accretive and $B : E \to E$ is θ-inversely-strongly accretive. A number of problems, which are related to evolution equations, convex programming, variational inequalities, split feasibility problems, minimization problems, inverse problems, and image processing, can be modeled via the inclusion problem. Many efforts have been devoted to constructing iterative algorithms for the solutions of the inclusion problem. The forward-backward splitting method is a popular one among the recent iterative algorithms. Forward-backward splitting method means an iteration involves only A as the forward step and B as the backward step, not the sum $A + B$. The classical forward-backward splitting iterative method is stated as follows: $x_1 \in H$, $x_{n+1} = (I + r_n A)^{-1}(x_n - r_n B x_n)$, $\forall n \in N$. Some of the related work can be found in [10, 12, 14, 15, 19, 20, 21, 22, 23] and the references therein.

In 2012, Ceng et al. [7] proposed the following iterative algorithm with a perturbed operator for approximating a zero point of the m-accretive mapping A in a real Hilbert space H:

$$\begin{cases} x_1 \in H, \\ y_n = \alpha_n x_n + (1-\alpha_n)(I+r_nA)^{-1}x_n, \\ x_{n+1} = \beta_n f(x_n) + (1-\beta_n)[(I+r_nA)^{-1}y_n - \lambda_n\mu_n W((I+r_nA)^{-1}y_n)], \quad \forall n \in N, \end{cases} \tag{1.1}$$

where $W : H \to H$ is a ϑ-strongly accretive and μ-strictly pseudo-contractive mapping with $\vartheta + \mu > 1$, $f : H \to H$ is a contraction, and $A : H \to H$ is m-accretive. Under some assumptions, $\{x_n\}$ was proved to be strongly convergent to the unique element $p_0 \in A^{-1}0$, which solves the following variational inequality: $\langle p_0 - f(p_0), p_0 - u\rangle \le 0$, $\forall u \in A^{-1}0$. Notice that the mapping W, which is called a perturbed operator, only plays a role in the construction of the iterative algorithm for selecting a particular zero point of A, but not involved in the variational inequality.

In [20], the study of the inclusion problem $0 \in Au + Bu$ was extended to the system of inclusion problems $0 \in A_iu + B_iu$, where $A_i : H \to H$ is m-accretive, and $B_i : H \to H$ is θ_i-inversely-strongly accretive for each $i \in N$. The algorithmm in [20] reads

$$\begin{cases} x_1 \in C, \\ u_n = Q_C(\alpha_n x_n + \beta_n a_n), \\ v_n = \tau_n u_n + \chi_n \sum_{i=1}^{\infty} \omega_i^{(2)} (I + r_{n,i}A_i)^{-1}(I - r_{n,i}B_i)(\frac{u_n+v_n}{2}) + \xi_n b_n, \\ x_{n+1} = \delta_n f(x_n) \\ +(1-\delta_n)(I - \zeta_n \sum_{i=1}^{\infty} \omega_i^{(1)} W_i) \sum_{i=1}^{\infty} \omega_i^{(2)} (I + r_{n,i}A_i)^{-1}(I - r_{n,i}B_i)(\frac{u_n+v_n}{2}), \quad \forall n \in N. \end{cases} \tag{1.2}$$

where $\sum_{i=1}^{\infty} \omega_i^{(1)} W_i$ is called a superposition perturbation, $W_i : H \to H$ is a perturbed operator in the sense of (1.1), that is, $W_i : H \to H$ is a ϑ_i-strongly accretive and μ_i-strictly pseudo-contractive mapping with $\vartheta_i + \mu_i > 1$, for each $i \in N$. The iterative sequence $\{x_n\}$ was proved to be strongly convergent to $p_0 \in \bigcap_{i=1}^{\infty}(A_i + B_i)^{-1}0$, which solves the variational inequality: $\langle p_0 - f(p_0), j(p_0 - u)\rangle \le 0$, $\forall u \in \bigcap_{i=1}^{\infty}(A_i + B_i)^{-1}0$.

In 2019, Wei *et al.* [21] injected some new ideas by proposing the following inertial forward-backward iterative algorithm for approximating the solution of the inclusion problems, considered in [20]

$$\begin{cases} x_0, x_1, e_1 \in H, \\ y_n = x_n + k_n(x_n - x_{n-1}), \\ w_n = \alpha_n x_n + \beta_n \sum_{i=1}^{\infty} \omega_{n,i}(I + r_{n,i}A_i)^{-1}(I - r_{n,i}B_i)y_n + \gamma_n e_n, \\ C_1 = H = Q_1, \\ C_{n+1} = \{p \in C_n : \|w_n - p\|^2 \le (1-\gamma_n)\|x_n - p\|^2 + \gamma_n\|e_n - p\|^2 \\ \ +k_n^2\|x_n - x_{n-1}\|^2 - 2\beta_n k_n\langle x_n - p, x_{n-1} - x_n\rangle\}, \\ Q_{n+1} = \{p \in C_{n+1} : \|x_1 - p\|^2 \le \|x_1 - P_{C_{n+1}}(x_1)\|^2 + \sigma_{n+1}\}, \\ x_{n+1} \in Q_{n+1}, \ n \in N. \end{cases}$$

The result that $x_n \to P_{\bigcap_{m=1}^{\infty} C_m}(x_1) \in \bigcap_{i=1}^{\infty}(A_i + B_i)^{-1}0$, as $n \to \infty$, was proved under some conditions. To set up the relationship between the limit of $\{x_n\}$, $P_{\bigcap_{m=1}^{\infty} C_m}(x_1)$, and the solution of variational inequalities, a mid-point inertial forward-backward iterative algorithm was presented as follows:

$$\begin{cases} z_0 = x_0, x_1, e_1 \in H, \\ z_n = \delta_n \lambda f(x_n) + (I - \delta_n F) x_n, \\ v_n = z_n + k_n(z_n - z_{n-1}), \\ w_n = \alpha_n v_n + \beta_n \sum_{i=1}^{\infty} \omega_{n,i} (I + r_{n,i} A_i)^{-1} (I - r_{n,i} B_i)(\frac{v_n + w_n}{2}) + \gamma_n e_n, \\ C_1 = H = Q_1, \\ C_{n+1} = \{p \in C_n : \|w_n - p\|^2 \leq \frac{2\alpha_n + \beta_n}{2 - \beta_n} \|z_n - p\|^2 + \frac{2\gamma_n}{2 - \beta_n} \|e_n - p\|^2 \\ \ + \frac{2\alpha_n + \beta_n}{2 - \beta_n} k_n \|z_n - z_{n-1}\|^2 - 2 \frac{2\alpha_n + \beta_n}{2 - \beta_n} k_n \langle z_n - p, z_{n-1} - z_n \rangle\}, \\ Q_{n+1} = \{p \in C_{n+1} : \|x_1 - p\|^2 \leq \|x_1 - P_{C_{n+1}}(x_1)\|^2 + \sigma_{n+1}\}, \\ x_{n+1} \in Q_{n+1},\ n \in N, \end{cases}$$

where f is a contraction, and F is a strongly positive linear bounded mapping. The result that $x_n \to P_{\bigcap_{m=1}^{\infty} C_m}(x_1) = P_{\bigcap_{i=1}^{\infty}(A_i+B_i)^{-1}0}(x_1)$, as $n \to \infty$, was proved under some conditions. Under the additional assumptions that $\widetilde{x} = P_{\bigcap_{i=1}^{\infty}(A_i+B_i)^{-1}0}(x_1)$ and $\widetilde{x} = P_{\bigcap_{i=1}^{\infty}(A_i+B_i)^{-1}0}(x_1)[\lambda f(\widetilde{x}) - F(\widetilde{x}) + \widetilde{x}]$, it was proved that $\widetilde{x}$ solves the variational inequality

$$\langle F\widetilde{x} - \lambda f(\widetilde{x}), \widetilde{x} - z \rangle \leq 0,\ \forall z \in \bigcap_{i=1}^{\infty} (A_i + B_i)^{-1} 0. \tag{1.3}$$

Very recently, a new forward-backward multi-choice iterative algorithm with superposition perturbations in a real Hilbert space was constructed in [22]. Some strong convergence theorems of common solutions of inclusion problems and variational inequalities were proved

$$\begin{cases} x_1, y_1, e_1, \varepsilon_1 \in H, \\ C_1 = H = Q_1, \\ u_n = \omega_n x_n + \varepsilon_n, \\ v_n = \beta_n u_n + (1 - \beta_n) \sum_{i=1}^{\infty} a_i (I + r_{n,i} A_i)^{-1} (I - r_{n,i} B_i) v_n, \\ z_n = \delta_n f(x_n) + (1 - \delta_n)(I - \zeta_n \sum_{i=1}^{\infty} c_i W_i) \sum_{i=1}^{\infty} a_i (I + r_{n,i} A_i)^{-1} (I - r_{n,i} B_i) v_n, \\ w_n = \alpha_n x_n + (1 - \alpha_n) \sum_{i=1}^{\infty} a_i (I + r_{n,i} A_i)^{-1} (I - r_{n,i} B_i) z_n, \\ x_{n+1} = \lambda_n \eta f(x_n) + (I - \lambda_n F) w_n + e_n, \\ C_{n+1} = \{p \in C_n : 2\langle \alpha_n x_n + (1 - \alpha_n) z_n - w_n, p \rangle \\ \ \leq \alpha_n \|x_n\|^2 + (1 - \alpha_n)\|z_n\|^2 - \|w_n\|^2\} \\ Q_{n+1} = \{p \in C_{n+1} : \|x_1 - p\|^2 \leq \|P_{C_{n+1}}(x_1) - x_1\|^2 + \sigma_{n+1}\},\ n \in N, \\ y_{n+1} \in Q_{n+1},\ n \in N, \end{cases} \tag{1.4}$$

It was shown that $x_n \to q_0 \in \bigcap_{i=1}^{\infty}(A_i + B_i)^{-1}0$, as $n \to \infty$, where q_0 satisfies the following variational inequalities:

$$\langle F q_0 - \eta f(q_0), q_0 - y \rangle \leq 0,\ \forall y \in \cap_{i=1}^{\infty} (A_i + B_i)^{-1} 0, \tag{1.5}$$

and

$$\langle q_0 - f(q_0), q_0 - y \rangle \leq 0,\ \forall y \in \cap_{i=1}^{\infty} (A_i + B_i)^{-1} 0. \tag{1.6}$$

In addition, $y_n \to P_{\bigcap_{m=1}^{\infty} C_m}(x_1) \in \bigcap_{i=1}^{\infty}(A_i + B_i)^{-1}0$, as $n \to \infty$.

Motivated by the above results, in particular, those in [11, 21, 22], a new iterative algorithm is constructed, which extends the corresponding results from a real Hilbert space to a real uniformly convex and q-uniformly smooth Banach space in this paper. The superposition perturbation is considered. Moreover, the strong convergent limit of the iterative sequence is proved to solve not only the inclusion problems but also two different types of generalized variational inequalities. Also, the application to capillarity systems is demonstrated.

To begin our study, the following tools are needed.

Lemma 1.1. [3, 18] *Let E is a real uniformly smooth and uniformly convex Banach space. Then the normalized duality mapping $J : E \to 2^{E^*}$ is single-valued and $J(-x) = -Jx$, for $x \in E$.*

Lemma 1.2. [23] *Let C be a nonempty, closed, and convex subset of a real Banach space E. Let $A : C \to E$ be a single-valued mapping, and let $B : E \to 2^E$ be m-accretive. Then $F((I + rB)^{-1}(I - rA)) = (A+B)^{-1}0$, where r is any positive real number.*

Lemma 1.3. [5] *Let E is a real uniformly convex Banach space, and let C be a nonempty, closed, and convex subset of E. Let $T_i : C \to C$ be a nonexpansive mapping for each $i \in N$ and $\sum_{i=1}^{\infty} a_i = 1$ for $\{a_i\} \subset (0,1)$. Then, $\sum_{i=1}^{\infty} a_i T_i$ is nonexpansive with $F(\sum_{i=1}^{\infty} a_i T_i) = \bigcap_{i=1}^{\infty} F(T_i)$ under the assumption that $\bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$.*

Lemma 1.4. [1] *Let E be a real uniformly convex and uniformly smooth Banach space with K being its nonempty, closed, and convex subset. Let $R_K : E \to K$ be the sunny generalized nonexpansive retraction. Then $\omega(x, R_K x) + \omega(R_K x, u) \leq \omega(x, u)$ for $x \in E$ and $u \in K$.*

Lemma 1.5. [2] *Let E be a real uniformly convex and uniformly smooth Banach space, and let C be a nonempty closed and convex subset of E. Then, for any $x \in E$ and $y \in C$, $\omega(y, \Pi_C x) + \omega(\Pi_C x, x) \leq \omega(y, x)$.*

Lemma 1.6. [13] *Let E be a real uniformly smooth and uniformly convex Banach space. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in E. If either $\{x_n\}$ or $\{y_n\}$ is bounded and $\omega(x_n, y_n) \to 0$ as $n \to \infty$, then $x_n - y_n \to 0$ as $n \to \infty$.*

Lemma 1.7. [8, 12] *Let E be a real q-uniformly smooth Banach space, where $q > 1$ is a real number, then the following inequality is true $\|x+y\|^q \leq \|x\|^q + q\langle y, j_q(x+y)\rangle$, where $x, y \in E$ and $j_q(x+y) \in J_q(x+y)$. Further, if E is smooth, then $\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y)\rangle$.*

Lemma 1.8. [4] *Let E be a real uniformly smooth and uniformly convex Banach space. Let $f : E \to E$ be a contraction with coefficient $k \in (0,1)$. Let $F : E \to E$ be a strongly positive linear bounded mapping with coefficient $\xi > 0$, and let $U : E \to E$ be a nonexpansive mapping. Suppose $0 < \eta \leq \xi/2k$ and $F(U) \neq \emptyset$. If, for each $t \in (0,1)$, $T_t : E \to E$ is defined by $T_t x := t\eta f(x) + (I - tF)Ux$, then T_t has a fixed point x_t for each $t \in (0, \|F\|^{-1}]$. Moreover, $x_t \to q_0$, as $t \to 0$, where $q_0 \in F(U)$, which satisfies the variational inequality: $\langle F q_0 - \eta f(q_0), j(q_0 - z)\rangle \leq 0$, $\forall z \in F(U)$.*

Lemma 1.9. [6] *Let E be a Banach space. Let $F : E \to E$ be a strongly positive linear bounded mapping with coefficient $\xi > 0$ and $0 < \rho \leq \|F\|^{-1}$. Then $\|I - \rho F\| \leq 1 - \rho\xi$.*

Lemma 1.10. [26] *Let $\{x_n\}$ and $\{b_n\}$ be two sequences of nonnegative real number sequences satisfying $x_{n+1} \leq (1-t_n)x_n + b_n$, $\forall n \in N$, where $\{t_n\} \subset (0,1)$ with $\sum_{n=1}^{\infty} t_n = +\infty$ and $t_n \to 0$, as $n \to \infty$. If $\limsup_{n\to\infty} \frac{b_n}{t_n} \leq 0$, then $\lim_{n\to\infty} x_n = 0$.*

Lemma 1.11. [11] *Let E be a real uniformly convex and q-uniformly smooth Banach space with constant K_q, where $q \in (1,2]$, and let $f : E \to E$ be a contraction with coefficient $k \in (0,1)$. Let $A_i : E \to E$ be a m-accretive mapping, and let $B_i : E \to E$ be a θ_i-inversely-strongly accretive mapping. Let $W_i : E \to E$ be ϑ_i-strongly accretive and μ_i-strictly pseudo-contractive with $\vartheta_i + \mu_i > 1$, for $i \in N$. Suppose $0 < r_{n,i} \le (\frac{q\theta_i}{K_q})^{\frac{1}{q-1}}$ for $i \in N$ and $n \in N$, $k_t \in (0,1)$ for $t \in (0,1), \sum_{n=1}^{\infty} c_n \|W_n\| < +\infty$, $\sum_{i=1}^{\infty} a_i = 1 = \sum_{i=1}^{\infty} c_i$ and $\bigcap_{i=1}^{\infty}(A_i + B_i)^{-1}0 \ne \emptyset$. If, for each $t \in (0,1)$, define $Z_t^n : E \to E$ by*

$$Z_t^n u = tf(u) + (1-t)(I - k_t \sum_{i=1}^{\infty} c_i W_i) \sum_{i=1}^{\infty} a_i (I - r_{n,i} A_i)^{-1}(I - r_{n,i} B_i)u.$$

Then Z_t^n has a fixed point u_t^n, that is,

$$u_t^n = tf(u_t^n) + (1-t)(I - k_t \sum_{i=1}^{\infty} c_i W_i) \sum_{i=1}^{\infty} a_i (I - r_{n,i} A_i)^{-1}(I - r_{n,i} B_i)u_t^n.$$

Moreover, if $\frac{k_t}{t} \to 0$, then $u_t^n \to p_0$, as $t \to 0$, where p_0 is the solution of variational inequality: $\langle p_0 - f(p_0), j(p_0 - z)\rangle \le 0$, $\forall z \in \bigcap_{i=1}^{\infty}(A_i + B_i)^{-1}0$.

Lemma 1.12. [9] *Let E be a real uniformly convex Banach space, and let $W : E \to E$ be ϑ-strongly accretive and μ-strictly pseudo-contractive with $\vartheta + \mu > 1$. Then, for any fixed number $\delta \in (0,1)$, $I - \delta W$ is a contraction with coefficient $[1 - \delta(1 - \sqrt{\frac{1-\vartheta}{\mu}})]$.*

Lemma 1.13. [12] *Let E be a real uniformly convex and q-uniformly smooth Banach space with constant K_q, where $q \in (1,2]$. Let $A : E \to E$ be an m-accretive mapping, and let $B : E \to E$ be a θ-inversely-strongly accretive mapping. Then $(I + rA)^{-1}(I - rB)$ is nonexpansive for $0 < r \le (\frac{q\theta}{K_q})^{\frac{1}{q-1}}$.*

2. New Iterative Algorithms and Strong Convergence Theorems

In this section, unless otherwise stated, we always assume that:

(1) E is a real uniformly convex and q-uniformly smooth Banach space with constant K_q, where $q \in (1,2]$;

(2) $A_i : E \to E$ is m-accretive and $B_i : E \to E$ is θ_i-inversely-strongly accretive, for each $i \in N$;

(3) $f : E \to E$ is a contraction with coefficient $k \in (0, \frac{1}{2}]$. If $\langle f(x) - x, j(y-x)\rangle = 0$, then $x = 0$ or $y = x$, for $x, y \in E$;

(4) $F : E \to E$ is a strongly positive linear bounded mapping with coefficient $\xi > 0$ and $\langle F(x) - \eta f(x) + f(y) - y, j(x-y)\rangle \ge 0$, for $x, y \in E$;

(5) $W_i : E \to E$ is ϑ_i-strongly accretive and μ_i-strictly pseudo-contractive, for $i \in N$;

(6) $\{e_n\} \subset H$ and $\{\varepsilon_n\} \subset E$ are the computational errors;

(7) $\{a_i\}$ and $\{c_i\}$ are two real sequences in $(0,1)$ with $\sum_{i=1}^{\infty} a_i = \sum_{i=1}^{\infty} c_i = 1$;

(8) $\{\alpha_n\}$, $\{\beta_n\}$, $\{\delta_n\}$, $\{\zeta_n\}$, $\{\omega_n\}$, $\{\tau_n\}$, $\{\chi_n\}$, and $\{\lambda_n\}$ are real sequences in $(0,1)$;

(9) $\{r_{n,i}\}$ is a real number sequence in $(0, +\infty)$;

(10) $\alpha_n + \tau_n + \chi_n \equiv 1$.

Theorem 2.1. *Let $\{x_n\}$ be a sequence generated by the following iterative algorithm:*

$$\begin{cases} x_1, y_1, e_1, \varepsilon_1 \in E,\ K_1 = E, \\ u_n = \omega_n x_n + \varepsilon_n, \\ v_n = \beta_n u_n + (1-\beta_n)\sum_{i=1}^{\infty} a_i (I + r_{n,i}A_i)^{-1}(I - r_{n,i}B_i)v_n, \\ z_n = \delta_n f(x_n) + (1-\delta_n)(I - \zeta_n \sum_{i=1}^{\infty} c_i W_i)\sum_{i=1}^{\infty} a_i (I + r_{n,i}A_i)^{-1}(I - r_{n,i}B_i)v_n, \\ w_n = \alpha_n x_n + \tau_n \sum_{i=1}^{\infty} a_i (I + r_{n,i}A_i)^{-1}(I - r_{n,i}B_i)z_n + \chi_n y_n, \\ x_{n+1} = \lambda_n \eta f(x_n) + (I - \lambda_n F)w_n + e_n, \\ K_{n+1} = \{p \in K_n : \|w_n - p\|^2 \le \alpha_n \|x_n - p\|^2 + \tau_n \|z_n - p\|^2 + \chi_n \|y_n - p\|^2\}, \\ y_{n+1} = R_{K_{n+1}}(x_1),\ n \in N, \end{cases} \tag{2.1}$$

where $R_{K_{n+1}}$ stands for the sunny generalized nonexpansive retraction from E onto K_{n+1}. If (i) $0 \in \bigcap_{i=1}^{\infty}(A_i + B_i)^{-1}0$;

(ii) $\mu_i + \vartheta_i > 1$, $\mu_i \in (0,1)$ and $\vartheta_i \in (0,1)$, for $i \in N$;

(iii) $0 < r_{n,i} \le (\frac{q\theta_i}{K_q})^{\frac{1}{q-1}}$, for $i, n \in N$;

(iv) $\alpha_n \to 0, \beta_n \to 0, \delta_n \to 0$, and $\zeta_n \to 0$, as $n \to \infty$;

(v) $0 < \eta < \frac{\xi}{2k}$;

(vi) $\sum_{i=1}^{\infty} c_i \|W_i\| < +\infty$; $\sum_{n=1}^{\infty} \|e_n\| < +\infty$, $\sum_{n=1}^{\infty} \|\varepsilon_n\| < +\infty$, $\sum_{n=1}^{\infty}(1-\omega_n) < +\infty$ and $\sum_{n=1}^{\infty} \chi_n < +\infty$;

(vii) $\frac{\delta_n}{\lambda_n} \to 0$, $\frac{\|e_n\|}{\lambda_n} \to 0$, $\frac{\|\varepsilon_n\|}{\lambda_n} \to 0$, $\frac{1-\omega_n}{\lambda_n} \to 0$, $\frac{\zeta_n}{\lambda_n} \to 0$ and $\frac{\alpha_n \chi_n}{\lambda_n} \to 0$, as $n \to \infty$;

$(viii)$ $\sum_{n=1}^{\infty} \lambda_n = +\infty$ and $\lambda_n \to 0$, as $n \to \infty$,

then $y_n \to q_0 \in \bigcap_{i=1}^{\infty}(A_i + B_i)^{-1}0$, as $n \to \infty$, where q_0 satisfies the following variational inequalities:

$$\langle Fq_0 - \eta f(q_0), j(q_0 - y)\rangle \le 0, \quad \forall y \in \cap_{i=1}^{\infty}(A_i + B_i)^{-1}0, \tag{2.2}$$

and

$$\langle q_0 - f(q_0), j(q_0 - y)\rangle \le 0, \quad \forall y \in \cap_{i=1}^{\infty}(A_i + B_i)^{-1}0. \tag{2.3}$$

Proof. We split the proof into eight steps.

Step 1. $\{v_n\}$ is well-defined.

For $s \in (0,1)$, defined $U_s : E \to E$ by $U_s x := su + (1-s)Tx$, $\forall x \in E$, where u is a fixed vector in E, and $T : E \to E$ is a fixed nonexpansive mapping. It is easy to check that $\|U_s x - U_s y\| = (1-s)\|Tx - Ty\| \le (1-s)\|x - y\|$. Thus U_s is a contraction, which ensures that there exists $x_s \in E$ such that $U_s x_s = x_s$, that is, $x_s = su + (1-s)Tx_s$. It follows from Lemma 1.13 and assumption (iii) that $(I + r_{n,i}A_i)^{-1}(I - r_{n,i}B_i) : E \to E$ is nonexpansive for $i, n \in N$. Since $\sum_{i=1}^{\infty} a_i = 1$, then from Lemmas 1.2 and 1.3, one has that $\sum_{i=1}^{\infty} a_i (I + r_{n,i}A_i)^{-1}(I - r_{n,i}B_i) : E \to E$ is nonexpansive for $n \in N$. Moreover, $F(\sum_{i=1}^{\infty} a_i (I + r_{n,i}A_i)^{-1}(I - r_{n,i}B_i)) = \bigcap_{i=1}^{\infty}(A_i + B_i)^{-1}0$. Consider T above as $\sum_{i=1}^{\infty} a_i (I + r_{n,i}A_i)^{-1}(I - r_{n,i}B_i)$, one can see that $\{v_n\}$ is well-defined.

Step 2. K_n is a nonempty closed subset of E.

It is easy to see from the construction of K_n that K_n is a closed subset of E. We are left to show that $K_n \neq \emptyset$. For this, it suffices to show that $\bigcap_{i=1}^{\infty}(A_i + B_i)^{-1}0 \subset K_n$, for $n \ge 2$. In fact,

$\forall p \in \bigcap_{i=1}^{\infty}(A_i+B_i)^{-1}0$, in view of Lemmas 1.2 and 1.3, one has

$$\begin{aligned}
&\|w_n-p\|^2\\
&\le \alpha_n\|x_n-p\|^2+\tau_n\sum_{i=1}^{\infty}a_i\|(I+r_{n,i}A_i)^{-1}(I-r_{n,i}B_i)z_n-p\|^2+\chi_n\|y_n-p\|^2\\
&\le \alpha_n\|x_n-p\|^2+\tau_n\|z_n-p\|^2+\chi_n\|y_n-p\|^2.
\end{aligned}$$

Then $K_n\neq\emptyset$, for $n\in N$.

Step 3. $\{u_n\}$,$\{v_n\}$, $\{z_n\}$, $\{w_n\}$, $\{x_n\}$, and $\{y_n\}$ are all bounded.

For $p \in \bigcap_{i=1}^{\infty}(A_i+B_i)^{-1}0$, one find from Lemma 1.4 that

$$\omega(x_1,y_{n+1})=\omega(x_1,R_{K_{n+1}}(x_1))\le\omega(x_1,p)-\omega(R_{K_{n+1}}(x_1),p)\le\omega(x_1,p),$$

which implies that $\{y_n\}$ is bounded. Note that

$$\|u_n-p\|\le\omega_n\|x_n-p\|+(1-\omega_n)\|p\|+\|\varepsilon_n\|, \tag{2.4}$$

and $\|v_n-p\|\le\beta_n\|u_n-p\|+(1-\beta_n)\|v_n-p\|$, implies that

$$\|v_n-p\|\le\|u_n-p\|. \tag{2.5}$$

In view of Lemma 1.12, one has

$$\begin{aligned}
&\|z_n-p\|\\
&\le\delta_n\|f(x_n)-f(p)\|+\delta_n\|f(p)-p\|\\
&\quad+(1-\delta_n)\|(I-\zeta_n\sum_{i=1}^{\infty}c_iW_i)\sum_{i=1}^{\infty}a_i(I+r_{n,i}A_i)^{-1}(I-r_{n,i}B_i)v_n-p\|\\
&\le\delta_nk\|x_n-p\|+\delta_n\|f(p)-p\|\\
&\quad+(1-\delta_n)\|(I-\zeta_n\sum_{i=1}^{\infty}c_iW_i)\sum_{i=1}^{\infty}a_i(I+r_{n,i}A_i)^{-1}(I-r_{n,i}B_i)(v_n-p)\|\\
&\quad+(1-\delta_n)\|(I-\zeta_n\sum_{i=1}^{\infty}c_iW_i)\sum_{i=1}^{\infty}a_i(I+r_{n,i}A_i)^{-1}(I-r_{n,i}B_i)p-p\|\\
&\le\delta_nk\|x_n-p\|+\delta_n\|f(p)-p\|+(1-\delta_n)[1-\zeta_n(1-\sum_{i=1}^{\infty}c_i\sqrt{\frac{1-\vartheta_i}{\mu_i}})]\|u_n-p\|\\
&\quad+(1-\delta_n)\zeta_n\sum_{i=1}^{\infty}c_i\|W_i\|\|p\|.
\end{aligned} \tag{2.6}$$

Note that

$$\|w_n-p\|\le\alpha_n\|x_n-p\|+\tau_n\|z_n-p\|+\chi_n\|y_n-p\|. \tag{2.7}$$

Fro Lemma 1.9, one has

$$\begin{aligned}
\|x_{n+1}-p\|&\le\lambda_n\|\eta f(x_n)-Fp\|+\|(I-\lambda_nF)(w_n-p)\|+\|e_n\|\\
&\le\lambda_n\eta k\|x_n-p\|+\lambda_n\|\eta f(p)-Fp\|+(1-\lambda_n\xi)\|w_n-p\|+\|e_n\|.
\end{aligned} \tag{2.8}$$

Combing with inequalities (2.4), (2.5), (2.6), (2.7), and (2.8), one has

$$
\begin{aligned}
&\|x_{n+1}-p\| \\
&\leq \{\lambda_n\eta k+(1-\lambda_n\xi)\alpha_n+(1-\lambda_n\xi)(1-\alpha_n)\delta_n k+(1-\lambda_n\xi)(1-\alpha_n)(1-\delta_n)[1 \\
&\quad -\zeta_n(1-\sum_{i=1}^{\infty}c_i\sqrt{\frac{1-\vartheta_i}{\mu_i}})]\omega_n\}\|x_n-p\|+\lambda_n\|\eta f(p)-F(p)\| \\
&\quad +\|e_n\|+(1-\lambda_n\xi)(1-\alpha_n)\delta_n\|f(p)-p\|+(1-\lambda_n\xi)(1-\alpha_n)(1-\delta_n)\zeta_n\sum_{i=1}^{\infty}c_i\|W_i\|\|p\| \\
&\quad +(1-\lambda_n\xi)(1-\alpha_n)(1-\omega_n)(1-\delta_n)[1-\zeta_n(1-\sum_{i=1}^{\infty}c_i\sqrt{\frac{1-\vartheta_i}{\mu_i}})]\|p\| \\
&\quad +(1-\lambda_n\xi)(1-\alpha_n)(1-\delta_n)[1-\zeta_n(1-\sum_{i=1}^{\infty}c_i\sqrt{\frac{1-\vartheta_i}{\mu_i}})]\|\varepsilon_n\| \\
&\quad \leq \{\lambda_n\eta k+(1-\lambda_n\xi)\alpha_n+(1-\lambda_n\xi)(1-\alpha_n)\delta_n k \\
&\quad +(1-\lambda_n\xi)(1-\alpha_n)(1-\delta_n)[1-\zeta_n(1-\sum_{i=1}^{\infty}c_i\sqrt{\frac{1-\vartheta_i}{\mu_i}})]\}\|x_n-p\| \\
&\quad +\lambda_n(\xi-\eta k)\frac{\|\eta f(p)-F(p)\|}{\xi-\eta k}+\|e_n\|+(1-\lambda_n\xi)(1-\alpha_n)\delta_n(1-k)\frac{\|f(p)-p\|}{1-k} \\
&\quad +(1-\lambda_n\xi)(1-\alpha_n)(1-\delta_n)\zeta_n(1-\sum_{i=1}^{\infty}c_i\sqrt{\frac{1-\vartheta_i}{\mu_i}})\frac{\sum_{i=1}^{\infty}c_i\|W_i\|\|p\|}{1-\sum_{i=1}^{\infty}c_i\sqrt{\frac{1-\vartheta_i}{\mu_i}}} \\
&\quad +(1-\omega_n)\|p\|+\|\varepsilon_n\|+\chi_n M \\
&\leq \max\{\|x_n-p\|,\frac{\|\eta f(p)-F(p)\|}{\xi-\eta k},\frac{\|f(p)-p\|}{1-k},\frac{\sum_{i=1}^{\infty}c_i\|W_i\|\|p\|}{1-\sum_{i=1}^{\infty}c_i\sqrt{\frac{1-\vartheta_i}{\mu_i}}}\}+\|e_n\| \\
&\quad +(1-\omega_n)\|p\|+\|\varepsilon_n\|+\chi_n M \\
&\leq \cdots \\
&\leq \max\{\|x_1-p\|,\frac{\|\eta f(p)-F(p)\|}{\xi-\eta k},\frac{\|f(p)-p\|}{1-k},\frac{\sum_{i=1}^{\infty}c_i\|W_i\|\|p\|}{1-\sum_{i=1}^{\infty}c_i\sqrt{\frac{1-\vartheta_i}{\mu_i}}}\} \\
&\quad +\sum_{i=1}^{n}\|e_i\|+\sum_{i=1}^{n}(1-\omega_i)\|p\|+\sum_{i=1}^{n}\|\varepsilon_i\|+M\sum_{i=1}^{n}\chi_i,
\end{aligned}
$$

where $M=\sup\{\|y_n-p\|:n\in N\}<+\infty$. The above estimation implies that $\{x_n\}$ is bounded. Following (2.4), (2.5), (2.6), and (2.7), we easily see that $\{u_n\}$, $\{v_n\}$, $\{z_n\}$, and $\{w_n\}$ are all bounded. Note that, for $p\in\bigcap_{i=1}^{\infty}(A_i+B_i)^{-1}0$,

$$\|\sum_{i=1}^{\infty}a_i(I+r_{n,i}A_i)^{-1}(I-r_{n,i}B_i)v_n-p\|\leq\|v_n-p\|.$$

Then $\{\sum_{i=1}^{\infty} a_i(I+r_{n,i}A_i)^{-1}(I-r_{n,i}B_i)v_n\}$ is bounded. Similarly, $\{\sum_{i=1}^{\infty} a_i(I+r_{n,i}A_i)^{-1}(I-r_{n,i}B_i)z_n\}$ is bounded. Since

$$\begin{aligned}&\|\sum_{i=1}^{\infty} c_iW_i\sum_{i=1}^{\infty} a_i(I+r_{n,i}A_i)^{-1}(I-r_{n,i}B_i)v_n\|\\ &\leq \sum_{i=1}^{\infty} c_i\|W_i\|\sum_{i=1}^{\infty}\|a_i(I+r_{n,i}A_i)^{-1}(I-r_{n,i}B_i)v_n\|,\end{aligned}$$

then $\{\sum_{i=1}^{\infty} c_iW_i\sum_{i=1}^{\infty} a_i(I+r_{n,i}A_i)^{-1}(I-r_{n,i}B_i)v_n\}$ is bounded.

Step 4. There exists $q_0\in\bigcap_{i=1}^{\infty}(A_i+B_i)^{-1}0$, which is a solution to variational inclusion (2.2).

In view of Lemmas 1.3 and 1.8, there exists z_t such that $z_t = t\eta f(z_t)+(I-tF)\sum_{i=1}^{\infty} a_i(I+r_{n,i}A_i)^{-1}(I-r_{n,i}B_i)z_t$ and $z_t\to q_0$, as $t\to 0$, where q_0 is the solution of (2.2).

Step 5. $\limsup_{n\to\infty}\langle \eta f(q_0)-Fq_0, j(x_{n+1}-q_0)\rangle\leq 0$, where q_0 is the same as that in Step 4.

Note that

$$\begin{aligned}\|z_n-v_n\|\leq{}& \delta_n\|f(x_n)\|+\zeta_n(1-\delta_n)\|\sum_{i=1}^{\infty} c_iW_i\sum_{i=1}^{\infty} a_i(I+r_{n,i}A_i)^{-1}(I-r_{n,i}B_i)v_n\|\\ &+\beta_n\|u_n-\sum_{i=1}^{\infty} a_i(I+r_{n,i}A_i)^{-1}(I-r_{n,i}B_i)v_n\|\\ &+\delta_n\|\sum_{i=1}^{\infty} a_i(I+r_{n,i}A_i)^{-1}(I-r_{n,i}B_i)v_n\|,\end{aligned}$$

which implies

$$\begin{aligned}&\|w_n-z_n\|\\ &\leq \alpha_n\|x_n-\sum_{i=1}^{\infty} a_i(I+r_{n,i}A_i)^{-1}(I-r_{n,i}B_i)z_n\|+\chi_n\|y_n-\sum_{i=1}^{\infty} a_i(I+r_{n,i}A_i)^{-1}(I-r_{n,i}B_i)z_n\|\\ &\quad+2\delta_n\|f(x_n)\|+2\beta_n\|\sum_{i=1}^{\infty} a_i(I+r_{n,i}A_i)^{-1}(I-r_{n,i}B_i)v_n-u_n\|\\ &\quad+2\zeta_n(1-\delta_n)\|\sum_{i=1}^{\infty} c_iW_i\sum_{i=1}^{\infty} a_i(I+r_{n,i}A_i)^{-1}(I-r_{n,i}B_i)v_n\|\\ &\quad+2\delta_n\|\sum_{i=1}^{\infty} a_i(I+r_{n,i}A_i)^{-1}(I-r_{n,i}B_i)v_n\|+\beta_n\|\sum_{i=1}^{\infty} a_i(I+r_{n,i}A_i)^{-1}(I-r_{n,i}B_i)v_n-u_n\|.\end{aligned}\tag{2.9}$$

Since $\{x_n\}$, $\{u_n\}$,$\{v_n\}$, $\{y_n\}$, $\{\sum_{i=1}^{\infty} a_i(I+r_{n,i}A_i)^{-1}(I-r_{n,i}B_i)v_n\}$, $\{\sum_{i=1}^{\infty} a_i(I+r_{n,i}A_i)^{-1}(I-r_{n,i}B_i)z_n\}$, and $\{\sum_{i=1}^{\infty} c_iW_i\sum_{i=1}^{\infty} a_i(I+r_{n,i}A_i)^{-1}(I-r_{n,i}B_i)v_n\}$ are all bounded, we find from (2.9) that $w_n-z_n\to 0$, as $n\to\infty$. Therefore,

$$\begin{aligned}&\|\sum_{i=1}^{\infty} a_i(I+r_{n,i}A_i)^{-1}(I-r_{n,i}B_i)w_n-w_n\|\\ &\leq \|w_n-z_n\|+\alpha_n\|x_n-\sum_{i=1}^{\infty} a_i(I+r_{n,i}A_i)^{-1}(I-r_{n,i}B_i)z_n\|\\ &\quad+\chi_n\|y_n-\sum_{i=1}^{\infty} a_i(I+r_{n,i}A_i)^{-1}(I-r_{n,i}B_i)z_n\|\to 0,\ n\to\infty.\end{aligned}\tag{2.10}$$

Let z_t be the same as that in Step 4. Then $\|z_t\| \le \|z_t - q_0\| + \|q_0\|$, which implies that $\{z_t\}$ is bounded. Observe that

$$\begin{aligned}
&\|z_t - w_n\|^2\\
&\le \|z_t - \sum_{i=1}^{\infty} a_i(I + r_{n,i}A_i)^{-1}(I - r_{n,i}B_i)w_n\|^2 + 2\langle \sum_{i=1}^{\infty} a_i(I + r_{n,i}A_i)^{-1}(I - r_{n,i}B_i)w_n - w_n, j(z_t - w_n)\rangle\\
&\le \|z_t - w_n\|^2 + 2\langle t\eta f(z_t) - tF(\sum_{i=1}^{\infty} a_i(I + r_{n,i}A_i)^{-1}(I - r_{n,i}B_i)z_t), j(z_t - \sum_{i=1}^{\infty} a_i(I + r_{n,i}A_i)^{-1}(I\\
&\quad - r_{n,i}B_i)w_n)\rangle + 2\|z_t - w_n\|\|\sum_{i=1}^{\infty} a_i(I + r_{n,i}A_i)^{-1}(I - r_{n,i}B_i)w_n - w_n\|.
\end{aligned}$$

Thus, $t\langle F(\sum_{i=1}^{\infty} a_i(I+r_{n,i}A_i)^{-1}(I-r_{n,i}B_i)z_t) - \eta f(z_t), j(z_t - \sum_{i=1}^{\infty} a_i(I+r_{n,i}A_i)^{-1}(I-r_{n,i}B_i)w_n)\rangle \le \|z_t - w_n\|\|\sum_{i=1}^{\infty} a_i(I+r_{n,i}A_i)^{-1}(I-r_{n,i}B_i)w_n - w_n\|$, which implies from (2.10) that $\lim_{t\to 0} \limsup_{n\to\infty}\langle F(\sum_{i=1}^{\infty} a_i(I+r_{n,i}A_i)^{-1}(I-r_{n,i}B_i)z_t) - \eta f(z_t), j(z_t - \sum_{i=1}^{\infty} a_i(I+r_{n,i}A_i)^{-1}(I-r_{n,i}B_i) w_n)\rangle \le 0$. Since $z_t \to q_0$ as $t \to 0$, then $\limsup_{n\to\infty}\langle Fq_0 - \eta f(q_0), j(q_0 - \sum_{i=1}^{\infty} a_i(I+r_{n,i}A_i)^{-1}(I - r_{n,i}B_i)w_n)\rangle \le 0$. Since $e_n \to 0$, $\lambda_n \to 0$ as $n \to \infty$, and $\{x_n\}$ and $\{w_n\}$ are bounded, then $x_{n+1} - w_n = \lambda_n(\eta f(x_n) - Fw_n) + e_n \to 0$, as $n \to \infty$. Based on the facts that $\sum_{i=1}^{\infty} a_i(I + r_{n,i}A_i)^{-1}(I - r_{n,i}B_i)w_n - w_n \to 0$ and $x_{n+1} - w_n \to 0$, one has $\limsup_{n\to\infty}\langle Fq_0 - \eta f(q_0), j(q_0 - x_{n+1})\rangle \le 0$.

Step 6. $x_n \to q_0$ as $n \to \infty$, where q_0 is the same as that in Steps 4 and 5.

Observe $\|u_n - q_0\|^2 \le \omega_n\|x_n - q_0\|^2 + 2\|\varepsilon_n\|\|u_n - q_0\| + 2(1-\omega_n)\|q_0\|\|u_n - q_0\|$. $\|v_n - q_0\|^2 \le \beta_n\|u_n - q_0\|^2 + (1-\beta_n)\|v_n - q_0\|^2$ ensures that $\|v_n - q_0\|^2 \le \|u_n - q_0\|^2$. In view of Lemmas 1.1, one has

$$\begin{aligned}
\|z_n - q_0\|^2 &\le (1-\delta_n)\|v_n - q_0\|^2 + 2\delta_n\langle f(x_n) - f(q_0), j(z_n - q_0)\rangle + 2\delta_n\langle f(q_0) - q_0, j(z_n - q_0)\rangle\\
&\quad + 2(1-\delta_n)\zeta_n\|\sum_{i=1}^{\infty} c_i W_i \sum_{i=1}^{\infty} a_i(I + r_{n,i}A_i)^{-1}(I - r_{n,i}B_i)v_n\|\|z_n - q_0\|\\
&\le (1-\delta_n)\|v_n - q_0\|^2 + 2\delta_n k\|z_n - x_n\|\|x_n - q_0\| + 2\delta_n k\|x_n - q_0\|^2 + 2\delta_n\langle f(q_0) - q_0,\\
&\quad j(z_n - q_0)\rangle + 2\zeta_n\|\sum_{i=1}^{\infty} c_i W_i \sum_{i=1}^{\infty} a_i(I + r_{n,i}A_i)^{-1}(I - r_{n,i}B_i)v_n\|\|z_n - q_0\|.
\end{aligned}$$

Note that $\|w_n - q_0\|^2 \le \alpha_n\|x_n - q_0\|^2 + \tau_n\|z_n - q_0\|^2 + \chi_n\|y_n - q_0\|^2$. Now, in view of Lemma 1.9 and the inequalities above, one has

$$\begin{aligned}
&\|x_{n+1} - q_0\|^2\\
&\le (1-\lambda_n\xi)\|w_n - q_0\|^2 + 2\|e_n\|\|x_{n+1} - q_0\| + 2\lambda_n\eta k\|x_n - q_0\|\|x_{n+1} - q_0\|\\
&\quad + 2\lambda_n\langle \eta f(q_0) - F(q_0), j(x_{n+1} - q_0)\rangle\\
&\le \{(1-\lambda_n\xi)\alpha_n + \lambda_n\eta k + 2(1-\lambda_n\xi)(1-\alpha_n)\delta_n k + (1-\lambda_n\xi)(1-\alpha_n)(1-\delta_n)\omega_n\}\|x_n - q_0\|^2\\
&\quad + 2\|e_n\|\|x_{n+1} - q_0\| + 2(1-\lambda_n\xi)(1-\alpha_n)(1-\delta_n)\|\varepsilon_n\|\|u_n - q_0\| + 2(1-\lambda_n\xi)(1-\alpha_n)(1-\delta_n)\\
&\quad (1-\omega_n)\|q_0\|\|u_n - q_0\| + 2\delta_n k(1-\lambda_n\xi)(1-\alpha_n)\|x_n - q_0\|\|x_n - z_n\| + (1-\lambda_n\xi)\alpha_n\chi_n\|y_n - q_0\|^2\\
&\quad + 2(1-\lambda_n\xi)(1-\alpha_n)\delta_n\|z_n - q_0\|\|f(q_0) - q_0\| + 2(1-\lambda_n\xi)(1-\alpha_n)\zeta_n\|z_n - q_0\|\|\sum_{i=1}^{\infty} c_i W_i \sum_{i=1}^{\infty}\\
&\quad a_i(I + r_{n,i}A_i)^{-1}(I - r_{n,i}B_i)v_n\| + \lambda_n\eta k\|x_{n+1} - q_0\|^2 + 2\lambda_n\langle \eta f(q_0) - F(q_0), j(x_{n+1} - q_0)\rangle.
\end{aligned}$$

Let $M' = \sup_n\{2\|x_{n+1}-q_0\|, 2\|u_n-q_0\|, 2k\|x_n-q_0\|\|x_n-z_n\|, 2\|u_n-q_0\|\|q_0\|, 2\|f(q_0)-q_0\|\|z_n-q_0\|, 2\|z_n-q_0\|\|\sum_{i=1}^{\infty} c_iW_i \sum_{i=1}^{\infty} a_i(I+r_{n,i}A_i)^{-1}(I-r_{n,i}B_i)v_n\|, \|y_n-q_0\|^2 : n \in N\}$. From Step 3, one has $M' < +\infty$. Therefore, it follows that

$$\begin{aligned}&(1-\lambda_n\eta k)\|x_{n+1}-q_0\|^2\\&\le \{(1-\lambda_n\xi)\alpha_n+\lambda_n\eta k+2(1-\lambda_n\xi)(1-\alpha_n)\delta_n k+(1-\lambda_n\xi)(1-\alpha_n)(1-\delta_n)\}\|x_n-q_0\|^2\\&\quad+[\|e_n\|+\|\varepsilon_n\|+(1-\omega_n)+2\delta_n+\zeta_n+\alpha_n\chi_n]M'+2\lambda_n\langle\eta f(q_0)-F(q_0), j(x_{n+1}-q_0)\rangle.\end{aligned}$$

Set $b_n^{(1)} = \frac{\lambda_n(\xi-2\eta k)}{1-\lambda_n\eta k}$, $b_n^{(2)} = \frac{M'}{1-\lambda_n\eta k}[\|e_n\|+\|\varepsilon_n\|+(1-\omega_n)+2\delta_n+\zeta_n+\alpha_n\chi_n]+\frac{2\lambda_n}{1-\lambda_n\eta k}\langle\eta f(q_0)-Fq_0, j(x_{n+1}-q_0)\rangle$. Then $\|x_{n+1}-q_0\|^2 \le (1-b_n^{(1)})\|x_n-q_0\|^2+b_n^{(2)}$. Based on assumptions (v), (vii), (viii), and Step 5, we see that $b_n^{(1)} \to 0$, as $n\to\infty$, $\sum_{n=1}^{\infty} b_n^{(1)} = +\infty$, and $\limsup_{n\to\infty}\frac{b_n^{(2)}}{b_n^{(1)}} \le 0$. It follows from Lemma 1.10 that $x_n \to q_0$, as $n\to\infty$.

Step 7. There exists $p_0 \in \bigcap_{i=1}^{\infty}(A_i+B_i)^{-1}0$, which is the solution of the variational inequality

$$\langle p_0-f(p_0), j(p_0-y)\rangle \le 0, \quad \forall y \in \cap_{i=1}^{\infty}(A_i+B_i)^{-1}0. \tag{2.11}$$

In fact, it follows from Lemma 1.11 that there exists u_t^n such that

$$u_t^n = tf(u_t^n)+(1-t)(I-k_t\sum_{i=1}^{\infty}c_iW_i)\sum_{i=1}^{\infty}a_i(I+r_{n,i}A_i)^{-1}(I-r_{n,i}B_i)u_t^n$$

and $u_t^n \to p_0$, as $t\to 0$, where p_0 is the solution of (2.11).

Step 8. $x_n \to p_0$, as $n\to\infty$, where p_0 is the same as that in Step 7.

It suffices to show that $p_0 = q_0$. Since $p_0 \in \cap_{i=1}^{\infty}(A_i+B_i)^{-1}0$, then (2.2) implies that $\langle Fq_0-\eta f(q_0), j(q_0-p_0)\rangle \le 0$. Since F is strongly positive linear bounded, f is a contraction, and $0<\eta<\frac{\xi}{2k}$, then

$$\begin{aligned}&\langle(Fq_0-\eta f(q_0))-(Fp_0-\eta f(p_0)), j(q_0-p_0)\rangle\\&=\langle F(q_0-p_0), j(q_0-p_0)\rangle+\eta\langle f(p_0)-f(q_0), j(q_0-p_0)\rangle\\&\ge \xi\|q_0-p_0\|^2-\eta k\|q_0-p_0\|^2 \ge 0.\end{aligned}$$

Therefore,

$$\langle Fp_0-\eta f(p_0), j(q_0-p_0)\rangle \le \langle Fq_0-\eta f(q_0), j(q_0-p_0)\rangle \le 0. \tag{2.12}$$

On the other hand, it follows from (2.11) and Lemma 1.1 that

$$\langle f(p_0)-p_0, j(q_0-p_0)\rangle \le 0, \tag{2.13}$$

which together with (2.12) yields $\langle Fq_0-\eta f(q_0)+f(p_0)-p_0, j(q_0-p_0)\rangle \le 0$. Following the condition imposed on F and f, we know that $\langle Fq_0-\eta f(q_0)+f(p_0)-p_0, j(q_0-p_0)\rangle = 0$. In view of (2.12) and (2.13), one has $\langle Fq_0-\eta f(q_0), j(q_0-p_0)\rangle = \langle f(p_0)-p_0, j(q_0-p_0)\rangle = 0$. Since $0 \in \bigcap_{i=1}^{\infty}(A_i+B_i)^{-1}0$, then $p_0 = 0$ or $p_0 = q_0$. If $p_0 = q_0$, then the result follows. If $p_0 = 0$, then $\langle Fq_0-\eta f(q_0), j(q_0-p_0)\rangle = 0$, which implies that $\langle Fq_0-\eta f(q_0), j(q_0)\rangle = 0$. Therefore, $\xi\|q_0\|^2 \le \langle Fq_0, j(q_0)\rangle = \eta\langle f(q_0), j(q_0)\rangle \le \eta k\|q_0\|^2$. Since $\xi > 2\eta k$, then $q_0 = 0$, which means that $p_0 = q_0 = 0$. Therefore, $x_n \to p_0 = q_0$, as $n\to\infty$. This completes the proof. □

Remark 2.2. If E reduces to a Hilbert space H, then K_{n+1} in Theorem 2.1 can be simplified as

$K_{n+1} = \{p \in K_n : 2\langle\alpha_n x_n+\chi_n y_n+\tau_n z_n-w_n, p\rangle \le \alpha_n\|x_n\|^2+\tau_n\|z_n\|^2+\chi_n\|y_n\|^2-\|w_n\|^2\}$, $n \in N$, which implies that K_{n+1} is a closed and convex subset H. Then $y_{n+1} = R_{K_{n+1}}(x_1) = P_{K_{n+1}}(x_1)$, for $n \in N$, where $P_{K_{n+1}}(x_1)$ means the metric projection of H onto K_{n+1}.

The following corollary is easy to derive.

Corollary 2.3. *Let H be a Hilbert space. Let $\{x_n\}$ be generated by the following iterative algorithm:*

$$\begin{cases} x_1, y_1, e_1, \varepsilon_1 \in H, \ K_1 = H, \\ u_n = \omega_n x_n + \varepsilon_n, \\ v_n = \beta_n u_n + (1-\beta_n)\sum_{i=1}^{\infty} a_i (I + r_{n,i}A_i)^{-1}(I - r_{n,i}B_i)v_n, \\ z_n = \delta_n f(x_n) + (1-\delta_n)(I - \zeta_n \sum_{i=1}^{\infty} c_i W_i)\sum_{i=1}^{\infty} a_i (I + r_{n,i}A_i)^{-1}(I - r_{n,i}B_i)v_n, \\ w_n = \alpha_n x_n + \tau_n \sum_{i=1}^{\infty} a_i (I + r_{n,i}A_i)^{-1}(I - r_{n,i}B_i)z_n + \chi_n y_n, \\ x_{n+1} = \lambda_n \eta f(x_n) + (I - \lambda_n F)w_n + e_n, \\ K_{n+1} = \{p \in K_n : 2\langle \alpha_n x_n + \chi_n y_n + \tau_n z_n - w_n, p\rangle \\ \le \alpha_n \|x_n\|^2 + \tau_n \|z_n\|^2 + \chi_n \|y_n\|^2 - \|w_n\|^2\}, \\ y_{n+1} = P_{K_{n+1}}(x_1), \ n \in N. \end{cases} \tag{2.14}$$

Then, under the assumptions of Theorem 2.1, $y_n \to q_0 \in \bigcap_{i=1}^{\infty}(A_i + B_i)^{-1}0$, as $n \to \infty$, where q_0 satisfies the following variational inequalities:

$$\langle Fq_0 - \eta f(q_0), q_0 - y\rangle \le 0, \quad \forall y \in \cap_{i=1}^{\infty}(A_i + B_i)^{-1}0, \tag{2.15}$$

and

$$\langle q_0 - f(q_0), q_0 - y\rangle \le 0, \quad \forall y \in \cap_{i=1}^{\infty}(A_i + B_i)^{-1}0. \tag{2.16}$$

Remark 2.4. Compare (2.1) (or (2.14)) with (1.2), one finds that iterative sequence y_n converges strongly not only to the solution of the inclusion problem but also to the solution of two generalized variational inequalities (2.2) (or (2.15)) and (2.3) (or (2.16)). This connects the study on the topics of iterative construction of zero points of the sum of accretive-type mappings and the iterative construction of the solutions of variational inequalities.

Remark 2.5. Comparing (2.1) with [22, (11)], one see that the corresponding work in [22] is extended from a Hilbert space H to a real uniformly convex and q-uniformly smooth Banach space in which the choice of y_{n+1} as the sunny generalized non-expansive retraction $R_{K_{n+1}}(x_1)$ plays an important role.

Remark 2.6. In (2.1) or (2.14), the superposition perturbation $\sum_{i=1}^{\infty} c_i W_i$ is considered in the construction of z_n, and a forward-backward splitting method is involved in the construction of v_n, z_n, and w_n, and a series of decreasing sets K_n, which is defined by employing the convexity of $\|\cdot\|^2$ and the relationship among iterative elements w_n, z_n, x_n, and y_n.

3. The Application to Capillarity Systems

The capillarity equation is a kind of important equation in capillarity phenomenon. It was studied as an example of m-d-accretive mappings, which were studied in [22] in a Hilbert space. In this section, we examine the following capillarity systems (see [22]) again and study in a different space

$$\begin{cases} -div[(1+\frac{|\nabla u^{(i)}|^{p_i}}{\sqrt{1+|\nabla u^{(i)}|^{2p_i}}})|\nabla u^{(i)}|^{p_i-2}\nabla u^{(i)}] \\ +\lambda_i(|u^{(i)}|^{q_i-2}u^{(i)}+|u^{(i)}|^{r_i-2}u^{(i)})+u^{(i)}(x)=f_i(x), \quad x\in\Omega \\ -<\nu,(1+\frac{|\nabla u^{(i)}|^{p_i}}{\sqrt{1+|\nabla u^{(i)}|^{2p_i}}})|\nabla u^{(i)}|^{p_i-2}\nabla u^{(i)}>=0, \quad x\in\Gamma,\ i\in N. \end{cases} \tag{3.1}$$

The discussion of (3.1) is under the following assumptions:

(1) Ω is a bounded conical domain in R^n ($n\in N$) with its boundary $\Gamma\in C^1$;
(2) ν is the exterior normal derivative of Γ;
(3) λ_i is a positive number, for $i\in N$;
(4) $p_i\in(\frac{2n}{n+1},+\infty)$, for $i\in N$. Moreover, if $p_i\geq n$, then $1\leq q_i,r_i<+\infty$, for $i\in N$. If $p_i<n$, then $1\leq q_i,r_i\leq\frac{np_i}{n-p_i}$, for $i\in N$;
(5) $|\cdot|$ denotes the norm in R^N, and $<\cdot,\cdot>$ denotes the inner-product.

Definition 3.1. [24] Define the mapping $B_i:W^{1,p_i}(\Omega)\to(W^{1,p_i}(\Omega))^*$ by $\langle v,B_iu\rangle=\int_\Omega<(1+\frac{|\nabla u|^{p_i}}{\sqrt{1+|\nabla u|^{2p_i}}})|\nabla u|^{p_i-2}\nabla u,\nabla v>dx+\lambda_i\int_\Omega|u(x)|^{q_i-2}u(x)v(x)dx+\lambda_i\int_\Omega|u(x)|^{r_i-2}u(x)v(x)dx$, for any $u,v\in W^{1,p_i}(\Omega)$, $i\in N$.

Definition 3.2. [24] For each $i\in N$, define the mapping $A_i:L^{p_i}(\Omega)\to 2^{L^{p_i}(\Omega)}$ by $D(A_i)=\{u\in L^{p_i}(\Omega)|\ \exists\ f\in L^{p_i}(\Omega)\ such\ that\ f\in B_iu\}$, $A_iu=\{f\in L^{p_i}(\Omega)\mid f\in B_iu\}$.

Lemma 3.3. [24] *For each $i\in N$, the mapping $A_i:L^{p_i}(\Omega)\to 2^{L^{p_i}(\Omega)}$ is m-accretive.*

Lemma 3.4. [25] *For each $i\in N$, define the mapping $S_i:L^{p_i}(\Omega)\to 2^{L^{p_i}(\Omega)}$ by $(S_iu)(x)=u(x)-f_i(x)$, for all $u(x)\in D(S_i)$, then S_i is θ_i-inversely-strongly accretive for $\theta_i\in(0,1]$ and $i\in N$.*

Lemma 3.5. [21, 22] *If, in (3.1), $f_i(x)\equiv\lambda_i(|k|^{q_i-1}+|k|^{r_i-1})sgnk+k$, where k is a constant, then $\{u^{(i)}(x)\equiv k:i\in N\}$ is the solution of capillarity system (3.1), and $\{k\}=\bigcap_{i=1}^{\infty}(A_i+B_i)^{-1}0$.*

Theorem 3.6. [9] *Let $f_i(x)\equiv\lambda_i(|k|^{q_i-1}+|k|^{r_i-1})sgnk+k$. Let A_i and B_i be defined in Definitions 3.1 and 3.2. Let $F:L^{p_i}(\Omega)\to L^{p_i}(\Omega)$ be a strongly positive linear bounded operator with coefficient $\xi>0$. Let $f:L^{p_i}(\Omega)\to L^{p_i}(\Omega)$ be a contraction with coefficient $k\in(0,1)$. Let $W_i:L^{p_i}(\Omega)\to L^{p_i}(\Omega)$ be ϑ_i-strongly accretive and μ_i-strictly pseudo-contractive mapping for $i\in N$. Let $\{y_n\}$ be generated as follows:*

$$\begin{cases} x_1,y_1,e_1,\varepsilon_1\in L^{p_i}(\Omega)\ chosen\ arbitrarily,\ K_1=L^{p_i}(\Omega), \\ u_n=\omega_nx_n+\varepsilon_n, \\ v_n=\beta_nu_n+(1-\beta_n)\sum_{i=1}^{\infty}a_i(I+r_{n,i}A_i)^{-1}(I-r_{n,i}B_i)v_n, \\ z_n=\delta_nf(x_n)+(1-\delta_n)(I-\zeta_n\sum_{i=1}^{\infty}c_iW_i)\sum_{i=1}^{\infty}a_i(I+r_{n,i}A_i)^{-1}(I-r_{n,i}B_i)v_n, \\ w_n=\alpha_nx_n+\tau_n\sum_{i=1}^{\infty}a_i(I+r_{n,i}A_i)^{-1}(I-r_{n,i}B_i)z_n+\chi_ny_n, \\ x_{n+1}=\lambda_n\eta f(x_n)+(I-\lambda_nF)w_n+e_n, \\ K_{n+1}=\{p\in K_n:\|w_n-p\|^2\leq\alpha_n\|x_n-p\|^2+\tau_n\|z_n-p\|^2+\chi_n\|y_n-p\|^2\} \\ y_{n+1}=R_{K_{n+1}},\ n\in N,i\in N. \end{cases}$$

Then, under the assumptions of Theorem 2.1, $y_n \to q_0(x) \in \bigcap_{i=1}^{\infty}(A_i + B_i)^{-1}0$, where $q_0(x)$ is not only the solution of capillarity system (3.1) but also a solution of variational inequalities (2.2) and (2.3).

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